A Dynamic Model of Optimal Retargeting*

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Abstract

A consumer searching for information on a product may be indicative that the consumer has some interest in that product, but is still undecided about whether to purchase it. Some of this consumer search for information is not observable to firms, but some may be observable. Once a firm observes a consumer searching for information on its product, the firm may then want to try to provide further information about the product to that consumer, a phenomenon which has been known in electronic commerce as retargeting. Firms may not observe all activities by a consumer in searching for information, may not be able to observe the information gained by consumers, and may not be able to observe whether a consumer stopped searching for information. A consumer could stop searching either because he received information of poor fit with the product, or because he bought the product (which may be unobservable to the firm), or because he exogenously lost interest in the product. This paper presents a dynamic model with these features characterizing the optimal advertising retargeting strategy by the firm. We find that a forward-looking firm can advertise more or less than a myopic firm to gain more information about whether the consumer is searching for information. We characterize how the optimal advertising retargeting strategy is affected by the ability of the firm to observe when the consumer purchases the product, when the firm is better able to observe the consumer search behavior, and by the informativeness of the signal received by the consumer. We find that better tracking of consumer search behavior could be beneficial for consumers, because it may reduce the length of time when a consumer receives retargeting. Finally, we also study the value of retargeting, we investigate what happens if the firm is able to recognize when purchases occur, and we consider the consumer's optimal search behavior and its implications for pricing.

1. Introduction

A consumer searching for information on a product may be indicative that the consumer has some interest in that product, but is still undecided about whether to purchase it. Some of this consumer search for information is not observable to firms, but some may be observable. Once a firm observes a consumer searching for information on its product, the firm may then want to try to provide further information about the product to that consumer, a phenomenon which has been known in electronic commerce as retargeting. This practice of providing targeted information to consumers searching for information has been present with salesforce behavior, but, with the development of the information technologies, has become more prominent because of better tracking of consumers' information gathering behavior. In fact, in recent years, we have observed firms sending online advertising when they see a consumer searching for information on a certain product. This is done through emails, display advertising of sites checked by that consumer, or with other forms of communication.

Obviously, firms may not observe all activities by a consumer in searching for information. For example, in the electronic world a firm may not be able to know about the off-line information gathering by consumers; even online, the consumer may be gathering information from sites where the firm does not track information. Even if a consumer is browsing in a site that has information on the product, the consumer may not be processing that information. A firm may also not know if a consumer is receiving favorable or unfavorable information, when it finds the consumer searching for information. Given the posterior search or purchase behavior by the consumer, the firm may infer to some degree whether the information obtained by the consumer was favorable or unfavorable, but does not know it first hand.

Moreover, the firm may not know if a consumer stopped searching for information, as it cannot observe all search occasions, and may not be able to observe whether and when a consumer purchases the product. In fact, anecdotal evidence seems to suggest that consumers continue to receive purchase-oriented advertising after purchasing the product, or, more generally, after they stop searching for information. A firm may not be able to observe whether a consumer purchased the product because that is done off-line, or is done on a site that is not monitored, or for which the information collected is not cross-checked with the retargeting information. Obviously, one can consider firms getting better at connecting consumer purchases with consumer search behavior and that affecting the optimal retargeting behavior.

This paper considers a model of these effects, taking into account the firm's beliefs about the likelihood of search behavior by the consumer, and the role of advertising. When the firm sees a signal that a consumer is searching for information, the firm learns that the consumer is searching for information, although it might not know whether the information received by the consumer is positive or negative or whether the consumer decided to purchase the product. When the firm does not see a signal that a consumer is searching for information, by Bayes' rule, it reduces its belief that the consumer is searching for information. Advertising is modeled as increasing the likelihood and frequency of the consumer learning information about the product. The optimal strategy calls for advertising when the firm has a sufficiently high belief that the consumer is considering the product. We can evaluate how the optimal strategy can be affected by different market forces, and how the optimal strategy changes when the firm can also observe if and when the consumer buys the product. We also characterize the length of time that a product is advertised without further information about the consumer's search behavior. The model also replicates the real-world experiences by consumers of receiving advertised after searching for information on a product, and continuing to receive that information, even after becoming disinterested in the product. We find that better tracking of consumer search behavior could be beneficial for consumers, because it may reduce the length of time when a consumer receives retargeting. Better tracking of consumer search behavior allows the firm to update faster that the consumer is not searching for information when not observing the consumer searching for information, and therefore the firms stops doing retargeting sooner.

We study also what happens if the information technology improves in such a way that firms are immediately able to recognize when purchases occur. In this case, firms stop retargeting when the consumer makes a purchase, but may continue still going on doing retargeting after the consumer receives a negative informative signal. In this case, we find that the threshold belief for the firm to do retargeting is now lower because of the extra benefit of waiting to find out whether the consumer purchased the product.

Although the main analysis considers only a benefit to the firm of a consumer purchase, we also consider a micro-model of consumer search and purchase behavior that rationalizes the assumptions in the main analysis. That setting formally considers consumer search behavior and its implications for pricing. We can show that if consumers are forward-looking and there is no dis-utility of receiving advertising, the firm can charge a higher price that allows consumers to search for information, because of the anticipated benefit of the additional information from retargeting.

Existing research has focused on understanding how consumers past purchase behavior could affect the future behavior of the firm towards those consumers. This has been considered in the context of advertising (Shen and Villas-Boas, 2018), pricing (e.g., Villas-Boas 1999 and 2003, Fudenberg and Tirole 2000, Fudenberg and Villas-Boas 2006, Shin and Sudhir 2010), and product design (e.g., Zhang 2011). However, in the real-world it seems that with the development of the information technologies, the most important practice is one of conditioning the behavior of firms on the search behavior of consumers, rather than on the purchase behavior of consumers.[1] There is also some related recent work on the search behavior of consumers for information (e.g., Branco, Sun, and Villas-Boas 2012, Ke, Shen, and Villas-Boas 2016, Fudenberg, Strack, and Strzalecki 2018, Gardete and Antill 2019), but this research has not considered how that search behavior affects the firm strategy.[2] There has also been some research on the significant empirical effectiveness of retargeting advertising, such as Manchanda, Dubé, Goh, and Chintagunta (2006), Lambrecht and Tucker (2013), Li and Kannan (2014), and Hoban and Bucklin (2015).

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 considers the case in which the firm does not observe whether and when the consumer purchases the product. Section 4 studies the value of retargeting, and Section 5 presents variations on the way retargeting works. Section 6 considers the case in which the firm observes consumer purchases. Section 7 presents some micro-modeling of the decision of a consumer to search for information and the pricing implications. Section 8 concludes.

2. The Model

Consider a consumer potentially searching for information on the purchase of a product. A consumer can be in either of two states: (1) searching for information on the product, or (2) not searching for information on the product. There could be several reasons that the consumer is not searching for information on the product: the consumer already bought the product, or the consumer received information that the product is a poor fit for his preferences, or the consumer realized that he no longer needs this type of product, or the consumer was never aware of this product.

The firm does not know which state the consumer is in, but, occasionally, if the consumer is searching for information, the firm learns that, and at that moment the consumer is in the state of searching for information. For now, we will assume that the firm does not know whether the consumer bought the product, and in Section 6 we will allow for that possibility.

[1] There is also some work exploring the possibility of firms offering exploding offers to deter consumers from searching further for alternatives (e.g., Armstrong and Zhou 2016), work on tracking consumers to practice intertemporal price discrimination (Öry 2016), and work on the design and price of information (Bergemann, Bonatti, and Smolin 2018).

[2] In related research, Ning (2018), considers the possibility of a seller knowing the information that the buyer is receiving, and making price offers conditional on that information.

Time is continuous, and, to simplify, we assume no discounting, as the real-world phenomena considered typically last only a few weeks at most.

Consumers in the state of not searching for information are not useful for the firm, as going forward they will not purchase the product, and we assume that there is no activity that the firm can do that makes consumers switch from no search to search. Consider also that the probability of a consumer switching from the no searching state to the searching state is so low, that, if a firm knew that a consumer was in the no searching state, it would never be profitable to advertise to that consumer as long as the firm does not see that the consumer is searching for information.[3]

Consumers in the searching for information state can either remain there or move to the not searching for information state, by either purchasing the product, receiving definitive information about the product and deciding that the product is a poor fit, or just losing interest in the product. Consider a continuous time framework where the hazard rate of the consumer receiving a signal, given that he is searching, is $p > 0$.[4] In the period of time dt, the probability of the consumer receiving a signal is $p\,dt$. This probability is greater if the firm is advertising to the consumer, which can be done at a cost $c\,dt$. This is the modeled main effect of retargeting. We denote $\widehat{p}$ as the hazard rate of the consumer receiving a signal if he is being advertised to, with $\widehat{p} > p$.[5] In Section 5 we consider retargeting also affecting the likelihood with which the firm is able to observe the consumer searching for information, and the likelihood of the consumer receiving a fully informative signal, whose role we now discuss.

If the consumer receives a signal, with probability $q \in (0, 1)$ that signal is fully informative about the fit of the product with the consumer.[6] We can think of q as small. Suppose that good or poor fit are equally likely; therefore, given that the consumer received a signal, with probability $q/2$ the consumer buys the product and moves to the no searching for information state, and with probability $q/2$ the consumer decides not to buy the product, but also moves to the no searching for information state. For now, we assume that the firm does not observe whether the consumer buys the product.

When the consumer receives definitive information about the product fit, the firm does not

[3]A consumer that switches from the no searching to searching state is assumed to have no information about the product fit independent of any prior experience with the product. This could explained, for example, by a change of preferences, when this change of state occurs

[4]Tables 1 and 2 in the Appendix present the notation used in the paper.

[5]This allows for a clear distinction between retargeting and not retargeting. This could also be seen as reducing the consumer search costs, and therefore increasing the rate at which the consumer receives information. In Section 8, we briefly discuss the case in which the extent of retargeting is a continuous variable, where p and c are increasing functions of the extent of retargeting.

[6]This set-up is similar to Bergemann and Välimäki (2006) in which a consumer fully learns the product fit at some constant hazard rate.

necessarily see that the consumer received this signal. In fact, the firm only sees that the consumer has seen some form of information with probability $\phi \in (0, 1)$, given that it occurred. Given that the firm observes the consumer searching for information, the belief by the firm x_t that the consumer is in the searching for information state at time t is $1-q$, as we know that with probability q the consumer receives a fully informative signal, and moves to the no searching for information state.

Finally, let us also consider that, when in the searching for information state, the consumer can also move exogenously to the no searching for information state because of losing interest in the product or category, and this occurs with hazard rate $\psi > 0$.

Note that, unconditional on any information, this structure yields a càdlàg process on beliefs (right continuous with left limits). Given this structure, we can use Bayes' rule to construct how the firm's belief about the consumer's search state evolves over time t when the firm does not receive any information that the consumer received a signal about product fit.

For example, consider that a firm observes a consumer searching for information, and then does not observe that consumer searching for information for some period. During that period, and as time passes, the firm puts more weight on the possibility that the consumer either received a fully informative signal, or decided to stop searching exogenously. That is, as time passes without observing the consumer searching for information, the firm has lower and lower beliefs that the consumer is still searching for information.

Formally, letting x_{t+dt} be the firm's beliefs conditional on not receiving any information during the period dt, one can obtain

$$x_t = (x_t + \frac{dx_t}{dt}\, dt)(1 - p\,\phi\, x_t\, dt) + (1-q)p\,\phi\, x_t\, dt + \psi\, x_t\, dt + p\, q\, x_t\, dt \tag{1}$$

as the beliefs at time t have to be consistent with what can occur in the future, and the beliefs are a supermartingale, falling in expected value, by $(\psi + pq)x_t\, dt$ in period dt, $E(\widetilde{x}_{t+dt}) = x_t - (\psi + pq)x_t\, dt$ where $\widetilde{x}_{t+dt}$ are the firm's unconditional (on information obtained in period dt) beliefs at time $t+dt$.[7] This yields a relation between the beliefs at time t, and what can occur and the beliefs at time $t + dt$. With likelihood $p\,\phi\, q\, x_t\, dt$ the firm knows for sure that the consumer is in the "searching" state and the belief that the consumer is searching jumps to $1 - q$, and with probability $(1 - p\,\phi\, x_t\, dt)$ the firm does not receive any information, and the belief that the consumer is searching goes to $x_{t+dt} = x_t + \frac{dx_t}{dt}\, dt$. Dividing by dt and making dt go to zero, we

[7] When there is advertising, the evolution of beliefs follows a similar expression to (1) with p replaced with $\widehat{p}$.

can then obtain

$$\frac{dx_t}{dt} = -p\phi x_t(1-x_t) - \psi x_t - p(1-\phi)qx_t. \tag{2}$$

With this information on how the firm beliefs evolve over time, we can set up the dynamic programming problem of when the firm should and should not retarget. Once we have the optimal retargeting strategy we can investigate how it is affected by the different market forces.

Before setting up the dynamic programming problem, let us discuss the role of the different parameters.

The hazard rate of receiving a signal p (without advertising) and $\widehat{p}$ (with advertising) measure the extent to which the consumer receives a signal about the quality of the product. When these hazard rates get infinitely large, the consumer is receiving signals almost all the time. This then makes the firm's belief that the consumer in in the searching state to decline very steeply without further information, but at the same time allows the firm to get more frequent information that the consumer is searching for information. These hazard rates being infinitely large also lead the consumer to make a decision about the product fit almost immediately. If, alternatively, these hazard rates are close to zero, then the beliefs of the firm regarding the consumer being in the searching state decline less steeply, but the consumer could be in the searching state for a long time and almost never get a signal, and the firm would rarely observe that the consumer is searching for information. The benefit of retargeting would be almost non-existent if $\widehat{p} \to p$. Obviously, when the difference $\widehat{p} - p$ increases, the benefit of retargeting is greater.

