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PREDICTING ADVERTISING PULSING POLICIES IN AN OLIGOPOLY: A MODEL AND EMPIRICAL TEST

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Given that firms pulse in advertising, should firms pulse in or out of phase? It is shown that out of phase maximizes the oligopoly profits and is also the Markov perfect equilibrium of the infinite horizon game. The basic intuition for this result comes from the following fact: it is more profitable to increase consideration when the competitor's consideration is lower.

Evidence from several product categories seems to support this theoretical result.

(Competitive Strategies; Competitive Advertising; Pulsing)

1. Introduction

When selling a product, a firm is restricted to the consumers that are informed (aware) about the existence of that product and consider buying it; the demand for the product that firm sells is only the fraction of total potential demand that considers buying the product.

One of the ways a firm can increase the number of consumers that have its product in their consideration set, is spending in advertising. On the other hand, if the advertising expenditures are too low, consideration fades down.[1]

The effect of advertising on consideration has been the object of intensive empirical research (Ebbinghaus 1913, Strong 1914, Zielske 1959, Rao 1970, Ackoff and Emshoff 1975, Simon 1982, Simonson and Winer 1990, to cite only some examples). One of the "stylized facts" that came out of this stream of research is that advertising affects consideration through an S-shaped response function. For small levels of advertising there are increasing marginal returns on consideration; for high levels of advertising these returns are decreasing.

Several authors have then used this "stylized fact" to justify the use of pulsing policies in advertising in the real world (Sasieni 1971, 1989; Lodish 1971; Mahajan and Muller 1986; Feinberg 1988). Looking at a monopolist these authors note that: (1) it is never optimal for the firm to advertise at the convex part of the response function and (2) for some range of the parameters it might be optimal for the firm to pulse (to alternate between advertising zero—or at a minimum level—and the efficient amount of advertising). In Figure 1 the efficient amount of advertising (advertising expenditures are denoted by u) is determined by the tangency between a line coming through the origin and the response function $g(u)$. The efficient amount of advertising is denoted by $\bar{u}$.

[1] Throughout the paper I use the word consideration for the fraction of the number of consumers that have the firm's product in their consideration set, i.e., consider buying the firm's product.

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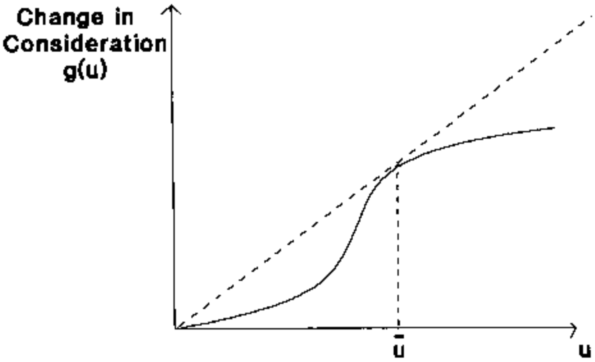

FIGURE 1. The S-Shaped Advertising Response Function.

But, what happens in an oligopoly? We would expect to observe uneven advertising policies, but should we observe synchronous or staggered (alternated) advertising rates by the duopolists?

This problem of the timing of the advertising expenditures in a competitive situation has been recognized by several authors. Among others, Wells, Burnett and Moriarty (1989) note that "a number of variables affect a *timing strategy*: consumer needs, the use cycle of the product and the degree of usage, and *competitive actions*" (p. 221). This timing problem can be especially important in the relancing of a product. "In crowded product segments the brand's *share of voice* can have a powerful influence on aperture opportunity. *Share of voice strategies* can shift *timing* and even media selection" (Wells, Burnett and Moriarty 1989, p. 231).

In this paper it is shown that for low interest rates the competitors should take turns at advertising. This is consistent with the somewhat known fact in the advertising industry that competitors, if pulsing, should not advertise at the same time.[2]

The intuition for this result comes from the specificities of the competition for consideration; the gains from having a greater consideration level are larger if the consideration level of the competitor is smaller. Then, the incentive for a firm to invest in advertising (in order to increase its own consideration level) is larger when the consideration level of the competitor is smaller. This results in an equilibrium where firms advertise in alternative periods.

In §2, the model of the duopoly case is presented and the main result of the paper is derived. Section 3 extends that result to the oligopoly case. The proofs to the propositions of §§2 and 3 that are not presented in the appendix are available upon request from the author. Section 4 reports some empirical evidence, and §5 concludes.

2. The Duopoly Case

Let us consider in this section a simple model of two competitors facing the advertising timing decision. Each of the firms manages only one product in the market under analysis and cares about the present discounted value of profits until infinity. Demand and cost conditions are stable over time.

Time is discrete.[3] The definition of the period for pulsing purposes is a complicated issue that has rarely been addressed (Feinberg 1988 uses a filter to derive the pulsing

[2] For example, the Hendry Corporation recommends to pulse only in alternate periods (if pulsing at all).

[3] The previous literature on this topic (Sasieni 1971, Feinberg 1988) has typically considered time to be continuous. This brings additional complexities, though not fundamental to obtain the major insights we are looking for in this work. This paper does not pretend to solve the complexities of using continuous time ("chattering can be the optimal policy," see Feinberg 1988), but its main contribution is in the analysis of advertising pulsing behavior in an oligopoly.

lapse of time; Mahajan and Muller 1986 try to suggest technological restrictions in the advertising industry) and that, undoubtedly, deserves further work. In this set-up, a period is the minimum lapse of time over which a firm makes decisions. This decision consists, simply, on the degree of intensity of the advertising expenditures.

In each period, both firms decide simultaneously (1) whether or not to advertise (to advertise at $\bar{u}$ or at zero), and (2) the value of the other market variables (i.e., price, quantity, location, product quality, sales force, etc.).

Advertising expenditures do not affect demand directly but only through consideration (C). The other variables in the marketing-mix (vector Z: price, quality of product, location, sales force, etc.) do affect demand directly. Furthermore, the values of the variables in Z in period t affect only demand in period t (i.e., demand in period $t + i$, with $i \neq 0$, is independent of Z in period t).

Consideration affects demand as only the fraction of the population that considers buying the product are potential customers for that firm.

Advertising affects consideration according to an S-shaped response function (Figure 1). In the case where the monopolist does not advertise in a certain period (zero advertising expenditures), the consideration level will fall in that period. In the case where the monopolist advertises very intensively in a certain period, the consideration level will rise in that period. Given this structure, demand is a function of advertising expenditures both today and in the past.

The analysis is restricted to Markov strategies; what firms do in each period only depends on what might affect profits today or in the future (in this case, the consideration levels of both firms at the beginning of the period) and, in any other way, independent of the history until that period. Note that past advertising expenditures affect profits today through the consideration level.

