

Well-posedness for degenerate elliptic PDE arising in optimal learning strategies

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We derive a comparison principle for a degenerate elliptic partial differential equation without boundary conditions which arises naturally in optimal learning strategies. Our argument is direct and exploits the degeneracy of the differential operator to construct (logarithmically) diverging barriers.

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1. Motivation

The purpose of this paper is the analysis of a degenerate elliptic partial differential equation (PDE) arising from a stochastic process in optimal learning strategies. Our main result establishes the uniqueness of viscosity solutions to this degenerate PDE *without* boundary conditions. The PDE we consider, cf. (2.2) for its precise form, shares key features with the following toy model

$$\max \left\{ \max_{1 \leq i \leq d} \left\{ -x_i^2(1-x_i)^2 u_{x_i x_i} + u \right\}, u - \varphi(x) \right\} = 0 \quad \text{in } (0, 1)^d,$$

a fully nonlinear degenerate elliptic PDE with obstacle φ . Note that no boundary conditions are imposed on the faces of the unit cube $(0, 1)^d$. An excellent framework to study such equations is the theory of viscosity solutions, starting from the fundamental work of Crandall, Ishii, and Lions [2], cf. [5] for an exposition of relevant parts of this field and the connection to stochastic differential equations. However, the additional degeneracy due to the vanishing of $-x_i^2(1-x_i)^2$ at the boundary and the absence of boundary conditions causes additional difficulties which we will handle here with care. Our proof is direct and elementary. The key idea is to exploit the degeneracy of the differential operator at the boundary to construct (logarithmically) diverging barriers. On a different note, regarding the regularity of the solution u across the free boundary and the more subtle question of the regularity of the free boundary itself, we refer the interested reader to the crucial work of Caffarelli [1], and the expository notes [3].

The model we analyze in this paper was introduced in the recent paper [4] by Ke and one of the authors: a decision maker may decide among d alternatives on which they can collect information, and ultimately invest. In this model, the payoff π_i of the i^{th} alternative is assumed to take either of the two values $\underline{\pi}_i < \bar{\pi}_i$. These random variables (π_1, \dots, π_d) are assumed to be independent. The

outside option has the deterministic payoff π_0 . As the process π_i is assumed to take only two values, it is characterized by the belief $X_i(t) = P(\pi_i(t) = \bar{\pi}_i | \mathcal{F}_t)$, where \mathcal{F}_t is a filtration representing the observed signals until time t . The decision maker's allocation policy $\mathcal{S} = \{I_t\}_{t>0}$ controls the stochastic differential equation (SDE)

$$dX_i = \frac{\bar{\pi}_i - \underline{\pi}_i}{\sigma_i^2} X_i (1 - X_i) \{[\pi_i - \underline{\pi}_i(1 - X_i) - \bar{\pi}_i X_i] dT_i + \sigma_i dW(T_i)\}, \quad (1.1)$$

where $T_i(t) = |\{s \in (0, t): I_s(X_i(s)) = 1\}|$ denotes the accumulated time alternative i has been investigated.

Let us briefly argue why this is sensible intuitively; we refer the interested reader to [4] for the derivation of (1.1) and more details: The larger the signal-to-noise ratio $\frac{\bar{\pi}_i - \underline{\pi}_i}{\sigma_i^2}$ is, the more likely will the decision maker update their belief according to a new signal. Also note that it makes sense that the right-hand side is decreasing in the noise σ_i : the larger the noise, the less likely will the decision maker trust the new signal and update their belief. The drift $\pi_i - \underline{\pi}_i(1 - X_i) - \bar{\pi}_i X_i$ is simply the difference of the prospected and expected value of alternative i . Finally, note that the prefactor $X_i(1 - X_i)$ indicates that the decision maker is most likely to update their belief if they were undecided in the first place, i.e., if $X_i = \frac{1}{2}$.

The expected payoff at a stopping time τ and under an allocation policy \mathcal{S} is

$$J(x; \mathcal{S}, \tau) := E \left[\max \left\{ \max_{1 \leq i \leq d} \{ \bar{\pi}_i X_i(\tau) + \underline{\pi}_i (1 - X_i(\tau)) \}, \pi_0 \right\} - \sum_{i=1}^d c_i T_i(\tau) \mid X(0) = x \right]. \quad (1.2)$$

The decision maker's objective is to maximize the payoff:

$$V(x) = \sup_{\mathcal{S}, \tau} J(x; \mathcal{S}, \tau). \quad (1.3)$$

In [4] it is argued that the value function V satisfies the Hamilton–Jacobi–Bellman PDE

$$\max \left\{ \max_{1 \leq i \leq d} \left\{ \frac{(\bar{\pi}_i - \underline{\pi}_i)^2}{2\sigma_i^2} x_i^2 (1 - x_i)^2 V_{x_i x_i} - c_i \right\}, g(x) - V \right\} = 0 \quad \text{in } Q, \quad (1.4)$$

where $Q = (0, 1)^d$ denotes the unit cube and $g(x) := \max \{ \max_i \{ \bar{\pi}_i x_i + \underline{\pi}_i (1 - x_i) \}, \pi_0 \}$.

