Towards an Information-Processing Theory of Loss Aversion

J. Miguel Villas-Boas
(University of California, Berkeley)

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Abstract

This paper considers a model where a risk-neutral individual can receive both a signal about whether an outcome is above a certain threshold (a reference point) and a continuous signal on the value of the outcome. The paper shows that, given the existence of these two signals for an outcome, the expected value function of the outcome exhibits diminishing sensitivities both above and below the reference point. Furthermore, in the examples considered, loss aversion occurs if the reference point is not too high. The paper shows how the informativeness of each signal affects the declining sensitivities and loss aversion effects, and how the model reduces to risk-neutral decision-making when the continuous signal on the value of the outcome is perfectly informative. The loss aversion effects occur for low reference points because the reference point is below the expected value of the outcome and because of the greater likelihood of receiving the signal that the outcome is above the reference point. The paper obtains the same result in a rational inattention framework because the individual may pay greater attention to the less likely low outcomes.
1. Introduction

There has been a large literature since Kahneman and Tversky (1979) documenting the existence of reference point effects, with losses affecting decision-makers more than gains, an effect known as loss aversion. This is documented in the context of lotteries, riskless choice (e.g., Thaler 1980, Tversky and Kahneman 1991, Koszegi and Rabin 2006), with both experiments and field evidence (e.g., Camerer 2000). See O’Donoghue and Sprenger (2018) for a recent survey of the evidence and variations on this general effect.

A general set-up of the theory, “prospect theory,” involves both diminishing sensitivities and loss aversion, where the utility with respect to some outcome \( x \), \( V(x) \) is normalized to zero at the reference point (with the outcomes also normalized, such that the reference point is set at zero), satisfying \( V''(x) < 0, V(x) < -V(-x) \), and \( V'(x) < V'(-x) \) for \( x > 0 \), and \( V''(x) > 0 \) for \( x < 0 \). The conditions \( V''(x) < 0 \) for \( x > 0 \) and \( V'' > 0 \) for \( x < 0 \) captures the existence of diminishing sensitivities. The condition \( V(x) < -V(-x) \) for \( x > 0 \) captures the existence of loss aversion. As noted in O’Donoghue and Sprenger (2018) most of the literature has focused on the loss aversion effects, using a two-part linear formulation, with \( V(x) = x \) for \( x > 0 \), and \( V(x) = \lambda x \) for \( x < 0 \), with \( \lambda > 1 \).

This paper takes the perspective that information processing on the value of an outcome is imperfect, and that individuals receive information (potentially also imperfect) about whether the outcome is above or below a certain threshold, which can be interpreted as a reference point. This later component captures the idea that individuals may find it easier to process information in terms of discrete categories than to process information about a continuous variable, and that information just above or below a threshold makes it easier to process information by simplifying it to only two categories. As further justification, the natural threshold is zero in some cases, and in those cases the payoff can be directly represented as a loss or a gain, which may be easier to process than the exact amount of the loss or the gain. The individual is assumed to be risk neutral, to focus on the effects of information processing, and to receive a noisy continuous signal about the value of the outcome, which we denote by \( s \), and a noisy binary signal as to whether the outcome is above or below the threshold, which we denote by \( S \). Given those signals, the individual updates her beliefs about the value of the outcome, and obtains an expected value of the outcome.

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1 Some research also considers the market equilibrium in settings in which consumers exhibit loss aversion (e.g., Orhun 2009, Amaldoss and He 2018).

2 Note that the normalization of \( x \) to zero at the reference point does not necessarily mean that, without normalization, the reference point is at zero.
Going one step further, given a value of the outcome and the data generating process for the signals, one can compute the expected value over the possible signals of the possible expectation of the value (given the signals) that the individual may form. Given the noisy signal about the value of the outcome, this expected value over the possible signals is less steep than one when the outcome is not close to the reference point, leading to the diminishing sensitivities effect. This is because the signal on the outcome is noisy, and the updated expected value of the outcome given the signal becomes less sensitive as the noisiness of the signal increases. Note also that the signal about whether the outcome is above or below the threshold makes the expected value over the possible signals steeper close to the threshold (which is an implication of prospect theory). Furthermore, under some conditions, we can obtain loss aversion effects, such that the expected value over signals increases less with the outcome for outcomes above the threshold than for outcomes below the threshold. In the examples considered, loss aversion effects can occur for low reference points because the expected value of the outcome is above the reference point, because of the greater likelihood of receiving a signal that the outcome is above the reference point, and because the individual may pay greater attention to understanding when the (less likely) low outcomes occur (in a rational inattention framework with binary actions). These results of loss aversion for low references points are consistent with the experimental evidence presented in Walasek and Stewart (2015), who show that loss aversion occurs when the space of gains is greater than the space of losses, and that gain premium (the opposite of loss aversion) occurs when the space of losses is greater than the space of gains.

These signals of information on an outcome capture the individual translating this numerical information into an imprecise internal representation (e.g., Bhui and Gershman 2018, Johnson, Häubl, and Keinan 2007). When making the decision the individual has only access to the noisy mental representation but may be somewhat aware of the coding mechanism, which can then be used to generate a posterior. This also means that the decision process is inherently stochastic, and therefore the individual has to take an expected value given the signals received (that is, given the noisy coding).

Some benchmarks are interesting to study in this set-up. First, if the continuous signal on the value of the outcome is fully informative, the model above reduces to expected utility theory. Second, if the continuous signal on the value of the outcome does not have any

Note that if the continuous signal has some noise, then the binary signal can provide some additional information.
information, the entire effect comes from the binary signal about whether the outcome is above or below the threshold. In that case, we can obtain diminishing sensitivities and loss aversion effects from the probability function of receiving a signal that the outcome is above or below the reference point. Third, if the signal about whether the outcome is above or below the threshold is fully informative, the expected value over the signals is a vertical graph at the reference point, which is the extreme case of the value function being steeper close to the reference point.

To give further intuition on the effect presented here related to loss aversion, consider an example in which an individual is given the option to accept or reject a gamble that pays either $15 or -$10 with equal probability, and may decide to reject such a gamble. The individual may receive signals on the value of each of these two outcomes and the value of not taking the gamble for that decision problem, and could come to the conclusion that -$10 is worse as it relates to $0 than $15 is good (with a reference point of $0). In particular, an individual may receive a binary signal and a continuous signal on the outcome $15, a binary signal and a continuous signal on the outcome -$10, and a binary signal and a continuous signal on the outcome $0, such that the encoded values of $15, -$10, and $0 for that decision problem satisfy $\frac{1}{2}v(S_{15}, s_{15}) + \frac{1}{2}v(S_{-10}, s_{-10}) < v(S_0, s_0)$, where $v(S_x, s_x)$ is the posterior expected value of the outcome (encoded value), after the signals received by the individual on outcome $x$. And then it may be that more individuals choose not to take the gamble than choose to take the gamble. This may occur because the individuals believe that payoffs below the reference point are less likely to occur, or the payoffs below the reference are closer to the reference point than the payoffs above the reference point. In the model considered here, this effect is obtained when the individuals have a lower reference point.

Another question is the extent to which the evaluation of the two outcomes is independent. In fact, an individual may have a good sense that the number 15 is 50% greater than the number 10, but when evaluating the benefit of the gain of $15, may use her perception of what a gain could be, and, when evaluating the effects of a loss, may use her perception of what a loss may be.

In some sense, the interpretation presented here can be seen as almost semantic, as when evaluating the “loss of $10” the individual gains information from both the word “loss” and the number “10.” The final evaluation depends on how much weight to put on each piece of information. Prospect theory gives a specific effect to the word “loss,” and this paper can be seen as providing an explanation about why the word “loss” has that effect.

Another example could be a consumer trying to evaluate the price of a product. The
consumer can receive a continuous signal (internal encoding) about the price and a binary signal about whether the price is “high” or “low” with respect to some reference point. When making a decision whether to purchase the product the consumer forms an expectation of what the price could be evaluated at given the noisy continuous signal of the price, and given the binary signal received about whether price is “high” or “low.”

The paper focuses on the issue of whether given conditions on the assumed particular signal structure, the individual can behave as if her value function over outcomes is consistent with diminishing sensitivities and loss aversion. That is, the signal structure is taken as exogenous throughout the paper, except in the section that considers a rational inattention framework in a binary action problem. One potential interesting question is how the signal structure can be chosen endogenously by the individual depending on the particular decision problem being faced, but that question is beyond the scope of this paper. The signal structure considered is composed of a binary signal about how the outcome relates to the reference point and of a continuous signal on the value of the outcome. By varying the degree of noisiness in each of these signals, the structure considered includes, as particular cases, the situation in which there is only the binary signal, or the situation in which there is no binary signal. That is, the signal structure allows for a general combination of the types of signals that may be common when modeling information processing.

In addition to formalizing an information processing signal structure of the problem, the paper can be seen as presenting three main results. First, under general conditions of the signal structure, the value function exhibits diminishing sensitivities. Second, in a rational inattention framework with binary actions, and with a general prior distribution over the outcome, the value function exhibits loss aversion if the reference point is sufficiently low. In this setting, there is only a binary signal about the relationship of the outcome with the reference point, but there is no continuous signal on the exact value of the outcome. Third, with a uniform prior distribution and particular assumptions about the signal structure, including a signal on the exact value of the outcome, the value function again exhibits loss aversion if the reference point is sufficiently low.

The existence of the reference point allows for information-processing in categories, in the extreme case of just two categories. One could think of existing further signals about finer categories, which may also affect the value function for individuals. The point presented here is that with just two categories and a continuous signal on the value of the outcome, one is able to obtain effects that exhibit diminishing sensitivities and loss aversion. In other words, the psychology of loss aversion forces a categorization, which, by definition, partitions
a large and complex set of objects (in this case, payoff realizations on the real line) into
simpler classifications (e.g., gain or loss). Categorization helps information processing by
making a coarse grouping but at the loss of distinctions within the group. The paper shows
that this categorization, modeled as information processing, induces coarse inferences by the
individual, which can generate loss aversion effects under some conditions (lower values of
the reference point).