The restriction to Markov strategies rules out all types of outcomes where the strategies are function of something that is irrelevant in terms of the profits today or in the future. In particular, it rules out the approaches to oligopoly behavior which rely on punishment stories just by themselves (examples in Friedman 1977, Green and Porter 1984, Brock and Scheinkman 1985 and Rotemberg and Saloner 1986). As pointed out by Maskin and Tirole (1988), there are several advantages of the Markov approach: (1) In the punishment stories approach, a firm cooperates only because every firm has cooperated in the past, although that fact is irrelevant for current or future profits (moreover, a firm reacts to whatever it did in the past). This "bootstrap" characteristic of the equilibria might not realistically represent business behavior. (2) Punishment stories' equilibria rely on the infinity of repetitions of the game while the results we present are extendible to long but finite horizons. (3) The punishment stories approach is plagued by an enormous number of equilibria while in this model we have at most two (i.e., the Markov restriction gives us much more predictive power).

In each period, the timing of events is the following: (1) firms observe the vector of consideration levels (one for each firm); (2) they decide simultaneously whether or not to advertise; (3) if firm i advertises, the consideration level of that firm increases to 1 in that period; otherwise its consideration takes the value it had at the beginning of the period; (4) given the consideration levels, the firms decide upon the other marketing variables; (5) the firms receive the payoffs of that period, corresponding to the advertising expenditures, consideration levels and other marketing variables; and (6) the consideration of each firm depreciates by some amount.

Two simplifying assumptions are made on the advertising response function: (1) the curve is sufficiently S-shaped such that it only makes sense to spend 0 or $\bar{u}$ in advertising;[4]

[4] In order to be precise let us define an S-shaped function and what it means for one function to be more S-shaped than another. $g(u)$ is an *S-shaped function* iff (1) $g(u)$ is continuous $\forall u \geq 0$, (2) $g(0) = 0$; (3) $g(u)$ is

and, (2) $\bar{u}$ is so powerful that when $\bar{u}$ is spent the consideration reaches its maximum level (say 1).[5] These assumptions allow us to worry only about two values of advertising expenditures (0 and $\bar{u}$), and about when to spend in advertising. However, they do not change the nature of the problem. (The first assumption is typically a result of the literature on S-shaped reaction functions and the second one simplifies the type of pulsing that results from the model.) Furthermore, it is assumed that in the periods of zero advertising the consideration fades down.

Let us now define the following notation: C_{it} is the consideration level of firm i in period t; $f(C_{it-1})$ is the consideration level of firm i in the beginning of ~~the~~ period t, with $f(\quad)$ such that $f(x) \leq x\ \forall x$.

Furthermore, the following variables and relations are defined: D_{it} takes only values zero or one (one if firm i advertises in period t, zero if not); $C_{it} = D_{it} + (1 - D_{it})f(C_{it-1})$, thus $C_{it} = 1$ if firm i advertises and $C_{it} = f(C_{it-1})$ otherwise; $\bar{u}$ is the cost of advertising if different from zero; $\pi^i(C_i, C_j)$ is the period profit for firm i (exclusive of advertising expenditures) given that firm i has consideration C_i and firm j consideration C_j. (This function incorporates the game, i.e., it is the Nash equilibrium of that game, in the other variables the firms are deciding upon, i.e., price, quantity, etc. These other variables do not affect the future payoffs of either firm. Given the Markov strategies, $\pi^i(\cdot)$ exists; $V_t^i(f(C_{it-1}), f(C_{jt-1}))$ is the net present value of the profits of firm i at the beginning of period t, if products i and j had, respectively, consideration C_{it-1} and C_{jt-1} in period t (as the conditions of the problem do not change through time we have that $V_t^i(\cdot) = V_k^i(\cdot) = V^i(\cdot)\ \forall t, k$); δ is the discount factor ($0 < \delta < 1$).[6]

In order to further simplify our task we assume that the firms are symmetric. Then $\pi^i(\cdot) = \pi^j(\cdot) = \pi(\cdot)$.

In order to formalize the equilibrium of the game, we define the following additional variables: $r(C_1, C_2)$ is the probability with which firm 1 advertises in a certain period if the consideration levels of firms 1 and 2 at the beginning of that period are respectively C_1 and C_2; $s(C_2, C_1)$ is the probability with which firm 2 advertises in a certain period if the consideration levels of firms 2 and 1 at the beginning of that period are respectively C_2 and C_1.

Then

$$V^1(C_1, C_2) = \max_{0 \leq r \leq 1} rs[\pi(1, 1) + \delta V^1(f(1), f(1)) - \bar{u}]$$

$$+ r(1 - s)[\pi(1, C_2) + \delta V^2(f(1), f(C_2)) - \bar{u}]$$

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weakly increasing in $u\ \forall u$; (4) $\exists u^*$, such that $g(u)$ is convex for $u \leq u^*$, and concave for $u \geq u^*$; and (5) $\lim_{u \to +\infty} g'(u) = 0$.

Consider $g_1(u)$ and $g_2(u)$, two S-shaped functions, such that there is one line that passes through the origin and is tangent at both $g_1(u)$ and $g_2(u)$ at $\bar{u}$, and $g_1(\bar{u}) = g_2(\bar{u})$. $g_1(u)$ is *more S-shaped than* $g_2(u)$ if

$$\int_0^{\bar{u}} g_1(u)\,du + \int_{\bar{u}}^{+\infty} [g_1(u) - g_1(\bar{u})]\,du < \int_0^{\bar{u}} g_2(u)\,du + \int_{\bar{u}}^{+\infty} [g_2(u) - g_2(\bar{u})]\,du.$$

The most S-shaped function is $g(u)$ such that (1) $g(u) = 0\ \forall u < \bar{u}$ and (2) $g(u) = g(\bar{u})\ \forall u \geq \bar{u}$. Assuming a response function like this one is a sufficient condition for only to make sense to spend 0 or $\bar{u}$ in advertising (but response functions close to this one give the same result).

[5] Notice that this does not mean that the effect of advertising is independent of the level of consideration immediately prior to advertising or the competitive consideration or advertising expenditures. Advertising expenditures lower than $\bar{u}$ can have an effect contingent on these variables but we are abstracting here from these effects (given that we use a very S-shaped response function).

[6] $\delta = 1/(1 + k)$ where k is the interest rate per period.

$$+ (1 - r)s[\pi(C_1, 1) + \delta V^1(f(C_1), f(1))]$$
$$+ (1 - r)(1 - s)[\pi(C_1, C_2) + \delta V^1(f(C_1), f(C_2))] \tag{1}$$

and similarly for firm 2 in order to calculate s.

The equilibrium is then characterized by $r(C_1, C_2)$ (which solves the maximization in (1)), $s(C_2, C_1)$, $V^1(C_1, C_2)$ (which satisfies (1)) and $V^2(C_2, C_1)$.