Clearly, the payoff satisfies $\pi_0 \leq J(x; \mathcal{S}, \tau) \leq \max_i \bar{\pi}_i$ and hence we have the uniform bounds for the value function

$$0 < \pi_0 \leq V(x) \leq \max_{1 \leq i \leq d} \bar{\pi}_i < \infty.$$

Let us briefly motivate why Lipschitz continuity is a natural concept for the value function. If the optimal stopping time τ is uniformly bounded by some finite time horizon T , the regularity $V \in C^{0,1}(Q)$ can be derived easily: Given two points $x, x' \in Q$, using the optimal allocation policy \mathcal{S} and stopping time τ of x for x' yields

$$V(x) - V(x') \leq J(x; \mathcal{S}, \tau) - J(x'; \mathcal{S}, \tau) = E[f(X_\tau) | X_0 = x] - E[f(X'_\tau) | X'_0 = x'],$$

where we momentarily defined the function $f(x) := \max \{ \max_{1 \leq i \leq d} \{ \bar{\pi}_i x_i + \underline{\pi}_i (1 - x_i) \}, \pi_0 \}$, which is Lipschitz continuous as the maximum of linear functions. Note that due to the choice of

the allocation policy, the last term in J , which is nonlocal in time cancels exactly. Hence, denoting by $\text{Lip}(f)$ the Lipschitz constant of f ,

$$V(x) - V(x') \leq \text{Lip}(f) E[|X_\tau - X'_\tau| | X_0 = x, X'_0 = x'],$$

so that we only need to appeal to the stability of the SDE (1.1) to obtain $V(x) - V(x') \leq C_T |x - x'|$, cf. [5, Theorem 2.4]. Interchanging the roles of x and x' proves the Lipschitz continuity of V . Our main result, Theorem 2.1, establishes the uniqueness in this class $C^{0,1}$.

2. Main result

Let us suppose for simplicity that all payoffs and noise levels are equal so that (1.4) becomes

$$\max \left\{ \max_{1 \leq i \leq d} \{x_i^2(1-x_i)^2 V_{x_i x_i} - c_i\}, g(x) - V \right\} = 0 \quad \text{in } Q. \quad (2.1)$$

In order to rewrite (2.1) in a more familiar form we use the change of variables $V(x) = b - e^{u(x)}$, where $b > \max_i \bar{\pi}_i$, which leads us to

$$\max \left\{ \max_{1 \leq i \leq d} \{-x_i^2(1-x_i)^2 (u_{x_i x_i} + u_{x_i}^2) - c_i e^{-u}\}, 1 - (b - g(x)) e^{-u} \right\} = 0 \quad \text{in } Q, \quad (2.2)$$

where g could now be any given continuous function on \bar{Q} such that $b - g(x) > 0$ for all $x \in \bar{Q}$ and $c_1, \dots, c_d > 0$ are given positive constants. Note that the condition on g is true in the above concrete example since $b > \max_i \bar{\pi}_i$.

Setting

$$F(x, r, p, A) := \max \left\{ \max_{1 \leq i \leq d} \{-x_i^2(1-x_i)^2 (A_{ii} + p_i^2) - c_i e^{-r}\}, 1 - (b - g(x)) e^{-r} \right\}$$

for $x \in Q$, $r \in \mathbb{R}$, $p \in \mathbb{R}^d$, and $A = (A_{ij})_{i,j=1}^d \in \mathfrak{S}(d) = \{M \in \mathbb{R}^{d \times d} : M^T = M\}$, the equation reads

$$F(x, u, Du, D^2u) = 0,$$

which can be formulated in the viscosity sense, see Definition 3.1 below.

The main result of this work is the following comparison theorem.

Theorem 2.1 *If $u, v \in C^{0,1}(Q)$ satisfy*

$$F(x, u, Du, D^2u) \leq 0 \quad \text{and} \quad F(x, v, Dv, D^2v) \geq 0 \quad \text{in the viscosity sense,} \quad (2.3)$$

then $u \leq v$ in Q .