The paper focuses on the analysis of information processing to generate a value function
of the outcomes, not formalizing a decision problem. But the information processing leads
to a value function which can exhibit declining sensitivities and loss aversion, with direct
effects on decision problems, which are discussed.

There has been recent evidence that losses may, in some situations, have a lower impact
than gains of the same size (e.g., Walasek and Stewart 2014, Gal and Rucker 2018). In
relation to that literature, this paper illustrates that the reference point may affect whether
or not losses have a greater impact than gains (see also Martin, Reimann, and Norton 2016,
Higgins and Liberman 2018, Wardley and Alberhasky 2021). Considering the signal above or
below a threshold as optimal for a decision-maker with limited attention in a binary action
problem, Woodford (2012) presents diminishing sensitivity effects in the value function. In
relation to that paper, this paper also considers the possibility of a signal on the value of
the outcome, and investigates loss aversion effects. See also Khaw, Li, and Woodford (2020)
on the possibility of errors in cognition leading to small-stakes risk aversion. There is also a
literature based on the individual having a discrete number of representations of the outcomes
which leads to steeper preferences of the individual in regions of the outcomes which are more
frequent (e.g., Robson 2001, Netzer 2009, Bhui and Gershman 2018). These preferences
can also be seen as ranking-based preferences, also known as “decision by sampling” (e.g.,
Friedman 1989, Stewart, Chater, and Brown 2006), and explain diminishing sensitivities if
the distribution of outcomes is bell-shaped, and loss aversion effects if the distribution of
outcomes is positively skewed. We discuss the relationship of the results presented here
to that work in Section 6. This paper is also related to work that accounts for errors in
perception of problems, and studies the existence of optimal biases in analysis to better
counteract those errors in perception (e.g., Steiner and Stewart 2016, Gossner and Steiner
2018), and can be seen in the spirit of offering additional foundations for the loss aversion
affects (e.g., Fudenberg 2006). Finally, another related literature is the one using contextual

An exception is subsection 3.2, which considers a rational inattention framework with a binary decision
problem.
deliberation and search for information to explain behavioral phenomena (e.g., Kuksov and Villas-Boas 2010, Guo 2016, 2022).\footnote{There is also a literature on endogenizing context-dependent preferences given consumer’s limited attention (e.g., Zhu and Dukes 2017).}

Note that “prospect theory” includes a (possibly non-linear) probability weighting function for probabilities of random events, and this possibility is not considered here, with the focus exclusively on loss aversion. Without a probability weighting function the results here do not explain some of the results on asymmetric responses to positive and negative prospects (Kahneman and Tversky, 1979) that depend on the existence of a non-linear probability weighting function. Note that similar ideas presented here on the information processing of probabilities may potentially yield a non-linear probability weighting function, but this issue is left for future research.

The remainder of the paper is organized as follows. The next section presents a general model, and Section 3 considers the benchmark in which the continuous signal about the value of the outcome provides no information (completely noisy continuous signal). Section 4 considers the benchmark in which the signal about whether the outcome is above or below the threshold is fully informative, and Section 5 considers a full model where both signals are partially informative. Section 6 presents some additional discussion of the results presented, and Section 7 concludes.

## 2. A General Model

Let $x$ be an outcome that the individual cares about. For example, this could be the amount received in a lottery, or the benefit of having a certain object. Let $H(x)$ and $h(x)$ be the cumulative and density distribution function of $x$, respectively, prior to the consumer receiving any signals, and with $x$ having support $[x, \bar{x}]$. The individual is assumed to be risk-neutral with respect to $x$ in her true utility, $U(x) = x$. This “true utility” is the utility in the case in which the individual has full information about $x$, by which we also mean that the individual has fully processed what $x$ represents. Risk-neutrality on $x$ allows us to focus on the curvature of the value to the individual of $x$ from the point of a view of a third party (e.g., an experimenter) who knows $x$, $V(x)$. The curvature on $V(x)$ will come from the imperfect information processing of $x$, generated by not fully informative signals.\footnote{Consider also that the third party knows both the prior probability distribution of $x$, and the conditional distribution of signals received by the individual on $x$, which are described below. These are needed to compute $V(x)$, as described below.} Let $X$ represent an outcome random variable, and $x$ be the realization of that random variable, which is
never known by the individual unless the individual has complete information (perfectly informative signals on that outcome).

Consider now the signal structure. As mentioned above, there is a binary signal about whether the outcome is above or below the reference point, and a continuous signal about the value of the outcome $x$. These signals and how they are used by the individual to infer the value of $x$ can be interpreted as the information processing by the individual in order to assign a value to $x$. Note that the reference point is considered as an exogenous parameter in the model, as it is noted in the literature that it can be manipulated by framing (see, for example, Tversky and Kahneman 1991, for a discussion). That is, the individual is assumed not to be able to fully design the information structure. We still discuss below what may occur if the individual can design the information structure (and endogenously choose the reference point).

Regarding the signal of whether the outcome is above or below the reference, let $r$ be the reference point of the individual, and denote by $S_1$ the case in which the individual receives a signal that the outcome $x$ is above or equal to the reference point, and by $S_0$ the case in which the individual receives a signal that the outcome $x$ is below or equal to the reference point.

For the signal of whether the outcome is above or below the reference point let

$$\Pr(S_1|x) = f(x, r)$$

where $f(x, r) \in [0, 1]$, increases in $x$, and decreases in $r$. By complementarity, we then have $\Pr(S_0|x) = 1 - f(x, r)$. Note that, given monotonicity on $x$ and given that $f(x, r)$ is bounded, we have that

$$\lim_{x \to +\infty} \frac{\partial f(x, r)}{\partial x} = \lim_{x \to -\infty} \frac{\partial f(x, r)}{\partial x} = 0,$$

and therefore $f(x, r)$ is S-shaped on $x$ (or has more than two inflection points). Note that this binary signal, $S_1$ or $S_0$, can be seen as just indicative of the relationship between the outcome $x$ and the reference point $r$, with the probability of either signal being monotonic in $x$ and with the opposite monotonicity in $r$, and it does not necessarily have to do with whether the outcome $x$ is above or below the reference point. Let us also assume that $\Pr(S_1|x = r) = Pr(X \geq r)$ which holds in a rational inattention framework, as shown in subsection 3.2 below.

7The Appendix presents a simple model illustrating the forces at work without formalizing the Bayesian updating given the signal structure.

8Note that this definition of the binary signal allows for both $S_1$ and $S_0$ to signal that the outcome is equal to the reference point. In the case of the perfectly informative binary signal one then has to define the probability of obtaining each of the signals if the outcome is equal to the reference point.

9In what follows in this section, loss aversion is obtained for low reference points if $\Pr(S_1|x = r) > 1/2$.
For the continuous signal of the outcome $x$, which we denote by $s$, we assume that it has some convex support in $\mathbb{R}$, with conditional density probability function $g(s|x)$, and conditional cumulative probability function $G(s|x)$, decreasing in $x$. We assume that signals $S_i$ and $s$ are independent given $x$.

2.1. Expected Value of Outcome Given Signals

We would like to compute the expected value of the outcome given the signals, $E(X|S_i, s)$ for $i = 0, 1$. We then need to obtain the conditional distribution of the outcome $x$ given the signals using Bayes’ rule,

$$h(x|S_1, s) = \frac{f(x,r)g(s|x)h(x)}{\int f(t,r)g(s|t)h(t) \, dt},$$

and obtain $h(x|S_0, s)$ in a similar way.

We can then obtain the expected value of the outcome given the signals as

$$E(X|S_i, s) = \int xh(x|S_i, s) \, dx. \quad (3)$$

Let $v(S_i, s) = E(X|S_i, s)$. One may find that $v(S_i, s)$ already has some of the properties of the value function of prospect theory, if we consider that function as a function of the signals received, for above and below the reference point. However, $v(S_i, s)$ depends on the signals observed, which are not considered in prospect theory. But we can go one step further and compute the expected value of $v(S_i, s)$ given the outcome $x$, which is discussed in the next subsection.

2.2. Expected Value over Signals

Consider now the expected value, given the outcome, of the expected value of the outcome given the signals. This can be understood as individuals being asked, for a given outcome $x$, what they expect the outcome to be given the signals that they may receive. Obtaining these for different draws of the signals, we can take the expected value over the signals. This can also be seen as the average evaluation of the outcome across individuals, given the signals that these individuals may receive.

when $r$ is sufficiently low, and there is no loss aversion if $\Pr(S_1|x = r) = 1/2$. In Sections 4 and 5 we consider the extreme case in which $\Pr(S_1|x = r) = 1/2$. 

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In fact, in the presentation of prospect theory in Kahneman and Tversky (1979), there
is the description of several experiments in which subjects were given the choice between
gambles, and reports of the fraction of subjects who prefer one gamble over the other. In the
example in the introduction, this could be, for instance, that 70% of the subjects preferred
not to take that gamble. The set-up presented here can justify these different choices by the
subjects, by the fraction of individuals who received signals on the possible outcomes such
that those signals make the individual choose not to take the gamble. We note that this
formulation is not exactly part of prospect theory, as it just presents the value function of
the outcome, but it can be seen as consistent with prospect theory. In fact, $V(x)$ would be
the value function that an experimenter would obtain from observing several decisions by
an individual, and can be seen as consistent with the value function estimated from those
decisions (e.g., Walasek and Stewart 2015). In other words, consider an individual who is
being subject to testing by an experimenter. The experimenter knows the exact value of
$x$ but does not see the random signals $s$ and $S_i$. On the other hand, the individual does
not observe $x$ directly (i.e., process the information of what $x$ means), but can make some
inference about $x$ by seeing $s$ and $S_i$. In an experimental setting the interval $[x, \bar{x}]$
would represent the set of all possible payoffs that experimenter could offer in a choice experiment,
and the realization of $x$ would be determined by the experimenter during the design stage
(but its effect unknown to the subjects). If the individual faces a choice between objects
with different $x$ (or gambles with different $x$), then the calculation of the expectation over
signals is similar to the probability of the individual choosing one of the options, which
can be obtained by the experimenter based on several observed choices by one or multiple
individuals.

