Online Appendix for "Following the Customers: Dynamic Competitive Repositioning" *

Z. Eddie Ning
(Cheung Kong Graduate School of Business)

J. Miguel Villas-Boas
(University of California, Berkeley)

December, 2020

*E-mail addresses: zhaoning@ckgsb.edu.cn and villas@haas.berkeley.edu.

1. Expected Duration of Differentiation and Industry Profits

Consider the duration of differentiation and co-positioning in the market. Starting from a point of differentiation we can see the expected duration going forward until when the firms choose to co-position, which occurs when the consumer preferences x reach either x^* or $1 - x^*$. Similarly, starting from a position of co-positioning we can consider the expected duration going forward until when a firm chooses to differentiate, which can start occurring when the consumer preferences x reach a distance $\underline{x}$ from where the firms are positioned.

Let $F_d(x)$ be the expected duration that firms continue to be differentiated if firms are differentiated and consumer preferences start at x. We know that for firms to continue to be differentiated we have that $x \in (1 - x^*, x^*)$. For a small time interval dt we can construct the equation of evolution of $F_d(x)$ as

$$F_d(x) = dt + E[F_d(x + dx)] \tag{1}$$

which results, by Itô's Lemma, in $F_d(x)'' = -2/\sigma^2$.

Using the fact that in equilibrium $F_d(x^*) = F_d(1 - x^*) = 0$, we can obtain

$$F_d(x) = \frac{1}{\sigma^2}[x(1 - x) - x^*(1 - x^*)]. \tag{2}$$

Co-positioned firms can first become differentiated when consumers preferences reach a distance $\underline{x}$ from where the co-positioned firms are located. Starting from this possible first instance of differentiation, the expected duration of differentiation is $F_d(\underline{x})$.

Consider now the period of time when firms are co-positioned. Let $F_s(x)$ be the expected duration that firms continue to be co-positioned if consumer preferences start at at a distance x from the firms' positioning, and one firm chooses to differentiate at the lowest possible x, $\underline{x}$. This is the lowest x for which there is a positive hazard rate of one of the firms repositioning, thus $F_s(x)$ captures the lowest expected duration of co-positioning.[1] We know that for firms to be co-positioned we have that $x < \underline{x}$. For a small time interval dt we can construct the

[1]This analysis considering the lowest expected duration of co-positioning gets sharper results and focuses on the case in which differentiation can be the highest. Considering the overall expected duration can be done numerically, and gets more complex because the consideration of the firms' mixed strategies yields a differential equation that is not solvable analytically as presented in the Appendix. Note also that in the next two sections, where we consider the cases of collusion and social welfare optimum, these computations give the exact expected duration, as in those cases one product moves with probability one at $\underline{x}$.

equation of evolution of $F_s(x)$ as

$$F_s(x) = dt + E[F_s(x + dx)] \tag{3}$$

which results, similarly, in $F_s''(x) = -2/\sigma^2$.

By assuming the lowest possible x at which a firm can choose to reposition, $\underline{x}$, as the state at which a firm repositions (to get the longest possible period of differentiation) we get $F_s(\underline{x}) = 0$. Furthermore, we have $F_s'(0) = 0$, as the process x has a reflecting boundary at zero.

We can then obtain

$$F_s(x) = \frac{1}{\sigma^2}[\underline{x}^2 - x^2]. \tag{4}$$

The lowest x for which differentiated firms become co-positioned is when consumer preferences reach a distance $(1-x^*)$ from where the co-positioned firms are located. Starting from this instance of co-positioning, the expected duration of co-positioning is $F_s(1-x^*)$.

For the long-run, we can get an overestimate of the fraction of time during which firms are differentiated, α_d, as $\frac{F_d(\underline{x})}{F_d(\underline{x})+F_s(1-x^*)}$ which yields

$$\alpha_d = x^* - \underline{x}. \tag{5}$$

From this, using (18) in the paper, we can obtain that for K small, the fraction of time with differentiated firms increases in the repositioning costs K, in the variability of consumer preferences σ^2, and in the importance of the attribute on which the firms can reposition δ. Considering the difference between the x^* and $\underline{x}$ curves, Figures 3-6 in the paper illustrate how the fraction during which firms are differentiated depends on K, δ, σ^2, and r. For the case when $K = .4, \delta = 4, r = .1$, and $\sigma^2 = .2$, the fraction of time when firms are differentiated is $\alpha_d = 16\%$.

Now we look at the comparative statics on industry profits when K is small. Consider the expected duration between one instance when one of the co-located firms repositions to differentiate to the next instance when one of the co-located firms repositions to differentiate. That is, two co-located firms first become differentiated when one firm repositions, then they co-locate again when the second repositioning takes places, then they become differentiated again when the third repositioning occurs. We can construct an overestimate of the expected duration between the first repositioning and the third repositioning, which will give us a lower bound on the frequency at which the repositioning cost is paid.

From before, we have $F_d(\underline{x})$, which is the maximum expected duration between the first and the second repositioning. Consider now the period of time between the second and third repositioning. Let $\widetilde{F}_s(x)$ be the expected duration that firms continue to be co-positioned if consumer preferences start at at a distance x from the firms' positioning, and one firm chooses to differentiate at the highest possible x, x^*. So $\widetilde{F}_s(x)$ captures an upper bound on the expected duration of co-positioning.

Similar to the derivation of $F_s(x)$, we can obtain

$$\widetilde{F}_s(x) = \frac{1}{\sigma^2}[x^{*2} - x^2]. \tag{6}$$

Starting from the first instance of co-positioning, when consumer preferences are at a distance of $(1 - x^*)$ from where the co-positioned firms are located, the expected duration of co-positioning is $F_s(1 - x^*)$.

We now have an overestimate of the expected duration between the first repositioning and the third repositioning as $F_d(\underline{x}) + \widetilde{F}_s(1 - x^*)$. Using equations (16)-(18) from Section 3 of the main text, we can obtain

$$F_d(\underline{x}) + \widetilde{F}_s(1 - x^*) = \frac{1}{\sigma^2}[x^* + \underline{x} - 1 + (x^* + \underline{x})(x^* - \underline{x})] = \frac{2}{\sigma^2}\sqrt[3]{\frac{9\sigma^2}{4\delta}}K^{1/3} + o(K^{2/3}). \tag{7}$$

which decreases in σ^2 and in δ for K small, and both derivatives are of the order $K^{1/3}$ for K small.

Then a lower bound on the industry repositioning cost per unit of time, without discounting, can be written as $2K/[F_d(\underline{x}) + \widetilde{F}_s(1 - x^*)]$, which increases in σ^2 and in δ for K small, and both derivatives are of the order $K^{2/3}$ for K small.

Now consider the industry flow profits per unit of time. When firms are co-located, the industry flow profit is 1. When firms are differentiated, the flow profit of a firm when the consumers are at a distance x from that firm is $\frac{1}{2}\left[1 + \frac{\delta(1-2x)}{3}\right]^2$. So the industry flow profits when firms are differentiated is $1 + \left[\frac{\delta(2x-1)}{3}\right]^2$, which is bounded above by $1 + \left[\frac{\delta(2x^*-1)}{3}\right]^2$ because $x \in (1 - x^*, x^*)$ if firms are differentiated.

We have $x^* - \underline{x}$ as an overestimate of the fraction of time during which firms are differentiated. So an upper bound on the industry flow profit per unit of time, without discounting, can be written as

$$1 + (x^* - \underline{x})\left[\frac{\delta(2x^* - 1)}{3}\right]^2 \tag{8}$$

Again using equations (16)-(18) from Section 3 of the main text, we can show that the expression $(x^* - \underline{x})\left[\frac{\delta(2x^*-1)}{3}\right]^2$ is of the order $K^{4/3}$ for K small, which is a smaller order than the lower bound on the industry repositioning cost per unit of time. Thus for K small, the effects of σ^2 and δ on repositioning costs per unit of time dominate their effects of flow profits per unit of time. The total industry profits per unit of time must then be increasing in σ^2 and in δ for K small. Because the duration $F_d(\underline{x}) + \widetilde{F}_s(1-x^*)$ approaches 0 as $K \to 0$, these results also hold under time discounting.

The results are summarized as follows:

Proposition A.1. *Consider K small. Then, the fraction of time with differentiated firms increases and the net present value of industry profits decreases in the repositioning costs, K, in the variability of consumer preferences, σ^2, and in the importance of the attribute on which the firms can reposition, δ.*

2. Collusion

We consider now the optimal collusive behavior of the two firms and compare it with the competitive case. We analyze first the case where collusion is only on the repositioning decisions, keeping the price equilibrium as competitive, and then consider when firms collude on both repositioning and prices. Note that the collusive case can also be seen as the case in which both products are carried by the same firm which is an useful benchmark to consider. This can also be seen as the effect of a merger between two competing firms. The case where collusion is only on repositioning decisions, keeping the price equilibrium as competitive, can also be seen as the case in which a firm carries both products, but the pricing decisions are controlled by two separate (and competing) divisions within the company. This could be seen as a case of a firm that has separate brand managers for each of their brands, who manage the tactical decisions (such as price), but where the strategic decisions (such as repositioning) are coordinated at the firm level. This case can also be seen as a benchmark where we study first the effect of just coordinating on the repositioning decisions, and then study the effect of coordinating on both repositioning and pricing.

2.1. Collusive Repositioning with Competitive Pricing

As noted in the paper, when the price equilibrium is competitive and firms are positioned in the same location, the profit for each firm is 1/2, independent of the location of the consumer preferences. Let $\pi_s(x)$, in this Section, represent the payoff for the sum of the profits of the two firms when firms are positioned at the same location, and consumers are at a distance x from the firms' location. Then, $\pi_s(x) = 1, \forall x$.

Also as noted in the paper, when the price equilibrium is competitive and firms are positioned at different locations, the profit of a firm when the consumers are at a distance x from that firm is $\frac{1}{2}\left[1 + \frac{\delta(1-2x)}{3}\right]^2$. Let $\pi_d(x)$, in this section, represent the payoff for the sum of the profits of the two firms when firms are positioned in different locations, and consumers are at a distance x from one of the firms' location. Then, $\pi_d(x) = 1 + \left[\frac{\delta(1-2x)}{3}\right]^2$.

Because $\pi_d(x) \geq \pi_s(x), \forall x$, with strict inequality almost everywhere, we then have that when firms collude on repositioning and compete on prices, they will never reposition again once firms are in different locations (that is, once they are differentiated). This captures the idea that firms like to be differentiated when competing on price, as known from the static differentiation literature. However, under competitive repositioning, as obtained in Section 3 in the paper, firms are not able to remain differentiated, and can remain in the same location for some period, depending on the evolution of consumer preferences. That is, there is more repositioning in the competitive market than if firms collude on repositioning but compete on prices.

2.2. Full Collusion

Consider now the case of collusion on both repositioning and prices. In this case, when firms are positioned in the same location, and consumers are at a distance x from the firms' location, collusive pricing requires that each firm prices such that the market is fully covered (given v large enough, as assumed above), and the marginal consumer is indifferent between buying and not buying the product. This results in a price for both firms of $v - \delta x - 1/2$, and industry profits of $\pi_s(x) = v - \delta x - 1/4$.

When firms are positioned in different locations, and consumers are at a distance x from one of the firms, say a distance x from Firm 1, the collusive pricing outcome would be for firms to price such that the market is covered, and a consumer who is indifferent between purchasing the product of either firm gets a surplus of zero. Then, there is going to be a z^*, which is the distance on attribute z from Firm 1, such that that consumer gets zero surplus and is indifferent between purchasing either product. That is, Firm 1 would charge a price of $p_1^* = v - \delta x - z^*$ and get a demand of z^* and Firm 2 would charge a price of $p_2^* = v - \delta(1-x) - (1-z^*)$ and get a demand of $(1-z^*)$. Maximizing the industry profits on z^* yields industry profits of

$$\pi_d(x) = v - \delta(1-x) - 1 + 2\left[\frac{2+\delta(1-2x)}{4}\right]^2. \tag{9}$$

In this Section, let $V_s(x)$ be the net present value of industry profits when firms are positioned at the same location, and consumer preferences are at a distance x from the firms' location, and let $V_d(x)$ be the net present value of industry profits when firms are positioned in different locations, and consumer preferences are at a distance x from the positioning of one of the firms. Then, similarly to the analysis in the paper we can obtain

$$V_s(x) = \frac{\pi_s(x)}{r} + A_s e^{\lambda x} + B_s e^{-\lambda x} \tag{10}$$

$$V_d(x) = \frac{\pi_d(x)}{r} + \frac{\pi_d''}{\lambda^2 r} + A_d e^{\lambda x} + B_d e^{-\lambda x}, \tag{11}$$

for some coefficients A_s, B_s, A_d, and B_d to be determined.

