

College Assignment as a Large Contest

Aaron L. Bodoh-Creed and Brent R. Hickman

November 29, 2016

Abstract

We develop a model of college assignment as a large contest wherein students with heterogeneous abilities compete for seats at vertically differentiated colleges through the acquisition of productive human capital. We use a continuum model to approximate the outcomes of a game with large, but finite, sets of colleges and students. By incorporating two common families of affirmative action mechanisms into our model, admissions preferences and quotas, we can show that (legal) admissions preference schemes and (illegal) quotas have the same sets of equilibria including identical outcomes and investment strategies. Finally, we explore the welfare costs of using human capital accumulation to compete for college admissions. We define the cost of competition as the welfare difference between a color-blind admissions contest and the first-best outcome chosen by an omniscient social planner. Using a calibrated version of our model, we find that the cost of competition is equivalent to a loss of \$91,795 in NPV of lifetime earnings.

Keywords: Affirmative action; welfare costs of competition; contests; approximate equilibrium.

JEL subject classification: D44, C72, I20, I28, L53.

1 Introduction

There are many economically salient features of the competition between students for admission to college. An ideal model would include heterogeneity amongst the colleges in terms of quality, allow for differences amongst the students in terms of ex-ante ability, and endogenize the decisions students make to compete for admission. For many policy questions, it is also necessary to allow for asymmetric admissions policies that grant preferential treatment to the children of alumni (so-called “legacy” status) or students from underrepresented demographic backgrounds (affirmative action). In the latter case, a further complication arises: a primary motivation for policies like affirmative action

is the idea of systematic a priori asymmetries between students due to socioeconomic factors (e.g., access to relevant resources like high-quality education). While many models include one or two of the above factors, providing a tractable model of the market that includes all four features—college heterogeneity, student heterogeneity, a priori cost asymmetry, and asymmetric allocation mechanisms—has proven difficult.

A primary reason for the difficulty is the dual role played by human capital (HC) in college admissions. First, a student’s HC is a durable asset which yields a direct economic return for future payoffs, leading to what we call a *productive channel* of investment incentives. This channel is present in the complete-information assortative matching model of Becker [4]. Second, because each student’s ex-ante ability is private information, colleges must rely on observable measures of HC production in order to separate high-ability students from their lower-ability counterparts when deciding who will be admitted to each college. Because of this, increasing HC yields an additional, indirect benefit since students who invest more also gain access to higher quality match partners in the college admissions market. The resulting *competitive channel* of incentives resembles the signaling incentives analyzed in the seminal model of Spence [62].

We model the college admissions market as a contest wherein colleges are rank-ordered and students compete for admission by endogenously choosing the level of HC to accrue prior to a rank-order admissions contest. While this model is difficult to solve when it includes a finite number of students and colleges, we show that a more tractable limit model with a continuum of colleges and students closely approximates the finite model with many agents. More formally, we show that the equilibria of the finite game must approach the unique equilibrium of the limit model as the number of students increases.

To understand why a large market setting might simplify things, consider a student (or a student’s parents) that is deciding how much effort to exert in school with an eye toward applying to colleges. If the student wants to ascertain whether she is likely to be admitted to a school with a given GPA and SAT score, she does not need to consider the other students who might also be applying or the strategies they are employing. Instead, she simply consults a college guide that describes the qualifications of previously admitted students. Since the aggregated choices of many market players produces a high degree of predictability in these (endogenous) admissions criteria, she can have confidence if she meets the criterion at a given school that admission is likely.¹

Now we turn to our first goal, the analysis of different mechanisms that have been

¹ Although this example and our main analysis are focused on college admissions, the basic insight regarding the tractability and applicability of limit games as approximations to real-world contests generalizes far outside the context of college admissions and affirmative action.

used to implement affirmative action schemes.² An affirmative action system implemented through a *quota* reserves separate pools of seats for different groups of students, and each student can only compete for the seats reserved for his or her group. Many different quota schemes are possible, in which different sets of college seats are reserved for each group. In an *admissions preference* scheme, all applicants compete for the same pool of seats, but the fashion in which the applications are ranked is dependent on the demographics of the applicant.

We show that given any equilibrium of any quota (admissions preference) system, one can design an admissions preference (quota) system with an equilibrium that results in the same school assignments and HC choices. The equivalence of the admissions preference and quota schemes throws new light on the legal arguments about the constitutionality of affirmative action. The U.S. Supreme Court has previously ruled that quota systems violate the 14th amendment of the U.S. Constitution, but certain forms of admissions preference systems are legal. Although we are not qualified to judge the legal arguments about affirmative action, the outcome equivalence of the two affirmative action mechanisms does help explain why the U.S. legal system has had difficulty drawing a sharp line between constitutional and unconstitutional forms of affirmative action.

Our second policy goal is to study the welfare effects on students of the costly competition for college admissions. As noted in the recent “Turning the Tide” ([64]) report produced by the Harvard Graduate School of Education, college applicants exert large amounts of real effort engaging in tasks with (what appears to be) the sole aim of improving their chances of being accepted at a desirable school. Education researchers have consistently shown that students in high performing schools suffer psychological costs during middle and high school (Luthar and Becker [48], Galloway et al. [32]), which suggests that the competitive pressures cause significant welfare losses. Presumably parents, who may serve as the source of much of the pressure on these students, are also likely to incur costs both in terms of time spent engaging with a child’s school work as well as inferior relationships within the family.

We define the *cost of competition* as the difference in average payoffs between the first-best outcome generated by an omniscient social planner and the outcome generated by a color-blind college admissions contest. In the first-best outcome, the omniscient social planner assortatively assigns students to schools by their type before investment occurs. Match utilities are later realized as a function of one’s HC and the quality of one’s match

²Although this paper focuses on affirmative action in the context of U.S. college admissions, the US government has mandated AA practices in various areas of the economy where it has influence, including education, employment, and procurement. AA has been widely implemented outside the United States in places such as Malaysia, Northern Ireland, India, and South Africa. For an in-depth discussion of AA implementations around the world, see Sowell [61].

partner. In such a scenario, only Beckerian productive incentives exist, and each student chooses the optimal level of investment to equate costs with the direct marginal benefit of HC given their assigned school. To achieve assortativity when the students' types are private information, colleges use the students' HC choices to produce a separating equilibrium wherein high-ability students can be distinguished from their lower-ability counterparts. In the equilibrium of the color-blind model we study below, students are assigned to schools assortatively by observed HC output levels after the investment stage, giving rise to additional Spencerian competitive incentives. Students now choose a wastefully high level of HC (i.e., above the Becker optimum from a first-best assignment) in order to keep their lower-ability competitors at bay.

We use a calibrated version of our model drawn from Hickman [38] to characterize the welfare implications of the dual role of HC investment. We find that the cost of competition per student is equivalent to a drop of \$91,795 in the net present value of lifetime income. Since the competition by each student imposes negative externalities on higher ability students, we find that students of above average ability suffer the largest losses. Interestingly, students of the very highest ability suffer lower losses as these students are naturally inclined to accrue large human capital levels even without competitive pressures.

Next we compute the second-best contest design. We use our limit model to formalize an optimal control problem that computes the welfare maximizing college assignment contest, subject to incentive compatibility and feasibility constraints of the contest structure. In order to limit the cost of competition, the second-best contest divides the colleges into an interval of low quality schools and an interval of high quality schools with one interval assigned to each demographic group. By limiting the scope of possible prizes, the planner limits the power of the competitive channel of incentives. Since the minority students have higher HC production costs, on average—due to socioeconomic factors and a priori resource allocations correlated with race—the second-best contest assigns the minority students seats at the worst colleges and nonminority students seats at the best colleges. We do not view this as a serious policy recommendation as it cuts against what is allowed by either law or a basic sense of fairness. However, on an academic level it serves to highlight the tension between welfare and fairness in this context.

The remainder of this paper has the following structure: in section 2, we briefly discuss the relation between this work and the previous literature on college admissions and AA. In section 3, we give an overview of the full model of competitive human capital investment and describe the college assignment contest we study. In section 4 we introduce the limit model with a continuum of students and prove that equilibria of the limit model are approximate equilibria of the finite model of section 3. In section 5 we prove that

quota and admissions preference schemes admit the same set of equilibrium outcomes and discuss the practical ramifications of this result. In section 6 we turn to the welfare analysis of the cost of competition. Finally, in section 7 we consider an extension to a model where the HC choices are observed by the colleges with noise.

2 Previous Literature

Since our paper is largely theoretical in nature, this literature review focuses on the theory-oriented part of the affirmative action literature. Although many of these papers make similar points to ours, our conclusions would be impossible without incorporating (1) heterogeneous colleges, (2) heterogeneous student quality, and (3) endogenous human capital accumulation choices by the students. Most of the papers above include 1 or 2 of these components, but no prior paper includes all 3. There is also a rich empirical literature on the effects of affirmative action programs, and we encourage the interested reader to see the summary provided in Hickman [38].

Previous economic theory has studied AA and effort incentives, but existing models exhibit important limitations. Fain [22] and Fu [31] study models in the spirit of all-pay contests with complete information (i.e., academic ability types are ex ante observable) where two students compete for a college seat.³ Extrapolating the two-player insights to real-world settings is difficult because the framework implicitly assumes that all minority students, even the most gifted ones, are at a disadvantage to even the least talented non-minorities. Franke [24] extends the contest idea to include more than 2 agents, but at the cost of focusing on a specific form of affirmative action program.

The bilateral matching literature has also touched on incentives under AA. An early example being Coate and Loury [18], which studies both human capital investment and achievement gaps. This paper considers two strategic groups of agents, firms and ex ante identical job applicants. Job applicants make a binary choice to either become qualified at some cost or remain unqualified given a privately known cost of becoming qualified, and the firms then observe a noisy signal of the potential employee's choice and decides to assign the applicant to one of two positions. Coate and Loury [18] provides conditions under which discriminatory equilibria exist, analyzes the effects of affirmative action on equilibrium outcomes, and demonstrates that the equilibrium beliefs about the qualifications of minority applications can be worsened under an affirmative action program.

Chan and Eyster [13] focuses on the effect of affirmative action bans on a single school

³Schotter and Weigelt [59] study a similar setting in the laboratory. Their results suggest that no equity-achievement tradeoff exists.

when admission can be conditioned on student traits correlated with race. Epple et al. [21] analyzes a similar question, but considers a set of vertically differentiated colleges. Both papers describe how colleges bias their admissions policy to encourage diversity. These papers assume student quality is fixed and exogenous, and so necessarily cannot say much about the general equilibrium effect of the admission policy changes on student incentives. Fryer et al. [28] partially addresses student incentives by including a binary effort choice in the spirit of Coate and Loury [17], but Fryer et al. [28] simplifies the setting by assuming colleges are homogenous.

Chade, Lewis, and Smith [12] studies a matching model of college admissions with heterogeneous colleges. However, academic achievement is exogenous, and the analysis focuses on the role of information frictions within the market (e.g., noisy signals of student ability) and the strategic behavior of colleges in setting admissions standards. Our framework is a frictionless matching market, but academic achievement is endogenous. In that sense, our work and Chade, Lewis, and Smith [12] may be considered complementary for understanding the role of market forces in college admissions.

Fryer [26] touches on our analysis of the equivalence of quota and admissions preference schemes. Fryer [26] studies a model of workplace affirmative action wherein firms that wish to maximize profit are subject to a goal for minority hiring imposed by the government and enforced by an auditor. Fryer [26] finds that firms facing a pool of applicants with few minorities will act as if they are subject to a quota on the number of minorities they must hire, which Fryer [26] argues implies an equilibrium equivalence between quota and hiring preference systems. In our setting the students are the strategic actors, and our equivalence result is in some ways stronger - not only is the racial balance at colleges in the two systems the same, but the endogenous HC choices of the students is identical across mechanisms.

Finally, our methodology analyzes approximate equilibria played by a large number of agents. Due to the broad scope of this literature, we provide only a brief survey and a sample of the important papers related to the topic. Early papers focused on conditions under which game-theoretic models could be used as strategic microfoundations for general equilibrium models (e.g., Hildenbrand [39] and [40], Roberts and Postlewaite [55], Otani and Sicilian [53], and Jackson and Manelli [43]). Other early papers focused on conditions under which generic games played by a finite number of agent approach some limit game played by a continuum of agents (e.g., Green [34] and [35], Sabourian [58]). A more recent branch of this literature applies these ideas to simplify the analysis of large markets and mechanisms with an eye to real-world applications (e.g., Swinkels [63]; Cripps and Swinkels [19]; McLean and Postlewaite [50]; Budish [8]; Kojima and Pathak [47]; Fudenberg, Levine and Pesendorfer [29]; Weintraub, Benkard and Van Roy

[65]; Krishnamurthy, Johari, and Sundarajan [45]; and Azevedo and Leshno [3]).⁴

Of these papers on approximate equilibria and large games, we would like to single out the contemporaneously developed Olszewski and Siegel [52] as particularly relevant. Olszewski and Siegel [52] uses the limit of a sequence of games played by a finite number of agents to prove that in large contests the outcome is approximately deterministic and assortative for very general prize and agent type spaces. Even though Olszewski and Siegel [52] does not define or study a limit game (as we do), they can use their results to characterize the equilibria in quasi-linear settings. It is not clear how useful the results of Olszewski and Siegel [52] are for characterizing equilibria outside of the quasi-linear setting. Finally, the productive human capital setting we study as our benchmark and the noisily observed human capital extension are outside of their model.