Formally, we would then define the value function over the outcome, $V(x)$, as

$$V(x) = f(x, r) \int v(S_1, s) g(s|x) \, ds + [1 - f(x, r)] \int v(S_0, s) g(s|x) \, ds. \quad (4)$$

Intuitively, if the outcome $x$ is large, we would expect the signal $s$ to indicate that $x$ is on
the large side, but conservatively because $s$ is not fully informative about $x$. This could be
a region of diminishing sensitivities for $x$. The same argument can be made for small $x$. For
$x$ close to the reference point $r$ the value function $V(x)$ could be moving relatively quickly
between $v(S_0, s)$ and $v(S_1, s)$, being relatively steep on $x$. Note that with $f(x, r)$ being S-
shaped in $x$ (or having more than two inflection points) we have diminishing sensitivities if the
signal $s$ is not informative about $x$. Note that the individual actually uses $v(S_i, s)$ for making
decisions, and $V(x)$ is the unconditional expectation of $v(S_i, s)$ from the experimenter’s perspective given that the experimenter knows $x$. From the experimenter’s perspective this is the value function that would explain the individual’s behavior and evaluation of the outcome. Note that the representation in (4) will yield in general that $V(r) \neq r$, but one can then make an additive transformation of $V(x)$, such that the transformed value equation is equal to the reference point when evaluated at the reference point, and the properties of diminished sensitivities and loss aversion of $V(x)$ would continue to hold.$^{10}$

One can see the function $V(x)$ as a weighted average of $x$ and of the expected value of $x$ given that the individual observed either $S_1$ or $S_0$. If the continuous signal $s$ is more informative the weight on $x$ would be greater; we investigate this idea in the following sections. As an extreme case, if $s$ is fully informative about $x$, then we have $V(x) = x$, which represents the expected utility framework.$^{11}$

Another extreme case is the case in which the continuous signal $s$ does not provide any information about the outcome $x$, and the individual has just to rely on the information whether $x$ is above or below the reference points. In that case we then have

$$V(x) = f(x, r)E(X|S_1) + [1 - f(x, r)]E(X|S_0),$$

from which we can immediately obtain that $V(x)$ is S-shaped if $f(x, r)$ is S-shaped in $x$ (as $E(X|S_1) > E(X|S_0)$).

Still another extreme case is one in which the signal about whether the outcome is above or below the reference is fully informative. In that case we would have

$$V(x) = \begin{cases} 
\int v(S_1, s)g(s|x)ds & \text{if } x > r \\
 f(r, r)\int v(S_1, s)g(s|x)ds + [1 - f(r, r)]\int v(S_0, s)g(s|x)ds & \text{if } x = r \\
 \int v(S_0, s)g(s|x)ds & \text{if } x < r 
\end{cases}$$

which has the value function vertical at the reference point. If the signal of whether the outcome is above or below the reference point is not perfectly informative this vertical segment is smoothed out (has a positive slope). In any case, this illustrates that the existence of the signal about whether the outcome is above or below the reference point creates a greater steepness of the value function close to the reference, and lower steepness further

$^{10}$We could have a transformed value function $\tilde{V}(x) = V(x) - V(r) + r$, which then yields $\tilde{V}(r) = r$.

$^{11}$This result of $V(x) = x$ when the continuous signal is fully informative is independent of how informative the binary signal is about whether the outcome is above or below the reference point.
away from the reference point. This lower steepness of the value function away from the reference point captures the idea of diminishing sensitivities, as we illustrate in the next sections.

Consider now, for this case of perfectly informative signal about whether outcome is above the reference, whether this value function exhibits loss aversion close to the reference point $r$. The value function would be seen as exhibiting loss aversion if $V(r^+) - V(r) < V(r) - V(r^-)$, where $V(r^+) = \lim_{x \downarrow r} V(x)$ and $V(r^-) = \lim_{x \uparrow r} V(x)$, which results in the condition

$$[1 - 2\Pr(X \geq r)][\int v(S_1, s)g(s|r)ds - \int v(S_0, s)g(s|r)ds] < 0. \quad (7)$$

Note that this condition is satisfied if $\Pr(X \geq r) > 1/2$, that is, if the reference point $r$ is sufficiently low; the condition is not satisfied if the reference point $r$ is sufficiently high. This points to the existence of loss aversion if the reference point is sufficiently low.

For example, if $x$ is normally distributed, there is loss aversion if the reference point $r$ is less than the mean, and there is no loss aversion if the reference point $r$ is greater than the mean. If we expect that reference points are lower than the unconditional median of the outcome, we should then observe loss aversion in practice. Stewart, Chater, and Brown (2006) shows evidence that real data may have this feature, looking at bank transactions and finding that that there are more small debits than small credits (in a case in which the reference point would be zero).

Note that this effect of the existence of loss aversion if the reference point is not too high can be seen as resulting from the the higher likelihood that the outcome is greater than the reference point in that case. This then causes $V(x)$ not to increase too much above the reference point when the outcome is close to the reference point - which leads to gains having smaller effects on $V(x)$ than losses.

The next sections consider several particular cases of this model.

2.3. Additional Motivation for Signal on Threshold with Binary Actions

One additional motivation for the existence of the signal on the threshold comes from the literature on rational inattention when the individual faces a binary action. Suppose that the individual has to take one of two actions based on the signal she receives about $x$. While taking into account the costs of information processing, what information-processing structure would allow the individual to make better decisions?
In order to consider this question one has to have a particular specification of the costs of information-processing. One particular way to consider those costs is the rational inattention framework, in which the costs of processing information are related to the difference between the entropy of the priors and the expected entropy of the posteriors (see, e.g., Sims 1998). Under this framework Matějka and McKay (2015) show that optimal information-processing behavior occurs when there are only two signals, one in which the individual takes one action, and the other in which the individual takes the other action, and that the probability of choosing each action is consistent with a generalization of the logit model.

In the set-up of this paper, suppose that the individual is considering whether to choose the alternative that gives uncertain value \( x \) or an alternative that gives value \( r \) for certain. Then, we have (from Matějka and McKay 2015)

\[
f(x, r) = \frac{\hat{e}_1 e^{\alpha x}}{\hat{e}_1 e^{\alpha x} + (1 - \hat{e}_1) e^{\alpha r}}
\]

for some \( \alpha > 0 \), and where \( \hat{e}_1 \) is the unconditional probability of the individual receiving the signal \( S_1 \), \( \hat{e}_1 = \int_x f(x, r) h(x) dx \), which is defined implicitly given (8). Note also that \( \lim_{x \to r} f(x, r) = \Pr(S_1 | x = r) = \hat{e}_1 \), as noted above.

This formulation assumes that the individual can perfectly design the information structure that she desires, and that the costs of that information structure are related to the change in entropy from the priors to the posteriors. In fact, the individual may not be able to perfectly choose the information structure of the signals that she receives. For example, the individual may also receive, for free, a signal \( s \) on the continuous value of the outcome \( x \), in which case the optimal function \( f(x, r) \) is no longer given by (8). Similarly, if the costs of information processing are not related to the change in entropy from the priors to the posteriors, the exact specification of the function \( f(x, r) \) may not be this generalized logit model. But this rational inattention framework points to the important incentive of the individual getting a signal which indicates that the individual take one action or the other. Roughly speaking, this points to the signal indicating whether the outcome is above or below a certain threshold, which can be interpreted as the reference point.

3. **Continuous Signal \( s \) Not Informative on Outcome**

Consider the extreme case when the signal \( s \) provides no information on the outcome. See also Woodford 2008, Yang 2011, Gentzkow and Kamenica 2014, Caplin, Dean, and Leahy 2019, 2021.
That is, all inferences about the value of the outcome come from the signal about whether the outcome is above or below the reference point. We consider first the case in which the probability of the individual receiving a signal that the outcome is above a certain threshold is an exponential model, and then consider the case in which that probability is the one of the rational inattention framework under a binary action.

3.1. Exponential Model

To obtain sharper results consider an exponential representation of the function $f(x, r)$

$$f(x, r) = \begin{cases} 
1 - \frac{1}{2}e^{-\alpha(x-r)} & \text{if } x \geq r \\
\frac{1}{2}e^{-\alpha(r-x)} & \text{if } x < r,
\end{cases}$$

(9)

where $\alpha$ is a constant which represents the degree of informativeness of the signal $S_i$. When $\alpha$ is greater the signal becomes more and more informative about whether the outcome is above or below the reference point. Note also that $f(r, r) = 1/2$.

We can then obtain that

$$v(S_1) = \int_r^x \frac{x}{2}e^{-\alpha(r-x)}h(x) \, dx + \int_r^x x[1 - \frac{1}{2}e^{-\alpha(x-r)}]h(x) \, dx$$

$$v(S_0) = \int_x^r \frac{x}{2}e^{-\alpha(r-x)}h(x) \, dx + \int_r^x x\frac{1}{2}e^{-\alpha(x-r)}h(x) \, dx$$

(10)

(11)

This yields that $v(S_1) > v(S_0)$ with both $v(S_1)$ and $v(S_0)$ increasing in the reference point $r$, and $v(S_1) - v(S_0)$ increasing in $\alpha$. We also have that $\lim_{\alpha \to 0} v(S_1) = \lim_{\alpha \to 0} v(S_0) = E(x)$ and $\lim_{\alpha \to \infty} v(S_1) = E(x|x > r)$ and $\lim_{\alpha \to \infty} v(S_0) = E(x|x < r)$.

Using the expected payoff of the outcome given the signals, we can then obtain the value function as

$$V(x) = \begin{cases} 
v(S_1) - \frac{1}{2}e^{-\alpha(x-r)}[v(S_1) - v(S_0)] & \text{if } x \geq r \\
v(S_0) + \frac{1}{2}e^{-\alpha(r-x)}[v(S_1) - v(S_0)] & \text{if } x < r,
\end{cases}$$

(12)

from which one immediately obtains diminishing sensitivities.