The optimum is characterized by an $\underline{x}$ and an x^* such that, when firms have the same positioning, and consumer preferences are at a distance $\underline{x}$, one of the firms repositions, and when firms have different positionings, and when the firm farther away from the consumer preferences is at a distance x^* from those consumer preferences, that firm repositions. Value-matching and smooth-pasting at $\underline{x}$ requires

$$V_s(\underline{x}) = V_d(1-\underline{x}) - K \tag{12}$$

$$V_s'(\underline{x}) = -V_d'(1-\underline{x}). \tag{13}$$

Value-matching and smooth-pasting at x^* requires

$$V_s(1-x^*) = V_d(x^*) + K \tag{14}$$

$$-V_s'(1-x^*) = V_d'(x^*). \tag{15}$$

Finally, symmetry of $V_d(x)$ at $x = 1/2$ requires $V_d'(1/2) = 0$. This condition plus (12)-(15) determine $\underline{x}$ and x^* (as described in the Appendix).

To get sharper results we can consider the case when $K \to 0$. In that case, we can obtain that $\underline{x}, x^* \to 1/2$. When the cost of repositioning converges to zero, the full collusion outcome also involves repositioning right away to the side of the market that is closer to the consumer preferences.

We can also obtain the speed of convergence when $K \to 0$. We can obtain that as $K \to 0$ we have

$$\frac{x^* - 1/2}{K^{1/3}} \to \sqrt[3]{\frac{3\sigma^2}{2\delta}} + \frac{1}{2}\sqrt[3]{\frac{\delta\sigma^4}{12}}K^{1/3}, \tag{16}$$

$$\frac{\underline{x} - 1/2}{K^{1/3}} \to \sqrt[3]{\frac{3\sigma^2}{2\delta}} - \frac{1}{2}\sqrt[3]{\frac{\delta\sigma^4}{12}}K^{1/3}, \tag{17}$$

$$\frac{x^* - \underline{x}}{K^{2/3}} \to \sqrt[3]{\frac{\delta\sigma^4}{12}}. \tag{18}$$

This structure of the limits of $\underline{x}$ and x^* is similar to the one in the competitive case considered in the previous section, and all the results stated in Proposition 2 in the paper also apply for the full collusion case considered in this section. More interestingly, we can compare the thresholds of repositioning in the competitive case with the ones in the collusion case.

Let $\underline{x}_{comp}$ and x^*_{comp} be the values of $\underline{x}$ and x^*, respectively, from the competitive equilibrium in the paper. Let $\underline{x}_{coll}$ and x^*_{coll} be the values of $\underline{x}$ and x^*, respectively, in the full collusion case. We can then obtain:

Proposition A.2. *For K small, we obtain $\underline{x}_{coll} < \underline{x}_{comp}, x^*_{coll} < x^*_{comp}$, and $x^*_{coll} - \underline{x}_{coll} > x^*_{comp} - \underline{x}_{comp}$.*

This shows that, under full collusion, when the cost of repositioning is small, firms reposition more frequently than in the competitive equilibrium case. In the competitive case, a firm repositions because of its private incentives to reposition. In the full collusion case, one firm repositions because of the incentives for the industry, which is able to capture the whole value of the repositioning. The whole value of the repositioning under collusive pricing is greater than the private incentives under price competition, and, therefore, the full collusion case results in more repositioning than the competitive case. Thus a monopoly owning both products would on average provide better products than two competing firms would.

Also interestingly, as $x^* - \underline{x}$ is greater in this case than in the case of competition, we have, by (5), that there is more product differentiation under collusion than in the competitive market case. On the other hand, static models of product positioning with price competition often imply that two products owned by competing firms are more differentiated than two products owned by a single firm. In a Hotelling model with quadratic transportation costs, the principle of maximum differentiation suggest that differentiation is higher under competition than under consolidation (d'Aspremont, Gabszewicz, and Thisse 1978). Similarly, Moorthy (1988) find that in a vertically differentiated market, two competing firms position farther apart than two products owned by a monopoly. Empirically, some papers find that after merger, products are repositioned to be more differentiated from each other (Berry and Waldfogel 2001, George 2007, Sweeting 2010). Our model with evolving consumer preferences offer a potential explanation for such outcome.

Figures 1-3 present numerically how x^* and $\underline{x}$ compare between the collusion and competitive case and evolve as a function of K, illustrating that the results in Proposition A.2 seem to hold for larger K. Similarly, Figures 4-9 illustrate how that comparison evolves as a function of δ, σ^2, and r.

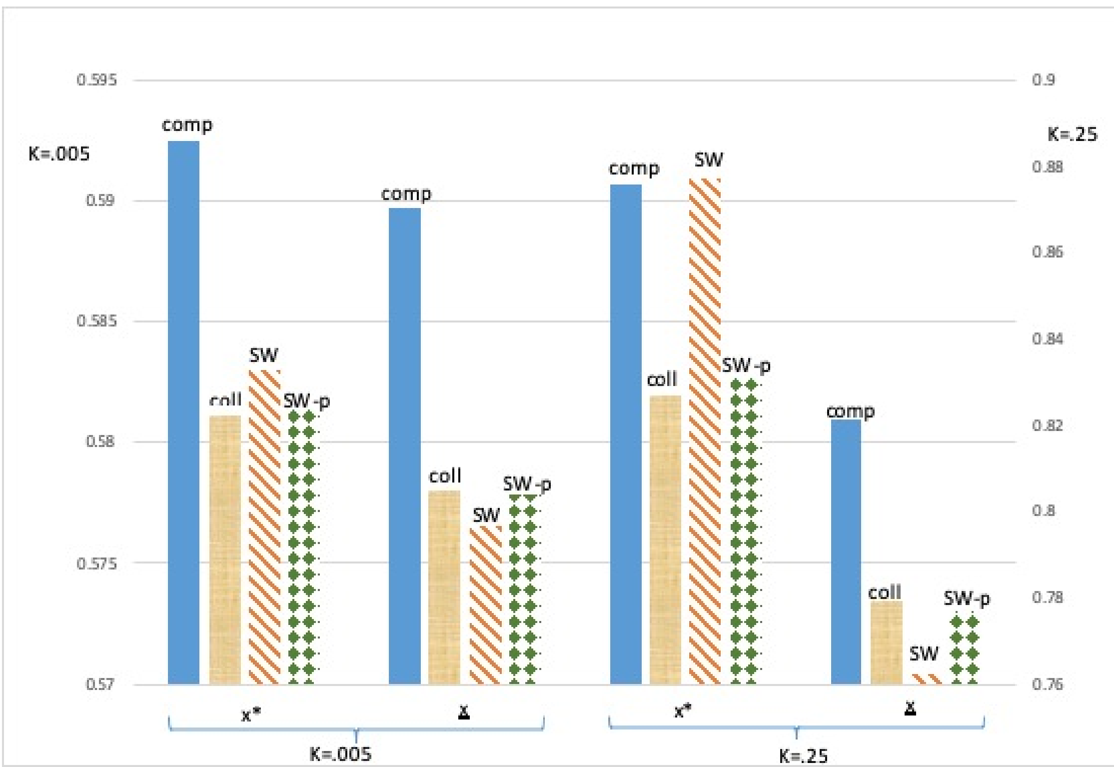

Figure 1: Comparison of x^* and $\underline{x}$ for K low ($K = .005$) and high ($K = .25$) for the competitive ("comp"), collusion ("coll"), social welfare ("SW"), and social welfare under competitive pricing ("SW-p") cases, for $\sigma^2 = .1$, $\delta = 1.5$, and $r = .1$. Note that the scale for the $K = .005$ case is on the left vertical axis, and the scale for the $K = .25$ case is on the right vertical axis.

3. Social Welfare

We consider now the optimal social welfare and compare it with the competitive and

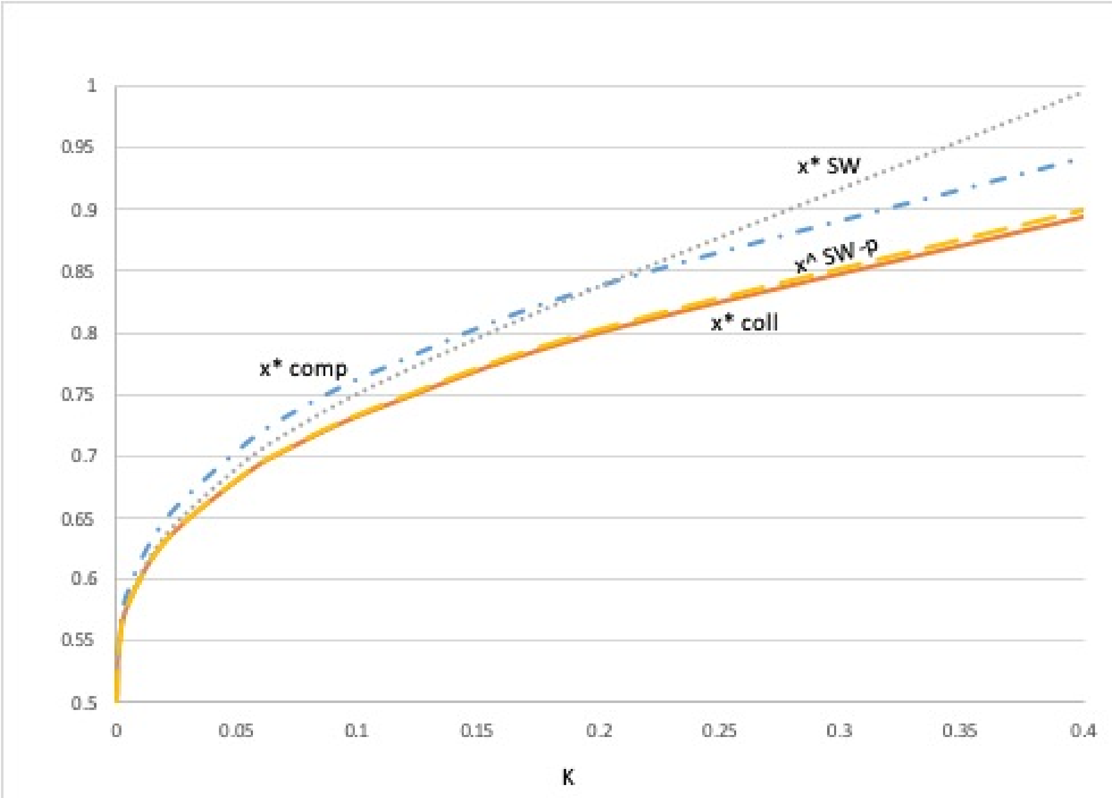

Figure 2: Evolution of x^* as a function of K for the competitive ("comp"), collusion ("coll"), social welfare ("SW"), and social welfare under competitive pricing ("SW-p") cases, for $\sigma^2 = .1, \delta = 1.5$, and $r = .1$.

collusive cases. We analyze first the case in which social welfare is only optimized on the repositioning decisions, keeping the price equilibrium as competitive, and then consider the full social welfare optimum.

3.1. Social Welfare under Competitive Pricing

Consider the question of what is optimal in terms of repositioning when firms continue to price competitively. This case would be relevant when the social planner could implement some regulation on the repositioning behavior, but would have to allow firms to price competitively.