Our work is also related to the literature on assortative marriage markets, Becker [4] providing an early example. To the best of our knowledge, all of these models assume transferable utility between agents paired in a match as well as complete information regarding the types of the agents. Since the complete information eliminates the welfare losses caused from competition in the college admissions contest, these models cannot address our questions.

3 The Finite Model

We model the market as a Bayesian game where the players are high school students characterized by an observable demographic classification—minority or non-minority—and a privately-known cost type that governs HC production. College seats are allocated to students according to a mechanism that is a function of the students’ HC choices and demographic classification. A student’s ex-post payoff is the utility derived from enrolling at a given college with her acquired human capital minus her investment cost.

In this section we lay out the model when the number of students and colleges are finite. Although we spend most of the paper working with a limit approximation of this model, the finite model and the limit approximation share many of the same underlying primitives described in this section.

⁴There also exists a literature on the relationship between large finite and nonatomic games that focuses on games of complete information and does not emphasize the use of nonatomic games as a framework for analyzing equilibria (examples include Housman [41], Khan and Sun [46], Carmona [9], and Carmona and Podczeck [10], Yang [66] and [11] amongst others). Kalai [44] studies large games show the equilibria of finite games are robust to modifications of the game form.

3.1 Agents, Actions, and Payoffs

The set of all students is denoted $\mathcal{K} = \{1, 2, \dots, K\}$, but there are two demographic subgroups, $\mathcal{M} = \{1, 2, \dots, K_{\mathcal{M}}\}$ (minorities) and $\mathcal{N} = \{1, 2, \dots, K_{\mathcal{N}}\}$ (non-minorities), where $K_{\mathcal{M}} + K_{\mathcal{N}} = K$ and the demographics of each student is observable.⁵ Each agent has a privately-known cost type $\theta \in [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_{++}$ that are drawn from follow group-specific distributions, $F_i(\theta)$, $i = \mathcal{M}, \mathcal{N}$. For convenience, we denote the unconditional type distribution for the overall population by $F_{\mathcal{K}}(\theta) = \frac{1}{K} [K_{\mathcal{M}}F_{\mathcal{M}} + K_{\mathcal{N}}F_{\mathcal{N}}]$. The reader should assume throughout that high values of θ are associated with students that have a high cost of accruing human capital.⁶

We denote the set of colleges $\mathcal{P}_{\mathcal{K}} = \{p_1, p_2, \dots, p_K\}$, where $p_k \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}_+$ denotes the quality level of the k^{th} college. We denote the k^{th} order statistic by $p(k : K)$ so that $\min_k\{\mathcal{P}_{\mathcal{K}}\} = p(1 : K)$ and $\max_k\{\mathcal{P}_{\mathcal{K}}\} = p(K : K)$. The colleges are passive, meaning each college accepts the students assigned to that college through the admissions contest.

Each agent's strategy space, $\mathcal{S} = [\underline{s}, \infty) \subset \mathbb{R}_{++}$, is the set of HC levels that can be chosen, and each students' choice of HC is observable to the colleges (e.g., through a standardized examination). Human capital \underline{s} is the minimum level required to participate in the market. In the current context, this would be minimum literacy and numeracy thresholds required to attend college. In skilled labor markets \underline{s} might represent a minimum level of education to participate in the market.

Agents value both college quality and HC. The gross match utility derived from being placed at college k for a student with type θ and HC s relative to not attending college is $U(p_k, s, \theta)$. In order to acquire HC $s \in \mathcal{S}$, an agent must incur a cost $C(s, \theta)$, which depends on both her unobservable type and her HC investment level. The total utility for a student of type θ that chooses human capital level s and is assigned to college k is

$$U(p_k, s, \theta) - C(s, \theta)$$

3.2 Allocation Mechanisms

We now describe the contest that allocates students to colleges. Letting s_i denote student i 's human capital level and \mathbf{s}_{-i} the vector of all other players' actions, we let $P_j(\cdot, \mathbf{s}_{-i}) : \mathcal{S} \rightarrow \mathcal{P}_{\mathcal{K}}$, $j = \mathcal{M}, \mathcal{N}$, be an assignment mapping that describes the college to which

⁵The assumption that there are two groups is not restrictive. Results for the two-group case extend straightforwardly to the case with $L > 2$ groups.

⁶Costs can arise in various ways: they could come from a labor-leisure tradeoff, where θ indexes one's preference for leisure; they could represent psychic costs, indexed by θ , involved in exerting effort to learn new concepts, where the more able students learn with the least effort; or costs could be interpreted as monetary investment required for study aides (e.g., computers, tutors, and private education), where θ represents the severity of the budget constraint that determines one's consumption-investment tradeoff.

student i is assigned given each possible HC choice. Note that we have deliberately allowed the assignment mapping to depend on student i 's demographic classification to reflect potential discrimination between members of the two groups. The ex post payoff for student i is then

$$\Pi_j^r(s_i, \mathbf{s}_{-i}; \theta) = U \left(P_j^r(s_i, \mathbf{s}_{-i}), s_i, \theta_i \right) - C(s_i, \theta_i), r \in \{cb, q, ap\} \text{ and } j = \mathcal{M}, \mathcal{N}$$

The *color-blind* admission rule involving no demographic discrimination is represented by the following assignment mapping:

$$P_{\mathcal{M}}^{cb}(s_i, \mathbf{s}_{-i}) = P_{\mathcal{N}}^{cb}(s_i, \mathbf{s}_{-i}) = P^{cb}(s_i, \mathbf{s}_{-i}) = \sum_{k=1}^K p(k : K) \mathbb{1}[s_i = s(k : K)]. \quad (1)$$

In the above expression, $\mathbb{1}$ is an indicator function equaling 1 if its argument is true and 0 otherwise. Note that P^{cb} yields a positive assortative matching.

We concentrate on two canonical forms of AA that have received attention due to wide implementation: quotas and admission preferences. A *quota* is the practice of reserving seats for each demographic group. Within the current modeling environment, this is equivalent to a set of $K_{\mathcal{M}}$ seats being reserved for minorities. Given the reserved seats, assignment occurs via a separate contest for each demographic group that yields an assortative assignment within each group.

Let $\mathcal{P}_{\mathcal{M}} = \{p_{\mathcal{M}1}, p_{\mathcal{M}2}, \dots, p_{\mathcal{M}K_{\mathcal{M}}}\}$ and $\mathcal{P}_{\mathcal{N}} = \{p_{\mathcal{N}1}, p_{\mathcal{N}2}, \dots, p_{\mathcal{N}K_{\mathcal{N}}}\}$ denote the sets of seats reserved for minorities and non-minorities, respectively, and let $\mathbf{s}_{\mathcal{M}} = \{s_{\mathcal{M}1}, s_{\mathcal{M}2}, \dots, s_{\mathcal{M}K_{\mathcal{M}}}\}$ and $\mathbf{s}_{\mathcal{N}} = \{s_{\mathcal{N}1}, s_{\mathcal{N}2}, \dots, s_{\mathcal{N}K_{\mathcal{N}}}\}$ denote the group-specific human capital profiles. Then a quota assignment mapping is represented by the functions

$$\begin{aligned} P_{\mathcal{M}}^q(s_i, \mathbf{s}_{-i}) &= \sum_{m=1}^{K_{\mathcal{M}}} p_{\mathcal{M}}(m : K_{\mathcal{M}}) \mathbb{1}[s_i = s_{\mathcal{M}}(m : K_{\mathcal{M}})], \text{ and} \\ P_{\mathcal{N}}^q(s_i, \mathbf{s}_{-i}) &= \sum_{n=1}^{K_{\mathcal{N}}} p_{\mathcal{N}}(n : K_{\mathcal{N}}) \mathbb{1}[s_i = s_{\mathcal{N}}(n : K_{\mathcal{N}})]. \end{aligned} \quad (2)$$

An *admission preference* system allows the students to compete against all of the other students, but the human capital choices of the members of each group are treated differently. In practice, many American admissions committees are thought to treat observed minority applicants' SAT scores (a commonly used measure of HC) as if they were actually higher when evaluating them against their non-minority competitors.⁷ More for-

⁷There has been a fair amount of empirical research estimating a substantial average admission preference for minorities at elite American colleges; e.g., Chung, Espenshade and Walling [16] and Chung and Espenshade [15]. Hickman [38] employs a similar empirical measure for the aggregate US college market

mally, an admission preference is a markup function $\tilde{S} : \mathcal{S} \rightarrow \mathbb{R}_+$ through which minority output is passed to produce a set of transformed HC levels, $\tilde{\mathbf{s}} = \{s_{N1}, \dots, s_{NK_N}, \tilde{S}(s_{M1}), \dots, \tilde{S}(s_{MK_M})\}$, and allocations are given by the following group-specific functions:

$$\begin{aligned} P_{\mathcal{M}}^{ap}(s_i, \mathbf{s}_{-i}) &= \sum_{k=1}^K p(k : K) \mathbb{1} [\tilde{S}(s_i) = \tilde{s}(k : K)], \quad \text{and} \\ P_{\mathcal{N}}^{ap}(s_i, \mathbf{s}_{-i}) &= \sum_{k=1}^K p(k : K) \mathbb{1} [s_i = \tilde{s}(k : K)]. \end{aligned} \tag{3}$$

Regardless of whether admissions are color-blind or follow some form of AA, ties between competitors are assumed to be broken randomly.

3.3 Model Assumptions

We now outline a series of assumptions on our model primitives. The assumptions serve three goals:

1. Establish the existence of a monotone equilibrium
2. Justify the use of a centralized, assortative matching structure
3. Insure the model is sufficiently well-behaved that our limit approximation is valid

Although we highlight how the assumptions tie into goals (1) and (2), we defer discussion of (3) until the next section.

Assumption 1. For each $(\theta, s) \in [\underline{\theta}, \bar{\theta}] \times \mathbb{R}_+$, $C_s(s, \theta) > 0$, $C_\theta(s, \theta) > 0$, and $C_{ss}(s, \theta) > 0$

Assumption 2. $U_p(p, s, \theta) > 0$, $U_s(p, s, \theta) \geq 0$, $U_\theta(p, s, \theta) \leq 0$, and $U_{ss}(p, s, \theta) \leq 0$

Assumption 1 states that costs are strictly increasing in human capital level s and cost type θ . Moreover, the cost function is convex in s for any given θ . Assumption 2 states that utility is differentiable, strictly increasing in college quality p , weakly increasing and concave in human capital level s and weakly decreasing in cost type θ . These assumptions imply that the individual decision problems have global maximums.

Assumption 3. $U(\underline{p}, \underline{s}, \bar{\theta}) - C(\underline{s}, \bar{\theta}) = 0$ and $\arg \max_s U(\underline{p}, s, \bar{\theta}) - C(s, \bar{\theta}) \leq \underline{s}$

Assumption 3 is a boundary condition for our model. First, the assumption requires that the lowest type of student is indifferent between participating in the college admissions contest and not attending college, which normalizes the utility of not attending

and finds evidence of a substantial admission preference even at lower-ranked colleges.

college to 0. This has the benefit of allowing us to interpret $U(p, s, \theta) - C(s, \theta)$ as the college premium of the students. Second, the assumption demands that the minimally qualified student who chooses to attend college does not have an interest in acquiring more HC, which is supported in the data we use to calibrate our model in section 6.

Assumption 4. *There exists \bar{s} such that $U(\bar{p}, \bar{s}, \underline{\theta}) - C(\bar{s}, \underline{\theta}) < U(\underline{p}, \underline{s}, \underline{\theta}) - C(\underline{s}, \underline{\theta})$*

Assumption 4 requires that there exist a human capital level so large that even the lowest cost type, the type that gets the largest value from human capital and has the lowest cost for acquiring human capital, would rather not invest in human capital at all and be assigned to the worst school. This implies that we can limit our analysis to human capital levels within $s \in [\underline{s}, \bar{s}]$. Unless otherwise stated, we simply let $\mathcal{S} = [\underline{s}, \bar{s}]$.

Assumption 5. $U_{ps}(p, s, \theta) \geq 0$ and $U_{p\theta}(p, s, \theta) \leq 0$

Assumption 5 requires complementarity between student HC, college quality, and student type. This assumption implies that positive assortative matching is efficient, which helps justify our use of the centralized rank-order mechanisms to model the market.