An increase in the cost of advertising, c, also makes retargeting less appealing. On the other hand, when c goes to zero, it becomes optimal to do retargeting for almost all levels of the firm's belief regarding whether the consumer is in the searching state, and the threshold belief above which the firm chooses to do retargeting, $\widehat{x}$, goes to zero.

Consider now the probability of a signal being informative given that it is received by the consumer, q. If this probability goes to one, almost every time that a consumer gets a signal, the consumer decides whether or not to buy the product, and there is therefore no incentive for a firm to do retargeting. If this probability is zero, then the consumer never gets true information about the value of the product, so the consumer would be in the searching state forever, and there would also be no incentive for a firm to do retargeting. In what follows, we can think of q as small, but nonzero, such that informative signals are not too frequent.

The role of ϕ is to measure the extent to which the firm is able to observe when the consumer is receiving a signal. When ϕ approaches one, the firm is able to observe almost all search occasions, and therefore each time the consumer searches the firm's belief that the consumer is searching keeps going to $1-q$. When $\phi=0$, the firm never has information if the consumer is searching. As the information technology improves and the firm is better able to track the consumer search behavior, we would expect that ϕ would increase.

The role of ψ is to allow for the possibility of the consumer dropping out of the search process exogenously. Given no discounting, ψ creates an incentive for the firm to advertise to the consumer when the belief that the consumer is in the searching state is sufficiently high, in order not to lose the consumer, and to accelerate the possibility of the consumer learning about the product fit. In this sense, $\psi>0$ plays the role of discounting in the model, such that there is an advantage in converting the consumer sooner rather than later. If $\psi=0$, the firm does not gain by converting the consumer sooner, and just chooses not to advertise, that is, not to do retargeting, for any beliefs. The greater is ψ, for some levels of ψ, the more important it may be to advertise now. Note also that ψ captures the effect that firms want to advertise to consumers when they are searching for information and in the market, due to the risk of the consumer exogenously losing interest in the product, and not because of discounting the future payoffs.

The current modeling of retargeting is one of providing informative advertising. Alternatively, we could consider the possibility of retargeting providing persuasive advertising. In that case, we could interpret q as the probability of the advertising being fully persuasive, and we would never have the possibility of the consumer leaving the searching state when receiving a signal. The belief dynamics would be exactly as above, and, in the computation of the expected payoff, the firm would get v with probability q when the consumer receives a signal, while in the case of informative advertising the expected payoff for the firm would be $v/2$ with probability q when the consumer receives a signal.

3. Optimal Retargeting

In order to compute the optimal retargeting strategy, we first present a result on the form of the optimal policy (with the proof in the Appendix).

Proposition 1: *There is a belief threshold $\widehat{x}$ such that the firm advertises for $x\geq\widehat{x}$ and does not advertise for $x<\widehat{x}$.*

From this we can now characterize the optimal retargeting strategy. Let $V(x)$ be the expected present value of the firm's profits if the firm has the belief $x<\widehat{x}$; let $\widehat{V}(x)$ be the expected present value of profits of the firm if the firm has the belief $x>\widehat{x}$; and let v be the profit for a firm when the consumer purchases the product. We restrict attention to the case in which it is optimal to do retargeting if the firm observes the consumer searching for information, $\widehat{x} < 1 - q$, which holds if the cost of retargeting c is low enough (presented in (**??**) in the Appendix). To derive the optimal retargeting policy, let us first consider the present value of profits when the belief of the firm is sufficiently low, such that the firm does not advertise, $x < \widehat{x}$. The Bellman equation of this problem can be written as,

$$V(x) = p\phi x\ dt\ \widehat{V}(1-q) + p\frac{q}{2}vx\ dt + (1 - p\phi x\ dt)[V(x) + V'(x)\frac{dx}{dt}\ dt], \tag{3}$$

as with probability $p\phi x\, dt$ the firm's beliefs jump to $1 - q$ at which point the firm moves to a retargeting region, with an expected present value of profits equal to $\widehat{V}(1-q)$, with probability $p\frac{q}{2}x\, dt$ the consumer receives an informative signal that the product is a good fit, and in that case the firm gets a payoff of v, and with probability $(1 - p\phi x\, dt)$ the firm does not receive any new information, and it then gets an expected payoff of $V(x_{t+dt})$ which can be approximated with a Taylor's expansion to $V(x_t) + V'(x_t)\frac{dx}{dt}\ dt$. Dividing by dt and substituting for $\frac{dx}{dt}$ from (2) one can obtain the differential equation

$$[p\phi(1-x) + \psi + pq(1-\phi)]V'(x) + p\phi V(x) = p\phi\widehat{V}(1-q) + p\frac{q}{2}v \tag{4}$$

which can be solved to obtain

$$V(x) = \widehat{V}(1-q) + \frac{qv}{2\phi} + C[B - p\phi x] \tag{5}$$

where $B = p\phi + \psi + pq(1-\phi)$, C is a constant to be determined below, $\widehat{V}(1-q)$ is the present value of profits when the belief that the consumer is searching is $1 - q$, which is determined by $V(0) = 0$.

When the firm is advertising, the Bellman equation becomes

$$\widehat{V}(x) = -c\ dt + \widehat{p}\phi x\ dt\ \widehat{V}(1-q) + \widehat{p}\frac{q}{2}vx\ dt + (1 - \widehat{p}\phi x\ dt)[\widehat{V}(x) + \widehat{V}'(x)\frac{dx}{dt}\ dt], \tag{6}$$

from which one can obtain along the same lines as above,

$$\widehat{V}(x) = \widehat{V}(1-q) + \frac{qv}{2\phi} + \widehat{C}[\widehat{B} - \widehat{p}\phi x] - \frac{c}{\widehat{B}} + c\frac{\widehat{B} - \widehat{p}\phi x}{\widehat{B}^2}\log\frac{\widehat{B} - \widehat{p}\phi x}{x} \tag{7}$$

where $\widehat{B} = \widehat{p}\phi + \psi + \widehat{p}q(1-\phi)$, and $\widehat{C}$ is a constant which is determined by (7) when $x = 1-q$.[8] To obtain C and $\widehat{x}$, one then makes $V(\widehat{x}) = \widehat{V}(\widehat{x})$, value matching at $\widehat{x}$, and $V'(\widehat{x}) = \widehat{V}'(\widehat{x})$, smooth pasting at $\widehat{x}$. Note that $C < 0$. We can obtain $\widehat{x}$ implicitly by (**??**), which is presented in the Appendix.

Note that we wrote the value function as a function of the beliefs of the firm regarding whether the consumer is searching for information. As these beliefs are uniquely determined by the extent of time since the firm observed the consumer searching for information, we could alternatively have used time since observation of the consumer searching for information instead of x_t in the value function.

To obtain some further insights, we concentrate on the case in which the cost of advertising, c, approaches zero, $c \to 0$. When this happens, the firm wants to advertise for almost all beliefs, that is, $\widehat{x} \to 0$. We can then obtain a measure of $\widehat{x}$ in comparison to c as

$$\lim_{c\to 0} \frac{\widehat{x}}{c} = 2\frac{\widehat{p}q+\psi}{\psi q v(\widehat{p}-p)}\frac{p\phi+\psi+pq(1-\phi)}{\widehat{p}\phi+\psi+\widehat{p}q(1-\phi)}. \tag{8}$$

From (8) we can obtain the following comparative statics.

Proposition 2: *Suppose that the firm does not detect when the consumer makes a purchase, and that $c \to 0$. Then, the region of beliefs for optimal retargeting, $x > \widehat{x}$, is increasing in the propensity for the consumer to receive signals with retargeting, $\widehat{p}$, the profit earned if the consumer purchases the product, v, the likelihood of the consumer receiving an informative signal given that he receives a signal, q, the likelihood of the firm observing that the consumer is searching, ϕ, and the likelihood of the consumer deciding to stop searching exogenously, ψ, for small ψ, and decreasing in the propensity of the consumer to receive signals without retargeting, p.*

Some of these results can be seen as one would expect: The firm wants to advertise more if the profit generated by a sale is greater (v), if advertising generates a higher chance of the consumer getting a signal ($\widehat{p}$), if there is a greater likelihood of signals being informative (q), and if the chance of a consumer receiving a signal without advertising is lower (p). Note also that, from (**??**), we can obtain that $\widehat{x}$ is a decreasing function of v/c for any c. We may consider that v is large for higher priced products such as an automobile or a house, and therefore, for such products we may expect that firms do retargeting for a longer period of time after a consumer makes a decision. And some anecdotal evidence seems to suggest that this is the case

[8] The constant $\widehat{C}$ is presented in the Appendix.

for consumers on the market for an automobile or for real estate. But another factor to consider is that, firms with such high v may bid up the cost of retargeting c, such that v/c may end up not being too large for such high priced items.

More interestingly, the firm wants to advertise more if it is better able to track that the consumer is searching for information (ϕ). One could potentially consider that with better tracking a firm would think that not observing the consumer searching for information might indicate that the consumer stopped being on the market. In fact, for $c \to 0$, when the firm has better tracking, the firm realizes that when it advertises it will be better able to discern whether the consumer is searching for information, and therefore chooses to still advertise for lower beliefs of the consumer being in the searching for information state.

Also interestingly, when the the likelihood of the consumer exogenously stopping the search process is greater (ψ), if ψ is small, the firm wants to advertise more. In this case, the firm realizes that in the future the consumer's interest may disappear, and wants to advertise more. Note that if $(\widehat{p} - p)[\phi + q(1-\phi)] > \widehat{p}q$ we can have the firm wanting to advertise in a smaller region of the beliefs when ψ increases, if ψ is sufficiently large. In this case, which can occur, for example, if the probability of the consumer receiving an informative signal is sufficiently small, if ψ is sufficiently large the firm realizes that the potential benefits of advertising leading to an informative signal are not too high, and the firm chooses to advertise in a smaller region of the belief space.

Given that we have $\widehat{x}$ we can compute the maximum amount of time that a consumer could be retargeted after being identified as searching for information. This can be obtained to be

$$\widehat{T} = \frac{1}{\widehat{B}} \log[\frac{1-q}{\psi + \widehat{p}q} \frac{\widehat{B} - \widehat{p}\phi\widehat{x}}{\widehat{x}}]. \tag{9}$$

One interesting comparative statics on $\widehat{T}$ is that it can be decreasing in ϕ, which we state in the next proposition.

Proposition 3: *Suppose that the cost of retargeting c is close to zero. Then, the maximum amount of time that a consumer could retargeted after being identified as searching for information, $\widehat{T}$, is decreasing in the firm's ability to track consumer search, ϕ.*

This proposition shows that the consumer could potential benefit from the firm having a greater ability to track consumer search, considering the consumer has dis-utility in receiving retargeting. When the firm has a greater ability to track consumer search, it updates faster that the consumer may not be searching for information, when it does not observe the consumer searching for information. Then, the firm may prefer to stop doing retargeting sooner, as its beliefs that the consumer is searching for information fall now more steeply.

Figure 1 presents the value function for some parameter values, and Figures 2 through 8 illustrate how the optimal threshold $\widehat{x}$ varies with the different parameters. As expected, as the cost of advertising c increases, the firm advertises less, and the maximum time of a consumer receiving advertising after having bought the product decreases. Figure 3 illustrates that the firm has a lower threshold $\widehat{x}$ to advertise when the consumer is more likely to get informative advertising, greater q, as shown in Proposition 2. More interestingly, the maximum time receiving advertising after purchases is not monotonic on q, increasing when q is small, as the firm is willing to advertise longer, and decreasing when q is large, as then the firm realizes that not observing the consumer search for a long time is more likely to mean that the consumer is not searching (the posterior beliefs that the consumer is searching for information decrease faster over time).

Figure 4 illustrates that the effect of the ability of the firm to track consumers has a monotonic effect on the optimal firm advertising strategy for $c > 0$. Increasing the ability to track if the consumer is searching for information makes the firm want to advertise more (lower $\widehat{x}$), and the time receiving advertising after purchase decreases. As shown in Proposition 3, a greater ϕ has an effect on the firm's beliefs declining faster, and this has a bigger impact on $\widehat{T}$ than the lower threshold $\widehat{x}$. As discussed above, this illustrates that improvements in information technologies leading to an increase in the tracking ability by firms of consumer search could actually be beneficial to consumers in reducing the length of time that consumers receive advertising. This could also potentially provide an incentive for consumers to credibly disclose to what extent they are searching for information.

As expected, the firm advertises more, and the consumer ends up spending more time receiving advertising after purchase, if the profit for the firm of a sale, v, is greater (Figure 5). Also as expected, and as illustrated in Figure 6, as the rate of receiving information without advertising increases, the firm advertises less, and the consumer spends less time receiving advertising after purchase. As illustrated in Figure 7, and as expected, when the rate at which the consumer receives information when being advertised to ($\widehat{p}$) increases, the firm is interested in advertising more for the same beliefs ($\widehat{x}$ is decreasing in $\widehat{p}$). More interestingly, when $\widehat{p}$ increases, the length of time for which a consumer continues to receive advertising after a purchase ($\widehat{T}$) first increases and then decreases. It increases for low $\widehat{p}$ because the firm now wants to advertise more. It decreases for high $\widehat{p}$ because in that case the firm updates more quickly that the consumer may not be in the searching for information state, and so it reaches the belief $\widehat{x}$ faster.