The model presented above is too general (and complex to solve). In order to answer the questions we are interested in, it is enough to consider a simplified version where firms advertise at least once in every pair of consecutive periods. This allows us to restrict the attention to only four states of the world: (1) firms 1 and 2 having advertised 1 period ago; (2) firms 1 and 2 having advertised 2 periods ago; (3) firm 1 having advertised 1 period ago and firm 2 having advertised 2 periods ago; and, (4) firm 2 having advertised 1 period ago and firm 1 having advertised 2 periods ago.

With the restriction of the analysis to equilibria where the duopoly subsists over time[7] and with assumptions F and D below, this objective is attained without loss of generality.

ASSUMPTION F. $f(C_{it-1}) = \text{Max}(C_{it-1} - \frac{1}{2}, 0)$.

ASSUMPTION D. If a firm stays a whole period with zero consideration level it cannot sell this product anymore.

Assumption F simplifies the number of consideration levels that are possible: complete consideration, 50% consideration and zero consideration. Assumption D guarantees us that zero consideration is a reflection barrier: once zero consideration is reached each firm advertises or exits the market (if the net present value of staying in is negative, which is assumed away).

The four states of the world to be considered (at the beginning of each period) are then $(C_1 = \frac{1}{2}, C_2 = \frac{1}{2})$, $(C_1 = \frac{1}{2}, C_2 = 0)$, $(C_1 = 0, C_2 = \frac{1}{2})$ and $(C_1 = 0, C_2 = 0)$.[8]

In each period, there are four types of consumers: (1) consumers that consider buying both products; (2) consumers that consider product 1 only; (3) consumers that consider product 2 only; and (4) consumers that do not consider either product. We assume probabilistic independence. (The consideration probability of product i is independent of whether product j is considered; see Silk and Urban (1978) and Hauser and Wernerfelt (1989) on the use of this assumption.)

In order to compute the payoffs of the duopolists in each state of the world, let us further assume that the only other marketing-mix variable (besides advertising) is price (P).

If all the consumers consider both products, demand for firm i is $Q^i(P_i, P_j)$. If all the consumers consider product i only, demand for firm i is $Q^i(P_i, \infty)$. The function $Q^i(\cdot)$ satisfies the obvious properties of demand functions in an oligopoly.[9]

[7] In the equilibria in which the duopoly subsists over time, the participation constraint for each firm must be satisfied: the parameters must be such that the equilibrium payoff for each firm must be greater than (or equal to) zero.

[8] Notice that at the beginning of each period no firm can have 100% consideration because it must have some depreciated amount from last period. The most it can be is $\frac{1}{2}$.

[9] Demand functions in an oligopoly satisfy the following very intuitive properties:

$$\frac{\partial Q^i}{\partial P_i} < 0, \tag{P1}$$

$$\frac{\partial Q^i}{\partial P_j} > 0, \tag{P2}$$

$$\left|\frac{\partial Q^i}{\partial P_i}\right| > \frac{\partial Q^i}{\partial P_j} \tag{P3}$$

In the general case (when there are four types of consumers in the market) demand for firm i is

$$Q_i = C_i \cdot (1 - C_j) \cdot Q^i(P_i, \infty) + C_i \cdot C_j \cdot Q^i(P_i, P_j).$$

Production costs are zero.[10]

The profit for firm i (exclusive of the advertising expenditures) is $\pi^i = P_i Q_i$.

In each period, there is a Bertrand game in prices given the consideration levels. Each firm $\text{Max}_{P_i}\ \pi^i$ given C_i, C_j and P_j.

We can then compute the Nash equilibrium of the price game[11] and obtain the function $\pi^i(C_i, C_j)$.

In state $(\frac{1}{2}, \frac{1}{2})$ the game to be played is (in matrix form):

		Firm 2: $D_2 = 1$	Firm 2: $D_2 = 0$
Firm 1	$D_1 = 1$	$\pi(1, 1) + \delta V^1(1/2, 1/2) - \bar{u}$, $\pi(1, 1) + \delta V^2(1/2, 1/2) - \bar{u}$	$\pi(1, 1/2) + \delta V^1(1/2, 0) - \bar{u}$, $\pi(1/2, 1) + \delta V^2(0, 1/2)$
	$D_1 = 0$	$\pi(1/2, 1) + \delta V^1(0, 1/2)$, $\pi(1, 1/2) + \delta V^2(1/2, 0) - \bar{u}$	$\pi(1/2, 1/2) + \delta V^2(0, 0)$, $\pi(1/2, 1/2) + \delta V^2(0, 0)$

Notation: $D_i = 1$ ($\equiv$ advertise); $D_i = 0$ ($\equiv$ no advertising)

The games to be played in states $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$ and $(0, 0)$ can be also represented by similar matrices.

Let us now introduce an assumption that helps us guarantee that no firm ever wants to exit the market. Its importance will be made clear below.

ASSUMPTION E. $\pi(1, \frac{1}{2}) + \pi(\frac{1}{2}, 1) > \bar{u}$.

In order to analyze the duopoly case, let us first derive one result on the payoff function.

RESULT 1. If $\partial Q^i/\partial P_j > P_i \cdot \partial^2 Q^i/\partial P_i \partial P_j$ and $|\partial^2 Q^i/\partial P_j^2|$ small,

$$\pi(1, \tfrac{1}{2}) - \pi(\tfrac{1}{2}, \tfrac{1}{2}) > \pi(1, 1) - \pi(\tfrac{1}{2}, 1).$$

PROOF. See Appendix 1.

Result 1 does not depend on Assumption F or E (the allowed consideration levels being only 1, $\frac{1}{2}$ and zero), but has a very simple interpretation: the benefits for a firm increasing its own consideration are greater when the competitor's consideration is lower (for a discussion of a similar result see Hauser and Wernerfelt 1989). The conditions of Result 1 are assumed throughout the paper.[12]

Using this result we can now proceed to analyze the duopoly case.

PROPOSITION 1. *If the loss from lowering the consideration level is larger than the efficient advertising expenditures (i.e., $\pi(1, 1) - \pi(\frac{1}{2}, 1) > \bar{u}$), the unique Markov perfect equilibrium is for firms to advertise, whatever the consideration levels of both firms are.*

[10] Having positive marginal production costs does not change the results and complicates unnecessarily the analysis.

[11] The price game is assumed to have an unique equilibrium which is in pure strategies. A sufficient condition for this to happen is

$$\left| 2\frac{\partial Q^i}{\partial P_i} + \frac{\partial^2 Q^i}{\partial P_i^2} \right| > \frac{\partial Q^i}{\partial P_j} + P_i \cdot \frac{\partial^2 Q^i}{\partial P_i \partial P_j}.$$

[12] Notice that the conditions of Result 1 are relatively mild. They basically consist on the first order effects of prices on demand to be more important than the second order effects.