Similarly to the analysis presented above, if firms have the same positioning, then both firms would price at 1. If firms are positioned in different locations then a firm with consumers at distance x would price at $1 + \delta\frac{1-2x}{3}$. That is, competitive pricing distorts the consumers toward the less desirable firm when firms are positioned in different locations, because the less desirable firm prices at a lower price. This is going to be a force for firms not to be

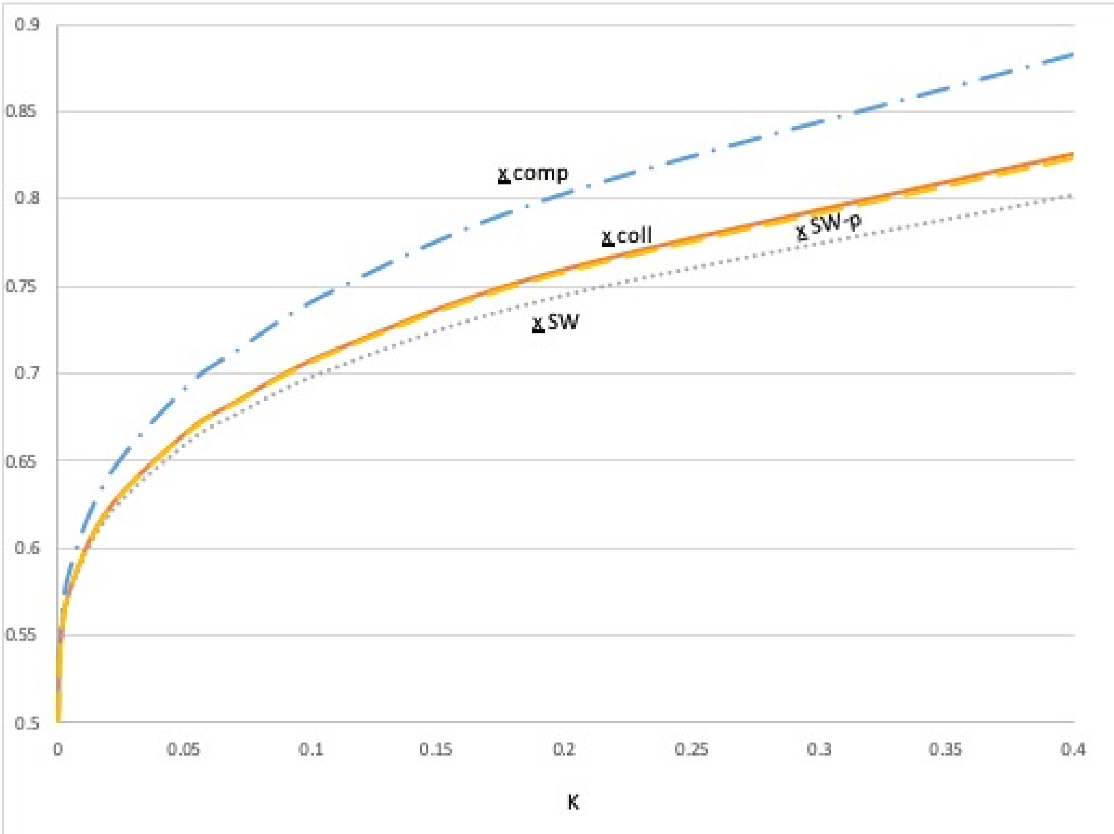

Figure 3: Evolution of $\underline{x}$ as a function of K for the competitive ("comp"), collusion ("coll"), social welfare ("SW"), and social welfare under competitive pricing ("SW-p") cases, for $\sigma^2 = .1, \delta = 1.5$, and $r = .1$.

positioned differently when maximizing social welfare. Note also that, as we assumed v large enough, there are no market expansion effects of firms pricing lower. That is, the social welfare only has to do with the costs of repositioning and the gross utility received by the consumers from the product that is allocated to them.

Given the demand allocation that results from these prices, we can then obtain the social welfare when firms have the same positioning, with consumers at a distance x, for which we use in this subsection the notation $\pi_s(x)$, and the social welfare when firms are positioned in different locations, with consumers at distance x from one of the firms, for which we use in this subsection the notation $\pi_d(x)$. This yields

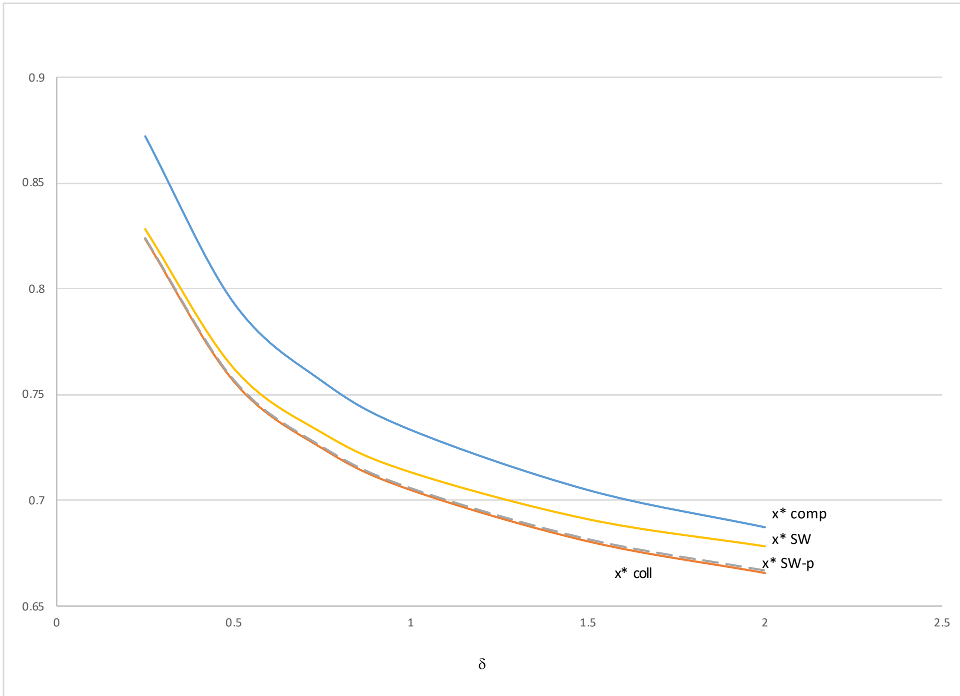

Figure 4: Evolution of x^* as a function of δ for the competitive ("comp"), collusion ("coll"), social welfare ("SW"), and social welfare under competitive pricing ("SW-p") cases, for $\sigma^2 = .1, K = .05$, and $r = .1$.

$$\pi_s(x) = v - \delta x - \frac{1}{4} \tag{19}$$

$$\pi_d(x) = v - \delta(1-x) - \frac{7}{10} + 5\left[\frac{9/5 + \delta(1-2x)}{6}\right]^2. \tag{20}$$

In this Section, let $V_s(x)$ be the expected net present value of social welfare payoffs when firms are positioned at the same location, and consumer preferences are at a distance x from the firms' location, and let $V_d(x)$ be the expected net present value of social welfare payoffs when firms are positioned in different locations, and consumer preferences are at a distance x from the positioning of one of the firms. Then we can obtain the expressions for the form of these value functions exactly as in the last section, (10) and (11), now with different functions $\pi_s(x)$ and $\pi_d(x)$, as described above.

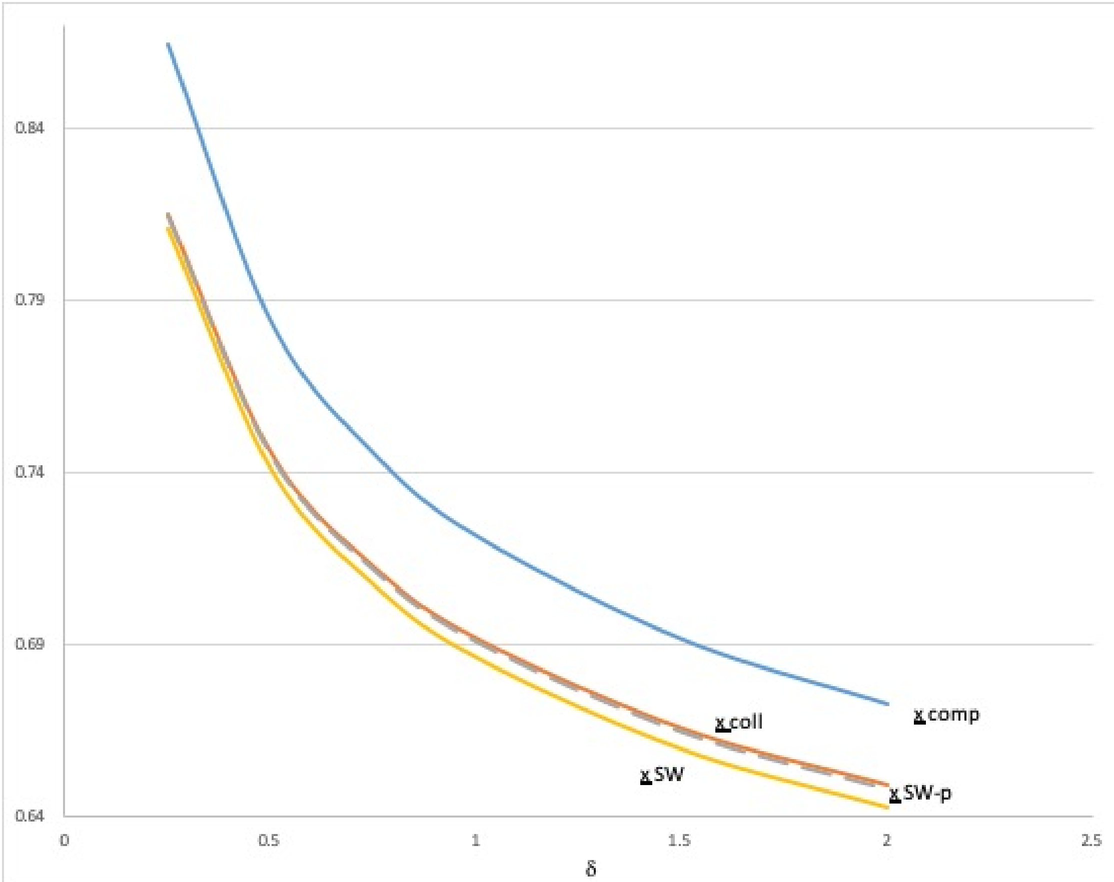

Figure 5: Evolution of $\underline{x}$ as a function of δ for the competitive ("comp"), collusion ("coll"), social welfare ("SW"), and social welfare under competitive pricing ("SW-p") cases, for $\sigma^2 = .1, K = .05$, and $r = .1$.

The optimum is also characterized by an $\underline{x}$ and an x^* (different $\underline{x}$ and x^*) such that, when firms have the same positioning, and consumer preferences are at a distance $\underline{x}$, one of the firms repositions, and when firms have different positionings, and when the firm farther away from the consumer preferences is at a distance x^* from those consumer preferences, that firm repositions. Value-matching and smooth-pasting at $\underline{x}$ and x^* require, as in the previous section, that (12)-(15) have to satisfied. Again, symmetry of $V_d(x)$ at $x = 1/2$ requires $V_d'(1/2) = 0$. These conditions, as in the previous Section, determine $\underline{x}$ and x^* (as described in the Appendix).

To get sharper results, we can consider the case when $K \rightarrow 0$. In that case, we can obtain again that $\underline{x}, x^* \rightarrow 1/2$. When the cost of repositioning converges to zero, the social welfare optimum subject to competitive pricing also involves repositioning right away to the side of the market that is closer to the consumer preferences.