Consider now the question of loss aversion. Note that $V(x) - V(r) = V(r) - V(r - (x - r))$, such that, with the current specification for $f(x, r)$, there is no loss aversion (or gain
premium).

However, one somewhat ad hoc approach could be to allow the individual to be more accurate at identifying that the outcome \( x \) is below the reference point \( r \), than at identifying that it is above the reference point. We would then obtain that the value function exhibits loss aversion. That is, if

\[
f(x) = \begin{cases} 
1 - \frac{1}{2} e^{-\alpha(x-r)} & \text{if } x \geq r \\
\frac{1}{2} e^{-\alpha'(r-x)} & \text{if } x < r
\end{cases}
\]  

(13)

and \( \alpha' > \alpha \) we obtain that the value function \( V(x) \) exhibits loss aversion. In the next sections we show that we can obtain loss aversion effects even if the function \( f(x, r) \) is symmetric on \( x \) around the reference point \( r \).

3.2. Rational Inattention Framework with Binary Actions

Consider now the rational inattention framework with binary actions. Suppose that the individual is considering whether to take an alternative that gives uncertain value \( x \) or an alternative that gives value \( r \) for certain.

The individual can also decide the information structure, which, as argued above, in this case reduces to choosing function \( f(x, r) \). Also as noted above, in this case the costs of processing information are assumed to be proportional to the difference between the entropy of the priors and the expected entropy of the posteriors. As shown in Matějca and McKay (2015) the optimal function \( f(x, r) \) in this case satisfies (8) noted above. In the same way as in the previous subsection we can then obtain \( v(S_1), v(S_0) \), and \( V(x) \),

\[
V(x) = \frac{\hat{c}_1 e^{\alpha x}}{\hat{e}_1 e^{\alpha x} + (1 - \hat{c}_1) e^{\alpha x}} [v(S_1) - v(S_0)] + v(S_0).
\]  

(14)

Also, using Proposition 3 in Matějca and McKay (2015), we can obtain that the unconditional probability of receiving the high signal \( S_1 \), \( \hat{c}_1 \), is decreasing in \( r \), and we can obtain \( \hat{c}_1 > 1/2 \) if and only if the reference point \( r \) is sufficiently small.

Note that, as in the previous subsection, \( V(x) \) exhibits diminishing sensitivities. Consider now the question of loss aversion. We know that if \( V(x) \) in (14) is concave at \( x = r \), we have that \( V(x) \) exhibits loss aversion close to \( x = r \). We can check that \( V(x) \) is concave at \( x = r \) if and only if \( \hat{c}_1 > 1/2 \), i.e., if and only if the reference point \( r \) is sufficiently small. We state this result in the following proposition.
Proposition 1. Suppose that the signal $s$ on the outcome is completely uninformative and consider the rational inattention framework with binary actions. Then, the value function $V(x)$ exhibits loss aversion if the reference point is sufficiently low.

Note that this result is obtained for any prior distribution $h(x)$ of the outcome. We will also replicate that result below in the case in which the individual receives also a continuous signal on the value of the outcome.

We can also interpret the condition of the reference point being sufficiently low as the reference point being sufficiently below the expected value of the outcome.

The effect obtained in the rational inattention framework can also be understood as the individual putting greater effort into determining whether the outcome $x$ is above or below the reference point $r$, for the values of the outcome close to the reference point. When the reference point is low, the individual is more likely to take the uncertain alternative $x$, and therefore to put more effort into trying to see if the outcome $x$ is below the reference point. That results in the value function being steeper below the reference point than above it, that is, the value function exhibits loss aversion.

As an example, we can consider the case in which the prior distribution of $x$ is uniform on the segment $[0, 1]$. In this case we can fully obtain $f(x, r)$. We can obtain the unconditional probability of the individual receiving signal $S_1, \hat{e}_1$, implicitly by

$$\hat{e}_1(e^{\alpha(1-\hat{e}_1)} - 1) - (1 - \hat{e}_1)(e^{\alpha r} - e^{\alpha(r-\hat{e}_1)}) = 0. \quad (15)$$

This yields that when $r = 1/2$ we have $\hat{e}_1 = 1/2$. And then $\hat{e}_1 > 1/2$ for $r < 1/2$. In this particular case the condition that $V(x)$ is concave at $x = \hat{e}, \hat{e} > 1/2$, is satisfied if the reference point $r$ is below the expected value of the outcome, $1/2$.

In this example, the condition for the the individual to exhibit loss aversion is that the reference point is below the expected value of the outcome, but that also has the effect that there is a lower likelihood of the outcome being below than above the reference point. It is interesting to investigate whether the existence of loss aversion is because of the reference point being below the expected vale of the outcomes or because of the lower likelihood of outcomes below the reference point. To investigate this question one has to consider a prior distribution of outcomes which is skewed, and, in particular, skewed to the right. To consider this we could have that the prior distribution of the outcome $x$ has a mass point at zero, and is otherwise uniformly distributed on the segment $[0, 1]$. Let $\gamma$ be the size of the mass
point at zero. Then the equivalent to (15) would now be

\[ \hat{e}_1 e^{\alpha (r - \hat{e}_1)} + (1 - \hat{e}_1) e^{\alpha (r - \hat{e}_1)} - e^{B [\hat{e}_1 + (1 - \hat{e}_1) e^{\alpha r}]} = 0, \tag{16} \]

where

\[ B = \alpha \gamma \hat{e}_1 - \frac{1 - \hat{e}_1}{1 - \gamma} \frac{e^{\alpha r} - 1}{\hat{e}_1 + (1 - \hat{e}_1) e^{\alpha r}}. \tag{17} \]

Consider now that the reference point \( r \) is set such that it is ex-ante equally likely that an outcome is below or above the reference point, \( H(r) = 1/2 \). In this particular case, this means that \( r = \frac{1 - 2 \gamma}{1 - \gamma} \), for \( \gamma \in [0, 1/2] \). The expected value of \( x \) can be computed in this case to be \( E[X] = \frac{1 - \gamma}{2} \), and we can obtain that the difference between the expected value of the outcome and the reference point, \( E[X] - r \), is increasing in the extent of right skewness of the prior distribution of \( x \), which in this case means increasing in the size of the mass at zero, \( \gamma \).

We can then obtain that \( \hat{e}_1 \) is locally increasing at \( \gamma = 0 \), which means that for \( \gamma \) small and positive we have \( \hat{e}_1 > 1/2 \), the individual exhibits loss aversion behavior. This shows that the reference point being below the expected value of the outcome leads to the existence of loss aversion behavior, which is important in the interpretation below of the experimental results of Walasek and Stewart (2015). Figure 1 illustrates how \( \hat{e}_1 \) increases with \( \gamma \), for all values of \( \gamma \).

We can consider the alternative adjustment in which we move the reference point to be equal to the expected value of the outcome as the skewness changes. In this case, the likelihood of outcomes above the reference point decreases as the distribution of outcomes become more skewed to the right. That is, in this case, \( r = E[X] = \frac{1 - \gamma}{2} \) and \( H(r) = \frac{1 + \gamma}{2} \). We can then obtain in this case that the individual exhibits the opposite of loss aversion when the distribution of outcomes becomes more positively skewed and the reference point is adjusted to be equal to the expected value of the outcome, which is a case in which the likelihood of outcomes above the reference point decreases. This result can then be consistent with Proposition 1 as a greater reference point in the conditions of that proposition means a lower likelihood of outcomes above the reference point. This discussion then suggests that the result in Proposition 1 is both because the reference point is below the expected value of the outcome, and because there is a greater likelihood of outcomes being above the reference point.\[13\]

\[13\]One could also consider the case in which the reference point is fixed and is not adjusted with the size of the mass point. In that case, a greater size of the mass point leads to \( V(x) \) exhibiting the opposite of loss
Figure 1: Example of $\hat{e}_1$ as a function of $\gamma$ for $\alpha = 5$ in the rational inattention model with mass point at zero and $H(r) = .5$.

We state these results in the following proposition.

**Proposition 2.** Suppose that the continuous signal $s$ on the outcome is completely uninformative, that the distribution of outcomes is uniform on $[0,1]$ except for a small size mass point at zero, and consider the rational inattention framework with binary actions. Then, if the reference point is set at $H(r) = 1/2$, the reference point is set below the expected value of the outcome, and the value function $V(x)$ exhibits loss aversion. Furthermore, if the reference point is set at the expected value of the outcome, $H(r) > 1/2$, and the value function exhibits the opposite of loss aversion.
4. Perfectly Informative Signal on the Relationship to the Reference Point

We now consider the benchmark in which the signal about whether the outcome is above or below the reference point is perfectly informative, and the continuous signal on the outcome provides some information.

To get sharper results (and to connect with the results in the next section), we concentrate on the case in which the prior distribution of \( x \) is uniform on \([0, 1]\) and the distribution of \( s \) given \( x \) has a mass point at \( x \) of size \( p \in [0, 1] \), and otherwise is distributed uniformly on \([0, 1] \). The parameter \( p \) is an index of how informative the continuous signal on the outcome \( x \) is. When \( p = 0 \) that signal is completely uninformative. When \( p = 1 \) that signal is completely informative. This can also be seen as a stylized model of inattention in which with probability \( p \) the individual fully pays attention and perceives \( x \) perfectly, while with a lapse probability \( 1 - p \), the individual receives a completely uninformative signal.

The signal about whether the outcome is above or below the reference points is fully informative, \( \Pr(S_1|x > r) = 1 \) and \( \Pr(S_1|x < r) = 0 \). When \( x = r \), we assume that both signals \( S_1 \) and \( S_0 \) are equally likely, as the limit of the exponential model of Section 3. That is, \( Pr(S_1|x = r) = 1/2 \).