This proposition simply shows that if $\bar{u}$ is small enough or $\pi(1, 1) - \pi(\frac{1}{2}, 1)$ is large enough, the unique equilibrium is, as in the monopoly case, for both firms to advertise in every period. In general, in cases where $\bar{u}$ is small or the penalty of lagging behind is large, firms advertise more often.

But this is not the most interesting situation, and from now on we will always assume $\pi(1, 1) - \pi(\frac{1}{2}, 1) < \bar{u}$ (i.e., both firms advertising in every period is not an equilibrium). We can then derive the main result of the paper.

PROPOSITION 2. *For δ close to 1 and $\bar{u}$ large, the only Markov perfect equilibria in pure strategies are characterized by alternations in advertising expenditures.*[13]

PROOF. See Appendix 1.

The main message of Proposition 2 is that, in a duopoly, firms should alternate the pulsing in advertising (if the efficient amount of advertising is large enough).

The basic intuition for Proposition 2 is the following one: firm i has a smaller increase in profit due to a raise in its consideration when the consideration of firm j is higher; then, if firm j is advertising, firm i should not also do so, and we obtain the staggered equilibrium.

The condition "δ close to 1" basically says that there is a $\bar{\delta}$, such that, for any δ greater than $\bar{\delta}$, the proposition holds. The condition "$\bar{u}$ large" is of this same type, and also guarantees that we are not in the case of Proposition 1.[14] These conditions on δ and $\bar{u}$ are stronger than the ones that are actually necessary for the proposition to hold.[15]

Notice also that these optimal advertising policies incorporate optimal pricing policies in each period. The evolution through time of the prices charged by the firms is positively correlated with the advertising expenditures; a firm charges a high price when it advertises and cuts prices when the competitor advertises. This has to do with the fact that when a firm advertises it has more monopoly power than when it is not advertising. This feature of the model is consistent with the widely observed phenomenon of firms promoting their products when the competitors launch important advertising campaigns. Figure 2 presents the behavior of prices and advertising expenditures through time. Proposition 3 summarizes the result.

PROPOSITION 3. *Under the conditions of Proposition 2, the prices and the advertising expenditures of each firm are positively correlated in a duopoly.*

Proposition 2 does not say anything in relation to mixed strategies; i.e., there might exist equilibria in mixed strategies which have firms advertising in a synchronized way. Proposition 4 below shows that even if we allow for mixed strategies, the system settles down in finite time to a path with alternation.

[13] These strategies are (i) $r(\frac{1}{2}, \frac{1}{2}) = 1$, $s(\frac{1}{2}, \frac{1}{2}) = 0$; $r(\frac{1}{2}, 0) = 0$, $s(0, \frac{1}{2}) = 1$; $r(0, \frac{1}{2}) = 1$, $s(\frac{1}{2}, 0) = 0$; $r(0, 0) = 1$, $s(0, 0) = 1$; or (ii) $r(\frac{1}{2}, \frac{1}{2}) = 0$, $s(\frac{1}{2}, \frac{1}{2}) = 1$; $r(\frac{1}{2}, 0) = 0$, $s(0, \frac{1}{2}) = 1$; $r(0, \frac{1}{2}) = 1$, $s(\frac{1}{2}, 0) = 0$; $r(0, 0) = 1$, $s(0, 0) = 0$.

[14] Notice also that $\bar{u}$ is constrained to satisfy the participation constraint (Assumption E).

[15] The conditions on δ and $\bar{u}$ that are necessary for Proposition 2 to hold are

$$\delta[\pi(1, 1) - \pi(\tfrac{1}{2}, 1)] - \pi(1, \tfrac{1}{2}) < -\bar{u}(1 - \delta),$$

$$[\pi(1, 1) - \pi(\tfrac{1}{2}, 1)] - \delta[\pi(1, 1) - \pi(1, \tfrac{1}{2})] < \bar{u},$$

$$[\pi(1, 1) - \pi(\tfrac{1}{2}, 1)] - \delta[\pi(\tfrac{1}{2}, 1) - \pi(\tfrac{1}{2}, \tfrac{1}{2})] < \bar{u},$$

$$[\pi(1, 1) - \pi(\tfrac{1}{2}, 1)] - \delta[\pi(1, \tfrac{1}{2}) - \pi(\tfrac{1}{2}, \tfrac{1}{2})] < \bar{u}(1 - \delta),$$

δ close to 1 and $\bar{u}$ close to

$$\pi(1, 1) + \pi(\tfrac{1}{2}, \tfrac{1}{2}) < \pi(\tfrac{1}{2}, 1) + \pi(1, \tfrac{1}{2})$$

satisfy these inequalities.

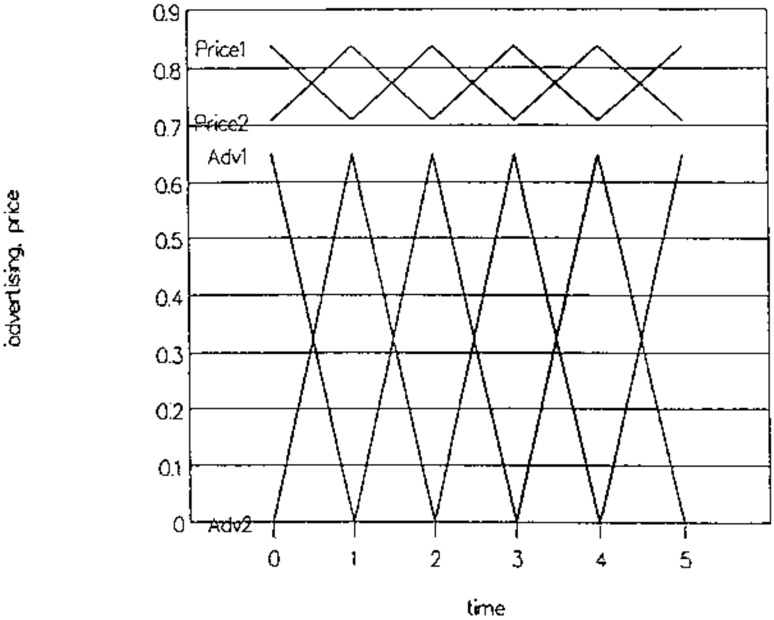

FIGURE 2. Advertising and Prices Evolution.

PROPOSITION 4. *For δ close to* 1 *and $\bar{u}$ large, the unique Markov perfect equilibrium in strictly mixed strategies is symmetric and settles down in finite time in a path with alternation of advertising expenditures.*[16]

As in Proposition 2, the conditions "δ close to 1" and "$\bar{u}$ large" are stronger than the ones needed to prove this proposition.

3. The Oligopoly Case

Let us now consider the oligopoly case. There are N firms in the market, which are assumed to subsist overtime. Firms are symmetric. Assumptions F and D hold, such that, every firm advertises at least every two periods. The payoff (exclusive of advertising expenditures) per period for each firm i is $\bar{\pi}^i(C_1, \ldots, C_N)$.