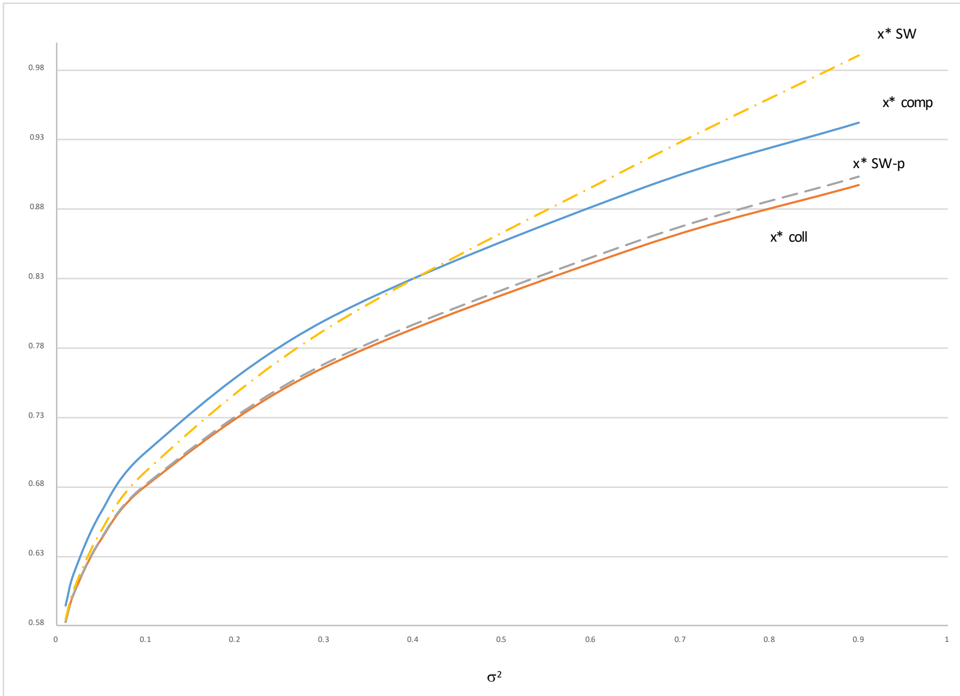

Figure 6: Evolution of x^* as a function of σ^2 for the competitive ("comp"), collusion ("coll"), social welfare ("SW"), and social welfare under competitive pricing ("SW-p") cases, for $K = .05, \delta = 1.5$, and $r = .1$.

We can also obtain that as $K \to 0$ we have

$$\frac{x^* - 1/2}{K^{1/3}} \to \sqrt[3]{\frac{3\sigma^2}{2\delta}} + \frac{5}{18}\sqrt[3]{\frac{2\delta\sigma^4}{3}}K^{1/3}, \tag{21}$$

$$\frac{\underline{x} - 1/2}{K^{1/3}} \to \sqrt[3]{\frac{3\sigma^2}{2\delta}} - \frac{5}{18}\sqrt[3]{\frac{2\delta\sigma^4}{3}}K^{1/3}, \tag{22}$$

$$\frac{x^* - \underline{x}}{K^{2/3}} \to \frac{5}{9}\sqrt[3]{\frac{2\delta\sigma^4}{3}}. \tag{23}$$

This structure of the limits of $\underline{x}$ and x^* is similar to the one in the competitive case considered in Section 3 in the paper and in the collusion case considered in the previous Section, and all the results stated in Proposition 2 also apply for the case of the social welfare optimum subject to competitive pricing considered in this Section. More interestingly, we can compare the thresholds of repositioning in the competitive case and in the collusion case with the ones in the case of the social welfare optimum subject to competitive pricing.

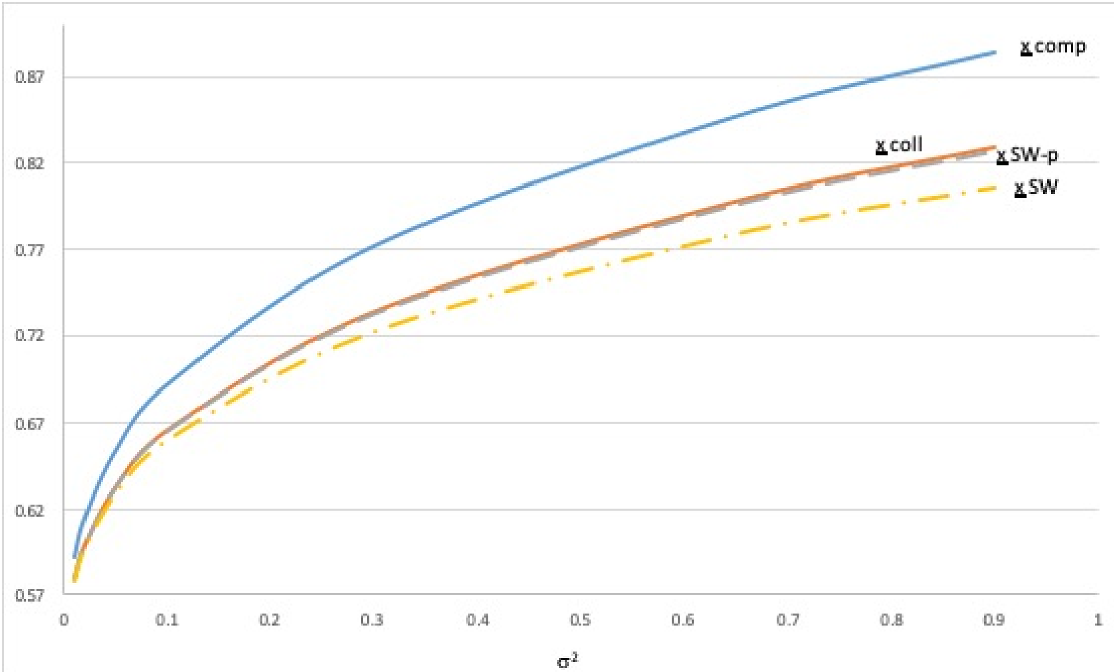

Figure 7: Evolution of $\underline{x}$ as a function of σ^2 for the competitive ("comp"), collusion ("coll"), social welfare ("SW"), and social welfare under competitive pricing ("SW-p") cases, for $K = .05, \delta = 1.5$, and $r = .1$.

Let $\underline{x}_{SW-p}$ and x^*_{SW-p} be the values of $\underline{x}$ and x^*, respectively, in the case of the social welfare optimum subject to competitive pricing. We can then obtain:

Proposition A.3. *For small K, we obtain $\underline{x}_{SW-p} < \underline{x}_{coll} < \underline{x}_{comp}, x^*_{coll} < x^*_{SW-p} < x^*_{comp}$, and $x^*_{SW-p} - \underline{x}_{SW-p} > x^*_{coll} - \underline{x}_{coll} > x^*_{comp} - \underline{x}_{comp}$.*

This shows that, under the social optimum subject to competitive pricing, when the cost of repositioning is small, firms reposition more frequently than in the competitive equilibrium case. In the competitive case, a firm repositions because of its private incentives to reposition. In the case of the social welfare optimum subject to competitive pricing, one firm repositions because of the incentives for social welfare, which includes the whole value of the repositioning, except for the mis-allocation resulting from competitive pricing. The whole value of the repositioning for social welfare is greater than the private incentives under price competition, and, therefore, the case of the social welfare optimum subject to competitive pricing results in more repositioning than the competitive case.

The relationship of this case to the full collusion case is also interesting. First, note that the thresholds of the social optimum under competitive pricing are closer to the thresholds under full collusion than to the thresholds in the competitive market equilibrium. The differ-

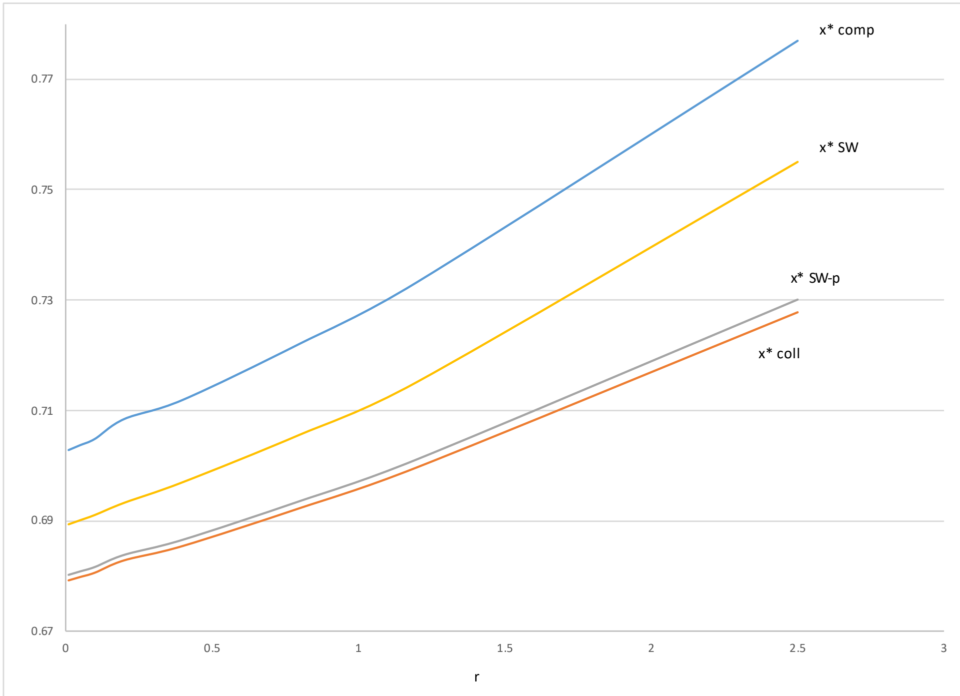

Figure 8: Evolution of x^* as a function of r for the competitive ("comp"), collusion ("coll"), social welfare ("SW"), and social welfare under competitive pricing ("SW-p") cases, for $\sigma^2 = .1, \delta = 1.5$, and $K = .05$.

ence between the thresholds of the case of the social welfare optimum subject to competitive pricing and the full collusion case are on the order of $K^{2/3}$, while the difference between the thresholds of the case of the social welfare optimum subject to competitive pricing and the competitive market equilibrium case are on the order of $K^{1/3}$, which is larger for K small. This would suggest that the outcome of full collusion is close to the social welfare optimum subject to competitive pricing, with the difference that the consumers would get a much larger surplus under the social welfare optimum subject to competitive pricing than in the full collusion case.

Second, the case of the social welfare optimum subject to competitive pricing has longer periods of firms being differentiated than in the full collusion case. In the full collusion case, the industry is not able to appropriate the utility generated to the infra-marginal consumers due to differentiation among products, which leads to the result where the firms are not differentiated enough from each other. In the case of the social welfare optimum subject to

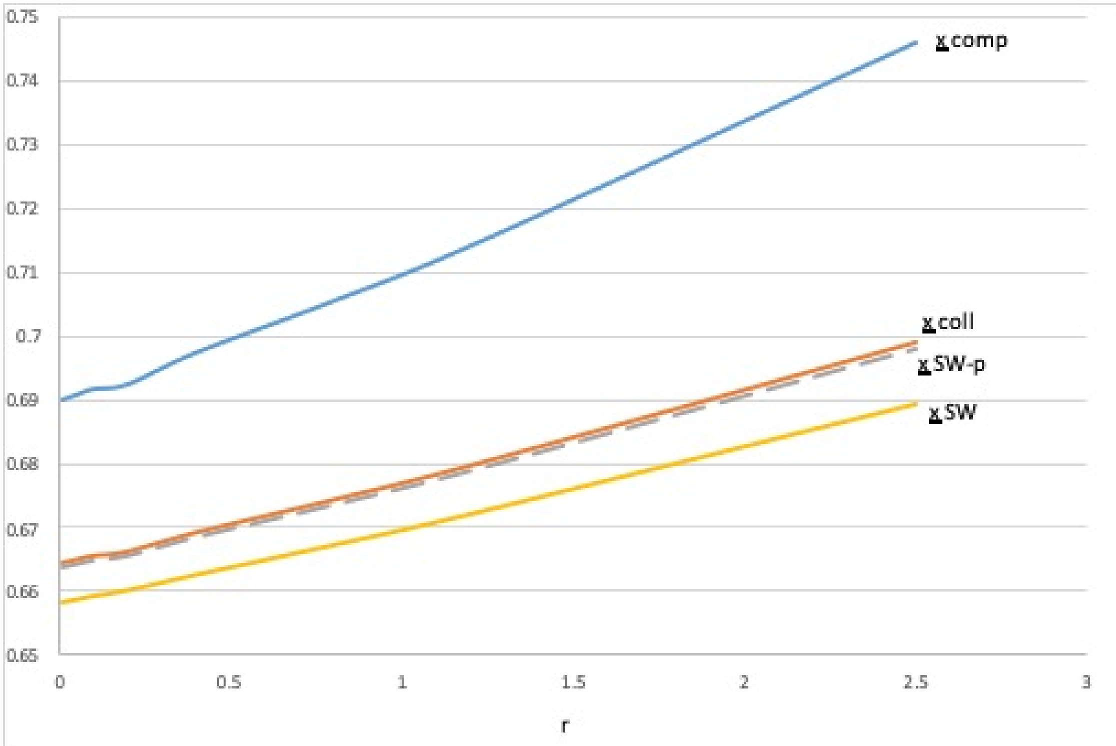

Figure 9: Evolution of $\underline{x}$ as a function of r for the competitive ("comp"), collusion ("coll"), social welfare ("SW"), and social welfare under competitive pricing ("SW-p") cases, for $\sigma^2 = .1, \delta = 1.5$, and $K = .05$.

competitive pricing, the social planner includes the utility generated to the infra-marginal consumers due to the products being differentiated, and therefore keeps the products differentiated for a longer period.