Using Bayes’ rule, we can then obtain the expected value of the outcome given the signals,

\[
E(X|s, S_1) = v(S_1, s) = \begin{cases} 
\frac{ps + (1-p)(1-r^2)/2}{p + (1-p)(1-r)} & \text{if } s > r \\
\frac{pr + (1-p)(1-r^2)}{p + 2(1-p)(1-r)} & \text{if } s = r \\
\frac{1 + r}{2} & \text{if } s < r.
\end{cases} \tag{18}
\]

\[
E(X|s, S_0) = v(S_0, s) = \begin{cases} 
\frac{r}{2} & \text{if } s > r \\
\frac{pr + (1-p)r^2}{p + 2(1-p)r} & \text{if } s = r \\
\frac{ps + (1-p)r^2/2}{p + (1-p)r} & \text{if } s < r
\end{cases} \tag{19}
\]

where we use the facts that \( E(X|X > r) = (1 - r^2)/2 \), \( E(X|X < r) = r^2/2 \), \( H(r) = r \), and \( \text{Prob}(s = r, S_1|x = r) = \text{Prob}(s = r, S_0|x = r) = p/2 \).

\footnote{\textsuperscript{14}A case with a normal distribution of \( x \) and of the signal \( s \) given \( x \) is presented in the Appendix.}
Note that this specification, with a perfectly informative signal on whether the outcome is above or below the reference point, leads to a discontinuity at the reference point $r$. This is because, when the individual knows that the outcome is on one side of the reference point, she uses that information to update that the expected value of the outcome (given the signals received) has to be strictly on that side of the reference point. Because that signal of the relationship of the outcome to the reference point is perfectly informative this translates into a discontinuity at the reference point of the value function, as noted below.

Taking the expected value over the signals given the outcome $x$, we can then obtain the value function,

$$V(x) = \begin{cases} 
px + (1-p)(1-r^2)/2 & \text{if } x > r \\
\frac{p}{p + (1-p)(1-r)} + (1-p)\frac{1+r}{2} & \text{if } x = r \\
\frac{p}{p + (1-p)r^2/2} + (1-p)\frac{r}{2} & \text{if } x < r.
\end{cases}$$

(20)

Consider whether this formulation satisfies the diminishing sensitivities property. In order to check this property in a value function that is not smooth everywhere (in this case the first derivative does not have a continuous first derivative at one point), we generalize the definition of diminishing sensitivities to be that $V(x)$ is a concave curve for $x \geq r$, and a convex curve for $X \leq r$. This captures the idea that the sensitivity of the value function to the outcome $x$ is smaller for values of $x$ further away from the reference point $r$. We use this definition of diminishing sensitivities in in the remainder of the paper in such situations. We can then then see that the value function (20) exhibits diminishing sensitivities as $V'(x) < 1$ for both $x > r$ and $x < r$, and it has a vertical discontinuity at $x = r$.

Consider now the question of loss aversion. To check that we can obtain that we have loss aversion if $V(r) - V(r^-) > V(r^+) - V(r)$, which holds if $r < 1/2$ (with the derivation presented in the Appendix). This is stated in the following proposition.

**Proposition 3.** Consider that the binary signal of whether the outcome is above or below the reference point is fully informative. Given that the continuous signal $s$ on the outcome is partially informative, $p \in (0,1)$, the value function exhibits loss aversion if and only if $r < 1/2$. If the continuous signal $s$ on the outcome is completely uninformative, $p \to 0$, or

\[\text{Note that if } r < 1/2 \text{ we also have } V(r+\delta) - V(r) < V(r) - V(r-\delta) \text{ for all } \delta > 0 \text{ as } V'(r+\delta) < V'(r-\delta).\]
completely informative, \( p \to 1 \), there are no loss aversion effects.

The condition of the reference point not being too high for the value function to exhibit loss aversion can be seen as relatively intuitive. If the reference point is not too high, the value function evaluated at \( x = r \) is relatively high. For example, when the individual receives the high signal \( S_1 \) (which occurs with a 50\% chance), she expects a relatively high payoff, in comparison to when she receives the low signal. This result is consistent with the one in the previous section, in which there was no signal on the value of the outcome, and we allowed the individual to choose the information structure when facing a binary action problem. The two results point to a mechanism that generates loss aversion with low reference points because of the greater space of payoffs above the reference point. In this section, this occurs because of the higher value function when evaluated at the reference point, \( x = r \). In the previous section, this occurred because of the greater probability of getting the high signal \( S_1 \) when the reference point is lower. Although the proposition is focused on conditions for loss aversion effects to be present, the reverse conditions (i.e., a sufficiently high reference point) would lead to the opposite of loss aversion, a gain premium.

In this particular case the condition for the existence of loss aversion is for the reference point to be below the expected value of the outcome, \( 1/2 \). In this particular case with a symmetric prior distribution of outcomes, this condition ends up being the same as having a lower likelihood of outcomes below the reference point. If the prior distribution is skewed, one can potentially distinguish between these two conditions related to the the reference point being low enough, as done in Proposition 2. That proposition shows that both the expected value of the outcome being above the reference point, and there existing a greater likelihood of outcomes above the reference point contribute for the value function to exhibit loss aversion effects.

The results present empirical implications that are not present in the description of the loss aversion effect in the literature, and which would be interesting to further explore. In fact, these results point to the existence of loss aversion if the reference point is sufficiently low, and to the opposite of loss aversion if the reference point is relatively high. Note that the experimental results in Walasek and Stewart (2015) can be seen as consistent with these results. Walasek and Stewart explain the experimental results with decision by sampling (e.g., Stewart, Chater, and Brown 2006), in which utility over an outcome is based on the ranking of that outcome. In Section 6, we further discuss the relationship of the results presented here to decision by sampling. Note also that the signal structure presented here
predicts variability in the actions of the individual depending on the realization of the signals. For example, the model can make predictions on the proportion of times an individual would choose a gamble, not only whether one is preferred over the other. In this dimension, one can see the model as more consistent with the data used to justify prospect theory (for example, that more subjects choose one gamble over another) than prospect theory itself. This variability of the realized signals has empirical implications such as the effects of the noisiness of the signals, on the variability of the realized choices. The model can also make predictions on how the proportions of choices across gambles would change with the reference point, for which prospect theory would not make any particular predictions.

Note also that the condition of $r < 1/2$ for the existence of loss aversion is the same as the one in the simple model in the Appendix. However, there are some significant differences between the model considered here and the simple model.

First, in the simple model, loss aversion occurs because $\Pr(S_1|x = r) > 1/2$ for $r < 1/2$, while here we have $\Pr(S_1|x = r) = 1/2$, and loss aversion occurs because $E(X|s = r, S_1)$ is relatively high compared with $E(X|s = r, S_0)$.

Second, in the simple model, we assume that the diminishing sensitivities had the same slope above and below the reference point (a slope of $\beta$). Here, we obtain that the slope of the diminishing sensitivities is flatter for outcomes above the reference point than for outcomes below the reference point if and only if $r < 1/2$, and the slope is the same above and below the reference point only for the particular case of $r = 1/2$.

Finally, note that, if the continuous signal $s$ on the outcome is either completely uninformative or completely informative ($p = 0$ or $p = 1$), then there are no loss aversion effects. This further clarifies that the continuous imperfect informative signal on the outcome is important to get loss aversion effects if the reference point is not too high and if the probability of getting the high signal $S_1$ when the outcome is at the reference point is not decreasing in the reference point.

5. **Both Signals are Partially Informative**

We now consider the case in which both the signal on the outcome and the signal about whether the outcome is above or below the reference point are partially informative.

We consider the stochastic structure of the signal on the outcome, as in the previous section.
For the signal about whether the outcome is above or below the reference point, we consider the exponential model structure used in Section 3. Recall that $\alpha$ is a parameter that captures the degree of informativeness of the binary signal about whether the outcome is above or below the reference point. If $\alpha \to \infty$ that signal is perfectly informative and we have the case of the previous section.

We can then derive the value function for this case, and illustrate how it displays diminishing sensitivities and loss aversion in a smooth way. (Details of the derivation are presented in the Appendix.)

Recall that $\tilde{e}_1$ is the probability of the individual receiving signal $S_1$, and let $\tilde{e}_0$ be the expected value of the outcome $x$ given that the individual received signal $S_1$ multiplied by $\tilde{e}_1$. Similarly, let $\tilde{e}_0$ be the probability of the individual receiving signal $S_0$, and let $\tilde{e}_0$ be the expected value of the outcome $x$ given that the individual received signal $S_0$ multiplied by $\tilde{e}_0$. We can obtain the values of $\tilde{e}_1, \tilde{e}_0, \tilde{e}_1$, and $\tilde{e}_0$ as a function of $\alpha$ and $r$ as $\tilde{e}_1 = 1 - r + \frac{1}{2\alpha}[e^{-\alpha(1-r)} - e^{-\alpha r}], \tilde{e}_1 = 1 - \frac{x^2}{2} - \frac{2e^{-\alpha r}}{2\alpha^2} + \frac{1+\alpha}{2\alpha^2}e^{-\alpha(1-r)}, \tilde{e}_0 = 1 - \tilde{e}_1$, and $\tilde{e}_0 = 1/2 - \tilde{e}_1$. When $\alpha$ converges to infinity, we have $\tilde{e}_1, \tilde{e}_0, \tilde{e}_1$, and $\tilde{e}_0$, converging to $1 - r, r, \frac{1-r^2}{2}$, and $r^2/2$, respectively.