The equilibrium of this large game is a function for each firm i: $\bar{r}_i(C_1, \ldots, C_N)$, which is the probability with which firm i advertises if the consideration levels at the beginning of that period are $C_1, \ldots, C_N$. Corresponding to these strategies, there is a value function for each firm i, $\bar{V}^i(C_1, \ldots, C_N)$, that gives the net present value of the profits of firm i at the beginning of a period in which the consideration levels of the N firms are $C_1, \ldots, C_N$.

Let us further assume that N and $\bar{u}$ are large. Let us define α_t as the fraction of firms that have consideration zero at the beginning of period t; $(1 - \alpha_t)$ of the firms have consideration $\frac{1}{2}$ (given Assumptions F and D, there are only firms with two consideration levels at the beginning of each period: $\frac{1}{2}$ and 0). If $\bar{u}$ is large enough, in equilibrium, firms advertise every two periods (intuition from the previous section). Given N large, if firm i deviates (or not) from its equilibrium strategy in period t, $\alpha_{t+1} = 1 - \alpha_t$.

Given N large, the payoff per period (exclusive of advertising expenditures) for firm i can be written as a function of the consideration level of firm i and the average consid-

[16] This equilibrium is characterized by the following strategies: $r(\frac{1}{2}, \frac{1}{2}) = s(\frac{1}{2}, \frac{1}{2}) = p$ with $0 < p < 1$; $r(\frac{1}{2}, 0) = s(0, \frac{1}{2}) = 0$, and $r(0, \frac{1}{2}) = s(\frac{1}{2}, 0) = r(0, 0) = s(0, 0) = 1$.

eration ($\bar{C}$): $\pi^i(C_i, \bar{C})$ (and given that the α fraction of firms that have consideration zero at the beginning of the period advertise, we have $\bar{C} = \frac{1}{2}(1 + \alpha)$). Furthermore, the strategy function ($\bar{r}_i(\cdot)$) and the value function ($\bar{V}^i(\cdot)$) can be also written as a function of C_i and $\bar{C}$: $r_i(C_i, \bar{C})$ and $V^i(C_i, \bar{C})$. Notice that the arguments of $\pi^i(\ \ )$ in period t are C_i and $\bar{C}$ during period t (C_{it} and $\bar{C}_t = \frac{1}{2}(1 + \alpha_t)$), while the arguments of $r_i(\ \ )$ and $V^i(\ \ )$ in period t are C_i and $\bar{C}$ in the beginning of period t ($f(C_{it-1})$ and $\frac{1}{2}(1 + \alpha_t)$).

Finally the assumption that guarantees us no exit (Assumption E) is now Assumption E′.

ASSUMPTION E′. $2 \cdot \pi(\frac{1}{2}, \frac{1}{2}) > \bar{u}$.

We can then generalize Proposition 2: there is a unique stable equilibrium (one in which the actions repeat themselves at least every two periods) with $\alpha_t = \frac{1}{2}\ \forall t$. Proposition 5 gives the result.

PROPOSITION 5. *Given N and $\bar{u}$ large and δ close to* 1, *the oligopoly case has an unique equilibrium in which the actions repeat themselves at least every two periods. Furthermore, in this equilibrium $\alpha_t = \frac{1}{2}\ \forall t$ (half the firms advertise in the odd periods and half the firms advertise in the even periods).*

4. Empirical Evidence

The basic prediction of the model presented in this paper is that there should exist a negative correlation in the advertising expenditures of the products in the same category.

But in the real world there are several factors that are not present in the model and that contradict its predictions. (Instead of the predicted negative correlation, these factors can explain positive correlations.)

First, there are seasonal factors. For several reasons, in some product categories there are strong seasonal effects which extend into the advertising expenditures. To the extent that these seasonal effects are common across brands in the same product category, we should expect positive correlations in the advertising expenditures.

Second, in every product category there are trends that extend into the advertising expenditures and that affect all brands. Also, for this reason, we should expect positive correlations in the advertising expenditures.

Finally, the model postulates a Markov behavior (firms' strategies are only function of payoff-dependent state variables). In the supergame literature, firms are not restricted to this behavior. One could then construct models where firms punish each other during recessions (as in Green and Porter 1984) by increasing the advertising expenditures; or where firms can cooperate less during booms (as in Rotemberg and Saloner 1986) by advertising more than the colluded oligopoly. In both cases, one should expect advertising expenditures to be positively correlated.

When analyzing the data on advertising expenditures one can control for the first factor (by introducing seasonal dummies) and the second factor (by simply detrending the data) but the third factor is difficult to account for.

So, finding negative correlation in the data on advertising expenditures should be interpreted as a support to the model presented above (although there might be other explanations: changes in consumer preferences, wrong definitions of segments, etc.).

In the next subsections the data that were used for the empirical test is described, the methodology of the test is presented and, finally, the results are reported and discussed.

4.1. *Data*

Data on eight consumer product categories and one service category were used to empirically test the theoretical results presented above. The consumer product categories included: Liquid Deodorizing Cleaners, Automatic Dishwashing Detergents, Toilet Soaps, Glass Cleaners, Hand Dishwashing Detergents, Skin Care Lotions, Pain Relievers, and Sleeping Aid Products. The service category was Credit Cards.

The number of brands chosen for analysis in each product category ranged from 3 to 12. Brands of each product category were selected based upon the share of voice[17] held by the respective brand and upon how strategically each brand was considered to behave. The number of brands in each product category can be seen in Table 1.

The data was provided by a leading advertising agency and consisted of advertising expenditures in network television (NBC, ABC and CBS) per brand, per month, between January 1988 and December 1989 (24 months). For the product categories that were chosen, advertising expenditures in network television account for about 75% of total advertising expenditures (other advertising channels are magazines, newspaper supplements, spot television, network radio, outdoors and cable TV networks).

4.2. *Methodology*

One of the implications of our theoretical model is that there is a negative correlation between any brand advertising expenditures and the advertising expenditures of the rest of the brands in that product category (if one controls for trend and seasonal effects, which are not included in the model). This is clearly the main message from the duopoly result (Propositions 2 and 4).

The message from the oligopoly result is also that there should be a negative correlation between any brand and the overall amount of advertising expenditures of the rest of the products in the category: the only way for this to happen for all the brands in the category and given Assumptions F and D (advertise at least every two periods) is for half the brands to advertise in every period. In fact, each brand's advertising expenditures will be negatively correlated with the overall amount of advertising expenditures of the rest of the brands. If the number of firms in the market is large, this correlation, although small in absolute value, is still undoubtedly negative. If Assumption F is dropped, we would then surely have greater absolute values for this correlation.