REMARK 2.2 As an immediate consequence, the viscosity solution of (2.2), without boundary conditions, is unique in the class $C^{0,1}(Q)$. We also refer to [4], where solutions to (2.2) are constructed explicitly in the case of two alternatives, i.e., $d = 2$. Since, by construction, V is Lipschitz and bounded, also $u(x) = \log(b - V(x))$ is Lipschitz and the problem is well-posed.

REMARK 2.3 There are two obvious difficulties:

1. There are no boundary conditions.
2. F is not uniformly elliptic.

We will see that these two properties are interconnected: The degeneracy at the boundary makes boundary conditions superfluous. Indeed, the underlying stochastic process (1.1) does not reach the boundary ∂Q in finite time. This is in fact due to the degeneracy: Even at the simpler example $dX_t = X_t dW_t$ one can see this effect. In this case, the solution is explicitly given by $X_t = e^{-\frac{1}{2}t + W_t} X_0$, which is positive a.s. if $X_0 > 0$.

Let us also mention that we believe our method of proof applies in more generality for degenerate elliptic differential operators whose normal components of the second derivatives degenerate quadratically towards the boundary.

Since $c_i > 0$ and $g(x) < b$, the function F is strictly monotonic increasing in r : For every $R > 0$ there exists a constant $\theta > 0$ such that for all $x \in Q$, $p \in \mathbb{R}^d$ and $A \in \mathcal{S}(d)$ we have

$$\theta(r - r') \leq F(x, r, p, A) - F(x, r', p, A) \quad \text{for all } r' \leq r \leq R. \quad (2.4)$$

For the rest of the paper we will assume for simplicity that $c_1 = \dots = c_d = 1$.

In the next section, we will prove Theorem 2.1 using the theory of viscosity solutions. Before doing so, let us illuminate the novelty in this work at the simpler example

$$\max_{1 \leq i \leq d} \{ -x_i^2(1 - x_i)^2 u_{x_i x_i} + u \} = 0 \quad \text{in } (0, 1)^d,$$

assuming that u and v are classical C^2 sub- and supersolutions, respectively, i.e.,

$$\max_{1 \leq i \leq d} \{ -x_i^2(1 - x_i)^2 u_{x_i x_i} + u \} \leq 0 \leq \max_{1 \leq i \leq d} \{ -x_i^2(1 - x_i)^2 v_{x_i x_i} + v \}.$$

We aim to show that $u \leq v$, without using boundary conditions. A classical computation shows that $u - v$ cannot have a positive interior maximum. Indeed, at such a point x_0 , it would hold $(u - v)_{x_i x_i} \leq 0$ and $u - v > 0$. Hence

$$\max_{1 \leq i \leq d} \{ -x_i^2(1 - x_i)^2 u_{x_i x_i} + u \} > \max_{1 \leq i \leq d} \{ -x_i^2(1 - x_i)^2 v_{x_i x_i} + v \},$$

a contradiction to the fact that u is a subsolution and v is a supersolution.

However, since we do not know whether $u \leq v$ on ∂Q , the above argument is not complete. To illustrate how to exploit the degeneracy of the differential operators $-x_i^2(1 - x_i)^2 \partial_{x_i}^2$ at the boundary to overcome the absence of boundary conditions, let us assume that again u and v are C^2 and let us assume that the slightly stronger inequality

$$\max_{1 \leq i \leq d} \{ -x_i^2(1 - x_i)^2 u_{x_i x_i} + u \} \leq \max_{1 \leq i \leq d} \{ -x_i^2(1 - x_i)^2 v_{x_i x_i} + v \} - \delta \quad (2.5)$$

holds for some fixed $\delta > 0$. The function

$$\Phi(x) := - \sum_{i=1}^d \{ \log x_i + \log(1 - x_i) \}$$

is non-negative and degenerates at the boundary in the sense that $\Phi(x) \rightarrow +\infty$ as $x \rightarrow \partial Q$, while $\Phi_{x_i x_i} = \frac{1}{x_i^2} + \frac{1}{(1-x_i)^2}$ guarantees that the differential operator applied to Φ is tame in the sense that $-x_i^2(1-x_i)^2 \Phi_{x_i x_i} \geq -2$ is uniformly bounded below. This suggests to modify the argument above with $u - v$ replaced by $u - v - \varepsilon \Phi$ for some $\varepsilon > 0$. In this case, since $u - v - \varepsilon \Phi \rightarrow -\infty$ as $x \rightarrow \partial Q$, the maximum has to occur at some point $x_{0,\varepsilon}$ in the interior of Q . If at this maximum $u - v > 0$, then

$$\max_{1 \leq i \leq d} \{ -x_i^2(1-x_i)^2 u_{x_i x_i} + u \} > \max_{1 \leq i \leq d} \{ -x_i^2(1-x_i)^2 v_{x_i x_i} + v \} - 2\varepsilon,$$

which contradicts (2.5) if $\varepsilon < \delta/2$.