Assumption 6. $U_{s\theta}(p, s, \theta) \leq 0$ and $C_{s\theta}(s, \theta) > 0$

Assumption 6 states that marginal benefits of HC are decreasing and marginal costs of HC are increasing in a student's type. Assumptions 5 and 6 is key for existence of a monotone pure-strategy equilibrium.

Assumption 7. $F_{\mathcal{M}}(\theta)$ and $F_{\mathcal{N}}(\theta)$ have continuous and strictly positive densities $f_{\mathcal{M}}(\theta)$ and $f_{\mathcal{N}}(\theta)$, on a common support $[\underline{\theta}, \bar{\theta}]$.

Assumption 7 is a standard regularity condition on the type distributions.

We require the following assumption on the markup functions to prove our approximation result for the admissions preference system. The assumption bounds the marginal markup applied to minority student human capital choices, which helps insure that minority student utility functions are continuous in the limit game.

Assumption 8. $\tilde{S}(\underline{s}) = \underline{s}$ and there exists $\lambda_1, \lambda_2 \in (0, \infty)$ such that for all $s \in \mathcal{S}$ we have

$$\lambda_2 > \tilde{S}'(s) > \lambda_1$$

In our model the agents' types are private information, and the agents choose their human capital level given knowledge of the number of competitors from each group $K_{\mathcal{M}}$ and $K_{\mathcal{N}}$, the distribution of student types in the economy, the equilibrium strategies, the set of college seats $\mathcal{P}_{\mathcal{K}}$, and the admission rule P_j^r , $r \in \{cb, q, ap\}$ and $j \in \{\mathcal{M}, \mathcal{N}\}$. A

Bayes-Nash equilibrium of the game $\Gamma(K_M, K_N, F_M, F_N, \mathcal{P}_K)$ is a set of HC accumulation strategies $\sigma_i^r : [\underline{\theta}, \bar{\theta}] \rightarrow \mathbb{R}_+$, $i \in \{1, \dots, K\}$, that generates optimal choices of human capital given that all other agents follow their equilibrium strategies.

Theorem 1. *In the college admissions game $\Gamma(K_M, F_M, K_N, F_N, \mathcal{P}_K)$ with $r \in \{cb, q, ap\}$, under assumptions 1–8 there exists a monotone pure-strategy equilibrium. Moreover, any equilibrium of the game must be strictly monotone with almost every type using pure strategies.*

4 The Limit Game

From the law of large numbers, we know that as the market becomes large (i.e., as $K \rightarrow \infty$), the distribution of realized types and college qualities approaches the population distribution almost surely. Intuition suggests if we compute an equilibrium of a limit model featuring a continuum of students and schools with types distributed as per the population distributions, then the equilibrium of this limit model ought to provide a good description of the outcomes in games with a sufficiently large, but finite, set of players.

To see how this simplifies the student’s problem (and as a result, our analysis), consider the plight of a college applicant in the United States. Do would-be college students go to elaborate lengths to determine who else is applying, where those students are applying, and what the other students’ qualifications are? Of course not - to determine whether an application is likely to be accepted, the would-be college student can simply look at data on the grades and SAT scores of currently enrolled students. In our model, this is akin to knowing the equilibrium mapping between human capital levels and school assignments.

We have repeatedly suggested that the equilibria of the limit game are easy to compute, which can be interpreted in two ways. First (and we think most importantly), the equilibria of the limit game are computed in a way that reflects the decision process described in the paragraph above. In other words, a student can discover his or her optimal action in the limit game through reasoning that we believe is typical of real-life behavior, which we think supports the plausibility of the limit model. Second, even using modern algorithms, it is difficult to numerically compute the equilibrium of the finite model when K is large. Our limit model can be used as a powerful analytical tool by researchers to compute counterfactual contest outcomes. In either case, our challenge is to prove that an equilibrium of the limit game actually does approximate an equilibrium of the more realistic finite agent game when K is large.

Much of the underlying mechanics of the convergence of the finite games to the relevant limit model are the same across the color-blind, admissions preference, or quota games. For each type of college assignment game we consider a sequence of finite games

denoted $\{\Gamma(K_{\mathcal{M}}, F_{\mathcal{M}}, K_{\mathcal{N}}, F_{\mathcal{N}}, \mathcal{P}_{\mathcal{K}})\}_{K_{\mathcal{N}}+K_{\mathcal{M}}=2}^{\infty}$, and we use the same utility function in the finite games and the limit game. We assume that $\frac{K_{\mathcal{M}}}{K} \rightarrow \mu \in (0, 1)$ as $K \rightarrow \infty$, which implies that μ is the asymptotic mass of the minority group.⁸ We let $F_{\mathcal{P}}^K(p)$ denote the distribution of college qualities in the K agent game, and we assume that there is a continuous cumulative distribution function (CDF) $F_{\mathcal{P}}(p)$ such that

$$\lim_{K \rightarrow \infty} \sup_{p \in [\underline{p}, \bar{p}]} \left\| F_{\mathcal{P}}^K(p) - F_{\mathcal{P}}(p) \right\| = 0 \quad (4)$$

Alternately, we could assume that college types are drawn in an independent fashion from $F_{\mathcal{P}}$ in each of the finite games, in which case equation 4 holds almost surely.

In the limit game there is a measure μ continuum of minority students with types distributed exactly as $F_{\mathcal{M}}$ and a measure $1 - \mu$ continuum of non-minority students with types distributed exactly as $F_{\mathcal{N}}$. Finally, there is a measure 1 continuum of college seats distributed exactly as $F_{\mathcal{P}}$. In order to describe the limit equilibria using an ODE, we must make the following regularity assumption on $F_{\mathcal{P}}$.

Assumption 9. $F_{\mathcal{P}}(p)$ has a continuous density $f_{\mathcal{P}}(p)$.

In the following subsections, we address the color-blind, quota, and admission preference games separately. In each subsection we describe the assignment mapping in the limit model and use the mapping to describe the equilibrium strategy in each game. In section 4.4 we provide conditions under which there exists an essentially unique PSNE of the limit game. We use the notation $G_j^r(s), j \in \{\mathcal{M}, \mathcal{N}\}$ and $r \in \{cb, q, ap\}$, to denote the equilibrium distributions of HC choices. We use $\psi_j^{ap}, j \in \{\mathcal{M}, \mathcal{N}\}$, to represent the inverse of the equilibrium strategy.

4.1 Color-blind Admissions Game

We denote the endogenous distribution of human capital levels chosen by the students as $G_{\mathcal{K}}^{cb}(s), j = \mathcal{M}, \mathcal{N}$. Since limiting payoffs do not depend on race, it follows that

⁸Although we assume throughout that the convergence is deterministic, one can think of the agents in the finite games as being created by nature by assigning each agent to group \mathcal{M} with probability $\mu \in (0, 1)$, after which nature draws a cost type for the student from the corresponding distribution.

$\sigma_{\mathcal{M}}^{cb}(\theta) = \sigma_{\mathcal{N}}^{cb}(\theta) = \sigma^{cb}(\theta)$.⁹ We can describe the match of students to colleges as:

$$\begin{aligned} P_{\mathcal{M}}^{cb}(s) &= P_{\mathcal{N}}^{cb}(s) = P^{cb}(s) = F_{\mathcal{P}}^{-1} \left(G_{\mathcal{K}}^{cb}(s) \right) \\ &= F_{\mathcal{P}}^{-1} \left(\mu G_{\mathcal{M}}^{cb}(s) + (1 - \mu) G_{\mathcal{N}}^{cb}(s) \right) \\ &= F_{\mathcal{P}}^{-1} \left(1 - \mu F_{\mathcal{M}} \left(\psi^{cb}(s) \right) - (1 - \mu) F_{\mathcal{N}} \left(\psi^{cb}(s) \right) \right). \end{aligned} \quad (5)$$

The intuition is simple: quantiles of the population HC distribution $G_{\mathcal{K}}^{cb}(s)$ are mapped into the corresponding quantiles of the distribution $F_{\mathcal{P}}$.

We now turn to describing the equilibrium of the color-blind limit game. Given that the equilibrium will, in the end, be assortative, we can write the endogenous assignment function as a function of agent type as follows

$$P_{\mathcal{M}}^{cb}(\theta) = P_{\mathcal{N}}^{cb}(\theta) = F_{\mathcal{P}}^{-1} \left(1 - \mu F_{\mathcal{M}}(\theta) - (1 - \mu) F_{\mathcal{N}}(\theta) \right)$$

If we describe the agent's problem in revelation mechanism form, then the decision problem reduces to choosing a type $\hat{\theta}$ to declare

$$\max_{\hat{\theta}} U \left(P^{cb}(\hat{\theta}), \sigma^{cb}(\hat{\theta}), \theta \right) - C \left(\sigma^{cb}(\hat{\theta}), \theta \right)$$

Manipulating the first order condition for this problem¹⁰ and using the fact that in equilibrium we have $\hat{\theta} = \theta$ yields the following differential equation for investment:

$$\begin{aligned} \frac{d\sigma^{cb}(\theta)}{d\theta} &= - \frac{U_p \left(P^{cb}[\sigma^{cb}(\theta)], \sigma^{cb}(\theta), \theta \right) \cdot f_{\mathcal{K}}(\theta)}{f_{\mathcal{P}} \left(F_{\mathcal{P}}^{-1} \left(1 - F_{\mathcal{K}}(\theta) \right) \right) \cdot \left(C_s \left(\sigma^{cb}(\theta), \theta \right) - U_s \left(P^{cb} \left(\sigma^{cb}(\theta) \right), \sigma^{cb}(\theta), \theta \right) \right)} \\ \sigma^{cb}(\bar{\theta}) &= \underline{s} \quad (\text{boundary condition}). \end{aligned} \quad (6)$$

The boundary condition comes from the fact that a player of type $\bar{\theta}$ will always be matched with the lowest seat in a monotone equilibrium, so she cannot do better than to simply choose HC level \underline{s} . Given the assumptions on the model primitives, the solution to Equation 6 is strictly decreasing.

⁹Theorists with experience in asymmetric auctions may find this statement puzzling, but one must keep in mind that it merely applies to *limiting* payoffs. In a two-player finite game, differing investment behavior arises from the fact that a minority competitor faces a profile of opponents of each type numbering $(K_{\mathcal{N}}, K_{\mathcal{M}} - 1)$, whereas a non-minority competitor faces profile $(K_{\mathcal{N}} - 1, K_{\mathcal{M}})$, and since costs are asymmetrically distributed across groups, a minority and a non-minority with the same private cost will have differing expectations of their standing in the distribution of realized competition. However, the difference between their expected ranks quickly vanishes as the number of players gets large.

¹⁰Importantly, this implies that the local incentive compatibility conditions are already "built in" to the differential equations. What remains is to show global incentive compatibility.

4.2 Quota Game

The empirical CDF of seats allocated to group $j \in \{\mathcal{M}, \mathcal{N}\}$ in the K -agent game is denoted Q_j^K . We assume that there exists a distribution of seats $Q_j(p)$ such that

$$\lim_{K \rightarrow \infty} \sup_{p \in [\underline{p}, \bar{p}]} \left\| Q_j^K(p) - Q_j(p) \right\| = 0$$

In the limit quota game a measure μ of minority students with types exactly distributed as $F_{\mathcal{M}}$ compete in a contest for a pool of seats exactly distributed as $Q_{\mathcal{M}}$. Similarly, a measure $1 - \mu$ of non-minority students with types exactly distributed as $F_{\mathcal{N}}$ compete in a separate contest for a pool of seats exactly distributed as $Q_{\mathcal{N}}$. The measures $Q_{\mathcal{M}}$ and $Q_{\mathcal{N}}$ are subject to the following feasibility constraint

$$\text{For all } p \text{ we have } \mu Q_{\mathcal{M}}(p) + (1 - \mu) Q_{\mathcal{N}}(p) = F_p(p)$$

Finally, we assume that the quota measures admit a density.

Assumption 10. $Q_j(p)$, $j \in \{\mathcal{M}, \mathcal{N}\}$, admits a density $q_j(p)$ on $[\underline{p}, \bar{p}]$.