When testing for the model, one must also account for the relative size effect of the brands competing in a certain product category; the theoretical model considered only the case of brands of the same size (share of voice) while the data presented some strong asymmetries in the relative size of the share of voice of the brands that were considered strategic players. The approach that was followed was to assume that the effect of advertising expenditures on consideration varied across brands; in particular, it was assumed that this effect was smaller for large firms than for small ones. In order to test the model, the correct solution was then to consider that the theoretical model held for standardized advertising expenditures.

After standardizing every series,[18] we have the following model:

$$S_{it} - \sum_{j=1}^{12} X_{tj}\gamma_j - \alpha_i T_t = \Theta \sum_{\substack{h=1,\\ h\neq i}}^{N} \{S_{ht} - \sum_{j=1}^{12} X_{tj}\gamma_j - \alpha_h T_t\} + \epsilon_{it} \tag{2}$$

for $i = 1, \ldots, N$ (N is the number of brands in the product category) and $t = 1, \ldots, 24$ (24 months of data per brand). S_{it} is the standardized version of the advertising expenditures of brand i at time t. X_{tj} is the element (t, j) of the matrix X of dimensions

[17] Share of voice of brand i is the advertising expenditures of brand i divided by the total advertising expenditures of the product category.

[18] If $\{Y_{it}\}$ is the series of advertising expenditures for brand i, the standardized series $\{Y'_{it}\}$ is constructed by computing

$$Y'_{it} = \frac{Y_{it} - \bar{Y}_i}{\sigma_i}, \qquad \text{where} \qquad \bar{Y}_i = \sum_{t=1}^{24} \frac{Y_{it}}{24} \qquad \text{and} \qquad \sigma_i = \left(\sum_{t=1}^{24} \frac{(Y_{it} - \bar{Y}_i)^2}{23}\right)^{1/2}.$$

(24×12). The matrix X includes the dummy variables to compute the seasonal effects: $X_{tj} = 1$ if $t = j$ or $t = j + 12$, and $X_{tj} = 0$ otherwise. γ_j is the seasonal effect of month j. Notice that the seasonal effects are considered equal across brands.

T_t is the generic element of the vector T of dimension 24. This vector accounts for the trend in advertising expenditures ($T_t = t$). α_i is the trend effect for brand i. Notice that the trend effects are different across brands so share changes during the period under analysis are allowed.

The LHS (Left-Hand Side) of the above expression represents the advertising expenditures net of the seasonal and trend effects. The RHS (Right-Hand Side) is composed of the parameter Θ times the sum across the other brands in the product category of the advertising expenditures net of the seasonal and trend effects plus an error term ϵ_{it}. Θ is assumed to be constant across brands[19] and is the parameter where the model will be tested; the theoretical model predicts Θ to be negative. ϵ_{it} is assumed to be normally distributed with $E(\epsilon_{it}) = 0\ \forall_{i,t}$, $E(\epsilon_{it}^2) = \sigma^2\ \forall_{i,t}$. Furthermore it is assumed that $E(\epsilon_{it}\epsilon_{j\tau}) = 0\ \forall_{i,t,j,\tau,i \neq j \text{ or } t \neq \tau}$.

The expression above can be transformed into

$$S_{it} = \sum_{j=1}^{12} X_{tj}\gamma'_j + \alpha'_i T_t + \Theta \sum_{\substack{h=1,\\ h\neq i}}^{N} S_{ht} + \epsilon_{it}, \qquad i = 1, \ldots N, \qquad t = 1, \ldots, 24 \quad (3)$$

where $\gamma'_j = \gamma_j[1 - (N-1)\Theta]$ and $\alpha'_i = \alpha_i - \sum_{h=1,h\neq i}^{N} \Theta\alpha_h$.

Estimating Equation (3) (or (3) jointly with all the other brands) by ordinary least squares would lead us to biased estimates of the parameters due to errors in variables: the variable $\sum_{h=1,h\neq i}^{N} S_{ht}$ is correlated with the error term ϵ_{it}.

In fact, Equation (3) can be interpreted as a system of simultaneous equations (one equation for each brand), which can be estimated through a Maximum Likelihood Estimation (MLE) method. (See Appendix 2 for the construction of the likelihood function $L(\gamma, \alpha, \Theta, \sigma^2)$, where γ is a vector of dimension 12, where γ'_j is the generic element and α is a vector of dimension N where α'_i is the generic element.)

Maximizing $L(\gamma, \alpha, \Theta, \sigma^2)$, ($L(\cdot) \equiv \log \mathcal{L}(\cdot)$) with respect to γ, α, Θ and σ^2 yields the MLE of these parameters ($\hat{\gamma}$, $\hat{\alpha}$, $\hat{\Theta}$ and $\hat{\sigma}^2$).

Testing the theoretical model is then testing if $\hat{\Theta}$ is statistically different from zero and negative.

The maximization of $L(\ \ )$ was implemented using the method of Gill et al. (1984).

4.3. *Results*

Table 1 presents the results for the nine product categories under study. $\hat{\Theta}$ is positive for one of the nine product categories (Pain Relievers). $\hat{\Theta}$ is statistically different from zero and negative at the 0.5% significance level for three product categories (Toilet Soaps, Glass Cleaners, and Skin Care Lotions), at the 5% significance level for four product categories (the previous three, plus Automatic Dishwashing Detergents), and at the 10% significance level for six product categories (the previous four, plus Liquid Deodorizing Cleaners and Hand Dishwashing Detergents).

The results look supportive of the theoretical model and seem to show the existence of advertising "wars" in the product category Pain Relievers (as explained above). An alternative explanation might be the wave of new products that were introduced during the sample period (due to changes in government regulation).

[19] Given that all the series were standardized assuming Θ to be constant across brands is the logical assumption. Furthermore, this assumption allows us to have greater power in the statistical tests of the model.

TABLE 1

Product Category	Number of Brands	$\hat{\theta}$	$\hat{\sigma}_{\theta}$	LL
Liquid Deod. Cleaners	3	−0.147	0.099	−14.44
Automatic Dishw. Det.	4	−0.152	0.079	−33.00
Toilet Soaps	7	−0.211	0.047	−55.70
Credit Cards	4	−0.093	0.075	−33.61
Glass Cleaners	3	−0.436	0.103	8.34
Hand Dishwashing Det.	3	−0.137	0.099	−18.62
Skin Care Lotions	7	−0.147	0.049	−55.25
Pain Relievers	12	0.252	0.014	−117.32
Sleeping Aid Products	4	−0.085	0.074	−37.14

LL is the value at the optimum of Log L plus $24N \log \sqrt{2\pi}$.

5. Concluding Remarks

It was shown that competitors in a duopoly (or oligopoly) should, under some reasonable conditions, advertise in alternate periods (if the only role of advertising is to increase consideration). The basic intuition for this result is that, when a firm is raising its consideration level, it has greater benefits if the other firm's consideration is lower. Furthermore, this equilibrium maximizes industry profits.