3. Proof via viscosity solutions

To turn the above heuristic argument for Theorem 2.1 into a valid proof, we will employ the theory of viscosity solutions. Let us first recall the definition of sub- and supersolutions.

DEFINITION 3.1 A continuous function $u \in C(Q)$ is called a subsolution (supersolution) of the PDE $F(x, u, Du, D^2u) = 0$ and we write

$$F(x, u, Du, D^2u) \leq 0 \ (\geq 0) \quad \text{in the viscosity sense} \quad (3.1)$$

if the following holds: Let $\zeta \in C^2(Q)$ be such that $\zeta - u$ has a local maximum (minimum) at x_0 , then

$$F(x, \zeta, D\zeta, D^2\zeta) \leq 0 \ (\geq 0) \quad (3.2)$$

at the point $x = x_0$. The function u is called a solution of $F(x, u, Du, D^2u) = 0$ and we write

$$F(x, u, Du, D^2u) = 0 \quad \text{in the viscosity sense}$$

if u is both a sub- and a supersolution.

We recall the following basic but handy characterization of sub- and supersolutions using sub- and superjets.

Lemma 3.2 *Let $u \in C(Q)$. Then*

$$F(x, u, Du, D^2u) \leq 0 \quad \text{in the viscosity sense}$$

if and only if for every $x_0 \in Q$ and any generalized superjet $(p, A) \in \bar{J}_+ u(x_0)$

$$F(x_0, u_0, p, A) \leq 0.$$

Let $v \in C(Q)$. Then

$$F(y, v, Dv, D^2v) \geq 0 \quad \text{in the viscosity sense}$$

if and only if for every $y_0 \in Q$ and any generalized subjet $(p, A) \in \bar{J}_- v(y_0)$

$$F(y_0, v(y_0), p, A) \geq 0.$$

Here and in the following, by $\bar{J}_+u(x_0)$, $\bar{J}_-v(y_0)$ we denote as usual the set of all generalized super- and subjets, respectively:

1. For $(p, A) \in \mathbb{R}^d \times \mathcal{S}(d)$, we have $(p, A) \in J_+u(x_0)$ if and only if (p, A) is a superjet of u at x_0 , i.e.,

$$u(x) \leq u(x_0) + p \cdot (x - x_0) + \frac{1}{2} (x - x_0) \cdot A (x - x_0) + o(|x - x_0|^2) \quad \text{as } x \rightarrow x_0.$$

The set of all generalized superjets $\bar{J}_+u(x_0)$ is simply given by the topological closure of this set.

2. Similarly, for $(p, A) \in \mathbb{R}^d \times \mathcal{S}(d)$, we have $(p, A) \in J_-v(y_0)$ if and only if (p, A) is a subjet of v at y_0 , i.e.,

$$v(y) \geq v(y_0) + p \cdot (y - y_0) + \frac{1}{2} (y - y_0) \cdot A (y - y_0) + o(|y - y_0|^2) \quad \text{as } y \rightarrow y_0.$$

The set of all generalized subjets $\bar{J}_-v(y_0)$ is the closure of $J_-v(y_0)$.

We first state and prove some lemmas which will be useful for the proof of the theorem. The first lemma exploits the degeneracy at the boundary by adding the penalization- or barrier-term discussed in the previous section which diverges at the boundary. Note that the barriers $u_\varepsilon \leq u$ and $v^\varepsilon \geq v$ defined below diverge as we approach the boundary ∂Q .