Each demographic group's contest results is an assortative match of types to the college seats allocated to that contest. These distinct contests yield group-specific assignment mappings of the form

$$P_j^q(s) = Q_j^{-1} \left(G_j^q(s) \right) = Q_j^{-1} \left(1 - F_j \left(\psi_j^q(s) \right) \right), \quad j \in \{\mathcal{M}, \mathcal{N}\}. \quad (7)$$

for values of s such that $Q_j^{-1} \left(G_j^q(s) \right)$ is well defined.¹¹ In cases where $Q_j^{-1} \left(G_j^q(s) \right)$ is not well defined, we let $P_j^q(s) = \sup \{ P_j^q(s') : s' < s \}$. As in the color-blind case, the quantiles of the group-specific human capital distributions are mapped into the corresponding quantiles of Q_j . The following differential equation describes the equilibrium when Q_j has a connected support.

$$\frac{d\sigma_j^q(\theta)}{d\theta} = - \frac{U_p \left(P_j^q \left(\sigma_j^q(\theta) \right), \sigma_j^q(\theta), \theta \right) \cdot f_j(\theta)}{q_j \left(Q_j^{-1} \left(1 - F_j(\theta) \right) \right) \cdot \left(C_s \left(\sigma_j^q(\theta), \theta \right) - U_s \left(P_j^q \left(\sigma_j^q(\theta) \right), \sigma_j^q(\theta), \theta \right) \right)}, \quad (8)$$

$$\sigma_j^q(\bar{\theta}) = \underline{s} \quad (\text{boundary condition}).$$

When Q_j does not have a connected support, then there will be jumps in the equilibrium strategies when $P_j^q \left(\sigma_j^q(\theta) \right)$ encounters the left edge of a gap in the support. The in-

¹¹If we allow Q_j to have a disconnected support, then Q_j need not be strictly monotone and the inverse will not exist.

interested reader is referred to online appendix B for a description of how to identify the location and size of the jumps. Between these jumps, equation 8 describes the equilibrium strategy.

4.3 Admissions Preference Game

As in the finite-agent model, an admission preference rule is modeled as a markup function $\tilde{S} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and seats are matched assortatively with transformed HC. In other words, minorities are repositioned ahead of non-minority counterparts with investment of $\tilde{S}(s)$ or less. The limiting assignment mapping for group \mathcal{M} is

$$\begin{aligned} P_{\mathcal{M}}^{ap}(s) &= F_{\mathcal{P}}^{-1} \left((1 - \mu)G_{\mathcal{N}} \left(\tilde{S}(s) \right) + \mu G_{\mathcal{M}}(s) \right) \\ &= F_{\mathcal{P}}^{-1} \left(1 - \left((1 - \mu)F_{\mathcal{N}} \left(\psi_{\mathcal{N}}^{ap} \left(\tilde{S}(s) \right) \right) + \mu F_{\mathcal{M}} \left(\psi_{\mathcal{M}}^{ap}(s) \right) \right) \right) \end{aligned} \quad (9)$$

and limiting allocations for group \mathcal{N} are given by

$$\begin{aligned} P_{\mathcal{N}}^{ap}(s) &= F_{\mathcal{P}}^{-1} \left((1 - \mu)G_{\mathcal{N}}(s) + \mu G_{\mathcal{M}} \left(\tilde{S}^{-1}(s) \right) \right) \\ &= F_{\mathcal{P}}^{-1} \left(1 - \left((1 - \mu)F_{\mathcal{N}} \left(\psi_{\mathcal{N}}^{ap}(s) \right) + \mu F_{\mathcal{M}} \left(\psi_{\mathcal{M}}^{ap} \left(\tilde{S}^{-1}(s) \right) \right) \right) \right) \end{aligned} \quad (10)$$

The intuition for the above expressions is similar as before: limiting mechanisms map the quantiles of a distribution into the corresponding college seat quantiles. For non-minorities, it is a mixture of the distributions of non-minority HC and subsidized minority HC. For minorities, it is a mixture of the distributions of minority HC and de-subsidized non-minority HC.

Unlike the color-blind or quota case, it is not possible to describe which kind of student obtains each seat in an admissions preference scheme without first solving for the equilibrium. This feature of the admissions preference mechanism makes it significantly harder to work with than the other two schemes. However, we prove in section 5 that the admissions preference and quota schemes are outcome equivalent - for any admissions preference markup function one can describe a quota scheme that generates the same school assignment and equilibrium human capital choices. Because of this equivalence, for the majority of the paper we work with color-blind and quota systems. However, for completeness, we now provide the differential equations describing the admissions

preference equilibrium when \tilde{S} is differentiable and $\tilde{S}(\underline{s}) = \underline{s}$.¹²

$$\begin{aligned} (\psi_{\mathcal{M}}^{ap})'(s) = & - \frac{C_s(s, \psi_{\mathcal{M}}^{ap}(s)) - U_s(P_{\mathcal{M}}^{ap}(s), s, \psi_{\mathcal{M}}^{ap}(s))}{U_p(P_{\mathcal{M}}^{ap}(s), s, \psi_{\mathcal{M}}^{ap}(s))} \cdot \frac{f_P(P_{\mathcal{M}}^{ap}(s))}{\mu f_{\mathcal{M}}(\psi_{\mathcal{M}}^{ap}(s))} \\ & - \frac{(1-\mu)f_{\mathcal{N}}(\psi_{\mathcal{N}}^{ap}(\tilde{S}(s)))}{\mu f_{\mathcal{M}}(\psi_{\mathcal{M}}^{ap}(s))} (\psi_{\mathcal{N}}^{ap})'(\tilde{S}(s)) \frac{d\tilde{S}(s)}{ds} \end{aligned} \tag{11}$$

and

$$\begin{aligned} (\psi_{\mathcal{N}}^{ap})'(s) = & - \frac{C_s(s, \psi_{\mathcal{N}}^{ap}(s)) - U_s(P_{\mathcal{N}}^{ap}(s), s, \psi_{\mathcal{N}}^{ap}(s))}{U_p(P_{\mathcal{N}}^{ap}(s), s, \psi_{\mathcal{N}}^{ap}(s))} \cdot \frac{f_P(P_{\mathcal{N}}^{ap}(s))}{(1-\mu)f_{\mathcal{N}}(\psi_{\mathcal{N}}^{ap}(s))} \\ & - \frac{\mu f_{\mathcal{M}}(\psi_{\mathcal{M}}^{ap}(\tilde{S}^{-1}(s)))}{(1-\mu)f_{\mathcal{N}}(\psi_{\mathcal{N}}^{ap}(s))} (\psi_{\mathcal{M}}^{ap})'(\tilde{S}^{-1}(s)) \frac{d\tilde{S}^{-1}(s)}{ds} \end{aligned}$$

4.4 Equilibrium Existence

Since the limit game is one of complete information, the relevant equilibrium concept is a Nash equilibrium. In a setting with a continuum of agents, the equilibrium is defined below. Our notation implies that all agents within each group play the same strategy, which theorem 4.4 implies must be the case in equilibrium.

Definition 1. $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$ is a **group-symmetric Nash equilibrium** of the limit game for system $r \in \{cb, q, ap\}$ if for $j \in \{\mathcal{M}, \mathcal{N}\}$ and all $\theta \in [\underline{\theta}, \bar{\theta}]$

$$\sigma_j(\theta) \in \arg \max_s U(P_j^r(s), s, \theta) - C(s, \theta)$$

given the endogenous $P_j^r(s)$ generated by $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$. The equilibrium is **essentially unique** if in any equilibrium all but a measure 0 set of types take actions as per $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$.

First note that in any equilibrium, the assignment mapping is strictly monotone increasing and admits a countable number of discontinuities. For group-symmetry to be violated, we would require a positive measure of agent-types to have multiple best responses to the assignment mapping. We use the theory of strict monotone comparative statics developed by Edlin and Shannon [20] to show that multiple best responses can only occur when the agent's HC choice places him at a discontinuity in the assignment mapping. Since the assignment mapping has a countable set of discontinuities, this can occur only for a measure 0 set of types (implying group-symmetry).

To prove essential uniqueness we argue that any group-symmetric equilibria must respect the boundary condition $\sigma_j(\bar{\theta})$ and the first order conditions captured by the dif-

¹² If \tilde{S} is not differentiable or $\tilde{S}(\underline{s}) \neq \underline{s}$, then the equilibrium may involve jumps. The interested reader is referred to appendix B for a description of how to identify the size and location of these jumps.

ferential equations describing the equilibrium. The difficulties we encounter are proving that any discontinuities in the equilibrium strategies must be uniquely defined. In the quota or color-blind system, it is straightforward to define the initial condition and prove that both the initial condition and any jumps in the equilibrium strategy are uniquely defined. This means the necessarily pure actions of the measure 1 of types that are not at discontinuities in σ_i^g are uniquely defined, which implies group-symmetry and essential uniqueness.

We require stronger conditions on admissions preference systems. It is straightforward to show the existence of an equilibrium, and any equilibrium of the admissions preference game is group symmetric and almost all agents use pure actions. However, we have not discovered a method for proving that the discontinuities or the boundary condition are uniquely defined without Assumption 8.

Theorem 2. *The equilibrium of any of our limit game affirmative action systems is essentially unique, almost all of the agents take pure actions, and the strategies are group-symmetric.*

4.5 Approximating the Finite Game with Limit Equilibria

Having defined the equilibria of the limit model using ODEs and proven that the equilibrium thus described exists and is essentially unique, we now argue that the equilibrium of the limit game is a useful approximation of the equilibria in games with a large, but finite, set of players. We use two notions of approximate equilibrium in this paper. Our first definition provides for an approximation in terms of incentives - agents that follow an ε -approximate equilibrium can gain at most ε by deviating. Intuitively, students lose little utility if they base their actions on the easy-to-solve limit game equilibrium strategy.

Definition 2. *Given $\varepsilon > 0$, an ε -approximate equilibrium of the K -agent game is a K -tuple of strategies $\sigma^\varepsilon = (\sigma_1^\varepsilon, \dots, \sigma_K^\varepsilon)$ such that for all agents, almost all types θ , and all human capital choices s' we have*

$$U\left(P_j^r(\sigma_i^\varepsilon(\theta), \sigma_i^\varepsilon(\theta)), \sigma_i^\varepsilon(\theta), \theta_i\right) - C(\sigma_i^\varepsilon(\theta), \theta_i) + \varepsilon \geq U\left(P_j^r(s', \sigma_i^\varepsilon(\theta)), s', \theta_i\right) - C(s', \theta_i)$$

A δ -approximate equilibrium provides a close approximation of the actual HC choices of each of the agents regardless of the exact equilibrium played.

Definition 3. *Given $\delta > 0$, a δ -approximate equilibrium of the K -agent game is a K -tuple of strategies $\sigma^\delta = (\sigma_1^\delta, \dots, \sigma_K^\delta)$ such that for any exact equilibrium of the K -agent game $\sigma = (\sigma_1, \dots, \sigma_K)$ we have for all agents and almost all types θ we have $\|\sigma_i^\delta(\theta) - \sigma_i(\theta)\| < \delta$.*

Our goal is to prove that the equilibrium of the limit game is a δ -approximate equilibria of admissions games with sufficiently many students. Proving this result amounts to

proving that the limit game is continuous in the appropriate sense.

Theorem 3. *Consider any admissions preference game or a quota game where $Q_{\mathcal{M}}$ and $Q_{\mathcal{N}}$ admit strictly positive PDFs over a connected support. Let σ_j^r , $j \in \{\mathcal{M}, \mathcal{N}\}$ and $r \in \{cb, q, ap\}$, denote an equilibrium of the limit game. Under assumptions 1 -9 and given $\varepsilon, \delta > 0$, there exists $K^* \in \mathbb{N}$ such that for any $K \geq K^*$ we have that σ_j^r is a ε -approximate equilibrium and a δ -approximate equilibrium of the K -agent game.*

When many students follow the equilibrium strategy of the limit game, then the realized distribution of human capital will (with high probability) be approximately the same as the distributions realized in the limit game. If the student utility functions are continuous, then these small differences have a negligible effect on the agent utility for each possible action (and so the maximum utility changes only slightly). This implies that $(\sigma_{\mathcal{M}}^r, \sigma_{\mathcal{N}}^r)$ is a ε -approximate equilibrium of the K -agent game.

Theorem 3 also implies that every equilibrium of the finite game must be close to the essentially unique equilibrium of the limit game. In order to prove our theorem, we need to rule out discontinuities in the utility functions of the agents with respect to s and the distribution of s across the population. Such a discontinuity can arise either through the contest structure or the endogenous equilibrium strategy. Once the model is shown to be sufficiently continuous, we can quickly show that the equilibrium correspondence is upper hemicontinuous in K . In other words, an exact equilibrium of the admissions game with many players must be close to some equilibrium of the limit game. However, it could be the case that there are equilibria of the limit game that are unlike any equilibrium of the finite game (i.e., lower hemicontinuity of the equilibrium correspondence might fail). To rule this out, we use the fact that the limit game has an essentially unique equilibrium to prove that the equilibrium correspondence is in fact continuous and $(\sigma_{\mathcal{M}}^r, \sigma_{\mathcal{N}}^r)$ is a δ -approximate equilibrium.