The evidence from several product categories looks supportive of the model. The results shown in this paper can be applied to problems where there exists a similar type of framework: In a model where consumers care for novelty, should firms introduce new products at the same time or not? In a model of technology adoption, should firms adopt technologies synchronously or not? In a model of investment in additional capacity, should firms invest at the same time or not?[20]

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Appendix 1

RESULT 1. If $\dfrac{\partial Q^i}{\partial P_j} > P_i \dfrac{\partial^2 Q^i}{\partial P_i \partial P_j}$ and $\left|\dfrac{\partial^2 Q^i}{\partial P_j^2}\right|$ small,

$$\pi(1, \tfrac{1}{2}) - \pi(\tfrac{1}{2}, \tfrac{1}{2}) > \pi(1, 1) - \pi(\tfrac{1}{2}, 1).$$

PROOF. Define $T^i(P_i, P_j) \equiv P_i \cdot Q^i(P_i, P_j)$. Then, notice that

$$\frac{\partial^2 \pi^i}{\partial C_i \partial C_j} = T^i(P_i, P_j) - T^i(P_i, \infty) + C_i \cdot \frac{\partial P_j}{\partial C_i} \cdot \left[T^i_2(P_i, P_j) + C_j T^i_{12}(P_i, P_j) \frac{\partial P_i}{\partial C_j} \right]$$

$$+ C_j \cdot \frac{\partial P_j}{\partial C_j} \cdot \left[T^i_2(P_i, P_j) + C_i T^i_{22}(P_i, P_j) \frac{\partial P_j}{\partial C_i} \right] + C_i C_j T^i_{22}(P_i, P_j) \frac{\partial^2 P_j}{\partial C_i \partial C_j}$$

where in equilibrium P_i and P_j are functions of C_i and C_j, T_i represents the derivative of the function $T(\cdot)$ to the ith argument, and T_{ij} represents the second derivative of the function T to the ith and jth arguments.

Differentiating the system of the FOC of each firm, one can also find that

$$\frac{\partial P_i}{\partial C_i} < 0, \qquad \frac{\partial P_i}{\partial C_j} < 0$$

if $\dfrac{\partial Q^i}{\partial P_j} > P_i \dfrac{\partial^2 Q^i}{\partial P_i \partial P_j} \forall P_i, P_j$.

Furthermore, given that $\left|\dfrac{\partial^2 Q^i}{\partial P_j^2}\right|$ is assumed to be small, we get that $\partial \pi^i / \partial C_i \partial C_j < 0$, which gets us Result 1. Q.E.D.

PROPOSITION 2. *For δ close to 1 and $\bar{u}$ large, the only Markov perfect equilibria in pure strategies are characterized by alternations in advertising expenditures.*

PROOF. Here we define the following variables in order to simplify notation: $\pi 1 \equiv \pi(1, 1)$, $\pi 2 \equiv \pi(1, \frac{1}{2})$, $\pi 3 \equiv \pi(\frac{1}{2}, 1)$ and $\pi 4 \equiv \pi(\frac{1}{2}, \frac{1}{2})$. First, we check that these strategies constitute a perfect equilibrium. (We check for (i) as (ii) is just the symmetric case.) We do this by checking that the prescribed strategies are a Nash equilibrium in each of the possible states of the world. Then, we show that no other strategies constitute a perfect equilibrium.

Given the postulated equilibrium strategies we can compute $V^1(\cdot)$ and $V^2(\cdot)$ for all the states of the world.

In order for (1, 0) to be a Nash equilibrium in state $(\frac{1}{2}, \frac{1}{2})$ we need

$$\pi 2 + \delta V^i(\tfrac{1}{2}, 0) - \bar{u} > \pi 4 + \delta V^1(0, 0) \qquad \text{and}$$

$$\pi 3 + \delta V^i(0, \tfrac{1}{2}) > \pi 4 + \delta V^2(\tfrac{1}{2}, \tfrac{1}{2}) - \bar{u}.$$

These conditions are respectively equivalent to

$$\delta(\pi 1 - \pi 3) - (\pi 2 - \pi 4) < -\bar{u}(1 - \delta) \qquad \text{and} \tag{A1}$$

$$(\pi 1 - \pi 3) + \delta(\pi 1 - \pi 2) < \bar{u}. \tag{A2}$$

Condition (A1) is true for δ close to 1 (using Result 1).

In order to confirm that Condition (A2) is satisfied, one can take the largest possible $\bar{u}$ (which is close to $\pi 2 + \pi 3$ for δ close to 1)[21] and δ close to 1. Then Condition (A2) becomes $\pi 1 < \pi 2 + \pi 3$ which is satisfied given Result 1 and $\pi 4 > 0$.

In state $(0, \frac{1}{2})$, in order for (1, 0) to be the Nash equilibrium Condition (A2) must hold.

In order to complete the proof that these strategies constitute an equilibrium, we just need to show that (0, 1) is a Nash equilibrium in state $(\frac{1}{2}, 0)$. Given Assumption D, (0, 1) is a Nash equilibrium in state $(\frac{1}{2}, 0)$ if

$$\pi 3 + \delta V^i(0, \tfrac{1}{2}) > \pi 1 + \delta V^1(\tfrac{1}{2}, \tfrac{1}{2}) - \bar{u}$$

which is equivalent to $\pi(1, 1) - \pi(\frac{1}{2}, 1) < \bar{u}$, which is assumed above.

Finally, we just need to prove that the equilibria presented constitute the only Markov perfect equilibria in pure strategies. For this to be true we first need to prove that (0, 0) is not a Nash equilibrium in state $(\frac{1}{2}, \frac{1}{2})$ (given that by Proposition 1, (1, 1) cannot be).

Suppose not, i.e., (0, 0) is a Nash equilibrium in state $(\frac{1}{2}, \frac{1}{2})$. Then

$$V^i(0, 0) = \frac{\pi 1 + \delta \pi 4 - \bar{u}}{1 - \delta^2} \qquad \forall i,$$

$$V^i(\tfrac{1}{2}, \tfrac{1}{2}) = \frac{\pi 4 + \delta \pi 1 - \delta \bar{u}}{1 - \delta^2} \qquad \forall i.$$

Furthermore,

$$\pi 4 + \delta V^i(0, 0) > \pi 2 + \delta V^2(\tfrac{1}{2}, 0) - \bar{u} \qquad \text{and} \tag{A3}$$

$$\pi 4 + \delta V^i(0, 0) > \pi 2 + \delta V^1(\tfrac{1}{2}, 0) - \bar{u}. \tag{A4}$$

In order to check Conditions (A3) and (A4), we need to know the values of $V^2(\frac{1}{2}, 0)$ and $V^1(\frac{1}{2}, 0)$. For this, we need to consider four cases in respect to the equilibria in states $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$: (1) (1, 1) is a Nash equilibrium in states $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$; (2) (0, 1) and (1, 0) are respectively the Nash equilibria in states $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$; (3) (1, 1) is the Nash equilibrium in state $(\frac{1}{2}, 0)$ and (1, 0) is the Nash equilibrium in state $(0, \frac{1}{2})$; and (4) (0, 1) is the Nash equilibrium in state $(\frac{1}{2}, 0)$ and (1, 1) is the Nash equilibrium in state $(0, \frac{1}{2})$.