We can then construct the expected value of the outcome as a function of the signals $E(X|s, S_i) \equiv g(s, S_i)$, and then obtain the value function as described in Section 3. We can obtain the value function as

$$V(x) = \begin{cases} 
(1 - \frac{1}{2}e^{-\alpha(x-r)})\left(p \frac{px(1 - \frac{1}{2}e^{-\alpha(x-r)}) + (1-p)\tilde{e}_1}{p(1 - \frac{1}{2}e^{-\alpha(x-r)}) + (1-p)\tilde{e}_1} + (1-p)A_1\right) + \\
\frac{1}{2}e^{-\alpha(x-r)}\left(p \frac{px\frac{1}{2}e^{-\alpha(x-r)} + (1-p)\tilde{e}_0}{p\frac{1}{2}e^{-\alpha(x-r)} + (1-p)\tilde{e}_0} + (1-p)A_0\right) 
\quad \text{if } x \geq r \\
\frac{1}{2}e^{-\alpha(r-x)}\left(p \frac{px\frac{1}{2}e^{-\alpha(r-x)} + (1-p)\tilde{e}_1}{p\frac{1}{2}e^{-\alpha(r-x)} + (1-p)\tilde{e}_1} + (1-p)A_1\right) + \\
(1 - \frac{1}{2}e^{-\alpha(r-x)})\left(p \frac{px(1 - \frac{1}{2}e^{-\alpha(r-x)}) + (1-p)\tilde{e}_0}{p(1 - \frac{1}{2}e^{-\alpha(r-x)}) + (1-p)\tilde{e}_0} + (1-p)A_0\right) 
\quad \text{if } x < r,
\end{cases}$$

(21)

where $A_1$ is the expected value of $v(s, S_1)$ given $x$ and given that $s \neq x$, and $A_0$ is the expected value of $v(s, S_0)$ given $x$ and given that $s \neq x$. We can get that $\lim_{\alpha \to \infty} A_1 = (1 + r)/2$ and $\lim_{\alpha \to \infty} A_0 = r/2$. The expressions of $A_1$ and $A_0$ can be solved and presented as a function of dilogarithms, as detailed in the Appendix.

When $\alpha$ converges to infinity, this value function converges to the value function in the
previous section, now with a positive slope close to the reference point. Given the results from the previous section, we also now have that, if \( r < \frac{1}{2} \) the value function exhibits loss aversion if \( \alpha \) is sufficiently large.

Figure 2 presents the value function for some parameter values, illustrating both loss aversion and diminishing sensitivities. Figure 3 illustrates how the value function varies with the degree of informativeness \( \alpha \) of the signal about whether the outcome is above or below the reference point, and how the value function approaches the limiting case of the previous section with a discontinuity at the reference point \( r \). Figure 4 illustrates how the value function varies with the reference point \( r \). Figure 5 presents examples of the value function for various degrees of informativeness of the signal of the outcome, \( p \), illustrating that, for large \( p \), we get closer to expected utility theory.
Figure 3: Example of value functions for $r = .25$, and $p = .3$, for various $\alpha$ ($\alpha = 10, 20, 40$).

6. Discussion

In this section we discuss the relationship to other explanations of loss aversion, potential ways to endogenize the signal structure, some testable empirical implications, and other issues with the examples considered.

6.1. Discrete Representation of Outcomes and Decision by Sampling

One way that has been presented to explain loss aversion and diminishing sensitivities is decision by sampling (e.g., Friedman 1989, Stewart, Chater, and Brown 2006). In decision by sampling, the utility of an outcome can also be seen as obtained by the ranking of that outcome in the space of all objects. If the distribution of outcomes is positively skewed, this results in a utility over outcomes that exhibits loss aversion properties. Furthermore, if the tails of the distribution of outcomes have lower density, the utility function also exhibits
Figure 4: Example of value functions for $p = .3$, and $\alpha = 20$, for various $r$ ($r = .25, .45, .65$).

diminishing sensitivities. As evidence consistent with this explanation of loss aversion, Stewart, Chater, and Brown (2006) look at movements in bank accounts and find that there are more small movements for debits than small movements for credits. When the reference point is zero, this leads to greater change in ranks within small debits than in small credits, leading the value function over bank account movements to be steeper over small debits than over small credits, consistent with loss aversion.

Note also that decision by sampling can be justified by assuming that the individual is restricted to using a discrete number of representations of the outcomes and organizes those representations in the form that is best suited to the problems being faced by the individual (e.g., Robson 2001, Netzer 2009, Bhui and Gershman 2018). As the number of

\[\text{Note that decision by sampling can also be used to explain hyperbolic discounting and overweighting of low probabilities. For market effects on the analysis of hyperbolic discounting see, for example, Jain (2009, 2019).}\]
Figure 5: Example of value functions for \( r = .25 \), and \( \alpha = 20 \), for various \( p \) (\( p = .3, .5, .7 \)).

representations goes to infinity, the value function converges, under some conditions, to the cumulative distribution of outcomes, which is decision by sampling.\(^{17}\)

In further evidence consistent with decision by sampling, Walasek and Stewart (2015) consider an experimental design in which subjects were offered to accept or reject a sequence of equal probability lotteries, in which the range of losses and gains was varied. Assuming the natural reference point at zero, that paper finds that, consistent with decision by sampling, the subjects exhibited loss aversion when the range of gains was greater than the range of losses, exhibited the opposite of loss aversion (i.e., gain premium) when the range of gains was smaller than the range of gains, and exhibited behavior that was not statistically different from loss aversion or gain premium when the range of gains was the same as the range of losses.

\(^{17}\)Netzer (2009) notes that, if the objective is to maximize the expected payoff, the slope of the value function does not vary as much as the slope of the cumulative distribution of outcomes.
If one sees the increase in the range of gains as the reference point decreasing and becoming lower than the expected value of the outcomes (relative decrease within the range of outcomes), we can interpret the results in Walasek and Stewart (2015) as consistent with the results here that loss aversion occurs when the reference point is low enough and below the expected value of the outcomes.\(^{18}\) That is, the results in Walasek and Stewart can be consistent with both decision by sampling and the information-processing theory presented here. It is therefore interesting to discuss the relationship between these two potential explanations for loss aversion, and the potentially different empirical implications across the two explanations.

Both decision by sampling and the model presented here use the full distribution of outcomes to affect the value function. In decision by sampling, this is done based on the rank of the different outcomes, while in the model presented here this occurs through the updating of beliefs given the signals that are obtained. One issue with decision by sampling is that differences in the outcome without change in rank do not affect preferences, while in the model considered here those changes would affect preferences.\(^{19}\) In this respect, it would be interesting to understand better the relationship of decision by sampling with a model with a signal structure on the outcomes.

Note also that decision by sampling does not offer a full explanation for the variability in the decisions except potentially for uncertainty in sampling or decision errors, while the model here directly explains the variability in the decisions with the uncertainty of the signals received.

Some empirical implications of the model presented here are the effects of the noisiness of the signals on the value function and the variability of decision-making, which do not have a direct relationship in decision by sampling.

6.2. Decision-Making Problem

The signal structure in the paper is taken as exogenous (except for the rational inattention results) to illustrate conditions under which the signal structure generates a value function that exhibits loss aversion, which is the main point of the paper. However, the

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\(^{18}\)Note that in Walasek and Stewart the likelihood of outcome above and below can be seen as remaining the same across conditions in that paper, which can be seen as being the conditions of Proposition 2. See also Alempaki et al. (2019) for further discussion on the analysis of those experimental conditions.

\(^{19}\)Note also that the range component of the range-frequency theory (which can also emerge from efficient coding, Bhui and Gershman 2018) would be affected by such differences.
signal structure may be, up to some point, endogenous to the problem facing the individual. Although interesting and important, this problem is beyond the scope of this paper.

In the particular example in the previous section, the individual could potentially choose the reference point and the degree of noisiness that she would be willing to accept in either the signal of the value of the outcome or the signal of whether the outcome is above or below the reference point. For example, it could be that, in more important decisions, the individual would prefer less noise on the signals received, such that, in some measure, the value function would exhibit less loss aversion, and potentially look more like the value function of an expected utility maximizer. One may also expect that in binary action problems (for example, buy or not buy) the individual may prefer to have a reference point at which the decision generates net positive value, and may choose to have less noise on the signal of whether the outcome is above or below the reference point. Similarly, in non-binary problems (for example, making a valuation for a sequence of decisions with different conditions), the individual may prefer to have less noise on the continuous signal on the value of the outcome, and worry less about the noise on the binary signal of whether the outcome is above or below the reference point.

6.3. Testable Empirical Implications

As noted above, the model considered has several empirical implications. One order of empirical implications regards the level of the reference point and the ex-ante distribution of the outcome.

Following up on the propositions above we would expect that if we would lower the reference point, we would expect subjects in an experiment to exhibit more loss aversion. By the same argument, if we would increase the reference point we would expect subjects to stop exhibiting loss aversion, and at some point to start exhibiting the opposite of loss aversion.

Similarly, we could also have an empirical implication on moving the ex-ante of distribution of outcomes, while keeping the reference point fixed. We would expect that if the mass of the distribution of outcomes moved to the right, the subjects would be more likely to exhibit loss aversion. By the same argument, if the mass of the distribution of outcomes moved to the left, we would expect the subjects to stop exhibiting loss aversion, and at some point to start exhibiting the opposite of loss aversion.
Another interesting empirical implication is that if the noise on the continuous signal is reduced, the subjects are likely to behave more like expected utility maximizers. This could potentially be done by facilitating information processing by the subjects, by, for example, explaining better the meaning of the outcomes, and the decision problem being faced. Related to the discussion above, with decision-makers potentially putting more effort in reducing the noise of the signals in more important tasks, one could also expect that less loss aversion (or the opposite of loss aversion) is exhibited when the importance of the task is increased.

As noted, above, more structurally, the model presented above presents an explanation of how individuals could exhibit some variation in their decision-making, and one can potentially test whether the observed variation in decision-making is consistent with the variation that is generated by the model after the fitting of the appropriate parameters. That is, the information-processing approach presented here can be matched structurally with data. This would be interesting to investigate in future work.

6.4. Functional Forms

The analysis in Sections 4 and 5 is done for a specific functional form for the signal structure. One may wonder whether the results carry through for other signal structures.

The result of diminishing sensitivities away from the reference can be expected to hold with other signal structures because the noise on the continuous signal on the outcome makes the value function flatter when farther away from the reference point, and the signal on whether the outcome is above or below the reference point makes the value function steeper closer to the reference point.

Consider now the result on loss aversion if the reference point is low. This result depends on how the signal about whether the outcome is above or below the threshold interacts with the signal on the outcome. One intuition for the result is that a low reference point generates a higher probability of receiving a signal that the outcome is above the reference point, with the result that the value function above the reference point is closer to the value function evaluated at the reference point than the value function below the reference point is to the value function evaluated at the reference point.