Finally, as illustrated in Figure 8, when the exogenous rate of the consumer dropping out of the search process, ψ, increases, and ψ is small, the firm wants to advertise more ($\widehat{x}$ is decreasing in ψ), as it wants to take advantage of the consumer searching for information. Note that for some parameter values as noted above, $(\widehat{p}-p)[\phi+q(1-\phi)] > \widehat{p}q$, (the case with solid lines in Figure 8) we can have $\widehat{x}$ increasing in ψ for large ψ, as the potential benefits of advertising are now weaker. More interestingly, when ψ increases, the length of time for which a consumer continues to receive advertising after a purchase ($\widehat{T}$) first increases and then decreases. It increases for low ψ because the firm now wants to advertise more. It decreases for high ψ because the firm's beliefs that the consumer is searching now fall more steeply, which yields $\widehat{T}$ to fall.

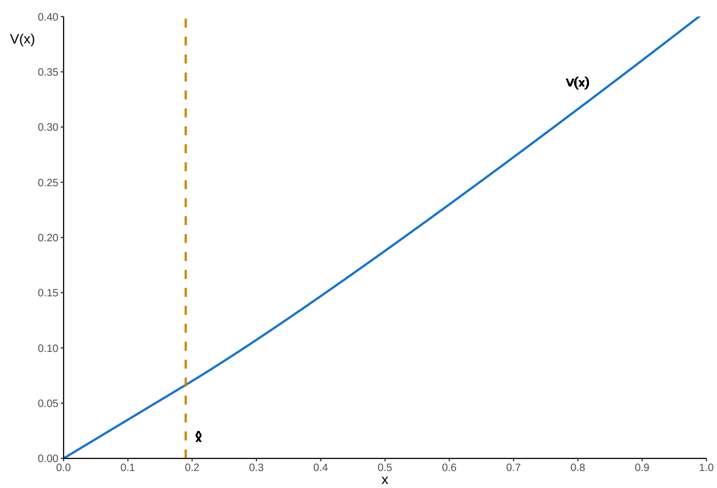

Figure 1: Value function for $v=2, c=.01, p=.4, \widehat{p}=1, q=.1, \phi=.5$, and $\psi=.1$. In this case we obtain $\widehat{x}\approx .19$.

The optimal policy accounts for the future effects of what the firm learns based on whether it advertises. It is interesting to compare this with the optimal policy if the firm were myopic. This myopic policy would be to advertise if the expected current benefit of advertising, $(\widehat{p}-p)\frac{q}{2}\,v\,x_t\,dt$, is greater than the cost of advertising, $c\,dt$. This would give a threshold of belief on searching of $x_m=\frac{2c}{vq(\widehat{p}-p)}$. Comparing this with the optimal policy for the case of small c, one can get that it could be larger or smaller. In fact, the optimal policy is to advertise more than in the myopic policy if the effect of advertising on the likelihood of receiving a signal is sufficiently large. In this case, if the firm advertises a little more, it can learn more precisely whether the consumer is in the searching for information state, and this is an effect that is not present in the myopic policy. On the other hand, if the effect of advertising on the likelihood of receiving a signal is not too large, the firm realizes that it will not lose too much in the future by not advertising, and advertises less than in the myopic case. Similarly, we can also obtain that the optimal policy is to

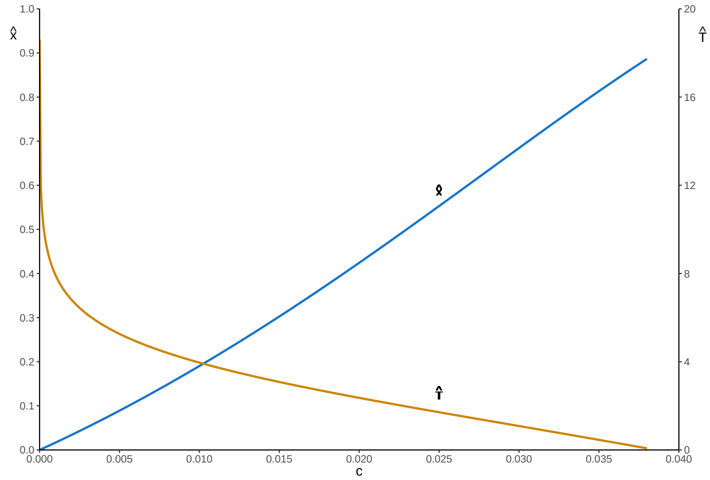

Figure 2: Evolution of $\widehat{x}$ and the maximum time receiving advertising without search, $\widehat{T}$, as a function of c for $v = 2, p = .4, \widehat{p} = 1, q = .1, \phi = .5$, and $\psi = .1$.

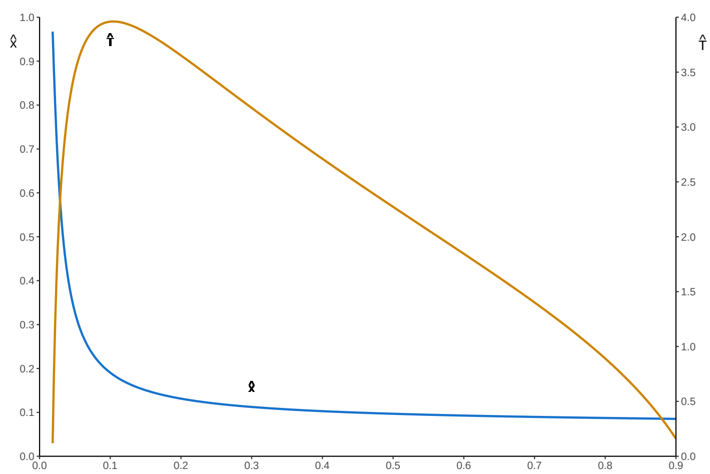

Figure 3: Evolution of $\widehat{x}$ and the maximum time receiving advertising without search, $\widehat{T}$, as a function of q for $v = 2, p = .4, \widehat{p} = 1, c = .01, \phi = .5$, and $\psi = .1$.

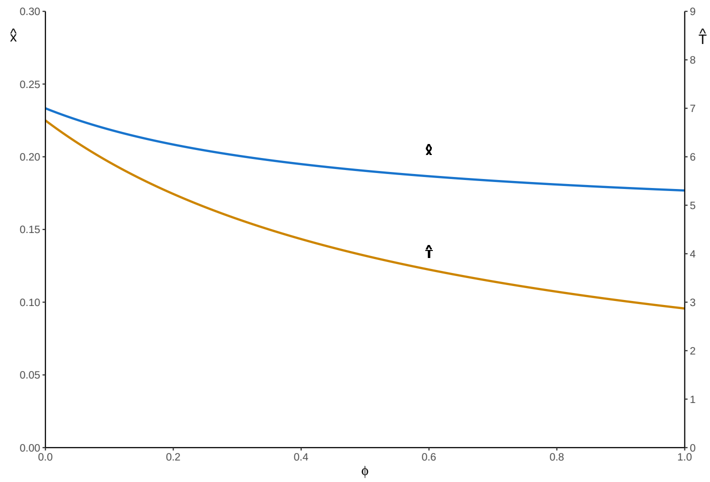

Figure 4: Evolution of $\widehat{x}$ and the maximum time receiving advertising without search, $\widehat{T}$, as a function of ϕ for $v = 2, p = .4, \widehat{p} = 1, c = .01, q = .1$, and $\psi = .1$.

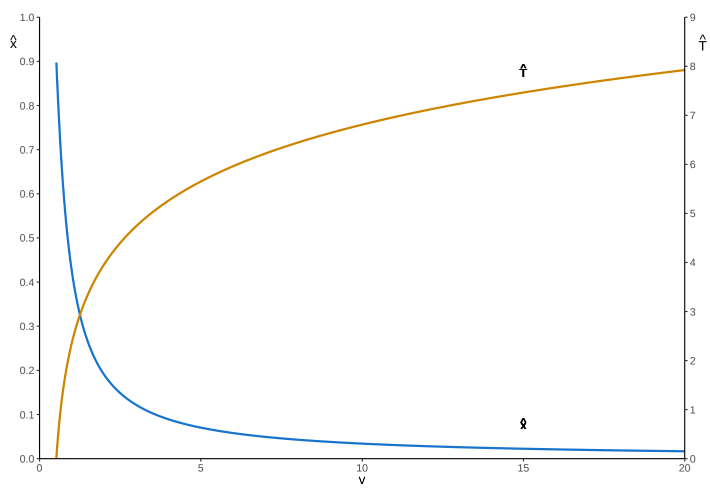

Figure 5: Evolution of $\widehat{x}$ and the maximum time receiving advertising without search, $\widehat{T}$, as a function of v for $\phi = .5, p = .4, \widehat{p} = 1, c = .01, q = .1$, and $\psi = .1$.

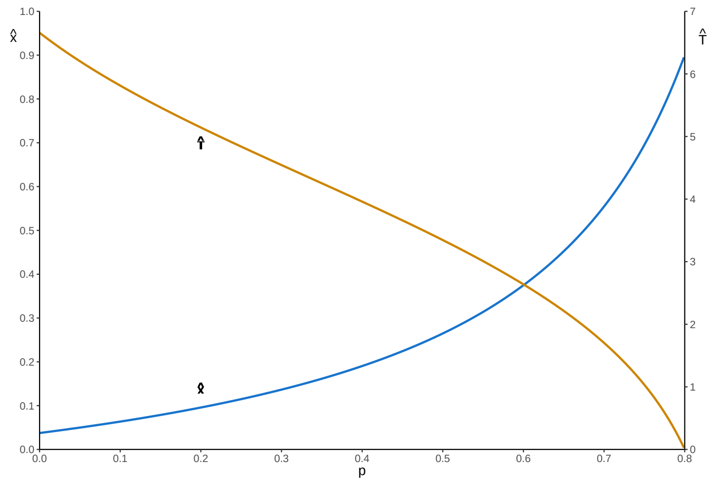

Figure 6: Evolution of $\widehat{x}$ and the maximum time receiving advertising without search, $\widehat{T}$, as a function of p for $\phi = .5, v = 2, \widehat{p} = 1, c = .01, q = .1$, and $\psi = .1$.

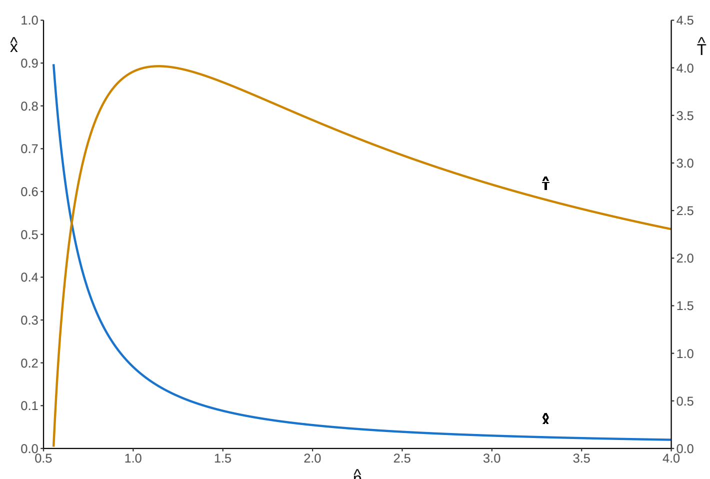

Figure 7: Evolution of $\widehat{x}$ and the maximum time receiving advertising without search, $\widehat{T}$, as a function of $\widehat{p}$ for $\phi = .5, v = 2, p = .4, c = .01, q = .1$, and $\psi = .1$.

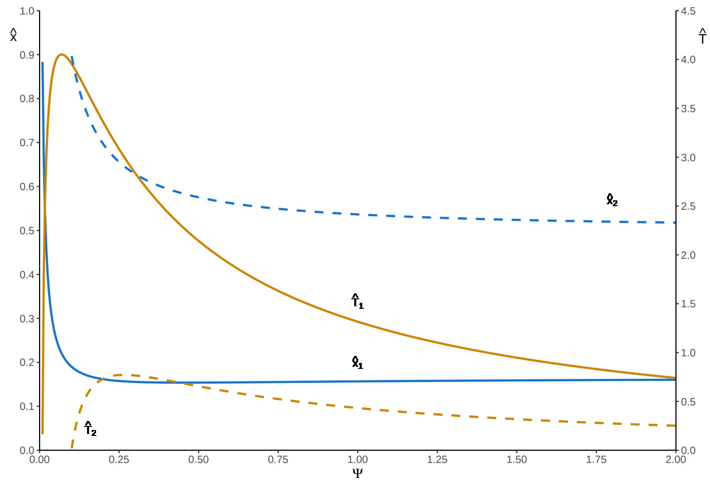

Figure 8: Evolution of $\widehat{x}$ and the maximum time receiving advertising without search, $\widehat{T}$, as a function of ψ for $\phi = .5, v = 2, \widehat{p} = 1, c = .01$, and $q = .1$. The solid lines have $p = .4$ and the dashed lines have $p = .8$.

advertise more than in the myopic policy if the likelihood of obtaining an informative signal (q) is low enough. In that case, a forward looking firm realizes that the consumer needs to receive signals for a longer period, and is willing to advertise more. On the other hand, if the likelihood of obtaining an informative signal is high enough, the myopic policy has a high incentive to advertise, which can be greater than in the optimal policy. The following proposition formalizes these results.

PROPOSITION 4: *Suppose $\psi, \phi > 0$. Then the optimal retargeting policy prescribes advertising for lower beliefs of consumer searching than the myopic policy if the effect of advertising is relatively high (high $\widehat{p}$) and $\psi\phi(1-q) > qp[\phi + q(1-\phi)]$ or if the likelihood of obtaining an informative signal, or the likelihood of receiving a signal without advertising, is sufficiently low (low q or low p).*

4. The Value of Retargeting

It is also interesting to evaluate the value for the firm of having the possibility of doing retargeting. This can be seen as the amount a firm is willing to invest to have the possibility of doing retargeting. In evaluating a policy of making retargeting unlawful, this is the payoff for the firm that is lost. The payoff for the consumer is evaluated in Section 7.