Lemma 3.3 *Let $\alpha, \varepsilon > 0$. For $u, v \in C^{0,1}(Q)$ let*

$$u_\varepsilon(x) := u(x) - \varepsilon \Phi(x), \quad v^\varepsilon(y) := v(y) + \varepsilon \Phi(y), \quad (3.3)$$

where

$$\Phi(x) := - \sum_{i=1}^d \{\log x_i + \log(1 - x_i)\} \geq 0, \quad (3.4)$$

and let $x_{\alpha,\varepsilon}, y_{\alpha,\varepsilon}$ be such that

$$u_\varepsilon(x_{\alpha,\varepsilon}) - v^\varepsilon(y_{\alpha,\varepsilon}) - \frac{\alpha}{2} |x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}|^2 = \sup_{(x,y) \in Q \times Q} \left\{ u_\varepsilon(x) - v^\varepsilon(y) - \frac{\alpha}{2} |x - y|^2 \right\}. \quad (3.5)$$

Then

$$\alpha |x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}| \leq 2 \max\{\text{Lip}(u), \text{Lip}(v)\}.$$

Here $\text{Lip}(u) := \sup_{x,y \in Q} \frac{|u(x) - u(y)|}{|x - y|}$ denotes the Lipschitz constant of the function u on Q .

Proof. Testing the maximality (3.5) with the pair $(x_{\alpha,\varepsilon}, x_{\alpha,\varepsilon})$ yields

$$u_\varepsilon(x_{\alpha,\varepsilon}) - v^\varepsilon(y_{\alpha,\varepsilon}) - \frac{\alpha}{2} |x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}|^2 \geq u_\varepsilon(x_{\alpha,\varepsilon}) - v^\varepsilon(x_{\alpha,\varepsilon})$$

and hence after reordering and using the definition of v^ε

$$\frac{\alpha}{2} |x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}|^2 \leq v^\varepsilon(x_{\alpha,\varepsilon}) - v^\varepsilon(y_{\alpha,\varepsilon}) \leq \text{Lip}(v) |x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}| + \varepsilon (\Phi(x_{\alpha,\varepsilon}) - \Phi(y_{\alpha,\varepsilon})).$$

However, we have no control over $\Phi(x_{\alpha,\varepsilon}) - \Phi(y_{\alpha,\varepsilon})$. To overcome this difficulty, we test the maximality as well with $(y_{\alpha,\varepsilon}, y_{\alpha,\varepsilon})$ instead of $(x_{\alpha,\varepsilon}, x_{\alpha,\varepsilon})$ and obtain

$$u_\varepsilon(x_{\alpha,\varepsilon}) - v^\varepsilon(y_{\alpha,\varepsilon}) - \frac{\alpha}{2} |x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}|^2 \geq u_\varepsilon(y_{\alpha,\varepsilon}) - v^\varepsilon(y_{\alpha,\varepsilon})$$

which yields

$$\frac{\alpha}{2} |x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}|^2 \leq u_\varepsilon(x_{\alpha,\varepsilon}) - u_\varepsilon(y_{\alpha,\varepsilon}) \leq \text{Lip}(u) |x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}| - \varepsilon(\Phi(x_{\alpha,\varepsilon}) - \Phi(y_{\alpha,\varepsilon})).$$

Since either $\Phi(x_{\alpha,\varepsilon}) - \Phi(y_{\alpha,\varepsilon}) \geq 0$ or $\Phi(x_{\alpha,\varepsilon}) - \Phi(y_{\alpha,\varepsilon}) < 0$, we may use our favorite among the two above estimates which yields

$$\frac{\alpha}{2} |x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}|^2 \leq \max\{\text{Lip}(u), \text{Lip}(v)\} |x_{\alpha,\varepsilon} - y_{\alpha,\varepsilon}|. \quad \square$$

Clearly, modifying a strong solution u of $F(x, u, Du, D^2u) = 0$ by the logarithmic barrier Φ given in (3.4) yields a solution $u^\varepsilon = u - \varepsilon\Phi$ to some modified equation $F_\varepsilon(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) = 0$. The following lemma states that this is also true in the case of viscosity solutions. Although the modified functions u_ε and v^ε diverge at the boundary, thanks to the degeneracy, the modified degenerate elliptic operators F_ε and $F_{-\varepsilon}$ are well-behaved close to the boundary.