It would be interesting to explore whether this result holds for other signal structures. In the signal structures that we explored, this result held for the simulations run.
In any case, the result on loss aversion can be seen more generally as illustrating that the value function can exhibit loss aversion effects under some conditions, which can make information processing effects consistent with the estimated loss aversion effects in the data.

Note also that the results for a rational inattention framework with a binary action indicate loss aversion if the reference point is not too high for any prior distribution of outcomes.

7. Conclusion

This paper presents a model of loss aversion and diminishing sensitivities in which the individual receives both a continuous signal on the outcome and a binary signal about whether the outcome is below or above a reference point. A partially informative continuous signal on the outcome, together with a signal about whether the outcome is above or below the reference point, generates diminishing sensitivities above and below the reference point. When the continuous signal on the outcome is informative, if the reference point is not too high, then the model generates loss aversion, because of both the expected value of the outcome being above the reference point, and there being a greater likelihood of receiving a signal that the outcome is above the threshold. Also, in a set-up with rational inattention under binary actions, the individual pays more attention to the outcomes below the reference point when the reference point is sufficiently low, leading to loss aversion when the reference point is low, which is consistent with the effect above.

The existence of a binary signal about whether the outcome is above or below a reference point captures the idea that it may be sometimes easier to put the outcome in categories than to process information about a continuous variable. The case of a reference point represents the simplest case of only two categories.

We could imagine that in some situations having more than two categories may be likely. For example, the individual could have four categories: above or below a reference point, extremely high, and extremely low. It would be interesting to explore such situations in experimental settings, and in a variation of the model presented here.\(^{20}\)

Another potentially interesting possibility is that the individual can decide the informativeness, at a cost, of the signals received. It would then be interesting to study the informativeness decisions as a function of the stakes in the problem considered.

\(^{20}\)Schley et al. (2021) consider the possibility of more than two categories in the context of the probability weighting function, arguing for the existence of multiple and separate convex regions.
The paper focuses on the case of one-dimensional loss aversion effects, but the results presented can be potentially applied to cases in which the objects being evaluated have more than one relevant dimension to be considered. The analysis of those cases could help in the case of considering the endowment effect, and it would be interesting to study in future research.
A Simple Model

This section presents a simple model that illustrates the forces at work, without formally modeling the probability distribution of the signals that the individual can receive.

Let us consider that \( S_1 \) and \( S_0 \) are fully informative. That is, \( \Pr(X > r|S_1) = 1 \) and \( \Pr(X < r|S_0) = 1 \). We also assume that \( \Pr(S_1|x = r) = \Pr(X \geq r) \) which holds in a rational inattention framework, as noted above.

Consider now that the value of \( x \) for the individual, \( V(x) \), is a weighted average of \( x \) and the expected value of \( x \) given that the individual observed either \( S_1 \) or \( S_0 \). That is,

\[
V(x) = \begin{cases} 
\beta x + (1 - \beta)E(X|X \geq r) & \text{if } x > r \\
\beta r + (1 - \beta)[\Pr(X \geq r)E(X|X \geq r) + \Pr(X < r)E(X|X < r)] & \text{if } x = r \\
\beta x + (1 - \beta)E(X|X < r) & \text{if } x < r
\end{cases}
\]

where \( \beta \) is the weight on \( x \). The idea would be that the weight on \( x \) would be greater if the signal received on the outcome \( x \) would be more informative. When \( \beta \to 1 \), the individual fully processes the information on \( x \) and behaves like an expected utility maximizer. When \( \beta \to 0 \), the signal on \( x \) does not provide any information, and the individual has just to rely on the information whether \( x \) is above or below the reference point.

This formulation of the value function above has diminishing sensitivities, as \( \beta < 1 \), and the value function is steeper (in fact, vertical) at the reference point.

Consider now whether this value function exhibits loss aversion. The value function would be seen as exhibiting loss aversion if \( V(r^+) - V(r) < V(r) - V(r^-) \), where \( V(r^+) = \lim_{x \to r^+} V(r) \) and \( V(x^-) = \lim_{x \to r^-} V(x) \), which results in the condition

\[
[1 - 2\Pr(X \geq r)] [E(X|X \geq r) - E(X|X < r)] < 0. \tag{ii}
\]

Note that this condition is satisfied if \( \Pr(X \geq r) > 1/2 \), that is, if the reference point \( r \) is sufficiently low; the condition is not satisfied if the reference point \( r \) is sufficiently high. This points to the existence of loss aversion if the reference point is sufficiently low.

As an example, consider the case where \( X \) is uniformly distributed on \([0, 1]\). Then \( E(X|X \geq r) = (1 + r)/2 \) and \( E(X|X < r) = r/2 \) and condition (ii) reduces to \( r < 1/2 \). That is, there is loss aversion if the reference point \( r \) is lower than \( 1/2 \).
Proof of Proposition 2

Consider first the case of $H(r) = 1/2$. We totally differentiate (17) with respect to $\hat{e}_1$ and $\gamma$, when $\gamma = 0$, under the constraint that $r = \frac{1 - 2\alpha}{1 - \gamma}$. Note that $\frac{dr}{d\gamma}_{\gamma=0} = -1/2$ and $\frac{d^2r}{d\gamma^2}_{\gamma=0} = -1$.

Let $w(\hat{e}_1, \gamma)$ denote the left hand side of (17). Then

$$\frac{\partial w}{\partial \hat{e}_1}_{\gamma=0} = 2(e^{\alpha/2} - 1) - \frac{\alpha}{2}(1 + e^{\alpha/2}),$$

which is negative for all $\alpha > 0$. Furthermore, we can obtain

$$\frac{\partial w}{\partial \gamma}_{\gamma=0} = 0$$

and

$$\frac{\partial^2 w}{\partial \gamma^2} = \alpha(1 - \hat{e}_1)e^{\alpha(r - \hat{e}_1)}(r'' + \alpha r^2) - \alpha(1 - \hat{e}_1)e^{e\alpha r}(r'' + \alpha r^2) - 2\alpha r'(1 - \hat{e}_1)e^{e\alpha r}\frac{\partial B}{\partial \gamma} - e^{B}[\hat{e}_1 + (1 - \hat{e}_1)e^{\alpha r}]\left(\frac{\partial B}{\partial \gamma}\right)^2.$$  

Evaluating at $\gamma = 0$ one obtains that $\frac{\partial^2 w}{\partial \gamma^2}$ is proportional to $\alpha e^{\alpha/2}$, and therefore greater than zero. We can then obtain that $\frac{d\hat{e}_1}{d\gamma} > 0$ for $\gamma > 0$ and small.

Second, consider the case of $r = \frac{1 - \gamma}{2}$. In this case we have $\frac{dr}{d\gamma}_{\gamma=0} = -1/2$ and $\frac{d^2r}{d\gamma^2}_{\gamma=0} = 0$. We can then also obtain that $\frac{\partial w}{\partial \gamma}_{\gamma=0} = 0$. Using $r'$ and $r''$ in (v) we can then obtain that $\frac{\partial^2 w}{\partial \gamma^2}_{\gamma=0}$ is proportional to $1 - e^\alpha + \alpha e^{\alpha/2}$ which is less than zero for any $\alpha > 0$. We can then obtain that in this case of $r = E(x)$ we have $\frac{d\hat{e}_1}{d\gamma} < 0$ for $\gamma > 0$ and small.

Perfectly Informative Signal on Relation to Reference Point with Normal Distribution on $x$ and $s$:

Consider the case of a perfectly informative signal on the relation to reference point, and that the prior distribution on $x$ is normal, and that the distribution on the signal $s$ given $x$ is also normal.

Let $E(x) = \mu$, and the variance of the prior of $x$ be $\sigma^2$. Furthermore, let $E(s|x) = x$, and the variance of $s$ given $x$ be $\omega^2$. Let $\tilde{\mu}(s)$ and $\tilde{\sigma}^2$ be the expected value and variance of $x$ given $s$. From standard calculations on the distribution of the conditional normal, we know

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that

\[ E(x|s) = \tilde{\mu}(s) = \frac{s\sigma^2 + \mu\omega^2}{\sigma^2 + \omega^2} \]  
\[ E[(x - \tilde{\mu}(s))^2|s] = \tilde{\sigma}^2 = \frac{\sigma^2\omega^2 + \omega^2}{\sigma^2 + \omega^2}. \]  

Given the calculations for expected value of truncated normal distributions we can then obtain:

\[ E(x|s, x > r) = \tilde{\mu}(s) + \tilde{\sigma}\phi\left(\frac{r - \tilde{\mu}(s)}{\tilde{\sigma}}\right) \bigg/ \left[1 - \Phi\left(\frac{r - \tilde{\mu}(s)}{\tilde{\sigma}}\right)\right] \]  
\[ E(x|s, x < r) = \tilde{\mu}(s) - \tilde{\sigma}\phi\left(\frac{r - \tilde{\mu}(s)}{\tilde{\sigma}}\right) / \Phi\left(\frac{r - \tilde{\mu}(s)}{\tilde{\sigma}}\right), \]

where \( \phi(\cdot) \) and \( \Phi(\cdot) \) are the density and cumulative distribution function, respectively, of the standard normal. From this we can then obtain the value function as

\[ V(x) = \begin{cases} 
E_s[E(x|s, x > r)|x] = \tilde{\mu}(x) + \tilde{\sigma}\phi\left(\frac{r - \tilde{\mu}(x)}{\tilde{\sigma}}\right) \bigg/ \left[1 - \Phi\left(\frac{r - \tilde{\mu}(x)}{\tilde{\sigma}}\right)\right] & \text{if } x \geq r \\
E_s[E(x|s, x < r)|x] = \tilde{\mu}(x) - \tilde{\sigma}\phi\left(\frac{r - \tilde{\mu}(x)}{\tilde{\sigma}}\right) / \Phi\left(\frac{r - \tilde{\mu}(x)}{\tilde{\sigma}}\right) & \text{if } x < r 
\end{cases} \]

Derivation of Loss Aversion Condition in the Case of Fully Informative Signal on whether Outcome is above Reference Point:

The condition for the existence of loss aversion is that \( V(r) - V(r^-) > V(r^+) - V(r) \), which can be reduced to \( V(r) > \frac{1}{2}[V(r^+) + V(r^-)] \). Note now that we can write \( V(r) \) as

\[ V(r) = \frac{1}{2}[V(r^+) + V(r^-)] + \frac{p}{2}\left(\frac{pr + (1-p)(1-r^2)}{p + 2(1-p)(1-r)} + \frac{pr + (1-p)r^2}{p + 2(1-p)r}\right) \]
\[ -\frac{2pr + (1-p)(1-r^2)}{2p + 2(1-p)(1-r)} - \frac{2pr + (1-p)r^2}{2p + 2(1-p)r} \]

from which we can obtain that \( V(r) > \frac{1}{2}[V(r^+) + V(r^-)] \) for \( p < 1 \) if and only if

\[ \frac{(1-r)^2}{[p + 2(1-p)(1-r)][p + (1-p)(1-r)]} > \frac{r^2}{[p + 2(1-p)r][p + (1-p)r]} \]

This condition can be reduced to

\[ \left(\frac{1}{1-r} - \frac{1}{r}\right) \left[p \left(\frac{1}{1-r} + \frac{1}{r}\right) + 3(1-p)\right] < 0 \]
which holds if and only if $r < 1/2$.