Figures 1-3 present numerically how x^* and $\underline{x}$ compare between the case of the social welfare optimum subject to competitive pricing, the collusion case, and the competitive case, and how they evolve as a function of K, illustrating that the results in Proposition A.3 in this Online Appendix seem to hold for larger K. Similarly, Figures 4-9 illustrate how that comparison evolves as a function of δ, σ^2, and r.

3.2. Social Welfare

Consider the question of what is optimal for social welfare in terms of repositioning. That is, now we consider not only the effects of the cost of repositioning on welfare, but also the effects of optimally allocating demand across the two products. Because there is no competitive pricing distorting consumers toward the less desirable product when products are positioned in different locations, this may then allow firms to be positioned in different locations for longer periods.

When products are positioned in the same location, demand is equally distributed between the two products. When products are positioned in different locations, the allocation of demand is just done to maximize social welfare, which results in a product located at a distance x from the consumer preferences having a demand of $\frac{1}{2} + \delta\frac{1-2x}{2}$.

Given this demand allocation, we can then obtain the social welfare when firms have the same positioning, with consumers at a distance x, for which we use in this subsection the notation $\pi_s(x)$, and the social welfare when firms are positioned in different locations, with consumers at distance x from one of the firms, for which we use in this subsection the notation $\pi_d(x)$. This yields

$$\pi_s(x) = v - \delta x - \frac{1}{4} \tag{24}$$

$$\pi_d(x) = v - \delta(1-x) - \frac{1}{2} + \left[\frac{1+\delta(1-2x)}{2}\right]^2. \tag{25}$$

As noted in the previous subsection, let $V_s(x)$ be the expected net present value of social welfare payoffs when firms are positioned at the same location, and consumer preferences are at a distance x from the firms' location, and let $V_d(x)$ be the expected net present value of social welfare payoffs when firms are positioned in different locations, and consumer preferences are at a distance x from the positioning of one of the firms. Then we can obtain the expressions for the form of these value functions exactly as in the last section, (10) and (11), now with different functions $\pi_s(x)$ and $\pi_d(x)$, as described above in this subsection.

The optimum is also characterized by an $\underline{x}$ and an x^* (different $\underline{x}$ and x^*) such that, when firms have the same positioning, and consumer preferences are at a distance $\underline{x}$, one of the firms repositions, and when firms have different positionings, and when the firm farther away from the consumer preferences is at a distance x^* from those consumer preferences, that firm repositions. Value-matching and smooth-pasting at $\underline{x}$ and x^* require, as in the previous section, that (12)-(15) have to be satisfied. Again, symmetry of $V_d(x)$ at $x = 1/2$ requires $V_d'(1/2) = 0$. These conditions, as in the previous Section, determine $\underline{x}$ and x^* (as described in the Appendix).

To get sharper results we can consider the case when $K \to 0$. In that case, we can obtain again that $\underline{x}, x^* \to 1/2$. When the cost of repositioning converges to zero, the social welfare optimum subject to competitive pricing also involves repositioning right away to the side of the market that is closer to the consumer preferences.

We can obtain that as $K \to 0$ we have

$$\frac{x^* - 1/2}{K^{1/3}} \to \sqrt[3]{\frac{3\sigma^2}{2\delta}} + \frac{1}{2}\sqrt[3]{\frac{2\delta\sigma^4}{3}}K^{1/3}, \tag{26}$$

$$\frac{\underline{x} - 1/2}{K^{1/3}} \to \sqrt[3]{\frac{3\sigma^2}{2\delta}} - \frac{1}{2}\sqrt[3]{\frac{2\delta\sigma^4}{3}}K^{1/3}, \tag{27}$$

$$\frac{x^* - \underline{x}}{K^{2/3}} \to \sqrt[3]{\frac{2\delta\sigma^4}{3}}. \tag{28}$$

This structure of the limits of $\underline{x}$ and x^* is similar to the one in the other cases considered, and all the results stated in Proposition 2 in the paper also apply for the case of the social welfare optimum. More interestingly, we can compare the thresholds of repositioning in the other cases with the ones in the social welfare optimum case.

Let $\underline{x}_{SW}$ and x^*_{SW} be the values of $\underline{x}$ and x^*, respectively, in the case of the social welfare optimum. We can then obtain.

Proposition A.4. *For small K, we obtain $\underline{x}_{SW} < \underline{x}_{SW-p} < \underline{x}_{coll} < \underline{x}_{comp}$, $x^*_{coll} < x^*_{SW-p} < x^*_{SW} < x^*_{comp}$, and $x^*_{SW} - \underline{x}_{SW} > x^*_{SW-p} - \underline{x}_{SW-p} > x^*_{coll} - \underline{x}_{coll} > x^*_{comp} - \underline{x}_{comp}$.*

This shows that, under the social welfare optimum, when the cost of repositioning is small, firms also reposition more frequently than in the competitive equilibrium case. In the competitive case, a firm repositions because of its private incentives to reposition. In the case of the social welfare optimum, one firm repositions because of the incentives for social welfare, which includes the whole value of the repositioning. We had already seen that this held for the case of the social welfare optimum subject to competitive pricing, and this should also obviously then hold without the constraint of competitive pricing. The whole value of the repositioning for social welfare is greater than the private incentives in the competitive market equilibrium.

The relationship of this case to the other cases considered above is also interesting. First, note that the thresholds of the social welfare optimum are closer to the thresholds under full collusion and in the case of social welfare optimum subject to competitive pricing, than to the thresholds in the competitive market equilibrium. The difference between the thresholds of the social welfare case and the cases of full collusion and social welfare subject to competitive pricing are on the order of $K^{2/3}$, while the difference between the thresholds of these cases and the competitive market equilibrium are on the order of $K^{1/3}$, which is larger for K small. This would suggest that the outcome of full collusion is close to the social welfare optimum.

The fact that the collusive thresholds approach the social optimum at a faster rate than the competitive thresholds do implies that, when K is small, social welfare is higher under collusion than under competition. One can show that, when firms are co-located, flow social welfare is the same whether firms are collusive or competitive. When firms differentiate, flow social welfare is higher under collusion than under competition, because more consumers will go to the firm that is closer on attribute x under collusion due to less price competition from the farther firm. Thus social welfare is higher under collusion even if collusive firms follow competitive firms' repositioning strategies.

Second, the case of the full social welfare optimum has longer periods of firms being differentiated than under the constraint of competitive pricing when consumer preferences. In the case of the social welfare optimum subject to competitive pricing, there was an incentive for firms to be in the same location because of the distortion in the allocation of demand due to competitive pricing. Because this incentive disappears without the constraint of competitive pricing, the optimum then has longer periods of differentiation between firms.

Figures 1-3 present numerically how x^* and $\underline{x}$ compare between the social welfare case and the other cases, and how that comparison evolves as a function of K, illustrating that the results in Proposition A.4 in this Online Appendix seem to hold for larger K. However, note that the effect of longer periods with differentiation gets stronger with larger K, such that at some point we have $x^*_{SW} > x^*_{comp}$. Similarly, Figures 4-9 illustrate how that comparison evolves as a function of δ, σ^2, and r.

4. Deterministic Trend

In this Section we investigate the possibility of existing a deterministic trend in the evolution of the dimension x, the dimension in which the firm can reposition. We consider the case of unbounded x_t presented in subsection 3.5 in the paper, with the fixed costs of reposition, K, small enough such that if a firm is positioned at n, it repositions to $n+1$ at some point, the market is fully covered, and both firms have always a strictly positive market share in equilibrium.

We consider the case in which there is only a deterministic trend in the evolution of the consumer preferences (and there is no random component), and then discuss the case in which there is both a deterministic trend and a random component in the evolution of the consumer preferences.

When there is only a deterministic component in the evolution of the consumer preferences, we have that the evolution of the consumer preferences is set as

$$dx = h\,dt, \tag{29}$$

where h is a parameter. We set $h > 0$ without loss of generality.

To construct the market equilibrium, let us consider the case in which $x \in [n, n+1]$, and note that now, as there is a deterministic trend we need to consider thresholds for the repositioning of firms from n to $n+1$, and thresholds from the repositioning of firms from $n+1$ to n. Given that $h > 0$ and there is no random component in the evolution of preferences, on the equilibrium path the only relevant thresholds will be the ones for the repositioning from n to $n+1$, which can be represented as $n + \underline{x}^+$ and $n + x^{*+}$, which correspond to the thresholds $\underline{x}$ and x^* in the previous Sections, where there was only a random component in the evolution of preferences, and there was no deterministic trend. Let also $n + \underline{x}^-$ and $n + x^{*-}$ be the corresponding thresholds for the repositioning of firms from $n+1$ to n.

As stated in subsection 3.5 in the paper, when the costs of repositioning K are small enough we will have $\underline{x}^+, \underline{x}^-, x^{*+}, x^{*-} \in [0, 1]$.

Analogously to the analysis in the paper the form of the equilibrium will be as follows. Suppose that we start from a situation in which both firms are positioned at n and x is close to n. Then, at some point x will reach $n+\underline{x}^+$, and for $x \in (n+\underline{x}^+, n+x^{*+})$ firms continuously mix with some hazard rate between staying in the same positioning and repositioning to $n+1$. When one of the firm repositions to $n+1$ (only one can reposition in period of time dt), the other firm stays put at n until x reaches $n + x^{*+}$ at which point that firm also repositions to $n+1$ and both firms are then positioned at $n+1$. As the consumer preferences continue to evolve, at some point x reaches $n + 1 + \underline{x}^+$ and the process re-starts on the repositioning of the firms from $n+1$ to $n+2$, and so on.

Given that we have the nature of the equilibrium for any n, we can then without loss of generality look at the case of $n = 0$. Note also that, because of the trend, we have to keep track in the value functions of where the firm is located, and the definition of x is the location of the consumer preferences instead of the distance to where the firm is located as considered in the previous Sections. Let then $V_{s0}(x)$ and $V_{s1}(x)$ be the value functions when the firms have the same positioning at 0 and 1, respectively. And let $V_{d0}(x)$ and $V_{d1}(x)$ be the value function of a firm positioned at 0 and 1, respectively, when the competitor is positioned in the other location.