Lemma 3.4 *Let $u, v \in C^0(Q)$ satisfy (2.3). Then u_ε and v^ε defined via (3.3) satisfy*

$$F_\varepsilon(x, Du_\varepsilon, D^2u_\varepsilon; u) \leq 0 \quad \text{and} \quad F_{-\varepsilon}(x, Dv^\varepsilon, D^2v^\varepsilon; v) \geq 0 \quad \text{in the viscosity sense,}$$

where for $x \in Q$, $p \in \mathbb{R}^d$, $A \in \mathfrak{S}(d)$ and $u: Q \rightarrow \mathbb{R}$,

$$F_\varepsilon(x, p, A; u(x)) := \max \left\{ \max_{1 \leq i \leq d} \left\{ - (x_i^2 (1 - x_i)^2 (A_{ii} + p_i^2) - e^{-u(x)} - f_\varepsilon(x_i, p_i)) \right\}, \right. \\ \left. 1 - (b - g(x))e^{-u(x)} \right\} \quad (3.6)$$

and

$$f_\varepsilon(x_i, p_i) := \varepsilon((1 - x_i)^2 + x_i^2) + \varepsilon^2(1 - 2x_i)^2 + 2\varepsilon x_i(1 - x_i)(2x_i - 1)p_i$$

Note that the u -variable in F_ε is frozen in the sense that we plug in the fixed function u , not the candidate u_ε . On the one hand, then the terms containing u become simply another x -dependence in the equation. On the other hand, we want to keep track of the frozen variable to remember the crucial strict monotonicity (2.4) in that variable. Note furthermore that the u -dependence is only pointwise.

Proof. We only show the statement for u as the one for v is completely analogous. Let $x_0 \in Q$ be fixed.

Note that there is a one-to-one correspondence between $J_+u_\varepsilon(x_0)$ and $J_+u(x_0)$: Let $(p_\varepsilon, A_\varepsilon) \in \bar{J}_+u_\varepsilon(x_0)$, i.e.,

$$u(x) - \varepsilon\Phi(x) \leq u(x_0) - \varepsilon\Phi(x_0) + p_\varepsilon^T(x - x_0) + \frac{1}{2}(x - x_0)^T A_\varepsilon(x - x_0) + o(|x - x_0|^2).$$

Developing Φ to second order around x_0

$$u(x) \leq u(x_0) + (p_\varepsilon^T + \varepsilon D\Phi(x_0))(x - x_0) + \frac{1}{2}(x - x_0)^T (A_\varepsilon + \varepsilon D^2\Phi(x_0))(x - x_0) + o(|x - x_0|^2),$$

i.e.,

$$(p_\varepsilon + \varepsilon D\Phi(x_0)^T, A_\varepsilon + \varepsilon D^2\Phi(x_0)) \in J_+u(x_0).$$

Therefore, if $(p_\varepsilon, A_\varepsilon) \in \bar{J}_+u_\varepsilon(x_0)$ is a generalized superjet of u at x_0 , then using the above argument for an approximating sequence, we obtain

$$(p_\varepsilon + \varepsilon D\Phi(x_0)^T, A_\varepsilon + \varepsilon D^2\Phi(x_0)) \in \bar{J}_+u(x_0).$$

Using the fact that u is a subsolution, i.e., (2.3), we obtain

$$F(x_0, u, p_\varepsilon + \varepsilon D\Phi(x_0)^T, A_\varepsilon + \varepsilon D^2\Phi(x_0)) \leq 0.$$

That means that at the point $x = x_0$ we have

$$\max \left\{ \max_{1 \leq i \leq d} \left\{ -x_i^2(1-x_i)^2 (A_{\varepsilon,ii} + p_{\varepsilon,i}^2) - x_i^2(1-x_i)^2 (\varepsilon \Phi_{x_i x_i} + 2\varepsilon \Phi_{x_i} p_{\varepsilon,i} + \varepsilon^2 \Phi_{x_i}^2) - e^{-u(x)} \right\}, 1 - (b - g(x))e^{-u(x)} \right\} \leq 0.$$

As $\Phi_{x_i} = -\frac{1}{x_i} + \frac{1}{1-x_i}$ and $\Phi_{x_i x_i} = \frac{1}{x_i^2} + \frac{1}{(1-x_i)^2}$, this is nothing else but (3.6). \square

One important ingredient of the proof is the by now classical doubling of variables first introduced by Jensen and then formulated by Ishii in (a more general form than) the following lemma.

Lemma 3.5 (Ishii's Lemma, see Theorem 3.2 in [2]) *Let $u, v \in C^0(Q)$, $\alpha > 0$ and suppose*

$$u(x_0) - v(x_0) - \frac{\alpha}{2}|x_0 - y_0|^2 = \max_{(x,y) \in Q \times Q} \left\{ u(x) - v(x) - \frac{\alpha}{2}|x - y|^2 \right\}.$$

Then there exist $A, B \in \mathcal{S}(d)$ such that

$$(\alpha(x_0 - y_0), A) \in \bar{J}_Q^+u(x_0), (\alpha(x_0 - y_0), B) \in \bar{J}_Q^-v(y_0), \quad (3.7)$$

and

$$-3\alpha \begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix} \leq \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \leq 3\alpha \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix}. \quad (3.8)$$

Here I_d and 0 denote the identity and zero matrix in \mathbb{R}^d , respectively. The inequalities in (3.8) are to be understood in the sense of symmetric matrices (or equivalently symmetric bilinear forms).