Consider this value of retargeting evaluated at a time when the firm observes a consumer searching for information. The belief that the consumer is searching for information is then $1-q$

at that time. The expected present value of profits with retargeting at that moment is then $\widehat{V}(1-q)$, which can be obtained from (5) as $\widehat{V}(1-q) = -\frac{qv}{2\phi} - CB$, given that $V(0) = 0$.

The maximum expected present value of profits with the possibility of retargeting can be obtained with $c \to 0$, which can be obtained to be

$$\lim_{c \to 0} \frac{\widehat{p}q(1-q)v}{2(\psi + \widehat{p}q)}. \tag{10}$$

Note that this limit is independent of ϕ as with $c \to 0$, the firm is retargeting forever in the limit, which means that the likelihood of the firm observing the consumer searching for information becomes irrelevant.

In order to obtain the value of retargeting we have to still compute the expected present value of profits when the belief that the consumer is searching for information is $1-q$ which can be obtained to be $\overline{V}(1-q) = \frac{pq(1-q)v}{2(\psi+pq)}$.

We can then obtain the value of retargeting when $c \to 0$ to be

$$\lim_{c \to 0} \widehat{V}(1-q) - \overline{V}(1-q) = \frac{q(1-q)v\psi(\widehat{p}-p)}{2(\psi+\widehat{p}q)(\psi+pq)}. \tag{11}$$

This yields the following comparative statics:

Proposition 5: *Suppose that the cost of retargeting is small. Then the value of retargeting is increasing in the propensity for the consumer to receive signals with retargeting, $\widehat{p}$, the profit earned if the consumer purchases the product, v, the likelihood of the firm observing that the consumer is searching, ϕ, and decreasing in the propensity of the consumer to receive signals without retargeting, p. The value of retargeting is increasing (decreasing) in the likelihood of the consumer receiving an informative signal given that he receives a signal, q, for $q < (>)\ q^*$ for a $q^* \in (0, 1/2)$, and in the likelihood of the consumer deciding to stop searching exogenously, ψ, for small $\psi < (>)\ \widehat{p}q$.*

The more interesting comparative statics are the ones on the probability of the consumer receiving an informative signal given that he receives a signal, q, and on the likelihood of the consumer deciding to stop searching exogenously, ψ. As discussed above when the probability of receiving an informative signal is close to zero, there is not much gain in retargeting, as the consumer is not likely to receive an informative signal. At the same time if the probability of receiving an informative signal is close to one, there is also not much of a benefit in retargeting, as when the firm observes the consumer searching for information, the consumer most likely also received a fully informative signal. The proposition shows that the value for retargeting is the highest for an intermediate value of q which lis ess than 1/2.

Also as discussed above, if the likelihood of the consumer deciding to stop searching exogenously is close to zero, there is not much gain in retargeting, as the consumer will end up receiving a fully informative signal even if the consumer is not being retargeted to. At the same time, if the likelihood of the consumer deciding to stop searching exogenously is very high, there is not much gain in retargeting as the consumer is likely to stop searching before receiving a fully informative signal, even when being retargeted to. We find that the ψ that maximizes the value of retargeting is $\psi = \widehat{p}q$.

Note that this value of retargeting is not the cost of retargeting c such that the firm chooses not to do any retargeting. That threshold cost of retargeting is presented in (**??**) in the Appendix. The greater c the less retargeting that the firms does, and the lower the value of retargeting.

One could also ask what is the value of information for the firm of learning for sure if a consumer is or is not searching for information. If a consumer is searching for information for sure, the expected presented value of profits is $\widehat{V}(1)$. If a consumer is not searching for information for sure, the expected value of profits is $V(0) = 0$. The value of perfect information is then $x\widehat{V}(1) - \widehat{V}(x)$ which is positive as $\widehat{V}(x)$ is strictly convex for $x > \widehat{x}$.[9] Note also that, given the convexity of the value function we have $\widehat{V}(1) - \widehat{V}'(1) < 0$ and $\widehat{V}(1) > V'(0) = \widehat{V}'(\widehat{x})$, which means that the value of perfect information is maximized at some $x^* \in (\widehat{x}, 1)$. That is, as time passes since the firms knows for sure that the consumer is searching for information, the value of perfect information for the firm first increases, until the belief reaches x^*, and then decreases. We can also get that for $c \to 0$ we have $V(1) < \widehat{V}'(1-q)$ such that $x^* \in (\widehat{x}, 1-q)$. That is, as time passes since the firm actually observes the consumer searching for information, the value of perfect information for the firm first increases, and then decreases

5. Variations on the Modeling of Retargeting

In this Section we consider the possibility that retargeting may not only increase the hazard rate of information received by the consumers, but can increase the likelihood of the firm observing when the consumers receive a signal, ϕ, and the likelihood of the consumers receiving a fully informative signal conditional on receiving a signal, q.

[9] The same holds for $x < \widehat{x}$ and in that case of value of perfect information would be $x\widehat{V}(1) - V(x)$.

5.1. Retargeting and Tracking Consumer Search

One possibility not considered above is that, by doing retargeting to a consumer, a firm may have a better chance at observing when the consumer receives a signal. For example, this could be because the firm becomes more active in monitoring that consumer search behavior, or because retargeting is more likely to increase the consumer clicking on the firm's advertisements, which are easier to track by the firm. In terms of the model above, this would mean that the parameter ϕ would be greater when the firm is retargeting, $\widehat{\phi} > \phi$.

In terms of the analysis above, considering this case could be done by replacing ϕ with $\widehat{\phi}$ in (7), with $\widehat{B} = \widehat{p}\widehat{\phi} + \psi + \widehat{p}q(1 - \widehat{\phi})$. The analysis would then lead to obtaining $\widehat{x}$ implicitly by (**??**), which is presented in the Appendix.

We can obtain that when retargeting does not lead to a greater hazard rate of the consumer receiving signals, $\widehat{p} = p$, then it is optimal for the firm not to do retargeting with $\widehat{\phi} > \phi$. That is, the ability to better track the consumer search does not lead by itself for retargeting to be optimal. In that case, by doing retargeting the firm would just incur the retargeting costs and not accelerate in any way the possibility of the consumer purchasing the product.

When the costs of retargeting approach zero, and $\widehat{p} > p$ we can obtain a measure of the ratio $\widehat{x}/c$ as

$$\lim_{c \to 0} \frac{\widehat{x}}{c} = 2 \frac{\widehat{p}q + \psi}{qv\psi(\widehat{p} - p)} \frac{p\phi + \psi + pq(1 - \phi)}{\widehat{p}\widehat{\phi} + \psi + \widehat{p}q(1 - \widehat{\phi})}. \tag{12}$$

From this we can obtain that, if $\widehat{p} > p$, a greater ability to track the consumer information search during retargeting can lead to greater retargeting. That is, if retargeting is occurring, the firm prefers to increase the region of beliefs where it occurs if it leads to greater tracking of consumer search information, because of the gain in the information by the firm.

5.2. Retargeting and Quality of Signals

Another possibility not considered above is that, by doing retargeting to a consumer, a firm may be able to increase the informativeness of the signals provided. For example, this could be because the retargeting of the firm could be more informative than the usual consumer information sources. In terms of the model above, this would mean that the parameter q would be greater when the firm is retargeting, $\widehat{q} > q$.

In terms of the analysis above, considering this case could be done by replacing q with $\widehat{q}$ in (7), with $\widehat{B} = \widehat{p}\phi + \psi + \widehat{p}\widehat{q}(1 - \phi)$. The analysis would then lead to obtaining $\widehat{x}$ implicitly by (**??**), which is presented in the Appendix.

We can obtain that, even when retargeting does not lead to a greater hazard rate of the consumer receiving signals, $\widehat{p} = p$, it is still optimal for the firm to do retargeting with $\widehat{q} > q$. That is, the ability by itself to get the consumers to receive more informative signals with retargeting leads retargeting to be optimal. In that case, by doing retargeting, the firm accelerates the possibility of the consumer purchasing the product before the consumers drops exogenously out of the search process.

When the costs of retargeting approach zero, with $\widehat{p} \geq p$ we can obtain a measure of the ratio $\widehat{x}/c$ as

$$\lim_{c \to 0} \frac{\widehat{x}}{c} = 2 \frac{\widehat{p}\widehat{q} + \psi}{v\psi(\widehat{p}\widehat{q} - pq)} \frac{p\phi + \psi + pq(1-\phi)}{\widehat{p}\phi + \psi + \widehat{p}\widehat{q}(1-\phi)}. \tag{13}$$

From this we can obtain that a greater informativeness of signals during retargeting leads to greater retargeting. That is, the firm prefers to increase the region of beliefs where retargeting occurs if it leads to greater signal informativeness, because of the acceleration in consumers deciding whether to purchase the product.

6. Recognizing Purchases

In the previous section it was assumed that the seller does not know when the consumer makes the purchase, and therefore, we have that the firm may continue to advertise even after a purchase. With improvements in tracking technologies, firms might have the ability to detect when consumers purchase the product, and therefore do not send further retargeting advertising. This Section considers this case, and compares the optimal strategy with the case in Section 3 in which the seller does not detect consumer purchases.

In this case, if a firm observes a consumer search for information (which happens with probability $p\phi\ dt$ when not advertising given that the consumer is searching for information), with probability $q/2$ the firm sees the consumer purchase the product. After observing the consumer search for information and not observing any purchase, the posterior belief that the consumer is still searching for information is $1 - \frac{q}{2-q}$, as the probability of no purchase given search is $1 - q/2$, and the probability of continuing to search for information given that a signal was received is $1 - q$.

The belief updating when the firm does not observe the consumer searching for information is also now different because the firm observes a product purchase which occurs with probability $p(1-\phi)\frac{q}{2}x_t\ dt$, if the firm does not observe consumer search, and given that the consumer is searching for information. As in Section 3, the beliefs at time t have to be consistent with what can occur in the future. As in Section 3, the beliefs are again a supermartingale, falling in expected value by $(\psi + pq)x_t\, dt$ in period dt, and we can obtain

$$x_t = (x_t + \frac{dx}{dt}dt)[1 - \phi\, p\, x_t\, dt - (1-\phi)\,\frac{q}{2}\,p\,x_t\,dt] + \phi\,(1-\frac{q}{2})\,(1-\frac{q}{2-q})\,p\,x_t\,dt + \psi\,x_t\,dt + q\,p\,dt. \quad (14)$$

We can then obtain that (2) changes in this case to

$$\frac{dx_t}{dt} = -p[\phi + (1-\phi)\frac{q}{2}]x_t(1-x_t) - \psi x_t - p(1-\phi)\frac{q}{2}x_t. \quad (15)$$

Let $\widetilde{x}$ be the threshold belief of the firm such that the firm only advertises for $x > \widetilde{x}$. With similar analysis as in Section 3 we can obtain that when the firm is not advertising, $x < \widetilde{x}$, the present value of profits of the firm can be obtained as

$$\widetilde{V}(x) = \widetilde{C}[B - Ax] + \frac{qv/2 + \phi(1-q/2)\widehat{\widetilde{V}}(1-\frac{q}{2-q})}{\phi + (1-\phi)q/2}, \quad (16)$$

where $A = p(\phi + (1-\phi)\frac{q}{2})$, $\widetilde{C}$ is a constant to be determined later, and B is, as defined above, $B = p\phi + p(1-\phi)q + \psi$.

For the region of beliefs where the firm advertises, $x > \widetilde{x}$, we can obtain, along the same lines as in Section 3, the present value of profits as

$$\widehat{\widetilde{V}}(x) = \widehat{\widetilde{C}}[\widehat{B} - \widehat{A}x] + c\frac{\widehat{B} - \widehat{A}x}{\widehat{B}^2} - \frac{c}{\widehat{B}} + \frac{qv/2 + \phi(1-q/2)\widehat{\widetilde{V}}(1-\frac{q}{2-q})}{\phi + (1-\phi)q/2}, \quad (17)$$

where $\widehat{A} = \widehat{p}[\phi + (1-\phi)q/2]$, $\widehat{\widetilde{C}}$ is a constant to be determined, and $\widehat{B}$ is, as defined above, $\widehat{B} = \widehat{p}\phi + \widehat{p}(1-\phi)q + \psi$. With the conditions $\widetilde{V}(0) = 0, \widetilde{V}(\widetilde{x}) = \widehat{\widetilde{V}}(\widetilde{x})$, and $\widetilde{V}'(\widetilde{x}) = \widehat{\widetilde{V}}'(\widetilde{x})$, and evaluating (17) at $x = 1 - \frac{q}{2-q}$, one can obtain the value of $\widetilde{x}$ and of the constants $\widetilde{C}$ and $\widehat{\widetilde{C}}$. The optimal $\widetilde{x}$ is determined implicitly by (**??**), presented in the Appendix.

For $c \to 0$ we can obtain $\widetilde{x} \to 0$, $\widetilde{x} < \widehat{x}$ and that $\frac{\widetilde{x}}{c}$ converges to the value in (8).

PROPOSITION 6: *For small c, $\widetilde{x} < \widehat{x}$, and $\frac{\widetilde{x}}{c} - \frac{\widehat{x}}{c} \to 0$ as $c \to 0$.*

We find that when the firm is able to recognize purchases when they occur, and the costs of doing retargeting are small, the firm has a threshold of beliefs that the consumer is searching for information in order to do retargeting, which is lower than the threshold in the case in which the firm does not recognize purchases immediately. In the case of recognizing purchases, the firm knows that if the firm waits longer without doing retargeting, the firm may learn that retargeting is not needed because the consumer purchased the product. On the other hand, when doing retargeting, the firm knows when the consumer made a purchase, and retargeting is no longer needed. It turns out that, when the costs of doing retargeting are small, the former effect dominates, and the threshold belief to do retargeting is lower when purchases are recognized. We could not find parameter values for which this result did not hold. Furthermore, when $c \to 0$, the two thresholds converge to zero at the same rate.