5 Mechanism Equivalence

In this section we make our argument that the quota and admissions preferences systems are outcome equivalent. In any equilibrium of any affirmative action scheme, the agents in the limit game respond optimally to the assignment mapping, $P_j(s)$, $j \in \mathcal{M}, \mathcal{N}$, that describes how HC choices lead to college assignments. We prove our equivalence result by showing that if an assignment mapping $P_j(s)$ is generated by some equilibrium of an admission preference (quota) system, then there exists an equilibrium of some quota (admissions preference) system that yields the same $P_j(s)$. Since the $P_j(s)$ are the same under each system, the optimal agent responses must also be the same. Note that the

notion of equivalence we use implies that not only are the same measures of minority and non-minority students assigned to each school, but the students at each school choose the same level of human capital under both systems.¹³

Theorem 4. $P_j(s) : \mathcal{S} \rightarrow \mathcal{P}, j \in \{\mathcal{M}, \mathcal{N}\}$ is the result of an equilibrium of some quota system if and only if there is an equilibrium of an admissions preference system that also yields these assignment functions and admits the same equilibrium strategies.

Theorem 4 is useful from a methodological perspective because it shows that there is no loss in generality from focusing solely on outcomes that can be realized using quota systems. Recall that under a quota system the equilibrium assignment of student types to colleges within each group is assortative, which means the only unknown endogenous quantity is the HC accumulation strategy. Given an admission preferences system, both the equilibrium human capital choices and the school assignment need to be computed, which makes the admissions preference schemes more difficult to study.

Our result bears a resemblance to the equivalence of quotas and tariffs in an international trade context (e.g., Bhagwati [5]), although our setting is complicated by the continuum of heterogeneous “goods” (i.e., college seats) being assigned and the continuum of endogenous “prices” (i.e., human capital levels) required to obtain the goods. Even without the insights from the international trade literature, we do not view it as surprising that the same assignment of student types to colleges can be generated using either type of mechanism. We find it more surprising that the endogenous human capital decisions can also be replicated, and (to the best of our knowledge) there is no analog of this result in the international economics literature. Of course some insights, such as the breakdown of the equivalence in the presence of aggregate uncertainty, are true both of the quota-tariff equivalence and our college admissions model.

As just hinted, the equivalence of admissions preference and quota schemes relies on the distribution of student types being fixed and known. If there are aggregate shocks to the distribution of student types, the equivalence will no longer hold unless the quota and admissions preference schemes are allowed to be functions of the realized distribution of applicant types. For example, under a quota scheme that is not responsive to the distribution of college students, the set of college seats reserved for minority student remains fixed even in the event that the minority student population is significantly better (or worse) than expected relative to the set of non-minority students. In contrast, an admissions preference scheme allows an unusually high quality minority applicant cohort to be allocated an unusually large number of seats at excellent schools. Large aggregate

¹³Note that we have not proven that any choice of $P_j(s)$ can be implemented by either a quota or an admissions preference scheme. For example, if $P_j(s)$ is strictly decreasing, then it cannot be implemented by any incentive compatible mechanism.

shocks in the distribution of undergraduate applicants seems unlikely, but it is easy to imagine aggregate shocks such as economic cycles that would affect the pool of applicants to MBA programs.

From a legal perspective, Theorem 4 throws light on why it has proven so difficult for the U.S. court system to draw a line between constitutionally permissible and impermissible affirmative action systems. The cornerstone of Supreme Court jurisprudence regarding affirmative action is the 1978 case *University of California Regents v. Bakke* [56]. Justice Powell's opinion established that the government has a compelling interest in encouraging diversity in university admissions founded on principles of academic freedom and a university's right to take what actions it feels necessary to provide a high quality education to its students. Given this compelling interest, universities are free to implement affirmative action programs, although these programs must be narrowly tailored and are subject to a rigorous "strict scrutiny" standard of review. In particular, Justice Powell's opinion clearly ruled that quotas violate the U.S. Constitution's 14th amendment's guarantee of equal protection because non-minority students cannot compete for the seats reserved for minority applicants.

The 2003 cases *Gratz et al. v. Bollinger et al.* and *Grutter v. Bollinger et al.* were the first affirmative action cases addressed by the Supreme Court following the ruling in *University of California Regents v. Bakke*. These cases turned on whether the University of Michigan admissions preference schemes are narrowly tailored. The justices in both of these cases looked to the outcomes to judge the extent to which the systems function as de facto quotas (and are hence unconstitutional).

Gratz et al. v. Bollinger et al. addressed whether the admissions preference scheme used by the University of Michigan College of Literature, Science, and the Arts (LSA) met the narrow-tailoring criteria. The admissions preference scheme used by the LSA attributed points to applicants based on (for example) academic performance, athletic ability, Michigan residency, and race. The court ruled the LSA admissions preference scheme unconstitutional for two reasons. First, the across-the-board attribution of points based solely on minority status was not individualized enough to qualify as narrowly tailored.¹⁴ Second, "...virtually all [minority freshman applicants] who are minimally qualified are admitted..." (*Gratz v. Bollinger et al.* [33], 539 U.S. 278), which means that these students are de facto not competing with non-minority applicants for admission. To summarize, the concurring justices argue in their opinion that although LSA did not formally use a quota, the results were functionally the same.

¹⁴In her dissent, Justice Ginsburg argued that LSA was simply articulating its policy clearly and unambiguously.

"If honesty is the best policy, surely Michigan's accurately described, fully disclosed College affirmative action program is preferable to achieving similar numbers through winks, nods, and disguises." (p. 305)

Grutter v. Bollinger et al. revolved around the admissions process of the University of Michigan Law School, which the Supreme Court ruled constitutional. The key difference between the LSA’s policy and the Law School’s is that each applicant to the law school is given individualized review without points attributed to particular traits of the applicant. However, if one reads the dissenting opinions, Justices Scalia and Rehnquist made separate arguments that the Law School admissions process was functionally equivalent to a quota:

“... the University of Michigan Law School’s mystical “critical mass” justification for its discrimination by race challenges even the most gullible mind. The admissions statistics show it to be a sham to cover a scheme of racially proportionate admissions.” (Scalia, p. 346 - 347 of *Grutter*)

“... the ostensibly flexible nature of the Law School’s admissions program that the Court finds appealing... appears to be, in practice, a carefully managed program designed to ensure proportionate representation of applicants from selected minority groups.” (Justice Rehnquist in *Grutter v. Bollinger* [36], 539 U.S. 385)

Theorem 4 implies that attempts to differentiate between unconstitutional quotas and constitutional admissions preferences on the grounds of the outcomes produced will prove futile. The delineation between admission preference and quota schemes must turn on some basis other than the implemented outcomes, such as the legality of the procedure underlying the outcome. As neither of the authors have legal training, it is beyond the scope of this project to say whether such a non-consequentialist legal basis for differentiating between admissions preference and quota systems exists.

6 The Welfare Cost of Competition

We now use our model to address the welfare cost of the competition for college seats. First, we would like to illustrate the source of the welfare loss due to competition. The limiting payoff for agent type θ with HC s is $\Pi^{cb}(s; \theta) = U(P^{cb}(s), s, \theta) - C(s; \theta)$. Differentiating, we get the following first-order condition:

$$\underbrace{\frac{\partial U}{\partial s}}_{\text{Productive Channel}} + \underbrace{\frac{\partial U}{\partial p} \cdot \frac{dP^{cb}(s)}{ds}}_{\text{Competitive Channel}} = \frac{dC}{ds}$$

The above expression states that the marginal cost of human capital investment (the right-hand side) must be exactly offset by the marginal benefits (the left-hand side), which can

be decomposed into two parts. First, there is the direct value accrued to the student of marginal increases to HC level s , represented by the term $\partial U/\partial s$; this is the *productive channel* of investment incentives. There is also the indirect benefit of marginally improving the college to which the student is assigned. An increase in one’s human capital level will improve the college quality at a rate of $dP^{cb}(s)/ds$, and the marginal utility of a change in college quality is $\partial U/\partial p$. The product of these two terms represents the *competitive channel* of investment incentives.

For the moment, suppose types were observable to a benevolent planner that could match students to colleges assortatively and allow them to invest ex post, in which case the first order condition (FOC) would be $U_s(P^{r*}(\theta), s, \theta) = C_s(s, \theta)$ where $P^{r*}(\theta) = F_p^{-1}(1 - F_K(\theta))$. We refer to the outcome generated by the social planner’s omniscient assignment and the ex post investments of the students as the *First-Best Outcome*. For concreteness, the following theorem compares the color-blind policy and the first-best outcome, but similar results could be generated for any of our admission schemes. The basic intuition for the result is clear: by shutting down the competitive channel, the first-best outcome reduces the incentive for students to “wastefully” acquire human capital.

Theorem 5. *HC investment for all types in the color-blind admissions scheme exceeds that in the first-best outcome.*

When considering what the second-best solution might look like in a world with incomplete information, one must balance two forces when evaluating the welfare effects of competition. First, color-blind competition allows the students to reveal their types in an incentive compatible fashion, which is necessary for an efficient, assortative match in an incomplete information setting. Second, the accumulation of human capital merely for the purpose of competing for a better college imposes negative externalities on the other students. If a high cost student increases her HC level to obtain a better college, then every student of lower cost must increase their HC choice as well if they wish to retain their college assignment.

It is not obvious how these forces balance in an incomplete information world. Should the students compete in a color-blind system that insures an assortative match? Or should society seek to minimize the negative externalities by dampening the competitive channel even though we may end up in a nonassortative match? We answer these questions using a model calibrated from real world data.

The remaining results we present in this section are numerical solutions of our model conducted using the estimated utility functions and type distributions from Hickman [38] to calibrate our model. The analysis of Hickman [38] is conducted using data from 1995-1996 application year. Hickman [38] uses the US News and World Report college quality

index as his metric for college quality, p , and the supply of all seats at quality level p (*i.e.*, within each school) is given by enrollment data from the Integrated Postsecondary Education Data Survey, conducted by the US Department of Education. A student's human capital level, s , is represented by the student's SAT score. The estimation of U uses a combination of the Baccalaureate and Beyond survey conducted by the U.S. Department of Education and the Current Population Survey executed by the U.S. Census Bureau. The Baccalaureate and Beyond survey collects data on entering college students and follows these students for 10 years post-graduation, which allows us to estimate the relationship between human capital, college placement, and salary 10 years post-graduation. We use the current population survey to estimate the salary growth rate path for the typical college graduate over the course of her lifetime. We combine these data sources to estimate the lifetime salary path of students as a function of their human capital when entering college and their college placement. Finally, we use these estimates to compute the net present value of lifetime salary as a function of human capital accrued before college and the college assignment. For each level of HC, Hickman [38] uses standard empirical auctions techniques to reverse engineer the value of θ that rationalizes that choice of HC. The results below pertain to a yearly discount factor of $\delta = 0.98$. All of our estimates were restricted to the population of male individuals in the respective surveys.

For the utility functions we use

$$\begin{aligned} U(p, s, \theta) &= \rho(p, s) * u(p) \\ \rho(p, s) &= -0.176 + 0.000774s - 0.00000049s^2 + 0.00076ps \\ u(p) &= 40134p^{0.536} \end{aligned}$$

where ρ represents the probability of graduating from college and u is the net present value of the lifetime college salary premium conditional on graduating. The cost function for the model is

$$C(s, \theta) = \theta e^{0.013(s-\underline{s})}, \underline{s} = 520$$

where a student's θ is inferred from his or her behavior under the status quo college admissions contest.¹⁵

The costs of competition can be interpreted in terms of net present value (NPV) of

¹⁵All numbers used in this calibrated comparative static exercise were taken from [38], whose identification and estimation strategy applies standard techniques from the auctions literature pioneered by Guerre, Perrigne, and Vuong [37]. Intuitively, the data contain two main components needed for identification. First, the supply and relative value of each seat on the college market characterize the distribution of gross payoffs on the table before the strategic investment game begins. Second, the distributions of SAT scores characterize the equilibrium level of competition for a given gross payoff. Therefore, one can in principle reverse engineer the private type θ which rationalizes each observed SAT score s (within each race group) as a best response to prevailing market conditions, including level of competition and stakes.

lifetime salary. When we consider the total effect, we use the statistic that 1,056,580 students newly enrolled in 4 year colleges in the fall of 1996 drawn from the Integrated Postsecondary Education System Dataset.

Although it is outside of the scope of this paper to provide a model that endogenizes school quality, it is well known that there are spillovers between students in many contexts that make school quality a function of which students enroll. The spillovers may be a function of student characteristics (e.g., Hoxby [42]) or student effort choices (e.g., Fruewirth [25]). In addition, if we take at face value the briefs filed by the universities in *Regent of the University of California v. Bakke* [56], *Gratz v. Bollinger* [33], and *Grutter v. Bollinger* [36], universities believe that student welfare is directly enhanced by diversity amongst the student body.