Proof of Theorem 2.1. Suppose for a contradiction that

$$0 < \delta = (u - v)(z) \quad \text{for some } z \in Q.$$

Let $\alpha, \varepsilon > 0$ be fixed and let u_ε and v^ε be given by (3.3). (One may think of $\alpha \gg 1$ and $\varepsilon \ll 1$.)

Step 1: By Bolzano–Weierstrass, the supremum

$$M_{\alpha,\varepsilon} := \sup_{(x,y) \in \mathcal{Q} \times \mathcal{Q}} \left\{ u_\varepsilon(x) - v^\varepsilon(y) - \frac{\alpha}{2} |x - y|^2 \right\}$$

is attained at some interior point $(x_{\alpha,\varepsilon}, y_{\alpha,\varepsilon}) \in \mathcal{Q} \times \mathcal{Q}$ since $u_\varepsilon(x) \rightarrow -\infty$ as $x \rightarrow \partial\mathcal{Q}$ and $v^\varepsilon(y) \rightarrow +\infty$ as $y \rightarrow \partial\mathcal{Q}$. Since $\bar{\mathcal{Q}}$ is compact, there exist sequences $\alpha_n \uparrow \infty$, $\varepsilon_n \downarrow 0$ such that

$$\begin{aligned} x_n &:= x_{\alpha_n, \varepsilon_n} \rightarrow \bar{x} \in \bar{\mathcal{Q}}, \\ y_n &:= y_{\alpha_n, \varepsilon_n} \rightarrow \bar{y} \in \bar{\mathcal{Q}}. \end{aligned}$$

Lemma 3.3 yields that

$$\alpha_n |x_n - y_n| \text{ stays bounded as } n \rightarrow \infty \quad (3.9)$$

and in particular, since $\alpha_n \rightarrow \infty$, this implies $|x_n - y_n| \rightarrow 0$, i.e., $\bar{x} = \bar{y}$.

Step 2: Since the maximizer (x_n, y_n) of M_n is an interior point, Ishii’s Lemma 3.5 furnishes the existence of two symmetric matrices $A_n, B_n \in \mathcal{S}(d)$ which together with $\alpha_n(x_n - y_n)$ contribute second-order sub- and superjets for u and v , respectively:

$$(\alpha_n(x_n - y_n), A_n) \in \bar{J}_+ u(x_n), \quad (\alpha_n(x_n - y_n), B_n) \in \bar{J}_- v(y_n)$$

and furthermore inequality (3.8) holds.

Step 3: It is straightforward to see that F_ε is strictly increasing in the frozen u -variable: For every $R > 0$ there exists $\theta > 0$ such that for all $r' \leq r \leq R$

$$\theta (r - r') \leq F_\varepsilon(x, p, A; r) - F_\varepsilon(x, p, A; r'). \quad (3.10)$$

Recall that the dependence of F_ε on u is pointwise. Hence by the maximality of (x_n, y_n)

$$\begin{aligned} \theta\delta &\leq \theta(u(x_n) - v(y_n)) \\ &\stackrel{(3.10)}{\leq} F_{\varepsilon_n}(x_n, \alpha_n(x_n - y_n), A_n; u(x_n)) - F_{\varepsilon_n}(x_n, \alpha_n(x_n - y_n), A_n; v(y_n)). \end{aligned}$$

By Lemma 3.4, the modified functions u_ε and v^ε are sub- and supersolutions (of the modified operators), respectively, so we obtain

$$F_{\varepsilon_n}(x_n, \alpha_n(x_n - y_n), A_n; u(x_n)) \leq 0 \leq F_{-\varepsilon_n}(y_n, \alpha_n(x_n - y_n), B_n; v(y_n)).$$

Combining the two above inequalities yields

$$0 < \theta\delta \leq F_{-\varepsilon_n}(y_n, \alpha_n(x_n - y_n), B_n; v(y_n)) - F_{\varepsilon_n}(x_n, \alpha_n(x_n - y_n), A_n; v(y_n)). \quad (3.11)$$