Figures 9-12 illustrate how $\widehat{x}$ and $\widetilde{x}$ evolve for the different parameters for the case of $c > 0$, showing that for the parameters considered we always have $\widehat{x} > \widetilde{x}$.

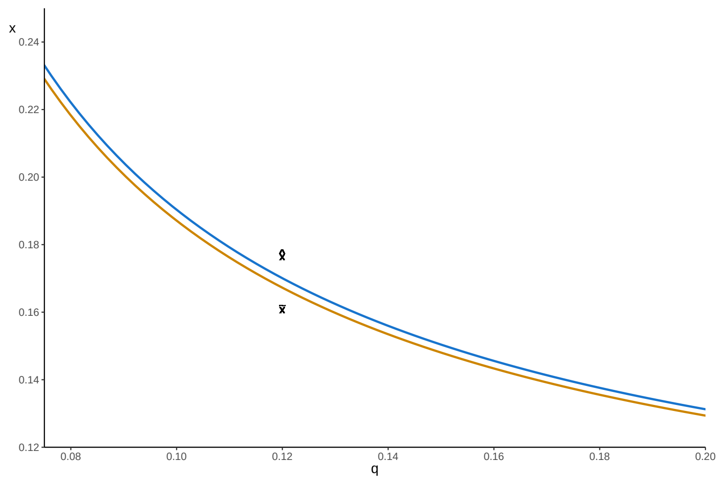

Figure 9: Evolution of $\widehat{x}$ and $\widetilde{x}$ as a function of q for $c = .01, v = 2, p = .4, \widehat{p} = 1, \phi = .5$, and $\psi = .1$.

It is also interesting to evaluate how the beliefs evolve in this case where purchases are recognized, and compare them with the case above when purchases are not recognized. In this case of purchases being recognized, we can obtain that the beliefs that the consumer is searching for information as a function of time, since the time at which the firm recognizes that the consumer is actually searching for information, as

$$x_t = \frac{2(1-q)\widehat{B}}{\widehat{p}(1-q)[\phi(2-q)+q] + [\psi(2-q)+\widehat{p}q]e^{\widehat{B}t}}. \tag{18}$$

Comparing this with the case above when purchases are not recognized, one can obtain that, without further information, the beliefs under purchase recognition are always above the beliefs

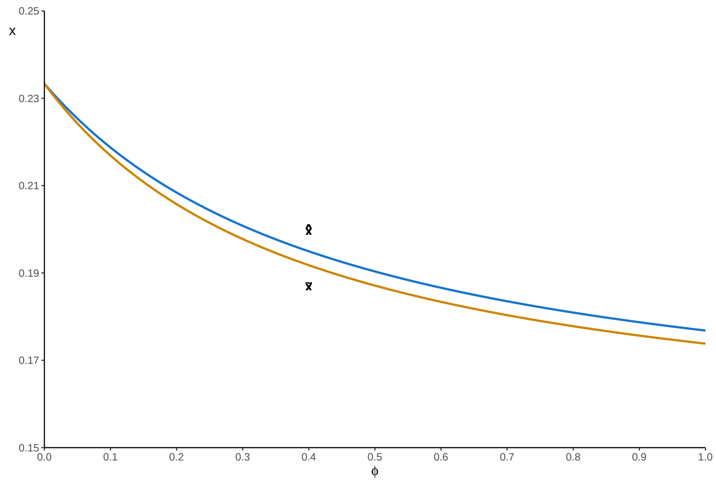

Figure 10: Evolution of $\widehat{x}$ and $\widetilde{x}$ as a function of ϕ for $c = .01, v = 2, p = .4, \widehat{p} = 1, q = .1$, and $\psi = .1$.

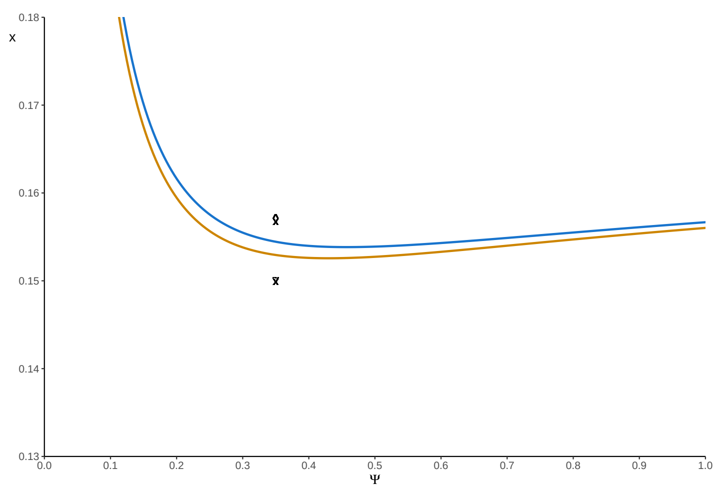

Figure 11: Evolution of $\widehat{x}$ and $\widetilde{x}$ as a function of ψ for $c = .01, v = 2, p = .4, \widehat{p} = 1, q = .1$, and $\phi = .5$.

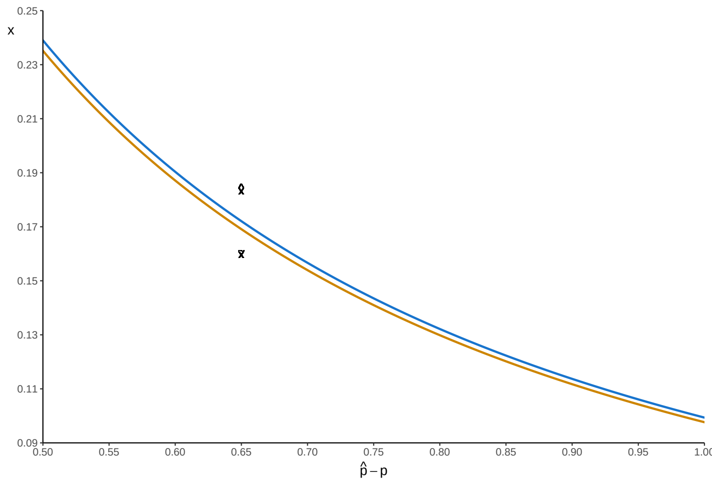

Figure 12: Evolution of $\widehat{x}$ and $\widetilde{x}$ as a function of $\widehat{p} - p$ for $c = .01, v = 2, p = .4, q = .1, \psi = .1$ and $\phi = .5$.

without purchase recognition. Figure **??** illustrates how these beliefs evolve over time under the two different conditions.

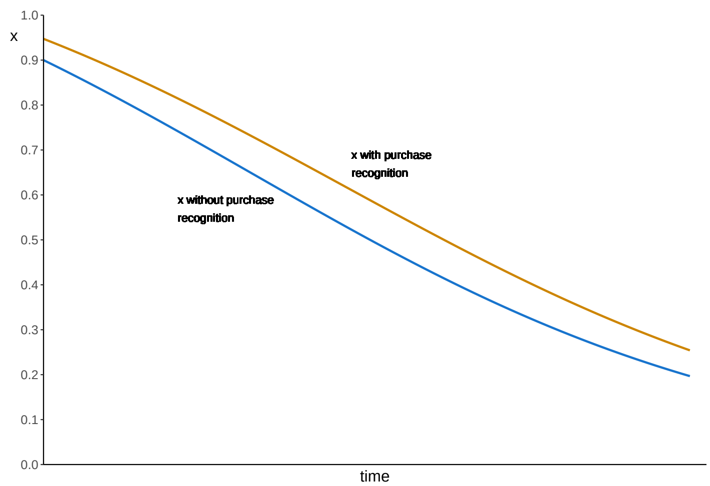

Figure 13: Evolution of the beliefs x over time, for the cases with and without purchase recognition, and if the firm does not get new information. This is presented for $v = 2, p = .4, \widehat{p} = 1, q = .1, \psi = .1$, and $\phi = .5$.

As in Section 3, in this case when purchases are recognized, we can also compute the maximum amount of time that a consumer could be retargeted after being identified as searching for information, which we can denote as $\widetilde{T}$, and is presented in the Appendix. For c close to zero, one can obtain that this length of time is greater than the maximum amount of time of retargeting in the case when purchases are not recognized. Recognizing purchases makes the extent of time that a consumer can receive retargeting without being in the state of search for information to be greater. This is because the firm now has a greater belief that the consumer is still searching for information, because the firm would observe whether a purchase had occurred.

7. Consumer Decision-Making and Pricing

7.1. Preliminaries

The analysis in the previous sections restricted attention to the firm's retargeting behavior, with just the assumption that the firm would receive a payoff of v if the consumer received an informative positive signal of the product fit. This section micro-models the consumer's search behavior and derives how v can be obtained from this behavior and the firm's pricing decision.

Suppose that the consumer can be in a state in which he knows that with equal probability he has either zero value for a product or some value ω per period of using the product. The product is a durable good with infinite life. By searching for information, with a process that is explained below, the consumer can determine, if he is in this state, whether the value is zero or ω. The consumer also knows that with hazard rate ψ he will switch from this state to a state in which he has zero value of the product forever. Working in terms of present value of benefits, the consumer can then have an overall gross benefit of getting the product of either zero or $w = \omega/\psi$. Suppose that the firm chooses a price P that cannot be customized, and cannot vary with time, and w has an ex-ante cumulative probability distribution function $F(w)$ with positive density on, and only on, $[\underline{w}, \overline{w}]$ with $\underline{w} > 0$. Note that one can obtain $F(w)$ directly from the cumulative probability distribution function on ω. We consider the situation where the consumer only learns w when he finds that he has a strictly positive benefit for the product. The case in which the consumer knows the value of w, while he is still uncertain whether the benefit of the product is either zero or w, is discussed briefly at the end of this section. Finally, let m be the marginal cost of production of the product, let P be the price charged by the firm, and let $\widetilde{P} = \arg\max_P (P - m)[1 - F(P)]$, the static monopoly price.[10]

A consumer can search for information at a cost of $\varepsilon\, dt$ for a period of length dt, where ε is assumed small. If the firm is not advertising, the consumer, if searching for information, receives a signal about the product fit in the period dt with probability $p\, dt$. If the firm is advertising, the consumer, if searching for information, receives a signal about the product fit in the period

[10] Suppose also that m is large enough such that the consumer's expected value ex-ante is below m. That is, it is not profitable to price such that the consumer purchases the product without searching for information.

dt with probability $\widehat{p}\,dt$. If the consumer is not searching for information, the consumer does not get any signal, whether or not the firm is advertising.[11]

Finally, let $S(P)$ be the expected consumer surplus conditional on the consumer receiving a positive informative signal of the product fit, and the firm charging price P. That is, $S(P) = \int_P^{\overline{w}}(w - P)\,dF(w)$. Consumers are assumed to be risk-neutral. Possible consumer discounting of the future is only considered through the hazard rate ψ at which their benefit of having the product is extinguished.

7.2. Myopic Consumers

Consider first the case of consumers being myopic with respect to any potential future advertising that results from the firm finding out that the consumer is searching for information. That is, consumers are unaware of the retargeting policy by the firm.

In that case, for a consumer to search for information, it must be that the expected benefit of searching for information is greater than the cost of searching for information. Consider the case in which the consumer is not receiving advertising. In that case, for the consumer to want to search for information, it must be the case that $\frac{q}{2}pS(P) \geq \varepsilon$. Define $\overline{P}$ as the price that makes this inequality bind, $S(\overline{P}) = \frac{2\varepsilon}{pq}$.

If the price is only checked each time the consumer gets a signal, given that the search costs are sunk, the firm would choose to charge the static monopoly price $\widetilde{P}$, and the consumer would only search for information if $\overline{P} \geq \widetilde{P}$, i.e., $S(\widetilde{P}) \geq \frac{2\varepsilon}{qp}$, and the analysis of the previous sections would carry through with $v = (\widetilde{P} - m)[1 - F(\widetilde{P})]$. In this setting, if $\overline{P} < \widetilde{P}$, no consumer would search for information, and the firm would not do any retargeting.

Potentially more interesting for the context being modeled, price could be seen as being freely checked (or fixed over time, and learned at the first search), and then the firm can use it to provide incentives for the consumer to search for information on product-fit. The remainder of the analysis considers this case.

As the expected profit from a consumer receiving a positive informative signal is $(P - m)[1 - F(P)]$, if ε is small enough we have $\frac{q}{2}pS(\widetilde{P}) > \varepsilon$, and the optimal price to charge is then $P^* = \widetilde{P}$. If ε is not small enough, then that inequality does not hold for $\widetilde{P}$ and we have then that the optimal price is $P^* = \overline{P}$. That is, we have that the optimal price $P^* = \min[\widetilde{P}, \overline{P}]$, which is

[11] We could also consider that when the firm is advertising the consumer gets a signal in period dt with probability between zero and $(\widehat{p} - p)\,dt$, and the main messages of the results below would still follow.

independent of the search costs ε for small ε, and decreasing in the search costs ε for large ε. Figure **??** illustrates how the optimal price P^* evolves with the search costs ε.