Since the first-best outcome, the color-blind contest, and the status quo contests all result in roughly assortative matches,¹⁶ it is plausible that the distribution of college qualities is the same in all three cases even when spillovers are considered. However, the second best contest results in a significantly nonassortative outcome, and we view these estimates as more speculative. Our goal when computing the second-best is to illustrate the tensions between efficiency and fairness, and we do not view the welfare value we compute as particularly informative.

6.1 The Cost of Competition

To estimate the welfare cost of competition we compare the first-best outcome with the color-blind contest. Note that both of these assignments are perfectly assortative. In the first-best assignment, the students' types are known by an omniscient social planner, and the students are assigned to colleges prior to choosing their HC level. In the color-blind match, the students' HC level is the tool used to rank and assign the students. The only difference in the outcome is that the students in the color-blind contest are pushed by the competitive channel to accrue more human capital. The *cost of competition* is the average welfare in the color-blind outcome minus the average welfare in the first-best outcome.

The following table presents the average welfare over the population in terms of the

¹⁶The affirmative action scheme used in the status quo economy result in some slight nonassortativity since the type of minority students at any given school is higher than the type of the nonminority student at that school. Given that the affirmative action effects are not too large and the minority students make up a small fraction of the population, the aggregate result is approximately assortative.

net present value of lifetime salary given a yearly discount factor of $\delta = 0.98$.

	NPV of Future Salary
College Premium Without Effort Costs	\$738,340
First-Best	\$702,341
Color-blind	\$610,546
Cost of Competition	-\$91,795

For a more granular perspective we have plotted the welfare cost of competition by the percentile of school quality to which those students are assigned. Figure 1 reveals where the burden of the cost of competition falls.

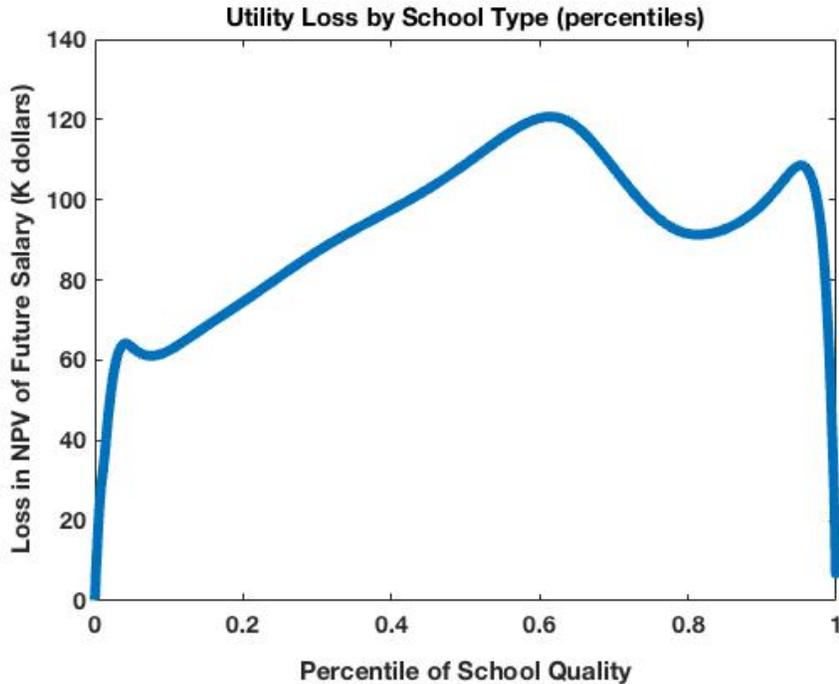


Figure 1:

There are two features of note to point out. The initial increase in welfare losses is because students impose negative externalities only on students at better schools. Intuitively, if a particular student increases his or her choice of human capital, students with lower cost types (i.e., assigned to better schools) must increase their own levels of human capital to retain their placement. The second feature of note is the eventual decline of the welfare losses, which is due to the long tail of students with very low costs of ac-

cumulating human capital. These students are naturally inclined, due to their low cost types, to accumulate large levels of human capital and are less adversely affected by the competitive channel.

We would like to close out this section with three observations. First, one should note that the welfare loss of \$91,795 is, in reality, spread across the entire time the student has been accumulating human capital in the 18 years before enrolling in college. Second, when multiplied by the over one million students that enroll in college each year, the total losses are almost \$100 billion per year. Finally, our estimates imply that 3.5 times as much human capital (in terms of NPV of future income) is accumulated per student in the color-blind contest as in the first-best outcome. In this sense, one can conclude that a large fraction of the human capital cost accrued by students is, in essence, wasteful signaling of underlying type.

6.2 Computing the Second Best

We now compute the second-best college assignment contest to discover what fraction of the welfare gains of the first-best can be recovered by admissions policy changes or a clever intervention on the part of the government. The optimal contest design problem takes the form of an optimal control problem. The control variable, $u(p) \in [0, 1]$, represents the fraction of seats at school p that are allocated to minority students. Because the index variable is the college, p , all of the variables in our problem must be written as functions of p . The state variables of our control problem are the equilibrium strategies of the students, $\sigma_M(p)$ and $\sigma_N(p)$, and the type of the student from each group that is assigned a seat at college p , $\theta_M(p)$ and $\theta_N(p)$. Our optimal control problem can be written

$$\max_u \int_p^{\bar{p}} \{u * [U(p, \sigma_M, \theta_M) - C(\sigma_M, \theta_M)] + (1 - u) * [U(p, \sigma_N, \theta_N) - C(\sigma_N, \theta_N)]\} f_P(p) dp$$

such that

$$\dot{\theta}_{\mathcal{M}}(p) = -\frac{u * f_{\mathcal{P}}(p)}{\mu f_{\mathcal{N}}(\theta_{\mathcal{M}}(p))} \quad (12)$$

$$\dot{\theta}_{\mathcal{N}}(p) = -\frac{(1-u) * f_{\mathcal{P}}(p)}{(1-\mu) f_{\mathcal{N}}(\theta_{\mathcal{N}}(p))} \quad (13)$$

$$\dot{\sigma}_{\mathcal{M}}(p) = \frac{U_p(p, \sigma_{\mathcal{M}}, \theta_{\mathcal{M}})}{C_s(\sigma_{\mathcal{M}}, \theta_{\mathcal{M}}) - U_s(p, \sigma_{\mathcal{M}}, \theta_{\mathcal{M}})} \quad (14)$$

$$\dot{\sigma}_{\mathcal{N}}(p) = \frac{U_p(p, \sigma_{\mathcal{N}}, \theta_{\mathcal{N}})}{C_s(\sigma_{\mathcal{N}}, \theta_{\mathcal{N}}) - U_s(p, \sigma_{\mathcal{N}}, \theta_{\mathcal{N}})} \quad (15)$$

$$\int_{\underline{p}}^{\bar{p}} u(p) f_{\mathcal{P}}(p) dp = \mu \quad (16)$$

$$\sigma_{\mathcal{M}}(\underline{p}) = \sigma_{\mathcal{N}}(\underline{p}) = 520, \theta_{\mathcal{M}}(\underline{p}) = \theta_{\mathcal{N}}(\underline{p}) = 1517 \quad (17)$$

The objective of the problem is a rewriting of the average social surplus using the index variable of the control problem, p . Equations 12 - 15 denote the laws of motion for the state variables, and equation 16 insures that enough seats are allocated to minorities so that the entire measure μ of minorities obtains a college seat. Equation 17 provides boundary conditions for our state variables from Hickman [38]. We impose the boundary condition $\sigma_j(\underline{p}) = \underline{s} = 520$, the lowest SAT score found in our data, regardless of whether or not both groups are assigned seats at the worst school, \underline{p} . We are, in effect, assuming that both groups are assigned at least a small fraction of the seats at every college.¹⁷

Since the objective function and the equations of motion are linear in u , we know immediately that the solution will have a bang-bang structure. In other words, the social surplus maximizing affirmative action scheme will involve complete segregation - all of the seats in each school will be allocated to one of the two groups. What we find is that the second-best outcome assigns all of the low quality college seats to minority students and reserves the best colleges for non-minority students, a result depicted in Figure 2.

There are two components to the logic underlying the second-best. First, the incentive to compete is driven by the difference in the quality of the prizes. To reduce this incentive, we must make the difference between the best and worst prize available to each group as small as possible, which can be done by breaking the prize space into two intervals. Second, we ought to assign the better prizes to the group that will reap larger complementarities from being matched to a high quality school. Since membership in the non-minority group is a signal that human capital accumulation costs are low, the second-best gives the majority of the seats at high quality colleges to non-minority students.

¹⁷We do this primarily so that the optimal control problems are tractable. An alternative to our approach would be to allow the boundary conditions for each group to be defined by indifference conditions.

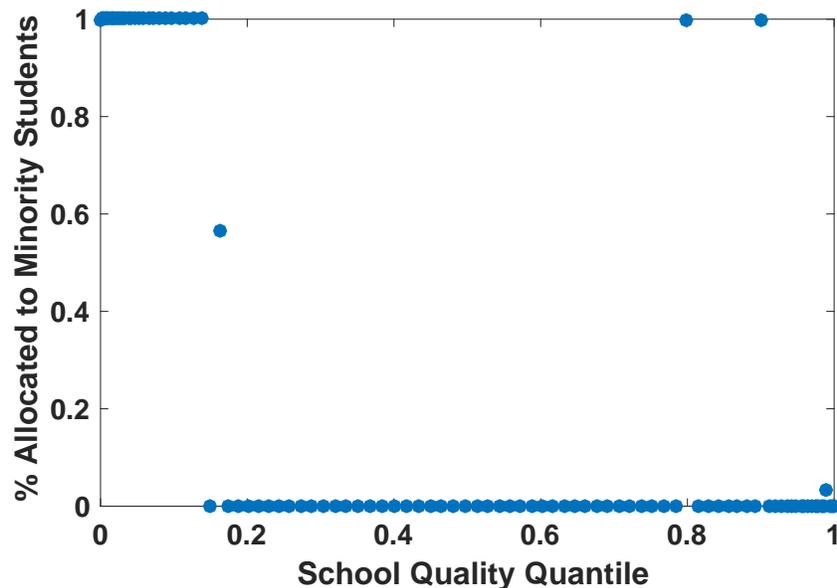


Figure 2

The conclusion we draw from our analysis is that the tension between fairness and welfare is even greater than a comparison between a color-blind and a proportional quota would suggest. In order to reduce the cost of competition, it is necessary to reduce the incentive for the students to compete with another. One way of doing this, as our second-best reveals, is to allow individuals to compete for only a narrow window of college assignments. In order to choose these narrow windows, any exogenous information available about the students' abilities ought to be brought to bear. In the context of our analysis, the students' minority status is the only useful piece of information. In real life the optimal policy would likely include using other signals of ability such as whether the students' parents are college educated. Again, these systems might be welfare improving in the sense of maximizing the average NPV of lifetime salary, but any system that used these signals would be unfair to the students and have the effect of perpetuating historical inequities to future generations.

7 Noisy Human Capital Measures

In the real world, students do not have complete control over the human capital level observed by colleges. For example, a student's investment in his or her studies might be highlighted by winning a prestigious academic award, which would boost the student's college placement relative to other students that made the same investment choice. We

model this by assuming that for each student, all colleges observe the same noisy human capital (NHC) measure $t = s + \varepsilon$, where s is the level of HC chosen by the student and ε refers to a mean 0 shock to the HC observation. Students are then matched to schools based on the NHC measure t . The important features of the shock is that (1) it is unobserved by the student until after the human capital choice has been made and (2) it is common across colleges. We assume throughout that ε has a uniformly continuous density function f_ε and CDF F_ε and has full support.

The noisy human capital values have two effects. First, and most obviously, it makes the outcomes of any particular action stochastic, which makes the analysis significantly more complicated. The primary difficulty is that we can no longer write down useful versions of the differential equations we used to characterize the equilibrium without noise. Second, the shocks to HC have the effect of placing students with different HC choices at the same school.

When discussing admissions preference systems, we assume that the mark-up function \tilde{T} is applied to the realized NHC values. We need the following assumption for the mark-up function to be well behaved.

Assumption 11. *There exists $\lambda_1, \lambda_2 \in (0, \infty)$ such that for all t we have*

$$\lambda_2 > \tilde{T}'(t) > \lambda_1$$

Given a vector \mathbf{t}_{-i} of the other players' NHC levels in the finite agent game, we let $P_j^r(\cdot, \mathbf{t}_{-i}) : \mathcal{T} \rightarrow \mathcal{P}_{\mathcal{K}}$, $j \in \{\mathcal{M}, \mathcal{N}\}$ and $r \in \{cb, q, ap\}$, be an assignment mapping that describes the college to which student i is assigned given each possible NHC realization. The ex post payoff for student i is then

$$\Pi_j^r(s_i, \mathbf{s}_{-i}; \theta) = E \left[U \left(P_j^r(t_i, \mathbf{t}_{-i}), s_i, \theta_i \right) \mid s_i, \mathbf{s}_{-i} \right] - C(s_i, \theta_i) \quad (18)$$

The expectation in equation 18 reflects uncertainty over both the types of the other agents and the values of ε drawn for each player. In the limit game, the analogous utility is

$$\Pi_j^r(s; \theta) = E \left[U \left(P_j^r(t), s, \theta \right) \mid s \right] - C(s, \theta) \quad (19)$$

The expectation in equation 19 reflects uncertainty about the agent's own value of ε .