**Derivation of Value Function for Case with Partially Informative Signals:**

For this case we have

$$\hat{e}_1 = \int_0^r \frac{1}{2} e^{-\alpha(r-x)} \, dx + \int_r^1 \left[ 1 - \frac{1}{2} e^{-\alpha(x)} \right] dx$$  \hspace{1cm} (xiv)

$$\tilde{e}_1 = \int_0^r \frac{1}{2} xe^{-\alpha(r-x)} \, dx + \int_r^1 x \left[ 1 - \frac{1}{2} e^{-\alpha(x-r)} \right] dx,$$  \hspace{1cm} (xv)

and

$$\hat{e}_0 = \int_0^r \left[ 1 - \frac{1}{2} e^{-\alpha(r-x)} \right] dx + \int_r^1 \frac{1}{2} e^{-\alpha(x-r)} \, dx$$  \hspace{1cm} (xvi)

$$\tilde{e}_0 = \int_0^r x \left[ 1 - \frac{1}{2} e^{-\alpha(r-x)} \right] dx + \int_r^1 \frac{1}{2} xe^{-\alpha(x-r)} \, dx.$$  \hspace{1cm} (xvii)

We can obtain the values of $\hat{e}_1, \hat{e}_0, \tilde{e}_1, \text{ and } \tilde{e}_0$ as a function of $\alpha$ and $r$ as presented in the main text.

This then yields the expected value of the outcome as a function of the signals as

$$E(X|s, S_1) = v(s, S_1) = \begin{cases} 
ps[1 - \frac{1}{2} e^{-\alpha(s-r)}] + (1 - p)\hat{e}_1 & \text{if } s \geq r \\
ps\frac{1}{2} e^{-\alpha(r-s)} + (1 - p)\hat{e}_1 & \text{if } s < r 
\end{cases}$$  \hspace{1cm} (xviii)

$$E(X|s, S_0) = v(s, S_0) = \begin{cases} 
ps\frac{1}{2} e^{-\alpha(s-r)} + (1 - p)\tilde{e}_0 & \text{if } s \geq r \\
ps\frac{1}{2} e^{-\alpha(s-r)} + (1 - p)\tilde{e}_0 & \text{if } s < r 
\end{cases}$$  \hspace{1cm} (xix)

With this function $v(s, S_i)$ we can then construct the value function $V(s)$ as described in Section 3. Let $A_1$ be the expected value of $v(s, S_1)$ given $x$ and given that $s \neq x$, and let
$A_0$ be the expected value of $v(s, S_0)$ given $x$ and given that $s \neq x$. That is,

$$A_0 = \int_0^r ps \frac{1}{2} e^{-\alpha (r-s)} + (1 - p) \tilde{e}_1 ds + \int_0^1 ps \frac{1}{2} e^{-\alpha (s-r)} + (1 - p) \tilde{e}_1 ds.$$  \hspace{1cm} (xx)

$$A_1 = \int_0^r ps \frac{1}{2} e^{-\alpha (r-s)} + (1 - p) \tilde{e}_1 ds + \int_0^1 ps \frac{1}{2} e^{-\alpha (s-r)} + (1 - p) \tilde{e}_1 ds.$$  \hspace{1cm} (xxi)

We can get that $\lim_{\alpha \to \infty} A_1 = (1 + r) / 2$ and $\lim_{\alpha \to \infty} A_0 = r / 2$. The expressions of $A_1$ and $A_0$ can be solved as follows.

Consider the first term of $A_1$. Note that it can be seen as being composed of $B_1$ and $B_2$

$B_1 = \int_0^r \frac{s}{1 + 2 \frac{1-p}{p} \tilde{e}_1 e^{\alpha (r-s)}} ds$ \hspace{1cm} (xxii)

$B_2 = \int_0^r \frac{(1 - p) \tilde{e}_1}{p \frac{1}{2} e^{-\alpha (r-s)} + (1 - p) \tilde{e}_1} ds$. \hspace{1cm} (xxiii)

Integrating $B_1$ by parts one obtains

$$B_1 = \frac{r^2}{2} + \frac{r}{\alpha} \ln \left(1 + 2 \frac{1-p}{p} \tilde{e}_1\right) + \frac{1}{\alpha^2} \left[ Li_2(-2 \frac{1-p}{p} \tilde{e}_1 e^{\alpha r}) - Li_2(-2 \frac{1-p}{p} \tilde{e}_1) \right]$$ \hspace{1cm} (xxiv)

where $Li_2(z)$ is the dilogarithm function (also known as Spence’s function), $Li_2(z) = -\int_0^z \frac{\ln(1-u)}{u} du$. Note also that if $|z| < 1$, we also have $Li_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2}$, and if $z \leq -1$, we have $Li_2(z) = -\frac{\pi^2}{6} - \frac{1}{2} [\ln(-z)]^2 - \sum_{k=1}^{\infty} \frac{1}{k^2 z^k}$, which allows us to compute approximations of $Li_2(z)$.

We can also obtain

$$B_2 = \frac{\tilde{e}_1}{\tilde{e}_1} \left[ r - \frac{1}{\alpha} \ln \frac{p + 2(1-p) \tilde{e}_1}{pe^{-\alpha r} + 2(1-p) \tilde{e}_1} \right].$$  \hspace{1cm} (xxv)

Using similar techniques on the second term of $A_1$ and on $A_0$ we can then obtain
We can then obtain the value function as presented in the main text.

\[
A_1 = \frac{r^2}{2} + \frac{r}{\alpha} \ln \left( 1 + 2 \frac{1 - p_0 e_1}{p} \right) + \frac{1}{\alpha^2} \left[ Li_2 \left( -2 \frac{1 - p_0 e_1 e^{\alpha r}}{p} \right) - Li_2 \left( -2 \frac{1 - p_0 e_1}{p} \right) \right] + \\
\frac{r e_1}{e_1} - \frac{e_1}{\alpha e_1} \ln \left( \frac{p + 2(1 - p) e_1}{p e^{-\alpha r} + 2(1 - p) e_1} \right) - \frac{p(1 - r^2)}{2[p + (1 - p) e_1]} - \\
\frac{(1 - p) e_1}{\alpha^2[p + (1 - p) e_1]} \left[ Li_2 \left( \frac{p}{2[p + (1 - p) e_1]} \right) - Li_2 \left( \frac{p e^{-\alpha(1-r)}}{2[p + (1 - p) e_1]} \right) \right] + \\
\frac{(1 - p) e_1}{\alpha[p + (1 - p) e_1]} \left[ r \ln \left( \frac{p + 2(1 - p) e_1}{2[p + (1 - p) e_1]} \right) - \ln \left( \frac{2 - e^{-\alpha(1-r)}}{2[p + (1 - p) e_1]} \right) \right] + \\
\frac{(1 - r)(1 - p) e_1}{p + (1 - p) e_1} + \frac{1}{\alpha[p + (1 - p) e_1]} \ln \left( \frac{p(2 - e^{-\alpha(1-r)} + 2(1 - p) e_1)}{p + 2(1 - p) e_1} \right) \quad (xxvi)
\]

\[
A_0 = \frac{p r^2}{2[p + (1 - p) e_0]} + \frac{r(1 - p) e_0}{\alpha[p + (1 - p) e_0]} \ln \left( \frac{p + 2(1 - p) e_0}{2[p + (1 - p) e_0]} \right) + \\
\frac{(1 - p) e_0}{\alpha^2[p + (1 - p) e_0]} \left[ Li_2 \left( \frac{p}{2[p + (1 - p) e_0]} \right) - Li_2 \left( \frac{p e^{-\alpha r}}{2[p + (1 - p) e_0]} \right) \right] + \\
\frac{(1 - p) r e_0}{p + (1 - p) e_0} - \frac{1}{\alpha} \ln \left( \frac{p + 2(1 - p) e_0}{p(2 - e^{-\alpha r}) + 2(1 - p) e_0} \right) + \\
\frac{1 - r^2}{2} - \frac{1}{\alpha} \ln \left( 1 + 2 \frac{1 - p_0 e_0 e^{\alpha(1-r)}}{p} \right) + \frac{r}{\alpha} \ln \left( \frac{p + 2(1 - p) e_0}{p} \right) + \\
\frac{1}{\alpha^2} \left[ Li_2 \left( -2 \frac{1 - p_0 e_0 e^{\alpha(1-r)}}{p} \right) - Li_2 \left( -2 \frac{1 - p_0 e_0 e^{\alpha(1-r)}}{p} \right) \right] + \\
\frac{(1 - r) e_0}{e_0} + \frac{e_0}{\alpha e_0} \ln \left( \frac{p e^{-\alpha(1-r)} + 2(1 - p) e_0}{p + 2(1 - p) e_0} \right) \quad (xxvii)
\]
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