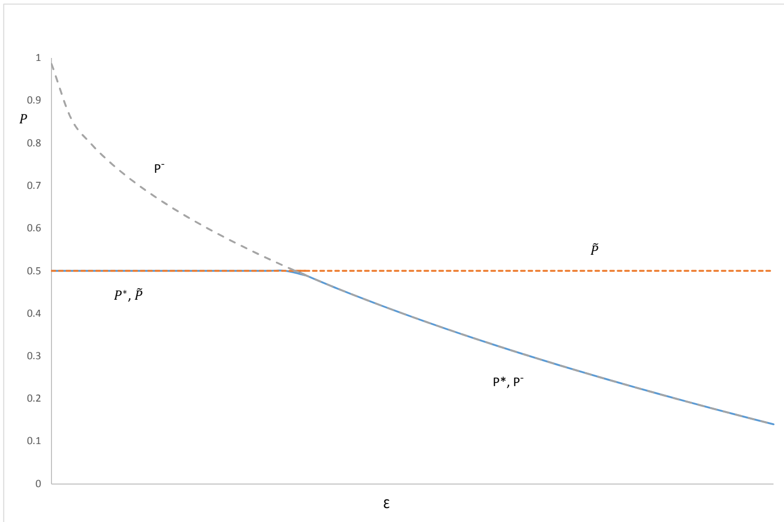

Figure 14: Optimal price P^* for the myopic consumers' case as a function of the search costs ε for the case in which $F(w) = w$ with support $[0, 1]$, and $m = 0, q = .2, p = 1$.

7.3. Forward-Looking Consumers

Consider now the case of forward-looking consumers, consumers who are aware that if they are observed searching for information, they will receive retargeting advertising that will provide more information about the product fit. Suppose that consumers are infinitely patient, but given that the consumer also knows that with hazard rate ψ he will switch from a state of having a potential positive benefit for the product to a state in which he has zero benefit of the product forever, the consumer has bounded benefit of having the product. The existence of hazard rate ψ works as discounting on the consumer benefits. This possibility of retargeting, given the assumed no hassle costs of receiving advertising, makes then the consumer more willing to search for information than in the case of myopic consumers, potentially allowing the firm to increase its price. To focus on the critical market forces, and simplify the analysis, let us also assume that the consumer knows when retargeting is occurring.

To analyze this situation, let W_n be the expected present value of benefits for the consumer when the firm is not retargeting, and let $W(t)$ be the expected present value of benefits for the consumer when the firm is retargeting for t time. From Section 3 we also have the maximum extent of time that a consumer is retargeted to without purchase as a function of v as $\widehat{T}$. The expected payoff for the consumer when searching for information without being retargeted to can be obtained as

$$W_n = \frac{q}{2}\, S(P)\, p\, dt - \varepsilon\, dt + (1 - B\, dt) W_n + \phi\, (1-q)\, W(0)\, p\, dt, \tag{19}$$

from which one can obtain[12]

$$BW_n = \frac{q}{2} pS(P) - \varepsilon + \phi(1-q)pW(0). \tag{20}$$

Similarly, if the consumer is searching for information, and is being retargeted to for a t period of time, we can now obtain his expected present value of payoffs as

$$W(t) = \frac{q}{2}\, S(P)\, \widehat{p}\, dt - \varepsilon\, dt + [1 - \widehat{B}\, dt][W(t) + W'(t)\, dt] + \phi\, (1-q)\, W(0)\, \widehat{p}\, dt, \tag{21}$$

from which we can obtain

$$W(t) = De^{\widehat{B}t} + \frac{1}{\widehat{B}}[\frac{q}{2}\widehat{p}S(P) - \varepsilon + \phi(1-q)\widehat{p}W(0)], \tag{22}$$

where D is a constant to be determined, and from which we can obtain

$$W(0) = \frac{D\widehat{B} + \frac{q}{2}\widehat{p}S(P) - \varepsilon}{q\widehat{p} + \psi}. \tag{23}$$

As the consumer's expected value of payoffs is continuous when the firm switches to stopping advertising, we have $W_n = W(\widehat{T})$, from which we can obtain, using (**??**), **??**, and (**??**), the value of D which is presented in the Appendix. We can obtain that $D < 0$, which yields that $W(t)$ is decreasing in t at an increasing rate. That is, the consumer is better off the longer he is likely to be receiving retargeting, and this benefit falls quickly when the maximum future time is reduced.

For search costs that are not too small, the optimal policy of the firm is to price such that a consumer not receiving retargeting is indifferent between searching and not searching for information, $W_n = 0$. As $W(t)$ is decreasing in t and $W(\widehat{T}) = W_n$, we then have that

[12]We remind that $B = p\phi + p(1-\phi)q + \psi$.

$W(t) > 0$ for $t \in [0, \widehat{T})$. From this we can obtain from (**??**) that $\frac{q}{2}\widehat{p}S(P) - \varepsilon > 0$; that is, when the period of retargeting starts, the consumer has a strictly positive current expected benefit of searching for information. This confirms that during the period of retargeting the consumer continues to want to search for information.

From (**??**) and $W(0) > 0$, we can also obtain that $\frac{q}{2}pS(P) - \varepsilon < 0$; that is, when the consumer is not receiving retargeting, the consumer has a strictly negative current expected benefit of searching for information. This is the positive effect of the consumer's search for information because the consumer is forward-looking.

To obtain further insights, note that for $c \to 0$, we have $\widehat{T} \to \infty$, from which we can obtain

$$W(0) \to \frac{\frac{q}{2}\widehat{p}S(P) - \varepsilon}{q\widehat{p} + \psi} \tag{24}$$

$$W_n \to \frac{1}{B}\{\frac{q}{2}pS(P) - \varepsilon + \frac{\phi(1-q)p}{q\widehat{p} + \psi}[\frac{q}{2}\widehat{p}S(P) - \varepsilon]\}. \tag{25}$$

Setting $W_n = 0$ to obtain the optimal price P^*, we get that when $c \to 0$, the optimal price when ε is small is obtained by

$$S(P^*) = \frac{2\varepsilon}{qp}\frac{B}{\widehat{B}}, \tag{26}$$

which leads to a higher price P^* than in the myopic consumers case. From this, one can also obtain that, as the search costs ε increase, P^* has to decrease, reducing then $v = (P^* - m)[1 - F(P^*)]$. That is, as search costs increase, the firm reduces the price to induce search, and earns a lower expected profit per consumer who receives a positive informative signal.

For general costs of retargeting c, note that P^* is increasing in $\widehat{T}$. For large $\widehat{T}$ this effect on P^* is small, as in that case P^* is close to (**??**). On the firm side, per the analysis in Section 3, we can obtain that for small c, the effect of v on $\widehat{T}$ is bounded, and bounded away from zero. As an increase in c makes the firm increase $\widehat{T}$ for a fixed v, we have that, in equilibrium, for c small, an increase in c leads to a decrease in $\widehat{T}$ and a small decrease in v (small decrease in P^*).[13] Figure **??** illustrates the change in the equilibrium for an increase in c for this case of small c.

The result that the consumer is willing to accept a higher price when being forward-looking depends on our assumption that that there was no dis-utility of receiving advertising. If receiving advertising creates dis-utility, that can be introduced in (**??**), which would lead to a lower price for the consumer to be willing to search for information. Letting η be the dis-utility per unit of

[13]The effects on v and P^* converge to zero as $c \to 0$.

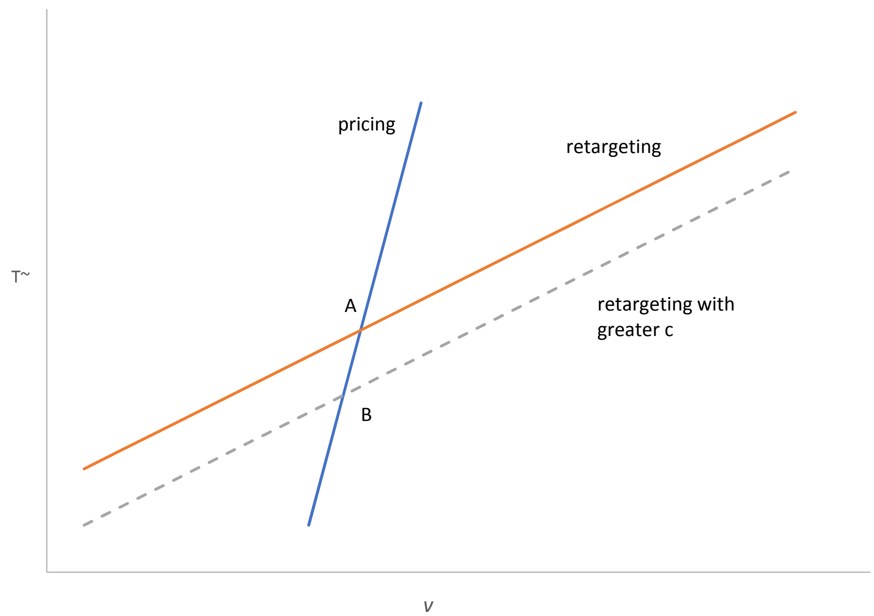

Figure 15: Equilibrium change in $\widehat{T}$ and v for an increase in the costs of retargeting c for small c: The retargeting best-response moves down, and the equilibrium moves from point A to point B.

time of receiving retargeting, (**??**) would change to

$$W(t) = \frac{q}{2}\,[S(P) - \eta\widehat{T} - t)]\,\widehat{p}\,dt - (\varepsilon + \eta)\,dt + [1 - \widehat{B}\,dt][W(t) + W'(t)\,dt] + \phi\,(1-q)\,W(0)\,\widehat{p}\,dt, \quad (27)$$

and the analysis would then proceed as above. If this dis-utility of receiving advertising, η, is sufficiently large, the price to induce consumer search for information may be lower than the price under myopic consumers.

Another interesting possibility not considered above is that the consumer, before engaging in search for information, may have a sense of his valuation for the product w in case of product-fit. This could be because the valuation in case of product-fit is related to some consumer characteristic that is common across all products. One such characteristic could be, for example, consumer income. In such a case, only consumers with a sufficiently large w will search for information with the intent of purchasing the product if they find a product fit. The choice of the price would then determine the threshold w for the consumer to search for information. The construction of v in this case would then just involve the margin obtained on the product.

8. Concluding Remarks

This paper considers the optimal retargeting strategy of a firm when the firm does not fully know whether the consumer is searching for information, but receives occasional signals when the consumer is searching for information. The consumer searches for information and at some point finds out whether the product is a good fit. The model captures the possibility of the consumer continuing to receive retargeting even after purchasing the product or after deciding that the product is not a good fit. The paper characterizes how the optimal policy is affected by the different parameters, and compares the optimal strategy with the case in which purchases are immediately recognized.

We also illustrate how consumers looking for information can be endogenized in the model, and illustrate how forward-looking consumers may potentially allow the firm to increase its price. This possibility allows for both the choices of price and retargeting to be endogenized, and illustrates how these two decisions interact, with a higher price leading to more retargeting, and more retargeting allowing for higher prices if consumers do not have a strong dis-utility for the advertising that is received.

The paper models retargeting as a discrete action that can either occur or not. Alternatively, one could think of retargeting as a continuous variable, such that a greater retargeting intensity is more costly but leads to more information to be provided to the consumers. This alternative model could potentially generate that the retargeting intensity is greater closer to the time that the firm receives information that the consumer is searching for information, and then slowly decreases over time. The analysis of such an alternative model is rather complex, but the main messages of the model considered here should carry over in that smoother case.

The model considered that the probability of a firm receiving information that the consumer is searching for information is exogenous. More generally, we could imagine the firm endogenously deciding on the intensity of monitoring whether the consumer is searching more information. This would be an interesting issue for future research.

Another interesting question to investigate is what happens when the consumer is searching for information to choose one of several products, and the retargeting decision is made by several firms. That problem would require considering how the different firms gain information on whether a consumer is searching for information, how retargeting by any set of firms affects how consumers receive information, and the threshold decisions for any number of firms to decide to do, or stop doing, retargeting. The surplus extracted by a firm after a product-fit, may also depend on the opportunity for the consumer of continuing to search for information. Such a model may also endogenize the probability of a consumer stopping the search process, because of the consumer purchasing the product of one of the competitors.

APPENDIX

Proof of Proposition 1: Consider the optimal policy, and note that the continuation payoff after the firm observes that the consumer is searching for information is independent of the belief that the consumer is searching for information immediately prior to the firm observing the consumer search, as the state-relevant payoff variable has the same value at that time. Let $\pi(t)$ be the expected future payoff for the firm if the firm does retargeting for a period of time t if the firm does not observe the consumer searching for information prior to t (in that contingency, the firm would go back to the optimal policy, after just observing that the consumer is searching for information), given that the consumer is in the search state. Note that $\pi(t)$ includes the possibility of the consumer dropping out of the search process before t (either because of ψ or because the consumer receives a negative informative signal on the product-fit), and the retargeting costs c that are incurred. Note also that the expected payoff for the firm under this strategy, given that the consumer is actually not in the search state, is $-ct$. Let $t^*(x)$ be the optimal t when the firm has a belief x that the consumer is in the search state.

Then, for it to be optimal for the firm to do retargeting when it has belief x, we must have that

$$x\pi(t^*(x)) + (1-x)(-ct^*(x)) \geq x\pi(0). \tag{i}$$

Consider now the belief $x' > x$, and suppose that the firm with that belief chooses the suboptimal strategy $t^*(x)$, as the maximum length of time that the firm does retargeting if the firm does not observe that the consumer is searching for information prior to $t^*(x)$. The expected future payoff for the firm would then be

$$x'\pi(t^*(x)) + (1-x')(-ct^*(x)) > x\pi(t^*(x)) + (1-x)(-ct^*(x)) \geq x\pi(0). \tag{ii}$$

This yields that at state x' it is optimal for the firm to do retargeting. As this was obtained for a general $x' > x$, we then have that there is a threshold $\widehat{x}$ such the firm does retargeting for $x \geq \widehat{x}$ and does not do retargeting for $x < \widehat{x}$.