The definitions of the assignment mappings P_j^r is similar to those in sections 3 and 4 with t playing the role of s . For example, the color-blind assignment mapping in the

finite game is

$$P_{\mathcal{M}}^{cb}(t_i, \mathbf{t}_{-i}) = P_{\mathcal{N}}^{cb}(t_i, \mathbf{t}_{-i}) = P^{cb}(t_i, \mathbf{t}_{-i}) = \sum_{k=1}^K p(k : K) \mathbb{1}[t_i = t(k : K)].$$

In the limit model, the color-blind assignment mapping is

$$P_{\mathcal{M}}^{cb}(t) = P_{\mathcal{N}}^{cb}(t) = F_{\mathcal{P}}^{-1} \left(\mu H_{\mathcal{M}}^{cb}(t) + (1 - \mu) H_{\mathcal{N}}^{cb}(t) \right) \quad (20)$$

where H_i^r is the equilibrium cumulative distribution function of NHC values.

For brevity, we quickly state analogs for our equilibrium existence and approximation results and focus the discussion on the differences between the claims and proof techniques for the noise-free benchmark model. First we use the supermodularity of the game to prove the existence of an equilibrium using techniques similar to those employed to prove Theorem 1.

Theorem 6. *In the finite agent college admissions game $\Gamma(K_{\mathcal{M}}, F_{\mathcal{M}}, K_{\mathcal{N}}, F_{\mathcal{N}}, \mathcal{P}_{\mathcal{K}})$ with $r \in \{cb, q, ap\}$, under assumptions 1–8 there exists a monotone pure-strategy equilibrium.*

We can extend the supermodularity techniques used to prove Theorem 6 to prove equilibria exist in the limit model with noise. In contrast, the proof of Theorem 2 was proven using an analysis of the ODEs characterizing the equilibrium of the limit model. Note that we cannot prove that the limit equilibrium is essentially unique since we cannot appeal to analogous ODEs in the model with noise.

Theorem 7. *There exists a monotone, pure strategy Nash equilibrium of our limit model in the color-blind, quota, or admissions preference systems.*

Finally, we obtain the following approximation result

Theorem 8. *Let σ_i^j , $i \in \{\mathcal{M}, \mathcal{N}\}$ and $j \in \{cb, q, ap\}$ denote an equilibrium of the limit game. We can choose K^* such that σ_i^j is a ε -approximate equilibrium of the K -agent game for any $K > K^*$.*

Our result is stronger than Theorem 3 in that we do not require that the quotas have a connected support. Theorem 3 is stronger in that we prove that the equilibrium of the limit game is also a δ -approximate equilibrium, which relied on the uniqueness of the limit game's equilibrium. It is, however, easy to prove that if we consider a sequence $\left\{ \left(\sigma_{\mathcal{M}}^{j,K}, \sigma_{\mathcal{N}}^{j,K} \right) \right\}_{K=1}^{\infty}$ where $\left(\sigma_{\mathcal{M}}^{j,K}, \sigma_{\mathcal{N}}^{j,K} \right)$ is an equilibrium of the K -agent game, then $\left\{ \left(\sigma_{\mathcal{M}}^{j,K}, \sigma_{\mathcal{N}}^{j,K} \right) \right\}_{K=1}^{\infty}$ converges to an equilibrium of the limit game (if it converges at all).

Finally, Theorem 4 on the equivalence of quota and admissions preference schemes extends to a model with noisy observations of human capital. To see this, note that if

one simply replaces the use of s in the proof of Theorem 4 with t , the argument applies immediately to the NHC model.

8 Conclusion

The purpose of this paper has been to introduce a new model of college admissions and use it to enrich the debate around the differences between quota and admissions preference systems as well as throw light on the welfare costs of the competition for admission, which anecdotal evidence suggests has ratcheted up in intensity in recent years. Modeling college admissions and affirmative action programs is challenging since one must consider the human capital investment decisions of students, heterogeneity in underlying quality on both sides of the market, and the decisions of universities given the information they are presented. Prior papers in the literature gain tractability by simplifying various components of the problem (e.g., assuming all colleges are homogeneous or student quality is innate). We instead consider a market with a continuum of agents, and the continuum approximation greatly simplifies the analysis and allows us to produce a number of novel results.

Our first application of the model is to study the difference between quota-based and admissions preference-based affirmative action systems, and we find that there is no difference in the equilibrium outcomes produced. It is not particularly surprising that one can achieve the same diversity levels at each institution under each scheme. We believe it is more surprising that the equivalence also holds for the human capital accumulation decisions of those admitted.

Our equivalence formalizes comments made in the jurisprudence regarding the difference between quotas and affirmative action schemes. The legality of affirmative action turns on what was perceived by Justice Powell as a difference between quotas and narrowly tailored admissions preference systems. Later opinions about the legality of different admissions preference schemes hinged on how closely the respective justice thought the admissions preference scheme mimicked the outcomes of a quota. Our analysis suggests that drawing a sharp line between quotas and admissions preference schemes based on outcomes may be futile. Not being legal scholars, we have little to say about the eventual legal ramifications for universities that employ color-sighted affirmative action programs.

We also analyze the welfare effects of competition for college admission using a calibration of our model drawn from Hickman [38] that used admissions data from 1996. In an incomplete information world, competition is necessary to achieve an efficient, assortative match between students and colleges. Our analysis suggests that competition

wipes out over \$91,000 of the NPV of the college salary premium for the average student, roughly 13% of the total college premium. Stated differently, the college admissions contest quadruples the average cost of preparing for college (again in terms of the NPV of future income).

Anecdotal evidence from news reports and books such as the *Battle Hymn of the Tiger Mother* by Amy Chua and *The Overachievers: The Secret Lives of Driven Kids* by Alexandra Robbins suggest that the pressures being placed on high school students may be increasing with time. Since our study focuses on individuals that graduated from college in 1996, we believe our estimates are a lower bound on the present costs of college competition. Studying the time trend of the college competition costs would allow us to address whether the costs are increasing over time and the effect of high achieving foreign students on student welfare.

Our final contribution is to analyze a model where the assignment to colleges is based on a noisy measure of human capital. We are at present estimating this model, and hope to use the results to speak to issues such as the relative contribution of human capital, innate ability, and college quality to labor market outcomes.

Three other interesting directions for further research exist. First, if we could incorporate a model of how student assignment endogenously influences college quality, we could discuss with more confidence the general equilibrium implications of massive changes in the assignment of students to schools.

Second, the current model focuses on student behavior, conditional on participation in the college market, but there is another interesting group of individuals to consider as well: those whose decision to attend college may be affected by a given policy. This question could be addressed by formalizing the “supply-side” comprised of potential colleges and firms who may enter the market and supply post-secondary education services or unskilled jobs. Such a model might illustrate how the marginal agent (i.e., the individual indifferent between attending college and entering the workforce) is affected by a given college admission policy. This would help to characterize the full effect of AA.

Finally, the eventual goal for this line of research should be to answer the question of how AA helps or hinders the objective of erasing the residual effects of institutionalized racism. This will require a dynamic model in which the policy-maker is not only concerned with short-term outcomes for students whose types are fixed, but also with the long-run evolution of the type distributions. Empirical evidence suggests that academic competitiveness is determined by factors such as affluence and parental education. If AA affects performance and outcomes for current minority students, then the next question is what effect it might have on their children’s competitiveness when the next generation enters high school? If a given policy produces the effect of better minority enrollment

and higher achievement in the short-run, then one might conjecture that a positive long-run effect will be produced. However, it seems evident that a long-run model is needed in order to give meaningful direction to forward-looking policy-makers. It is our hope that the theory developed here will help in the development of a model that addresses these important questions in the future.

References

- [1] Aliprantis, C. and K. Border (2006) *Infinite Dimensional Analysis: A Hitchhiker's Guide, Third edition*, Springer-Verlag Inc.: New York.
- [2] Athey, S. (2001) "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69 (4), pp. 861-869.
- [3] Azevedo, E. and J. Leshno (2014) "A Supply and Demand Framework for Two-Sided Matching Markets," *mimeo*.
- [4] Becker, G. (1973) "A Theory of Marriage: Part I," *Journal of Political Economy*, 81 (4), pp. 813-846.
- [5] Bhagwati, J. (1965) "On the equivalence of tariffs and quotas" in R.E. Baldwin et al., *Trade, growth and the balance of payments - Essays in honors of G. Haberler*, Rand McNally, pp. 53 - 67.
- [6] Bodoh-Creed, A. (2011) "Approximation of Large Games with Applications to Uniform Price Auctions," *Journal of Economic Theory*, in press.
- [7] Bodoh-Creed, A. (2013) "Efficiency and information aggregation in large uniform-price auctions," *Journal of Economic Theory*, 148 (6), pp. 2436 - 2466.
- [8] Budish, E. (2008) "The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes," *Journal of Political Economy*, 119 (6), pp. 1061 - 1103.
- [9] Carmona, G. (2004) "Nash Equilibria of Games with a Continuum of Players," *mimeo*.
- [10] Carmona, G. and K. Podczeck (2009) "On the Existence of Pure-Strategy Equilibria in Large Games," *Journal of Economic Theory*, p.1300 - 1319.
- [11] Carmona, G. and K. Podczeck (2011) "Approximation and Characterization of Nash Equilibria of Large Games," *mimeo*.

- [12] Chade, H; G. Lewis; and L. Smith (2011) "Student Portfolios and the College Admissions Problem," *Review of Economic Studies*, 81 (3), pp. 971 - 1002.
- [13] Chan, J. and E. Eyster (2003) "Does Banning Affirmative Action Lower College Student Quality?" *The American Economic Review*, 93 (3) , pp. 858 - 872.
- [14] Chua, A. (2011) *Battle Hymn of the Tiger Mother*, Penguin Books: London.
- [15] Chung, C. and T. Espenshade (2005) "The Opportunity Cost of Admission Preferences at Elite Universities," *Social Science Quarterly*, 86 (20), pp. 293-305.
- [16] Chung, C.; T. Espenshade; and J. Walling (2004) "Admission Preferences for Minority Students, Athletes, and Legacies at Elite Universities," *Social Science Quarterly*, 85 (5), pp. 1422-1446.
- [17] Coate, S. and G. Loury (1993) "Anti-Discrimination and the Problem of Patronization," *American Economic Review*, 83 (2), pp. 92-98.
- [18] Coate, S. and G. Loury (1993) "Will Affirmative-Action Policies Eliminate Negative Stereotypes?" *The American Economic Review*, 83 (5) , pp. 1220-1240.
- [19] Cripps, M. and J. Swinkels (2006) "Efficiency of Large Double Auctions," *Econometrica*, 74, pp. 47-92.
- [20] Edlin, A. and C. Shannon (1998) "Strict Monotonicity in Comparative Statics," *Journal of Economic Theory*, 81, pp. 201 - 219.
- [21] Epple, D.; R. Romano; and H. Sieg (2003) "The Practice and Proscription of Affirmative Action in Higher Education: An Equilibrium Analysis," *NBER Working Paper*.
- [22] Fain, J. (2009) "Affirmative Action Can Increase Effort," *Journal of Labor Research*, 30 (2), pp. 168-175.
- [23] Fisher v. University of Texas at Austin et al. (2012) 570 U.S. (slip op.), Supreme Court of the United States, Legal Information Inst., Cornell U. Law School (Web, 19 April 2015).
- [24] Franke, J. (2008) "Does Affirmative Action Reduce Effort Incentives? A Contest Game Analysis," Technische Universitat Dortmund, Department of Economics and Social Science, *mimeo*.
- [25] Fruehwirth, J. (2013) "Identifying peer achievement spillovers: Implications for desegregation and the achievement gap," *Quantitative Economics*, 4 (1), pp. 85 - 124.