We can then have the Bellman equations of the different value functions as follows:

$$V_{si}(x) = \frac{1}{2}dt + e^{-r\,dt}[V_{si}(x) + V_{si}'(x)h\,dt] \text{ for i=0,1} \tag{30}$$

$$V_{d0}(x) = \frac{1}{2}\left[\frac{3+\delta(1-2x)}{3}\right]^2 dt + e^{-r\,dt}[V_{d0}(x) + V_{d0}'(x)h\,dt] \tag{31}$$

$$V_{d1}(x) = \frac{1}{2}\left[\frac{3+\delta(2x-1)}{3}\right]^2 dt + e^{-r\,dt}[V_{d1}(x) + V_{d1}'(x)h\,dt]. \tag{32}$$

Solving the resulting differential equations, together with the condition that the expected present value of profits is the same at $x = 0$ and $x = 1$, $V_{s0}(0) = V_{s1}(1)$, and value matching and smooth pasting at $\underline{x}^+$ and x^{*+} as in Section 3 in the paper, $V_{s0}(\underline{x}^+) = V_{d1}(\underline{x}^+) - K$, $V_{s0}'(\underline{x_0}^+) = V_{d1}'(\underline{x}^+)$, $V_{d0}(x^{*+}) = V_{s1}(x^{*+}) - K$, $V_{d0}'(x^{*+}) = V_{s1}'(x^{*+})$, and $V_{d1}(x^{*+}) = V_{s1}(x^{*+})$, we can determine the equilibrium thresholds (derivation presented in the Appendix) as

$$\underline{x}^+ = \frac{1}{2} + \frac{3}{2\delta}(\sqrt{1+2rK} - 1) \tag{33}$$

$$x^{*+} = \frac{1}{2} + \frac{3}{2\delta}(1 - \sqrt{1-2rK}), \tag{34}$$

and the hazard rate mixing probability when both firms are positioned at 0 and $x \in (\underline{x}^+, x^{*+})$ is determined analogously to the analysis in Section 3.

We can check directly that $x^{*+} > \underline{x}^+$, as it was assumed in the computation of the equilibrium, and that $\underline{x}^+, x^{*+} \to 1/2$, as $K \to 0$, when the repositioning costs go to zero, both firms reposition relatively quickly to the location closest to the consumer preferences. Note also that when K goes to zero, $\underline{x}^+$ and x^{*+} converge to $1/2$ at the same speed, while these thresholds converge to $1/2$ slower than K going to zero, in the case of Section 3 in the paper, where the evolution of preferences is only determined by the random component. That is, for the costs of repositioning, K, small, the thresholds for repositioning are farther away from $1/2$ in the only random component case than in the only deterministic trend case. This can be understood by the fact that in the only deterministic trend case the evolution of preferences is moving away from 0 for sure, which makes the firms reposition sooner, while in the only random component case, it is possible that the consumer preferences return to 0.

Note also that the difference $x^{*+} - \underline{x}^+$ represents an overestimate of the fraction of time that the products are differentiated, which also occurred in the only random component case. Noting that this difference $x^{*+} - \underline{x}^+$ converges to zero at the speed of K^2 when K goes to zero, which is faster than the speed of convergence in the only random component case (speed of $K^{2/3}$), we can obtain that there is less differentiation in the only deterministic trend case than in the only random component case.

It is also interesting to observe that the thresholds $\underline{x}^+$ and x^{*+} are independent of the evolution of preferences parameter h. That is, a faster evolution of preferences, greater h, does not affect these thresholds, but only affects the speed with which the firms reposition because the consumer preferences evolve faster to reach these thresholds, and changes the hazard rate mixing probability when both firms are positioned at 0 and $x \in (\underline{x}^+, x^{*+})$.

Finally, note that firms are slower to reposition, and stay differentiated for a longer period, when the discount rate is greater, and when the importance of the repositioning attribute, δ, is lower. We summarize these results in the following proposition.

Proposition A.5. *Consider that the costs of repositioning K are small. Then, the only deterministic trend case results in lower differentiation and lower thresholds for repositioning than the only random component case. Furthermore, in the only deterministic trend case the threshold to reposition are increasing in the repositioning costs, K, and in the discount rate, r, and decreasing in the importance of the repositioning attribute, δ.*

As we are in a case in which the evolution of preferences is only deterministic, if firms start positioned at 0 and $x < \underline{x}^+$, firms will never reposition from 1 to 0 on the equilibrium path. However, if firms start positioned at 1 it could be that they want to reposition to 0 if $x < \underline{x}^+$. This possibility was allowed for Section 3 in the paper of the only random component case, in which it was important to understand when firms wanted to reposition from 1 to 0, which occurred always on the equilibrium path. In fact, this possibility of repositioning from 1 to 0, can be important also when there is both a deterministic trend and a random component in the evolution of consumer preferences, in which case that repositioning may occur on the equilibrium path for any starting position, and computing it for the only deterministic trend case gives some insights for that more general model.

We then consider the reverse thresholds when firms resposition from from 1 to 0. Note that in the deterministic case, reverse repositioning can only happen in some early periods, if x_0 is low and at least one firm is positioned at 1 initially. Otherwise, because $\underline{x}^+$ and x^{*+} are greater than $\frac{1}{2}$, firms never reposition reversely once they begin repositioning in the direction of the consumer trend. We can obtain that the repositioning thresholds $\underline{x}^-$ and x^{*-} can be obtained by value matching at these thresholds,[2] $V_{s1}(\underline{x}^-) = V_{d0}(\underline{x}^-) - K$ and

[2]We do not have smooth-pasting conditions at these reverse thresholds. The smooth-pasting condition, which require the two value functions to have the same derivative, comes from the requirement that the

$V_{s0}(x^{*-}) - K = V_{d1}(x^{*-})$. From this we can obtain (see Appendix)

$$\lim_{K \to 0} \frac{\underline{x}^{-} - 1/2}{\sqrt{K}} = -\sqrt{\frac{3h}{\delta}}, \tag{35}$$

$$\lim_{K \to 0} \frac{x^{*-} - 1/2}{\sqrt{K}} = -\sqrt{\frac{3h}{\delta}}, \tag{36}$$

That is, the thresholds for the firms to move from 1 to 0 converge more slowly to 1/2 than K converging to zero. We can then obtain that, for K small, when $h > 0$, the thresholds for firms to move from 1 to 0 are farther away from the mid-point 1/2, than the thresholds for firms to move from 0 to 1.

We could also consider that the evolution of consumer preferences has both a random component and a deterministic component, $dx = h\,dt + \sigma^2\,dW$, where W represents the standardized Brownian motion. That case would get the composition of the effects of the only random component case considered in the previous sections, and of the only deterministic case considered in this Section. In particular, as in the only random component case the thresholds for firms to reposition move more quickly away from 1/2 than in the only deterministic case as the repositioning costs K increase from zero, we have then that the effects of the only random component case over the only deterministic trend case would be to increase the thresholds at which firms want to reposition.

player should prefer moving at the threshold to delaying an infinitesimal dt. In the case of deterministic trend, for reverse thresholds, this does not impose additional restriction on the shape of the value function, because x_t never reaches that threshold again once a player delays for dt. Consider x^{*-} for example, the firm at 1 should weakly prefer moving immediately than waiting. This means

$$V_{s0}(x^{*-}) \geq \frac{1}{2}\left[\frac{3 + \delta(2x-1)}{3}\right]^2 dt + e^{-r\,dt}[V_{d1}(x^{*-}) + V_{d1}'(x^{*-})h\,dt]$$

with value-matching and (32). This becomes $V_{d1}(x^{*-}) \geq V_{d1}(x^{*-})$, which is automatically satisfied. Similarly, there is no smooth-pasting condition at $\underline{x}^{-}$.

APPENDIX

Expected Duration of Firms in the Same Location. In the main text we considered the expected duration of firms in the same location when one firm repositions when x reaches $\underline{x}$. We now consider this expected duration, accounting for the fact firms decide to reposition with mixed strategies with hazard rate $\mu(x)$ for $x \in (\underline{x}, x^*)$. From the analysis in the main text, using $F_s'(0) = 0$, we have that

$$F_s(x) = a_0 - \frac{x^2}{\sigma^2} \tag{i}$$

for $x \in (0, \underline{x})$ where a_0 is a constant to be determined. For $x \in (\underline{x}, x^*)$ we have that the evolution of the expected duration $F_s(x)$ has to satisfy

$$F_s(x) = dt + [1 - \mu(x)\, dt]^2 E[F_x(x + dx)]. \tag{ii}$$

Using Itô's Lemma, this yields the differential equation

$$2\mu(x)F_s(x) = 1 + \frac{\sigma^2}{2}F_s''(x). \tag{iii}$$

Solving for (iii) together with the conditions $F_s(\underline{x}^-) = F_s(\underline{x}^+), F_s'(\underline{x}^-) = F_s'(\underline{x}^+)$ (smoothness of the expected duration function at $\underline{x}$), and $\lim_{x \to x^*} F_s(x) = 0$, we can obtain a_0 and the full characterization of $F_s(x)$ for $x \in [0, x^*)$, which can be obtained numerically.

Socially Optimal and Collusive Outcomes: We consider four cases for comparison: (1) socially optimal outcome, (2) collusion where firms maximize joint profit, (3) socially optimal repositioning with competitive pricing, and (4) collusive repositioning with competitive pricing. For the case of socially optimal outcome, the flow utilities are:

$$\pi_s(x) = v - \delta x - \frac{1}{4}$$
$$\pi_d(x) = v - \delta(1 - x) - \frac{1}{2} + \left[\frac{1 + \delta(1 - 2x)}{2}\right]^2$$

For the case of collusion or monopoly, the flow utilities are:

$$\pi_s(x) = v - \delta x - \frac{1}{2}$$
$$\pi_d(x) = v - \delta(1-x) - 1 + 2\left[\frac{2+\delta(1-2x)}{4}\right]^2$$

For the case of socially optimal repositioning with competitive pricing, the flow utilities are:

$$\pi_s(x) = v - \delta x - \frac{1}{4}$$
$$\pi_d(x) = v - \delta(1-x) - \frac{7}{10} + 5\left[\frac{9/5+\delta(1-2x)}{6}\right]^2$$

For the case of collusive repositioning with competitive pricing, the flow utilities are:

$$\pi_s(x) = 1$$
$$\pi_d(x) = 1 + \left[\frac{\delta(1-2x)}{3}\right]^2$$

Note that in the case of collusive repositioning with competitive pricing, the flow utility under different positions is always higher than the flow utility under same position. Thus if firms have different positions, they should never relocate. Below we first consider the other three cases. As noted in the text, the general solution to the decision maker's value functions has:

$$V_s(x) = \frac{\pi_s(x)}{r} + A_s e^{\lambda x} + B_s e^{-\lambda x}$$
$$V_d(x) = \frac{\pi_d(x)}{r} + \frac{\pi_d''}{\lambda^2 r} + A_d e^{\lambda x} + B_d e^{-\lambda x}$$

for $\lambda = \sqrt{\frac{2r}{\sigma^2}}$ and some coefficients A_s, B_s, A_d, and B_d.