Presentation of $\widehat{C}$ in (7): By making $x = 1-q$ in (7) one obtains

$$\widehat{C} = \frac{c}{(\psi + \widehat{p}q)\widehat{B}} - \frac{qv}{2\phi(\psi + \widehat{p}q)} - \frac{c}{\widehat{B}^2} \log \frac{\psi + \widehat{p}q}{1-q}. \tag{iii}$$

Table 1: Notation

Variable	Description
p	hazard rate of consumer receiving product information if no retargeting
$\widehat{p}$	hazard rate of consumer information if firm is doing retargeting
c	cost per unit of time of firm doing retargeting
q	probability of consumer receiving fully informative signal given that he received product information (with no retargeting if in subsection 5.2)
$\widehat{q}$	probability of consumer receiving fully informative signal given that he received product information if firm is doing retargeting and this probability is different than the case of no retargeting (it only appears in subsection 5.2
ϕ	probability of firm observing that consumer is searching for information conditional on consumer receiving product information (with no retargeting if in subsection 5.1)
$\widehat{\phi}$	probability of firm observing that consumer is searching for information conditional on consumer receiving product information if firm is doing retargeting and this probability is different than the case of no retargeting (it only appears in subsection 5.1)
x	firm belief that the consumer is in the state of searching for information
ψ	hazard rate of consumer exogeneously moving to the state of not searching for information
$\widehat{x}$	endogenous belief threshold, when firm does not recognize when purchases occur, such that firm does retargeting for $x \geq \widehat{x}$ and does not do retargeting for $x < \widehat{x}$
$\widetilde{x}$	endogenous belief threshold, when firm recognizes when purchases occur, such that firm does retargeting for $x \geq \widetilde{x}$ and does not do retargeting for $x < \widetilde{x}$
x_m	endogenous belief threshold, when firm is myopic and does not recognize when purchases occur, such that firm does retargeting for $x \geq x_m$ and does not do retargeting for $x < x_m$
$V(x)$	value function of firm in region of beliefs of no retargeting if firm does not recognize when purchases occur
$\widehat{V}(x)$	value function of firm in region of beliefs of retargeting if firm firm does not recognize when purchases occur
$\widetilde{V}(x)$	value function of firm in region of beliefs of no retargeting if firm recognizes when purchases occur
$\widehat{\widetilde{V}}(x)$	value function of firm in region of beliefs of retargeting if firm firm recognizes when purchases occur
v	payoff for firm if consumer purchases the product
$\widehat{T}$	maximum amount of time that a consumer could be retargeted when firm does not recognize when purchases occur
$\widetilde{T}$	maximum amount of time that a consumer could be retargeted when firm recognizes when purchases occur
B	$p\phi + \psi + p(1-\phi)q$
$\widehat{B}$	$\widehat{p}\phi + \psi + \widehat{p}(1-\phi)q$
A	$p(\phi + (1-\phi)q/2)$
$\widehat{A}$	$\widehat{p}(\phi + (1-\phi)q/2)$
$C, \widehat{C}, \widetilde{C}, \widehat{\widetilde{C}}$	constants in firm value functions

Table 2: Notation of variables that only appear in Section **??**

Variable	Description
ω	utility per unit of time for consumer of product if product is of any value to consumer
w	present value of product for consumer if product is of any value ($= \omega/\psi$; distributed with cumulative distribution function $F(w)$)
m	marginal cost of production
P	price charged by firm
$S(P)$	expected consumer surplus given price P, conditional on consumer receiving a positive informative signal
$\widetilde{P}$	static monopoly price ($= \arg\max_P (P-m)[1-F(P)]$)
ε	search cost per unit of time for consumer
$\overline{P}$	product price such that myopic consumer is indifferent between searching and not searching for information
P^*	optimal price
$W(t)$	expected present value of utility for consumer when searching for information and receiving retargeting for t time since firm observed the consumer search for information
W_n	expected present value of utility for consumer when searching for information and not receiving retargeting
D	constant in consumer value function
η	dis-utility of receiving retargeting per unit of time

Derivation of Optimal Policy for $c \to 0$ and the Firm not Detecting Consumer Purchases:

From (5) we can obtain $V'(x) = -Cp\phi$ and

$$\widehat{V}'(x) = -\widehat{C}\widehat{p}\phi - \frac{\widehat{p}\phi c}{\widehat{B}^2}\log\frac{\widehat{B} - \widehat{p}\phi x}{x} - \frac{c}{x}\frac{1}{\widehat{B}}. \tag{iv}$$

Using $V(\widehat{x}) = \widehat{V}(\widehat{x})$ and evaluating $\widehat{V}(x)$ at $x = 1-q$ yields the following condition that determines the value of $\widehat{x}$:

$$2B(\widehat{p}q + \psi) + (\widehat{p} - p)\psi\widehat{x}\left[-\frac{qv\widehat{B}}{c} + 2\phi + 2\phi\frac{\widehat{p}q + \psi}{\widehat{B}}\log\frac{(\widehat{B} - \widehat{p}\phi\widehat{x})(1-q)}{\widehat{x}(\widehat{p}q + \psi)}\right] = 0. \tag{v}$$

When $c \to 0$ in (**??**) we can then obtain (8).

To check the condition on c such that we have $\widehat{x} < 1-q$, note that by using $\widehat{x} = 1-q$ in (**??**) we can obtain an upper bound on c, which is

$$c < \frac{1}{2}\frac{(\widehat{p}-p)\psi(1-q)qv\widehat{B}}{B(\widehat{p}q+\psi)+\phi(\widehat{p}-p)\psi(1-q)} \leq \frac{1}{2}vq(1-q)(\widehat{p}-p). \tag{vi}$$

Proof of Proposition 2: Differentiating (8) with respect to $\widehat{p}, p, v, \psi, \phi$, and q, one obtains the results in the proposition.

Proof of Proposition 5: Direct differentiation of (11) gets the comparative statics with respect to $p, \widehat{p}, v, q$, and ψ. The comparative static with respect to ϕ requires more work. From $V(x) = 0$ one can obtain $\widehat{V}(1-q) = -\frac{qv}{2\phi} - CB$ where

$$C = \widehat{C}\frac{\widehat{p}}{p} + c\frac{\widehat{p}}{p\widehat{B}^2}\log\frac{\widehat{B}-\widehat{p}\phi\widehat{x}}{\widehat{x}} + \frac{c}{\widehat{B}p\phi\widehat{x}} \tag{vii}$$

from $V'(\widehat{x}) = \widehat{V}'(\widehat{x})$, and

$$\widehat{C} = \frac{1}{\psi+\widehat{p}q}\left(\frac{c}{\widehat{B}} - \frac{qv}{2\phi}\right) - \frac{c}{\widehat{B}^2}\log\frac{\psi+\widehat{p}q}{1-q} \tag{viii}$$

from the evaluation of $\widehat{V}(x)$ and $x = 1-q$.

We can then obtain

$$\begin{aligned}\widehat{V}(1-q) \quad = \quad & \frac{qv\widehat{p}(1-q)}{2(\psi+\widehat{p}q)} + \frac{qv\psi(\widehat{p}-p)}{2\phi p(\psi+\widehat{p}q)} - \frac{cB}{p\widehat{B}}\frac{\widehat{p}\phi\widehat{x}+\psi+\widehat{p}q}{\phi\widehat{x}(\psi+\widehat{p}q)} \\ & + \frac{cB\widehat{p}}{p\widehat{B}^2}\log\frac{\widehat{x}(\psi+\widehat{p}q)}{(1-q)(\widehat{B}-\widehat{p}\phi\widehat{x})}\end{aligned} \tag{ix}$$

From (**??**) we can write $\widehat{x} = c(D_1 + D_2)$ with

$$D_1 = 2\frac{\psi+\widehat{p}q}{qv\psi(\widehat{p}-p)}\frac{B}{\widehat{B}} \tag{x}$$

and

$$D_2 = \frac{2\phi\widehat{x}}{qv\widehat{B}}\left[1 + \frac{\psi+\widehat{p}q}{\widehat{B}}\log\frac{(1-q)(\widehat{B}-\widehat{p}\phi\widehat{x})}{\widehat{x}(\psi+\widehat{p}q)}\right]. \tag{xi}$$

The term D_1 is the term that does not go to zero when $c \to 0$, and is independent of $\widehat{x}$, and $D_2 \to 0$ when $c \to 0$.

We can then obtain

$$\frac{\partial\widehat{V}(1-q)}{\partial\phi} = \frac{B}{p\phi^2\widehat{B}}\left[\frac{1}{D_1+D_2}-\frac{1}{D_1}\right]+c\frac{\widehat{p}\psi(\widehat{p}-p)(1-q)}{\widehat{B}^2p(\psi+\widehat{p}q)}+\frac{B}{\widehat{B}p\phi(D_1+D_2)}\left[\frac{\psi(\widehat{p}-p)(1-q)}{B\widehat{B}}+\frac{D_1'+D_2'}{D_1+D_2}\right]$$
$$-c(1-q)\widehat{p}\frac{\widehat{p}p(\phi+q-\phi q)+\psi(2\widehat{p}-p)}{p\widehat{B}^3}\log\frac{\widehat{x}(\psi+\widehat{q}q)}{(1-q)(\widehat{B}-\widehat{p}\phi\widehat{x})}+c\frac{B\widehat{p}}{p\widehat{B}^2}\left[\frac{\widehat{B}(D_1'+D_2')}{(\widehat{B}-\widehat{p}\phi\widehat{x})(D_1+D_2)}+\frac{\widehat{p}\widehat{x}}{\widehat{B}-\widehat{p}\phi\widehat{x}}\right] \quad \text{(xii)}$$

which reduces to

$$\begin{aligned}\frac{\partial\widehat{V}(1-q)}{\partial\phi} \quad = \quad & \frac{B}{p\phi^2\widehat{B}}\frac{-D_2}{D_1(D_1+D_2)}+c\frac{\widehat{p}\psi(\widehat{p}-p)(1-q)}{\widehat{B}^2p(\psi+\widehat{p}q)}+\frac{B}{\widehat{B}p\phi}\frac{D_1D_2'-D_1'D_2}{D_1(D_1+D_2)^2}\\ & -c(1-q)\widehat{p}\frac{\widehat{p}p(\phi+q-\phi q)+\psi(2\widehat{p}-p)}{p\widehat{B}^3}\log\frac{\widehat{x}(\psi+\widehat{q}q)}{(1-q)(\widehat{B}-\widehat{p}\phi\widehat{x})}\\ & +c\frac{B\widehat{p}}{p\widehat{B}^2}\left[\frac{\widehat{B}(D_1'+D_2')}{(\widehat{B}-\widehat{p}\phi\widehat{x})(D_1+D_2)}+\frac{\widehat{p}\widehat{x}}{\widehat{B}-\widehat{p}\phi\widehat{x}}\right]. \qquad \text{(xiii)}\end{aligned}$$

Note now that

$$D_2' = \frac{2\widehat{x}}{qv\widehat{B}}\left[1+\frac{\psi+\widehat{p}q}{\widehat{B}}\log\frac{(1-q)(\widehat{B}-\widehat{p}\phi\widehat{x})}{\widehat{x}(\psi+\widehat{p}q)}\right]+G, \quad \text{(xiv)}$$

where

$$G = \frac{2\phi\widehat{x}'}{qv\widehat{B}}\left[1+\frac{\psi+\widehat{p}q}{\widehat{B}}\log\frac{(1-q)(\widehat{B}-\widehat{p}\phi\widehat{x})}{\widehat{x}(\psi+\widehat{p}q)}\right]-\frac{2\phi\widehat{x}'}{qv(\widehat{B}-\widehat{p}\phi\widehat{x})}. \quad \text{(xv)}$$

When $c\to 0$ the biggest terms in $\frac{\partial\widehat{V}(1-q)}{\partial\phi}$ are of the order $\widehat{x}\log\frac{1}{\widehat{x}}$. Dividing by this expression we obtain

$$\lim_{c\to 0}\frac{1}{\widehat{x}\log\frac{1}{\widehat{x}}}G = -2\frac{(1-q)(\widehat{p}-p)\psi\phi(\psi+\widehat{p}q)}{B\widehat{B}^3qv} \quad \text{(xvi)}$$

and we can also obtain that

$$\lim_{c\to 0}\frac{1}{\widehat{x}\log\frac{1}{\widehat{x}}}(D_1G-D_1'D_2) = 0. \quad \text{(xvii)}$$

This then yields

$$\lim_{c\to 0}\frac{1}{\widehat{x}\log\frac{1}{\widehat{x}}}\frac{\partial\widehat{V}(1-q)}{\partial\phi} = \widehat{p}(1-q)\frac{\widehat{p}p(\phi+q-\phi q)+\psi(2\widehat{p}-p)}{pD_1\widehat{B}^3} > 0. \quad \text{(xviii)}$$

Optimal Policy for Variations in the Modeling of Retargeting: Consider the case of optimal retargeting when purchases are not recognized and $\widehat{\phi} > \phi$ and $\widehat{q} > q$. The case of Subsection 5.1 is the case of $\widehat{q} = q$. The case of Subsection 5.2 is the case of $\widehat{\phi} = \phi$.