In Case 1, we have

$$V^i(\tfrac{1}{2}, 0) = V^i(0, \tfrac{1}{2}) = V^i(0, 0) = \frac{\pi 1 + \delta \pi 4 - \bar{u}}{1 - \delta^2} \qquad \forall i \qquad \text{and}$$

$$V^i(\tfrac{1}{2}, 0) > \pi 3 + \delta V^i(0, \tfrac{1}{2}). \tag{A5}$$

(A5) is in this case equivalent to

$$(\pi 1 - \pi 3) - \delta(\pi 3 - \pi 4) > \bar{u},$$

Given δ close to 1, $\bar{u}$ large and Result 1, this condition implies $\pi 2 + \pi 3 - 2\pi 3 > \pi 2 + \pi 3$, which is never satisfied.

[21] This results from the participation constraint on each firm: the net present value of future profits must be greater than zero.

In Case 2, we have

$$V^i(\tfrac{1}{2}, 0) = \frac{\pi 3 + \delta\pi 2 - \delta\bar{u}}{1 - \delta^2} \qquad \forall i,$$

$$V^i(0, \tfrac{1}{2}) = \frac{\pi 2 + \delta\pi 3 - \bar{u}}{1 - \delta^2} \qquad \forall i.$$

but then Condition (A3) becomes the opposite of Condition (A1), which is true given Result 1.

In Case 3, we have

$$V^1(\tfrac{1}{2}, 0) = V^i(0, 0), \qquad V^1(0, \tfrac{1}{2}) = \pi 2 + \delta V^i(0, 0) - \bar{u}.$$

But then, in order for (1, 1) to be a Nash equilibrium in state $(\frac{1}{2}, 0)$ we would need

$$V^1(\tfrac{1}{2}, 0) > \pi 3 + \delta V^1(0, \tfrac{1}{2})$$

which is equivalent to

$$(\pi 1 - \pi 3) - \delta(\pi 2 - \pi 4) > \bar{u}(1 - \delta)$$

which is not true given Result 1 and δ close to 1. (A similar argument works for Case 4.) Q.E.D.

Appendix 2

Equation (3) can be interpreted as a system of simultaneous equations (one equation for each brand).

$$\begin{cases} S_{1t} = \sum_{j=1}^{12} X_{tj}\gamma'_j + \alpha'_1 T_t + \Theta \sum_{\substack{h=1 \\ h\neq 1}}^{N} S_{ht} + \epsilon_{1t}, \\ S_{2t} = \sum_{j=1}^{12} X_{tj}\gamma'_j + \alpha'_2 T_t + \Theta \sum_{\substack{h=1 \\ h\neq 2}}^{N} S_{ht} + \epsilon_{2t}, \\ \vdots \\ S_{Nt} = \sum_{j=1}^{12} X_{tj}\gamma'_j + \alpha'_N T_t + \Theta \sum_{\substack{h=1 \\ h\neq N}}^{N} S_{ht} + \epsilon_{Nt} \end{cases} \tag{A6}$$

for $t = 1, \ldots, 24$.

This system of equations can be estimated through a Maximum Likelihood Estimation (MLE) method. The likelihood function for this system is

$$L(\gamma, \alpha, \Theta, \sigma^2) = \prod_{t=1}^{24} \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} \operatorname{Exp}\left\{-\frac{1}{2}\frac{\epsilon_{it}^2}{\sigma^2}\right\}$$

where γ is a vector of dimension 12 where γ'_j is the generic element, and α is a vector of dimension N where α'_i is the generic element. Changing variables from ϵ_{it} to S_{it} for $i = 1, \ldots, N$ and $t = 1, \ldots, 24$, one has to compute the Jacobian of ϵ_{it} as a function of S_{it} for $i = 1, \ldots, N$ and $t = 1, \ldots, 24$. Notice that $\partial\epsilon_{it}/\partial S_{j\tau} = 0$ for $t \neq \tau$.

Furthermore,

$$\frac{\partial \epsilon_{it}}{\partial S_{jt}} = \frac{\partial \epsilon_{i\tau}}{\partial S_{j\tau}} \qquad \forall i, j, t, \tau.$$

Then, let us compute the Jacobian J of dimension $(N \times N)$ and with generic element $\partial\epsilon_{it}/\partial S_{jt}$, which is independent of t. J is then equal to

$$J = \begin{bmatrix} 1 & -\Theta & -\Theta & \cdots & & -\Theta \\ -\Theta & 1 & -\Theta & \cdots & & -\Theta \\ -\Theta & -\Theta & 1 & \cdots & & -\Theta \\ \vdots & \vdots & -\Theta & & \vdots & \vdots \\ & & \vdots & & & \\ \vdots & \vdots & \vdots & & 1 & -\Theta \\ -\Theta & -\Theta & -\Theta & \cdots & -\Theta & 1 \end{bmatrix}.$$

The likelihood function can be then transformed into

$$\mathrm{L}(\gamma, \alpha, \Theta, \sigma^2) = \prod_{t=1}^{24} \mathrm{ABS}(|J|) \sum_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{Exp}\left\{-\frac{1}{2}\mathrm{v}\left(S_{it} - \Theta \sum_{\substack{j=1,\\ j\neq 1}}^{N}, S_{jt} - \sum_{j=1}^{12} X_{jt}\gamma_j' - \alpha_t T_t\right)^2 \Big/ \sigma^2\right\}$$

where $|J|$ is the determinant of J and $\mathrm{ABS}(y)$ is the absolute value of y. Notice that

$$|J| = 1 - \sum_{i=1}^{N} \Theta^i (i-1) C_{N-i}^{N} \qquad \text{where} \qquad C_k^N = \frac{N!}{k!(N-k)!}.$$

Finally, the log-likelihood function is

$$L(\gamma, \alpha, \Theta, \sigma^2) = \log \mathrm{L} = N \log[\mathrm{ABS}(|J|)] - 24N \log(\sqrt{2\pi}) - 24N \log \sigma$$
$$- \frac{1}{2\sigma^2} \sum_{t=1}^{24} \sum_{i=1}^{N} \left(S_{it} - \Theta \sum_{\substack{j=1,\\ j\neq i}}^{N} S_{jt} - \sum_{j=1}^{12} X_{jt}\gamma_j' - \alpha_t T_t\right)^2.$$

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