When firms are on the same side, let $\underline{x}$ denote the threshold where one firm relocates. It must then be that we have value-matching and smooth-pasting at $\underline{x}$, which yields

$$V_s(\underline{x}) = V_d(1-\underline{x}) - K \tag{iv}$$
$$V_s'(\underline{x}) = -V_d'(1-\underline{x}) \tag{v}$$

When firms are on different sides, a firm repositions when its distance to consumers reaches

x^*. The value-matching and smooth-pasting conditions at x^* are:

$$V_s(1-x^*) = V_d(x^*) + K \tag{vi}$$

$$-V_s'(1-x^*) = V_d'(x^*). \tag{vii}$$

Finally, by the symmetry of $V_d(x)$ around $x = \frac{1}{2}$, we get $V_d'(\frac{1}{2}) = 0$, which yields:

$$A_d e^{\lambda} = B_d \tag{viii}$$

Equations (iv), (v), (vi), and (vii) form the following system of equations:

$$\frac{\pi_s(\underline{x})}{r} + A_s e^{\lambda \underline{x}} + B_s e^{-\lambda \underline{x}} = \frac{\pi_d(1-\underline{x})}{r} + \frac{\pi_d''}{r\lambda^2} + A_d e^{\lambda(1-\underline{x})} + B_d e^{-\lambda(1-\underline{x})} - K \tag{ix}$$

$$\frac{\pi_s'(\underline{x})}{\lambda r} + A_s e^{\lambda \underline{x}} - B_s e^{-\lambda \underline{x}} = -\frac{\pi_d'(1-\underline{x})}{\lambda r} - A_d e^{\lambda(1-\underline{x})} + B_d e^{-\lambda(1-\underline{x})} \tag{x}$$

$$\frac{\pi_s(1-x^*)}{r} + A_s e^{\lambda(1-x^*)} + B_s e^{-\lambda(1-x^*)} = \frac{\pi_d(x^*)}{r} + \frac{\pi_d''}{r\lambda^2} + A_d e^{\lambda x^*} + B_d e^{-\lambda x^*} + K \tag{xi}$$

$$-\frac{\pi_s'(1-x^*)}{\lambda r} - A_s e^{\lambda(1-x^*)} + B_s e^{-\lambda(1-x^*)} = \frac{\pi_d'(x^*)}{\lambda r} + A_d e^{\lambda x^*} - B_d e^{-\lambda x^*} \tag{xii}$$

Subtracting (x) from (ix) and using (viii), we get:

$$\frac{\pi_s(\underline{x})}{r} - \frac{\pi_s'(\underline{x})}{r\lambda} + 2B_s e^{-\lambda \underline{x}} = \frac{\pi_d(1-\underline{x})}{r} + \frac{\pi_d''}{r\lambda^2} + \frac{\pi_d'(1-\underline{x})}{r\lambda} + 2B_d e^{-\lambda \underline{x}} - K$$

$$2B_s = 2B_d + \left[\frac{\pi_d(1-\underline{x}) - \pi_s(\underline{x})}{r} + \frac{\pi_d''}{r\lambda^2} + \frac{\pi_d'(1-\underline{x})}{r\lambda} + \frac{\pi_s'(\underline{x})}{r\lambda} - K\right] e^{\lambda \underline{x}} \tag{xiii}$$

Adding (xii) to (xi) and using (viii) gives:

$$\frac{\pi_s(1-x^*)}{r} - \frac{\pi_s'(1-x^*)}{r\lambda} + 2B_s e^{-\lambda(1-x^*)} = \frac{\pi_d(x^*)}{r} + \frac{\pi_d''}{r\lambda^2} + \frac{\pi_d'(x^*)}{r\lambda} + 2B_d e^{-\lambda(1-x^*)} + K$$

$$2B_s = 2B_d + \left[\frac{\pi_d(x^*) - \pi_s(1-x^*)}{r} + \frac{\pi_d''}{r\lambda^2} + \frac{\pi_d'(x^*)}{r\lambda} + \frac{\pi_s'(1-x^*)}{r\lambda} + K\right] e^{\lambda(1-x^*)} \tag{xiv}$$

Similarly, adding (x) to (ix) and using equation (viii gives

$$\frac{\pi_s(\underline{x})}{r} + \frac{\pi_s'(\underline{x})}{r\lambda} + 2A_s e^{\lambda \underline{x}} = \frac{\pi_d(1-\underline{x})}{r} + \frac{\pi_d''}{r\lambda^2} - \frac{\pi_d'(1-\underline{x})}{r\lambda} + 2A_d e^{\lambda \underline{x}} - K$$

$$2A_s = 2A_d + \left[\frac{\pi_d(1-\underline{x}) - \pi_s(\underline{x})}{r} + \frac{\pi_d''}{r\lambda^2} - \frac{\pi_d'(1-\underline{x})}{r\lambda} - \frac{\pi_s'(\underline{x})}{r\lambda} - K\right] e^{-\lambda \underline{x}} \tag{xv}$$

Subtracting (xii) divided by λ from (xi) gives

$$\frac{\pi_s(1-x^*)}{r} + \frac{\pi_s'(1-x^*)}{r\lambda} + 2A_s e^{\lambda(1-x^*)} = \frac{\pi_d(x^*)}{r} + \frac{\pi_d''}{r\lambda^2} - \frac{\pi_d'(x^*)}{r\lambda} + 2A_d e^{\lambda(1-x^*)} + K$$

$$2A_s = 2A_d + \left[\frac{\pi_d(x^*) - \pi_s(1-x^*)}{r} + \frac{\pi_d''}{r\lambda^2} - \frac{\pi_d'(x^*)}{r\lambda} - \frac{\pi_s'(1-x^*)}{r\lambda} + K\right] e^{-\lambda(1-x^*)} \tag{xvi}$$

If we define $f(x) = \pi_d(x) - \pi_s(1-x) + \pi_d''(x)/\lambda^2$, then we can obtain from (xiii) and (xiv):

$$e^{\lambda(x^*+\underline{x}-1)} = \frac{f(x^*) + f'(x^*)/\lambda + rK}{f(1-\underline{x}) + f'(1-\underline{x})/\lambda - rK} \tag{xvii}$$

and obtain from (xv) and (xvi):

$$e^{\lambda(x^*+\underline{x}-1)} = \frac{f(1-\underline{x}) - f'(1-\underline{x})/\lambda - rK}{f(x^*) - f'(x^*)/\lambda + rK} \tag{xviii}$$

These two conditions determine the thresholds $\underline{x}$ and x^*. Note that we can write the conditions for competitive equilibrium from equations (14) and (15) in the paper in the exact same form.

Limit as $K \to 0$ for Socially Optimal and Collusive Outcomes: For the socially optimal case, we have

$$f(x) = \left[\frac{1+\delta(1-2x)}{2}\right]^2 - \frac{1}{4} + 2\frac{\delta^2}{\lambda^2}.$$

For the collusive case, we have

$$f(x) = 2\left[\frac{2+\delta(1-2x)}{4}\right]^2 - \frac{1}{2} + \frac{\delta^2}{\lambda^2}.$$

For socially optimal repositioning with competitive pricing, we have

$$f(x) = 5\left[\frac{9/5+\delta(1-2x)}{6}\right]^2 - \frac{9}{20} + \frac{10}{9}\frac{\delta^2}{\lambda^2}.$$

More generally, we can write

$$f(x) = a\left[\frac{b+\delta(1-2x)}{c}\right]^2 - a\left(\frac{b}{c}\right)^2 + 8\frac{a}{c^2}\frac{\delta^2}{\lambda^2}. \tag{xix}$$

For future use, note that in the competitive case $a = 1/2$ and $b = c = 3$.

Let $p^* = \frac{b+\delta(1-2x^*)}{c}$, and $\underline{p} = \frac{b+\delta(2\underline{x}-1)}{c}$. Furthermore, let $G = e^{\lambda(x^*+\underline{x}-1)} = e^{\frac{c\lambda}{2\delta}(\underline{p}-p^*)}$. We can re-write (xvii) and (xviii) as:

$$G\left[\underline{p}^2 - \left(\frac{b}{c}\right)^2\right] - \left[p^{*2} - \left(\frac{b}{c}\right)^2\right] + \frac{4\delta}{c\lambda}(p^* - G\underline{p}) + \frac{8\delta^2}{c^2\lambda^2}(G-1) = rK(1+G)/a \quad \text{(xx)}$$

$$G\left[p^{*2} - \left(\frac{b}{c}\right)^2\right] - \left[\underline{p}^2 - \left(\frac{b}{c}\right)^2\right] + \frac{4\delta}{c\lambda}(Gp^* - \underline{p}) + \frac{8\delta^2}{c^2\lambda^2}(G-1) = -rK(1+G)/a \quad \text{(xxi)}$$

Subtracting (xxi) from (xx), and dividing by $(1+G)(p^* + \underline{p})$, we get:

$$(\underline{p} - p^*) - \frac{4\delta}{c\lambda}\frac{G-1}{G+1} = \frac{2rK}{a(p^* + \underline{p})}$$

or

$$\log G - 2\frac{G-1}{G+1} = \frac{c\lambda}{a\delta}\frac{rK}{p^* + \underline{p}} \quad \text{(xxii)}$$

As $K \to 0$ in (xxii), $G \to 1$, which implies $\underline{p} = p^*$, or $\underline{x} + x^* = 1$, in the limit.

Adding (xx) and (xxi), and dividing by $p^* - \underline{p}$, we obtain:

$$\frac{G-1}{p^* - \underline{p}}\left[\underline{p}^2 + p^{*2} - 2\left(\frac{b}{c}\right)^2 + \frac{16\delta^2}{c^2\lambda^2}\right] + \frac{4\delta}{c\lambda}(G+1) = 0 \quad \text{(xxiii)}$$

With $\frac{G-1}{p^*-\underline{p}} \to -\frac{c\lambda}{2\delta}$ as $G \to 1$ and $p^* - \underline{p} \to 0$, we obtain

$$\underline{p}^2 + p^{*2} = 2\left(\frac{b}{c}\right)^2 \quad \text{(xxiv)}$$

which implies that as $K \to 0$, $\underline{p}, p^* \to \frac{b}{c}$. So, we then have that both $\underline{x}$ and x^* approach 1/2 in the limit.

Consider now the question of the speed of convergence. Let $y = p^* + \underline{p}$. Then, we can write

$$x^* = \frac{1}{2} + \frac{2b - cy}{4\delta} + \frac{1}{2\lambda}\log(G) \tag{xxv}$$
$$\underline{x} = \frac{1}{2} + \frac{cy - 2b}{4\delta} + \frac{1}{2\lambda}\log(G) \tag{xxvi}$$
$$x^* - \underline{x} = \frac{2b - cy}{2\delta} \tag{xxvii}$$

Again we have from (xviii) in the paper that

$$\lim_{G\to 1} \frac{\log(G) - 2\frac{G-1}{G+1}}{(G-1)^3} = \frac{1}{12}$$

Then, from (xxii), we can obtain

$$\lim_{K\to 0} \frac{(G-1)^3}{K} = \frac{6c^2}{ab}\frac{r\lambda}{\delta} \tag{xxviii}$$

Noting that $p^* = \frac{y}{2} - \frac{\delta}{c\lambda}\log(G)$ and $\underline{p} = \frac{y}{2} + \frac{\delta}{c\lambda}\log(G)$ we can obtain from (xxiii) that

$$\lim_{K\to 0} \frac{y - 2(\frac{b}{c})}{(G-1)^2} = -\frac{1}{6bc}\frac{\delta^2\sigma^2}{r}. \tag{xxix}$$

Using this plus (xxviii), we can write (16) and (17) in the paper for this general case, as $K \to 0$, as:

$$\frac{x^* - 1/2}{K^{1/3}} = \left(\frac{c^2}{ab}\right)^{1/3} \frac{1}{2\lambda}\left(\frac{6r\lambda}{\delta}\right)^{1/3} + \left(\frac{c^2}{ab}\right)^{2/3} \frac{\delta\sigma^2}{24br}\left(\frac{6r\lambda}{\delta}\right)^{2/3} K^{1/3} \tag{xxx}$$
$$\frac{\underline{x} - 1/2}{K^{1/3}} = \left(\frac{c^2}{ab}\right)^{1/3} \frac{1}{2\lambda}\left(\frac{6r\lambda}{\delta}\right)^{1/3} - \left(\frac{c^2}{ab}\right)^{2/3} \frac{\delta\sigma^2}{24br}\left(\frac{6r\lambda}{\delta}\right)^{2/3} K^{1/3} \tag{xxxi}$$

which implies that

$$\lim_{K\to 0} \frac{x^* - 1/2}{K^{1/3}} = \left(\frac{c^2}{ab}\right)^{1/3} \left(\frac{3\sigma^2}{8\delta}\right)^{1/3} \tag{xxxii}$$
$$\lim_{K\to 0} \frac{x^* - \underline{x}}{K^{2/3}} = \frac{1}{2b}\left(\frac{c^2}{ab}\right)^{2/3} \left(\frac{\delta\sigma^4}{3}\right)^{1/3}. \tag{xxxiii}$$

Equations (xxxii) and (xxxiii) then allow us to compare the thresholds across different scenarios for K small. Let us use $\underline{x}_{comp}$ and x^*_{comp} to denote the thresholds from the competitive equilibrium. Let $\underline{x}_{SW}$ and x^*_{SW} denote the thresholds from the socially optimal outcome. Let $\underline{x}_{coll}$ and x^*_{coll} denote the thresholds from the collusive outcome. And let $\underline{x}_{SW-p}$ and x^*_{SW-p} denote the thresholds from the case of socially optimal repositioning with competitive price. We have that, for K close to zero:

$$\underline{x}_{SW} < \underline{x}_{SW-p} < \underline{x}_{coll} < \underline{x}_{comp}$$
$$x^*_{coll} < x^*_{SW-p} < x^*_{SW} < x^*_{comp}$$

This comparison shows that competition leads to less repositioning compared to both the socially optimal and the collusion case.