Step 4: We claim that

$$\begin{aligned} & F_{-\varepsilon_n}(y_n, \alpha_n(x_n - y_n), B_n; v(y_n)) - F_{\varepsilon_n}(x_n, \alpha_n(x_n - y_n), A_n; v(y_n)) \\ & \leq 3\alpha_n |x_n - y_n|^2 + \alpha_n^2 |x_n - y_n|^3 + 8\varepsilon_n (1 + \alpha_n |x_n - y_n| + \varepsilon_n) + \omega_g(|x_n - y_n|), \end{aligned} \quad (3.12)$$

which will conclude the proof of the theorem. Here ω_g denotes the modulus of continuity of the given function g . Indeed, if the claim is true, then by (3.9) we obtain

$$\limsup_{n \rightarrow \infty} F_{-\varepsilon_n}(y_n, \alpha_n(x_n - y_n), B_n; v(y_n)) - F_{\varepsilon_n}(x_n, \alpha_n(x_n - y_n), A_n; v(y_n)) \leq 0,$$

a contradiction to the strict positivity (3.11).

We are left with proving (3.12). To this end we will use the second inequality in (3.8), which simply means

$$\xi^T A \xi - \eta^T B \eta \leq 3\alpha |\xi - \eta|^2 \quad \text{for all } \xi, \eta \in \mathbb{R}^d. \quad (3.13)$$

In order to prove (3.12), let $\varepsilon > 0$, $x, y \in Q$, $p \in \mathbb{R}^d$, $A, B \in \mathcal{S}(d)$ be given s.t. (3.13) holds. For notational simplicity set $\sigma_i(x) := (1 - x_i) x_i$.

We distinguish two cases.

Case 1: $F_{-\varepsilon}(y, p, B; r) = 1 - (b - g(y))e^{-r}$. In this case

$$\begin{aligned} F_{-\varepsilon}(y, p, B; r) - F_{\varepsilon}(x, p, A; r) & \leq 1 - (b - g(y))e^{-r} - \left(1 - (b - g(x))e^{-r}\right) \\ & \leq \omega_g(|x - y|)e^{-r}. \end{aligned}$$

Case 2: $F_{-\varepsilon}(y, p, B; r) > 1 - (b - g(y))e^{-r}$. By definition of $F_{-\varepsilon}$, there exists an index $j \in \{1, \dots, d\}$ such that

$$F_{-\varepsilon}(y, p, B; r) = -(\sigma_j(y)e_j)^T B \sigma_j(y)e_j + \sigma_j^2(y)p_j^2 - e^{-r} - f_{-\varepsilon}(y_j, p_j).$$

Hence

$$\begin{aligned} & F_{-\varepsilon}(y, p, B; r) - F_{\varepsilon}(x, p, A; r) \\ & \leq (\sigma_j(x)e_j)^T A \sigma_j(x)e_j - (\sigma_j(y)e_j)^T B \sigma_j(y)e_j \\ & \quad + (\sigma_j^2(y) - \sigma_j^2(x))p_j^2 - f_{-\varepsilon}(y_j, p_j) + f_{\varepsilon}(x_j, p_j). \end{aligned}$$

Applying (3.13) to the first right-hand side term (with the collinear vectors $\xi = \sigma_j(x)e_j$ and $\eta = \sigma_j(y)e_j$), we obtain

$$\begin{aligned} F_{-\varepsilon}(y, p, B; r) - F_{\varepsilon}(x, p, A; r) & \leq 3\alpha (\sigma_j(x) - \sigma_j(y))^2 + |\sigma_j^2(x) - \sigma_j^2(y)| |p|^2 \\ & \quad + \sup_{t \in (0,1)} (|f_{-\varepsilon}(t, p_j)| + |f_{\varepsilon}(t, p_j)|). \end{aligned}$$

Since the gradient $D\sigma_j(x) = (1 - 2x_j)e_j^T$ is bounded by 1 and $0 \leq \sigma_j(x) \leq 1$, we obtain $|\sigma_j(x) - \sigma_j(y)| \leq |x - y|$ and $|\sigma_j^2(x) - \sigma_j^2(y)| \leq |\sigma_j(x) + \sigma_j(y)||\sigma_j(x) - \sigma_j(y)| \leq 2|x - y|$. Furthermore,

$$\sup_{t \in (0,1)} (|f_{-\varepsilon}(t, p_j)| + |f_{\varepsilon}(t, p_j)|) \leq 2\varepsilon (2 + 2|p| + \varepsilon).$$

and therefore (3.12) holds. This concludes the proof of Theorem 2.1. \square

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