By making $V(\widehat{x}) = \widehat{V}(\widehat{x})$ we get

$$\widehat{V}(1-q) - \widehat{V}(1-\widehat{q}) + \frac{v}{2}(\frac{q}{\phi} - \frac{\widehat{q}}{\widehat{\phi}}) + C[B - p\phi\widehat{x}] = \widehat{C}[\widehat{B} - \widehat{p}\widehat{\phi}\widehat{x}] - \frac{c}{\widehat{B}} + c\frac{\widehat{B} - \widehat{p}\widehat{\phi}\widehat{x}}{\widehat{B}^2}\log\frac{\widehat{B} - \widehat{p}\widehat{\phi}\widehat{x}}{\widehat{x}}, \quad \text{(xix)}$$

where $\widehat{B} = \widehat{p}[\widehat{\phi} + (1-\widehat{\phi})\widehat{q}] + \psi$, and from which we can obtain, solving for C,

$$C = \widehat{C}\frac{\widehat{B} - \widehat{p}\widehat{\phi}\widehat{x}}{B - p\phi\widehat{x}} - \frac{c}{\widehat{B}(B - p\phi\widehat{x})} + c\frac{\widehat{B} - \widehat{p}\widehat{\phi}\widehat{x}}{\widehat{B}^2(B - p\phi\widehat{x})}\log\frac{\widehat{B} - \widehat{p}\widehat{\phi}\widehat{x}}{\widehat{x}} - \frac{v}{2(B - p\phi\widehat{x})}(\frac{q}{\phi} - \frac{\widehat{q}}{\widehat{\phi}}) - \frac{\widehat{V}(1-q) - \widehat{V}(1-\widehat{q})}{B - p\phi\widehat{x}}. \quad \text{(xx)}$$

By making $V'(\widehat{x}) = \widehat{V}'(\widehat{x})$ we get

$$Cp\phi = \widehat{C}\widehat{p}\widehat{\phi} + c\frac{\widehat{p}\widehat{\phi}}{\widehat{B}^2}\log\frac{\widehat{B} - \widehat{p}\widehat{\phi}\widehat{x}}{\widehat{x}} + \frac{c}{\widehat{B}\widehat{x}}, \quad \text{(xxi)}$$

from which, solving for C, we can get

$$C = \widehat{C}\frac{\widehat{p}\widehat{\phi}}{p\phi} + c\frac{\widehat{p}\widehat{\phi}}{p\phi\widehat{B}^2}\log\frac{\widehat{B} - \widehat{p}\widehat{\phi}\widehat{x}}{\widehat{x}} + \frac{c}{\widehat{B}p\phi\widehat{x}}. \quad \text{(xxii)}$$

By making the left hand side of (**??**) equal to the left hand side of (**??**) we can obtain

$$\left[\widehat{C} + \frac{c}{\widehat{B}^2}\log\frac{\widehat{B} - \widehat{p}\widehat{\phi}\widehat{x}}{\widehat{x}}\right]\left[\frac{\widehat{p}\widehat{\phi}}{p\phi} - \frac{\widehat{B} - \widehat{p}\widehat{\phi}\widehat{x}}{B - p\phi\widehat{x}}\right] + \frac{c}{\widehat{B}}\left[\frac{1}{p\phi\widehat{x}} + \frac{1}{B - p\phi\widehat{x}}\right] + \frac{v}{2(B - p\phi\widehat{x})}(\frac{q}{\phi} - \frac{\widehat{q}}{\widehat{\phi}}) + \frac{\widehat{V}(1-q) - \widehat{V}(1-\widehat{q})}{B - p\phi\widehat{x}} = 0. \quad \text{(xxiii)}$$

Note now that to obtain $\widehat{C}$ we can evaluate $\widehat{V}(x)$ at $x = 1 - \widehat{q}$ to obtain

$$\widehat{C} = \frac{1}{\psi + \widehat{p}\widehat{q}}\left(\frac{c}{\widehat{B}} - \frac{\widehat{q}v}{2\widehat{\phi}}\right) - \frac{c}{\widehat{B}^2}\log\frac{\psi + \widehat{p}\widehat{q}}{1 - \widehat{q}}, \quad \text{(xxiv)}$$

which we will use in (**??**). Note now that from the evaluation $\widehat{V}(x)$ at $x = 1-q$ one can obtain

$$\widehat{V}(1-q)-\widehat{V}(1-\widehat{q}) = \frac{\widehat{q}v}{2\widehat{\phi}}+\widehat{C}[\psi+\widehat{pq}-\widehat{p\phi}(\widehat{q}-q)]-\frac{c}{\widehat{B}}+c\frac{\psi+\widehat{pq}-\widehat{p\phi}(\widehat{q}-q)}{\widehat{B}^2}\log\frac{\psi+\widehat{pq}-\widehat{p\phi}(\widehat{q}-q)}{1-q}. \tag{xxv}$$

Substituting for $\widehat{C}$ from (**??**) we can obtain

$$\widehat{V}(1-q)-\widehat{V}(1-\widehat{q}) = -\frac{\widehat{p\phi}(\widehat{q}-q)}{\psi+\widehat{pq}}\left(\frac{c}{\widehat{B}}-\frac{\widehat{q}v}{2\widehat{\phi}}\right)+c\frac{\psi+\widehat{pq}-\widehat{p\phi}(\widehat{q}-q)}{\widehat{B}^2}\log\left[\frac{\psi+\widehat{pq}-\widehat{p\phi}(\widehat{q}-q)}{\psi+\widehat{pq}}\frac{1-\widehat{q}}{1-q}\right]. \tag{xxvi}$$

Using (**??**) and (**??**) in (**??**), and multiplying by $2\phi\widehat{\phi}p\widehat{x}\widehat{B}(\psi+\widehat{pq})(B-p\phi\widehat{x})/c$ one obtains that $\widehat{x}$ it is determined by

$$\begin{aligned}\left[2\widehat{\phi}-\widehat{q}\widehat{B}\frac{v}{c}+2\widehat{\phi}\frac{\psi+\widehat{pq}}{\widehat{B}}\ln\frac{(\widehat{B}-\widehat{p\phi}\widehat{x})(1-\widehat{q})}{(\psi+\widehat{pq})\widehat{x}}\right](\widehat{p\phi}-p\phi)\psi\widehat{x}+2\widehat{\phi}B(\psi+\widehat{pq})+\\ \frac{v}{c}p\widehat{x}\widehat{B}(\psi+\widehat{pq})(\widehat{\phi}q-\phi\widehat{q})-\widehat{p\phi}(\widehat{q}-q)(2\widehat{\phi}-\frac{\widehat{q}v\widehat{B}}{c})p\phi\widehat{x}+\\ +2\phi\widehat{\phi}p\widehat{x}\frac{(\psi+\widehat{pq})(\psi+\widehat{pq}-\widehat{p\phi}(\widehat{q}-q))}{\widehat{B}}\log\left[\frac{\psi+\widehat{pq}-\widehat{p\phi}(\widehat{q}-q)}{\psi+\widehat{pq}}\frac{1-\widehat{q}}{1-q}\right]=0.\end{aligned} \tag{xxvii}$$

Proof of Proposition 6: By making $\widetilde{V}(\widetilde{x}) = \widehat{\widetilde{V}}(\widetilde{x})$ one obtains

$$-\widetilde{C}A\widetilde{x} = \widehat{\widetilde{C}}(\widehat{B}-\widehat{A}\widetilde{x})-\frac{c}{\widehat{B}}+c\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widehat{B}^2}\log\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widetilde{x}}+\overline{V}, \tag{xxviii}$$

where we use $\widetilde{V}(0) = 0$, and where $\overline{V} = \frac{qv/2+\phi(1-q/2)\widehat{\widetilde{V}}(1-q/(2-q))}{q/2+\phi(1-q/2)}$.

By making $\widetilde{V}'(\widetilde{x}) = \widehat{\widetilde{V}}'(\widetilde{x})$, and multiplying throughout by $\widetilde{x}$, one obtains

$$-\widetilde{C}A\widetilde{x} = -\widehat{\widetilde{C}}\widehat{A}\widetilde{x}-c\frac{\widehat{A}\widetilde{x}}{\widehat{B}^2}\log\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widetilde{x}}-\frac{c}{\widehat{B}}. \tag{xxix}$$

Subtracting (**??**) from (**??**) one obtains

$$\overline{V}+\widehat{\widetilde{C}}\widehat{B}+\frac{c}{\widehat{B}}\log\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widetilde{x}}=0, \tag{xxx}$$

which will be used below.

Note now that from (**??**) one can obtain

$$\widetilde{C} = \widehat{\widetilde{C}}\frac{\widehat{p}}{p} + c\frac{\widehat{p}}{p\widehat{B}^2}\log\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widetilde{x}} + \frac{c}{A\widehat{B}\widetilde{x}}. \tag{xxxi}$$

Noting that we can write $\widetilde{V}(x) = \widetilde{C}(B - Ax) + \overline{V}$, the condition $\widetilde{V}(\widetilde{x}) = \widehat{\widetilde{V}}(\widetilde{x})$ can be reduced to

$$\widetilde{C} = \widehat{\widetilde{C}}\frac{\widehat{B}-\widehat{A}\widetilde{x}}{B-A\widetilde{x}} - \frac{c}{\widehat{B}(B-A\widetilde{x})} + c\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widehat{B}^2(B-A\widetilde{x})}\log\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widetilde{x}}. \tag{xxxii}$$

Equalizing the left hand sides of (**??**) and (**??**), one obtains

$$\left[\widehat{\widetilde{C}} + \frac{c}{\widehat{B}^2}\log\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widetilde{x}}\right]\left[\frac{\widehat{p}}{p} - \frac{\widehat{B}-\widehat{A}\widetilde{x}}{B-A\widetilde{x}}\right] + \frac{c}{\widehat{B}}\left[\frac{1}{A\widetilde{x}} + \frac{1}{B-A\widetilde{x}}\right] = 0. \tag{xxxiii}$$

Multiplying (**??**) by $pA(B - A\widetilde{x})\widetilde{x}\widehat{B}$, one obtains

$$\left[\widehat{\widetilde{C}} + \frac{c}{\widehat{B}^2}\log\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widetilde{x}}\right]A\widehat{B}(\widehat{p}-p)\psi\widetilde{x} + pcB = 0, \tag{xxxiv}$$

Now, to obtain $\widehat{\widetilde{C}}$, note by evaluating $\widehat{\widetilde{V}}(x)$ at $x = 1 - \frac{q}{2-q}$ we can obtain

$$\frac{q}{2}(\widehat{\widetilde{V}}(1-\frac{q}{2-q}) - v) = \frac{\widehat{A}}{\widehat{p}}\left[\widehat{\widetilde{C}}(\psi + \frac{\widehat{p}q}{2-q}) + c\frac{\psi + \widehat{p}q/(2-q)}{\widehat{B}^2}\log\frac{\psi(2-q)+\widehat{p}q}{2(1-q)} - \frac{c}{\widehat{B}}\right]. \tag{xxxv}$$

Using (**??**) one can obtain

$$\widehat{\widetilde{V}}(1-\frac{q}{2-q}) - v = -\frac{q+\phi(2-q)}{\phi(2-q)}\left[\widehat{\widetilde{C}}\widehat{B} + \frac{c}{\widehat{B}}\log\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widetilde{x}} + v\right]. \tag{xxxvi}$$

Using this in (**??**), solving for $\widehat{\widetilde{C}}$, and using it in (**??**), one obtains that $\widetilde{x}$ can be implicitly determined by

$$2B(\widehat{p}q+\psi)+(\widehat{p}-p)\psi\widetilde{x}\left[-\frac{qv\widehat{B}}{c}+\phi(2-q)+\phi\frac{\psi(2-q)+\widehat{p}q}{\widehat{B}}\log\frac{2(\widehat{B}-\widehat{A}\widetilde{x})(1-q)}{\widetilde{x}[\psi(2-q)+\widehat{p}q]}\right]=0. \tag{xxxvii}$$

Making $c\to 0$, one can obtain that $\frac{\widetilde{x}}{c}$ converges to (8). Writing c as a function of $\widehat{x}$ from (**??**) as $c=f(\widehat{x})$, and c as a function of $\widetilde{x}$ from (**??**) as $c=g(\widetilde{x})$, one can obtain that $\frac{\partial\widehat{x}}{\partial c}=\frac{\partial\widetilde{x}}{\partial c}>0$ and $\frac{\partial^2\widehat{x}}{\partial c^2}>\frac{\partial^2\widetilde{x}}{\partial c^2}$ at $c\to 0$. This then means that $\widehat{x}>\widetilde{x}$ for c close to zero,

Analysis of the Maximum Time of Retargeting when Purchases Are Recognized, $\widetilde{T}$:

Along the lines of Section 3, we can obtain $\widetilde{T}$ as a function of $\widetilde{x}$ as

$$\widetilde{T}=\frac{1}{\widehat{B}}\log\frac{2(1-q)}{\psi(2-q)+\widehat{p}q}\frac{\widehat{B}-\widehat{A}\widetilde{x}}{\widetilde{x}}. \tag{xxxviii}$$

To compare $\widehat{T}$ and $\widetilde{T}$, note that when $c\to 0$, we have

$$e^{\widehat{B}(\widehat{T}-\widetilde{T})}\to\frac{\psi(2-q)+\widehat{p}q}{2(\psi+\widehat{p}q)}, \tag{xxxix}$$

from which we can get $\widetilde{T}>\widehat{T}$.

Presentation of D in (**??**): Using $W_n=W(\widehat{T})$, (**??**), (**??**), and (**??**), we can obtain

$$\begin{aligned} D \;=\; & -(\widehat{p}-p)\left[\psi\frac{q}{2}S(P)(q\widehat{p}+\psi)+\psi\phi(1-q)[\frac{q}{2}\widehat{p}S(P)-\varepsilon]+(q\widehat{p}+\psi)[\phi+(1-\phi)q]\varepsilon\right]\Big/\Big[(q\widehat{p}+\\ & \psi)B\widehat{B}e^{\widehat{B}\widehat{T}}+(\widehat{p}-p)\psi\phi(1-q)\widehat{B}\Big]. \end{aligned} \tag{xl}$$

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