Derivation of Equilibrium in the Only Deterministic Trend Case:

Solving the differential equations determined by (30)-(32) one obtains

$$V_{s0}(x) = \frac{1}{2r} + C_1 e^{\alpha x} \tag{xxxiv}$$
$$V_{s1}(x) = \frac{1}{2r} + C_2 e^{\alpha x} \tag{xxxv}$$
$$V_{d0}(x) = C_3 e^{\alpha x} + \frac{1}{2r}\left[\frac{3+\delta(1-2x)}{3}\right]^2 - \frac{2\delta}{3\alpha r}\left[\frac{3+\delta(1-2x)}{3}\right] + \frac{4\delta^2}{9r\alpha^2} \tag{xxxvi}$$
$$V_{d1}(x) = C_4 e^{\alpha x} + \frac{1}{2r}\left[\frac{3+\delta(2x-1)}{3}\right]^2 + \frac{2\delta}{3\alpha r}\left[\frac{3+\delta(2x-1)}{3}\right] + \frac{4\delta^2}{9r\alpha^2}, \tag{xxxvii}$$

where $\alpha = r/h$ and C_1, C_2, C_3 and C_4 are constants to be determined. The condition $V_{s0}(0) = V_{s1}(1)$ determines $C_2 = C_1 e^{-\alpha}$. Value matching and smooth pasting at $\underline{x}^+$ and x^{*+}, and $V_{s1}(x^{*+}) = V_{d1}(x^{*+})$ yields

$$\widetilde{C}_4 e^{\alpha \underline{x}^+} + \frac{1}{2}\left[\frac{3+\delta(2\underline{x}^+ - 1)}{3}\right]^2 + \frac{2\delta}{3\alpha}\left[\frac{3+\delta(2\underline{x}^+ - 1)}{3}\right] + \frac{4\delta^2}{9\alpha^2} - rK = \frac{1}{2} + \widetilde{C}_1 e^{\alpha \underline{x}^+} \quad \text{(xxxviii)}$$

$$\widetilde{C}_4 e^{\alpha \underline{x}^+} + \frac{2\delta}{3\alpha}\left[\frac{3+\delta(2\underline{x}^+ - 1)}{3}\right] + \frac{4\delta^2}{9\alpha^2} = \widetilde{C}_1 e^{\alpha \underline{x}^+} \quad \text{(xxxix)}$$

$$\widetilde{C}_3 e^{\alpha x^{*+}} + \frac{1}{2}\left[\frac{3+\delta(1-2x^{*+})}{3}\right]^2 - \frac{2\delta}{3\alpha}\left[\frac{3+\delta(1-2x^{*+})}{3}\right] + \frac{4\delta^2}{9\alpha^2} = \frac{1}{2} + \widetilde{C}_1 e^{\alpha(x^{*+}-1)} - rK \quad \text{(xl)}$$

$$\widetilde{C}_3 e^{\alpha x^{*+}} - \frac{2\delta}{3\alpha}\left[\frac{3+\delta(1-2x^{*+})}{3}\right] + \frac{4\delta^2}{9\alpha^2} = \widetilde{C}_1 e^{\alpha(x^{*+}-1)} \quad \text{(xli)}$$

$$\widetilde{C}_4 e^{\alpha x^{*+}} + \frac{1}{2}\left[\frac{3+\delta(2x^{*+}-1)}{3}\right]^2 + \frac{2\delta}{3\alpha}\left[\frac{3+\delta(2x^{*+}-1)}{3}\right] + \frac{4\delta^2}{9\alpha^2} = \frac{1}{2} + \widetilde{C}_1 e^{\alpha(x^{*+}-1)} \quad \text{(xlii)}$$

where $\widetilde{C}_1 = rC_1, \widetilde{C}_3 = rC_3$, and $\widetilde{C}_4 = rC_4$.

From (xxxviii) and (xxxix) we can obtain

$$\left[\frac{3+\delta(2\underline{x}^+ - 1)}{3}\right]^2 = 1 + 2rK, \quad \text{(xliii)}$$

from which we can obtain (33). Similarly, from (xl) and (xli) we can obtain

$$\left[\frac{3+\delta(1-2x^{*+})}{3}\right]^2 = 1 - 2rK, \quad \text{(xliv)}$$

from which we can obtain (34).

Derivation of $\underline{x}^-$ and x^{*-} in the Only Deterministic Trend Case:

The condition $V_{d0}(\underline{x}^-) - K = V_{s1}(\underline{x}^-)$ yields

$$e^{\alpha \underline{x}^-}(\widetilde{C}_1 e^{-\alpha} - \widetilde{C}_3) = \frac{1}{2}\left[\frac{3+\delta(1-2\underline{x}^-)}{3}\right]^2 - \frac{1}{2} - \frac{2\delta}{3\alpha}\left[\frac{3+\delta(1-2\underline{x}^-)}{3}\right] + \frac{4\delta^2}{9\alpha^2} - rK. \quad \text{(xlv)}$$

Using (xl) and (34), we can obtain

$$e^{\alpha x^{*+}}(\widetilde{C}_1 e^{-\alpha} - \widetilde{C}_3) = \frac{4\delta^2}{9\alpha^2} - \frac{2\delta}{3\alpha}\sqrt{1-2rK}. \quad \text{(xlvi)}$$

Using (xlvi) in (xlv) one obtains

$$\frac{4\delta^2}{9\alpha^2}\left[e^{-\frac{3\alpha}{2\delta}(1+Z-\sqrt{1-2rK})}-1\right]+\frac{2\delta}{3\alpha}\left[1+Z-\sqrt{1-2rK}e^{-\frac{3\alpha}{2\delta}(1+Z-\sqrt{1-2rK})}\right]-Z-\frac{Z^2}{2}+rK=0, \tag{xlvii}$$

where $Z=\delta(1-2\underline{x}^-)/3$. Note from (xlvii) that $\lim_{K\to 0}\underline{x}^-=1/2$. Taking the left hand side of (xlvii) as $f(Z,rk)$, we can obtain $\frac{\partial f}{\partial(rK)}_{|K=0}=2-\frac{2\delta}{3\alpha}$ and

$$\frac{\partial f}{\partial Z}=\frac{2\delta}{3\alpha}(1-e^{-\frac{3\alpha}{2\delta}(1+Z-\sqrt{1-2rK})})+\sqrt{1-2rK}e^{-\frac{3\alpha}{2\delta}(1+Z-\sqrt{1-2rK})}-1-Z, \tag{xlviii}$$

from which we can obtain $\lim_{K\to 0}\frac{\partial Z}{\partial K}=\infty$. Furthermore, we can obtain

$$\lim_{K\to 0}\frac{\partial f}{\partial Z}\frac{1}{1+Z-\sqrt{1-2rK}}=-\frac{3\alpha}{2\delta}. \tag{xlix}$$

We can then obtain that when K converges to zero, we have $\frac{\partial Z}{\partial(rK)}=\frac{4\delta}{3\alpha(1+Z-\sqrt{1-2rK})}$. As Z converges to zero as K goes to zero, we have that when K goes to zero, $\frac{\partial Z}{\partial(rK)}\to\frac{Z}{rK}$, which can then be used to find that wen K goes to zero,

$$Z\to\frac{\sqrt{1-2rK}-1+\sqrt{(\sqrt{1-2rK}-1)^2+16\delta rK/(3\alpha)}}{2}. \tag{l}$$

As

$$\lim_{K\to 0}\frac{\sqrt{1-2rK}-1+\sqrt{(\sqrt{1-2rK}-1)^2+16\delta rK/(3\alpha)}}{\sqrt{rK}}=4\sqrt{\frac{\delta}{3\alpha}}, \tag{li}$$

we have that

$$\lim_{K\to 0}\frac{Z}{\sqrt{rK}}=2\sqrt{\frac{\delta}{3\alpha}}, \tag{lii}$$

from which we can get (35).

The condition $V_{d1}(x^{*-})=V_{s0}(x^{*-})-K$ yields

$$e^{\alpha x^{*-}}(\widetilde{C}_1-\widetilde{C}_4)=\frac{1}{2}\left[\frac{3+\delta(2x^{*-}-1)}{3}\right]^2-\frac{1}{2}+\frac{2\delta}{3\alpha}\left[\frac{3+\delta(2x^{*-}-1)}{3}\right]+\frac{4\delta^2}{9\alpha^2}+rK. \tag{liii}$$

Using (xxxviii) and (33), we can obtain

$$e^{\alpha\underline{x}^+}(\widetilde{C}_1-\widetilde{C}_4)=\frac{4\delta^2}{9\alpha^2}+\frac{2\delta}{3\alpha}\sqrt{1+2rK}. \tag{liv}$$

Using (liv) in (liii) one obtains

$$\frac{4\delta^2}{9\alpha^2}\left[e^{\frac{3\alpha}{2\delta}(1+\widetilde{Z}-\sqrt{1+2rK})}-1\right]-\frac{2\delta}{3\alpha}\left[1+\widetilde{Z}-\sqrt{1+2rK}e^{\frac{3\alpha}{2\delta}(1+\widetilde{Z}-\sqrt{1+2rK})}\right]-\widetilde{Z}-\frac{\widetilde{Z}^2}{2}-rK=0, \tag{lv}$$

where $\widetilde{Z}=\delta(2x^{*-}-1)/3$. Note from (lv) that $\lim_{K\to 0}x^{*-}=1/2$. Taking the left hand side of (lv) as $\widetilde{f}(\widetilde{Z},rk)$, we can obtain $\frac{\partial\widetilde{f}}{\partial(rK)}_{|K=0}=-2-\frac{2\delta}{3\alpha}$ and

$$\frac{\partial\widetilde{f}}{\partial\widetilde{Z}}=\frac{2\delta}{3\alpha}(e^{\frac{3\alpha}{2\delta}(1+\widetilde{Z}-\sqrt{1+2rK})}-1)+\sqrt{1+2rK}e^{\frac{3\alpha}{2\delta}(1+\widetilde{Z}-\sqrt{1+2rK})}-1-\widetilde{Z}, \tag{lvi}$$

from which we can obtain $\lim_{K\to 0}\frac{\partial\widetilde{Z}}{\partial K}=-\infty$. Furthermore, we can obtain

$$\lim_{K\to 0}\frac{\partial\widetilde{f}}{\partial\widetilde{Z}}\frac{1}{1+\widetilde{Z}-\sqrt{1+2rK}}=\frac{3\alpha}{2\delta}. \tag{lvii}$$

We can then obtain that when K converges to zero, we have $\frac{\partial\widetilde{Z}}{\partial(rK)}=\frac{4\delta}{3\alpha(1+\widetilde{Z}-\sqrt{1+2rK})}$. As $\widetilde{Z}$ converges to zero as K goes to zero, we have that when K goes to zero, $\frac{\partial\widetilde{Z}}{\partial(rK)}\to\frac{\widetilde{Z}}{rK}$, which can then be used to find that wen K goes to zero,

$$\widetilde{Z}\to\frac{\sqrt{1+2rK}-1-\sqrt{(\sqrt{1+2rK}-1)^2+16\delta rK/(3\alpha)}}{2}. \tag{lviii}$$

As

$$\lim_{K\to 0}\frac{\sqrt{1+2rK}-1-\sqrt{(\sqrt{1+2rK}-1)^2+16\delta rK/(3\alpha)}}{\sqrt{rK}}=-4\sqrt{\frac{\delta}{3\alpha}}, \tag{lix}$$

we have that

$$\lim_{K\to 0}\frac{\widetilde{Z}}{\sqrt{rK}}=-2\sqrt{\frac{\delta}{3\alpha}}, \tag{lx}$$

from which we can get (36).