- [26] Fryer, R. (2009) "Implicit Quotas," *Journal of Legal Studies*, 38, pp. 1 - 20.
- [27] Fryer, R. and G. Loury (2005) "Affirmative Action and Its Mythology," *Journal of Economic Perspectives*, 19 (3), pp. 147 - 162.
- [28] Fryer, R.; G. Loury; and T. Yuret (2007) "An Economic Analysis of Color-Blind Affirmative Action," *The Journal of Law, Economics, & Organization*, 24 (2), pp. 319 - 355.
- [29] Fudenberg, D.; D. Levine; and W. Pesendorfer (1998) "When are Nonanonymous Players Negligible?" *Journal of Economic Theory*, 79, pp. 46-71.
- [30] Fryer, R.; and G. Loury (2005) "Affirmative Action and its Mythology," *Journal of Economic Perspectives*, 19 (3), pp. 147-162.
- [31] Fu, Q. (2006) "A Theory of Affirmative Action in College Admissions," *Economic Inquiry*, 44 (3), pp. 420-428.
- [32] Galloway, M.; J. Conner; and D. Pope (2013) "Nonacademic Effects of Homework in Privileged, High-Performing High Schools," *The Journal of Experimental Education*, 81 (4), pp. 490 - 510.
- [33] Gratz v. Bollinger (2003) 539 U.S. 244, Supreme Court of the United States, Legal Information Inst., Cornell U. Law School (Web, 6 April 2015).
- [34] Green, E. (1980) "Noncooperative Price Taking in Large Dynamic Markets," *Journal of Economic Theory*, 22, pp. 155-181.
- [35] Green, E. (1984) "Continuum and Finite-Player Noncooperative Models of Competition," *Econometrica*, 52, pp. 975-993.
- [36] Grutter v. Bollinger (2003) 539 U.S. 306, Supreme Court of the United States, Legal Information Inst., Cornell U. Law School (Web, 6 April 2015).
- [37] Guerre, E.; Perrigne, I.; and Vuong Q. (2000) "Optimal Nonparametric Estimation of First-Price Auctions," *Econometrica*, 68, pp. 525-574.
- [38] Hickman, B. (2014) "Pre-College Human Capital Investment and Affirmative Action: A Structural Policy Analysis of US College Admissions," *mimeo*.
- [39] Hildenbrand, W. (1974) "On Economies with Many Agents," *Journal of Economic Theory*, 2, pp. 161-188.
- [40] Hildenbrand, W. (1974) *Core and Equilibria of a Large Economy*, Princeton University Press: Princeton, New Jersey.

- [41] Housman, D. (1988) "Infinite Player Noncooperative Games and the Continuity of the Nash Equilibrium Correspondence," *Mathematics of Operations Research*, 13, pp. 488 - 496.
- [42] Hoxby, C. (2000) "Peer Effects in the Classroom: Learning From Gender and Race Variation," *NBER Working Paper*.
- [43] Jackson, M. and A. Manelli (1997) "Approximately Competitive Equilibria in Large Finite Economies," *Journal of Economic Theory*, 77, pp. 354 - 376.
- [44] Kalai, E. (2005) "Large Robust Games," *Econometrica*, 72, pp. 1631-1665.
- [45] Krishnamurthy, I.; R. Johari; and M. Sundarajan (2014) "Mean Field Equilibria of Dynamic Auctions with Learning," *Management Science*, 60 (12), pp. 2949-2970.
- [46] Khan, M. and Y. Sun (1997) "Non-Cooperative Games with Many Players," *Handbook of Game Theory with Economic Applications*, Vol. 3, North Holland: New York.
- [47] Kojima, F. and P. Pathak (2007) "Incentives and Stability in Large Two Sided Matching Markets," *American Economic Review*, 99, pp. 608 - 627.
- [48] Luthar, S. and B. Becker (2002) "Privileged but Pressured? A Study of Affluent Youth," *Child Development*, 73 (5), pp. 1593 - 1610.
- [49] McAdams, D. (2003) "Isotone Equilibrium in Games of Incomplete Information," *Econometrica*, 71 (4), pp. 1191-1214.
- [50] McLean, R. and A. Postlewaite (2002) "Informational Size and Incentive Compatibility," *Econometrica*, 70, pp. 2421-2453.
- [51] Jencks, C. and M. Phillips (1998) *The Black-White Test Score Gap*, Brookings, Washington D.C.
- [52] Olszewski, W. and R. Siegel (2014) "Large Contests," *mimeo*.
- [53] Otani, Y. and J. Sicilian (1990) "Limit Properties of Equilibrium Allocations of Walrasian Strategic Games," *Journal of Economic Theory*, 51, pp. 295-312.
- [54] Robbins, Alexandra (2007) *The Overachievers: The Secret Lives of Driven Kids*, Hachette Books: New York.
- [55] Roberts, D. and A. Postlewaite (1976) "The Incentives for Price-Taking Behavior in Large Exchange Economies," *Econometrica*, 44, pp. 115 - 127.

- [56] Regent of the University of California v. Bakke (1978) 438 U.S. 265, Supreme Court of the United States, Legal Information Inst., Cornell U. Law School (Web, 6 April 2015).
- [57] Rochet, J. and P. Choné (1998) "Ironing, Sweeping, and Multidimensional Screening," *Econometrica*, 66 (4), pp. 783 - 826.
- [58] Sabourian, H. (1990) "Anonymous Repeated Games with a Large Number of Players and Random Outcomes," *Journal of Economic Theory*, 51 (1), pp. 92 - 110.
- [59] Schotter, A.; and K. Weigelt (1992) "Assymmetric Tournaments, Equal Opportunity Laws and Affirmative Action: Some Experimental Results," *Quarterly Journal of Economics*, 107 (2), pp. 511-539.
- [60] Schuette, Attorney General of Michigan v. Coalition to Defend Affirmative Action, Integration and Immigration Rights and Fight For Equality By Any Means Necessary et al. (2013) 572 U.S. (slip op.), Supreme Court of the United States, Legal Information Inst., Cornell U. Law School (Web, 19 April 2015).
- [61] Sowell, T. (2004) *Affirmative Action Around the World*, Yale University Press.
- [62] Spence, M. (1973) "Job Market Signaling," *The Quarterly Journal of Economics*, 87 (3), pp. 355 - 374.
- [63] Swinkels, J. (2001) "Efficiency of Large Private Value Auctions," *Econometrica*, 69, pp. 37-68.
- [64] "Turning the Tide: Inspiring Concern for Others and the Common Good through College Admissions." Retrieved from <http://mcc.gse.harvard.edu/collegeadmissions> on 27 February 2016.
- [65] Weintraub, G.; Benkard, B.L.; and Van Roy, B. (2008) "Markov Perfect Industry Dynamics With Many Firms," *Econometrica*, 76 (6), pp. 1375-1411.
- [66] Yang, J. (2011) "Connections between Finite- and Infinite-Player Games: Normal- and Extended-form Analyses," *mimeo*.

A Proofs

We break our proofs into three sections. Section A.1 contain the proofs of the majority of our results. Section A.2 contains the proof of Theorem 3, which is significantly more involved than our other arguments. Finally, section A.3 contain the proofs for our claims

regarding the extension of our model to a setting where human capital is observed with noise.

A.1 Proofs

The remaining results in the paper are produced roughly in the order they appear in the main body.

Theorem 1. *In the college admissions game $\Gamma(K_M, F_M, K_N, F_N, \mathcal{P}_K)$ with $r \in \{cb, q, ap\}$, under assumptions 1–8 there exists a monotone pure-strategy equilibrium. Moreover, any equilibrium of the game must be strictly monotone with almost every type using pure strategies.*

Proof. Consider the following gridding of the strategy space $\mathcal{S}_T = \left\{ \underline{s}, \underline{s} + \frac{(\bar{s}-\underline{s})}{T}, \underline{s} + \frac{2(\bar{s}-\underline{s})}{T}, \dots, \bar{s} \right\}$ where T is a natural number, and we refer to the game with actions restricted to \mathcal{S}_T as the T -gridded game. Theorem 1 of Athey [2, Theorem 3] implies the existence of a monotone PSNE on \mathcal{S}_T for any T . Consider a sequence of such monotone PSNE. Since these strategies are monotone and bounded, Helly's selection theorem implies that a subsequence of $(\sigma_i^T)_{T=1}^\infty$ converges to σ_i^* for all i , and for the duration we will let $(\sigma_i^T)_{T=1}^\infty$ denote the convergent subsequence. From Lemma 4 of Athey [2, Theorem 3], we can focus on equilibria where $(\sigma_i^T)_{T=1}^\infty$ (and as a result σ_i^*) have the form $\sigma_i^T(\bar{\theta}) = \sigma_i^*(\bar{\theta}) = \underline{s}$.

Our first step is to prove an analog to Lemma 3 for the case where the action space has been discretized.

Lemma 1. *σ_i^* must be strictly monotone.*

Proof. First consider the possibility that for agent i there is an interval of the form $[\theta_1, \bar{\theta}]$ where $\sigma_i^*(\theta) = \underline{s}$ for all $\theta \in [\theta_1, \bar{\theta}]$. Since player i chooses \underline{s} with positive probability ρ in the limit game, for any $\delta > 0$ we can choose T sufficiently large that player i chooses an action within $[\underline{s}, \underline{s} + \delta]$ with probability at least ρ in the T -gridded game.

Note that in any equilibrium we must have for agents $j \neq i$ that $\sigma_j^*(\bar{\theta}) = \underline{s}$. If these player-types choose \underline{s} , then they have a positive probability of tying with agent i and receive (at random) one of the worst school assignments available. By deviating to $\underline{s} + \delta$ for an arbitrarily small cost (since δ can be chosen to be arbitrarily small), player j with type $\bar{\theta}$ can with probability ρ instead receive the second-worst (or better) school assignment, which is a discrete improvement. Therefore it cannot be the case that player j with type $\bar{\theta}$ finds it optimal to choose \bar{s} when T is sufficiently large. From this contradiction we conclude that there cannot exist an interval of the form $[\theta_1, \bar{\theta}]$ where $\sigma_i^*(\theta) = \underline{s}$ for all $\theta \in [\theta_1, \bar{\theta}]$ for any player.

Now we turn to the possibility of violations of strict monotonicity at higher human capital levels. Suppose for some i we have that σ_i^* is not monotone, which means there

exists $s \in (\underline{s}, \bar{s}]$ and an interval $[\theta_1, \theta_2]$ such that $\sigma_i^*(\theta) = s > \underline{s}$ for $\theta \in [\theta_1, \theta_2]$. Let ρ denote the probability with which player i plays s . From the convergence of $(\sigma_i^T)_{T=1}^\infty$, for any $\delta > 0$ there exists T^* sufficiently large that for all $T > T^*$ and $\theta \in [\theta_1, \theta_2]$ we have $\sigma_i^T(\theta) \in (s - \delta, s + \delta)$.

We now argue that no player $j \neq i$ can find it optimal in equilibrium to play $s \in (s - \delta, s + \delta)$ for sufficiently large T where $\delta > 1/T$. To see this, note that if $\sigma_j^*(\theta) \in (s - \delta, s + \delta)$, then by choosing $\tilde{s} = s + \delta$, player j has a probability ρ of discretely improving his outcome by surpassing player i 's effort level. Since δ (and hence the cost of this deviation) can be chosen to be arbitrarily small and the benefit remains bounded strictly above 0, the existence of such a profitable deviation proves our claim.

Having established that no agent $j \neq i$ chooses s within $s \in (s - \delta, s + \delta)$ with positive probability, we know that there is no competitive channel of incentives active at HC level s . This means that in the agent must be indifferent across all choices of s in $(s - \delta, s + \delta)$. However, since U and C are strictly convex in s , this is impossible. From this contradiction we can conclude that σ_i^* is strictly monotone. \square

Now we use Lemma 1 to argue that the utility in the T -gridded game converges to the utility received in the game with the continuum of actions. Let $\Pi_i(s, \sigma_{-i}, \theta)$ denote the expected utility of agent i with type θ playing action s when the other agents choose according to σ_{-i} .

Lemma 2. Consider $s_T \rightarrow s$ when $T \rightarrow \infty$ where $s_T \in \mathcal{S}_T$. Then $\Pi_i(s_T, \sigma_{-i}^T, \theta) \rightarrow \Pi_i(s, \sigma_{-i}^*, \theta)$.

Proof. First decompose $\Pi_i(s_T, \sigma_{-i}^T, \theta) \rightarrow \Pi_i(s, \sigma_{-i}^*, \theta)$ into two parts

$$\begin{aligned} \Pi_i(s_T, \sigma_{-i}^T, \theta) - \Pi_i(s, \sigma_{-i}^*, \theta) &= [\Pi_i(s_T, \sigma_{-i}^*, \theta) - \Pi_i(s, \sigma_{-i}^*, \theta)] + \\ &\quad [\Pi_i(s_T, \sigma_{-i}^T, \theta) - \Pi_i(s_T, \sigma_{-i}^*, \theta)] \end{aligned} \quad (21)$$

From Lemma 1 we conclude that $\Pi_i(\circ, \sigma_{-i}^*, \theta)$ is continuous since the distribution of college assignments must vary continuously with s since σ_{-i}^* are strictly monotone, $U(p, s, \theta)$ and $C(s, \theta)$ are continuous in s and θ , and F_M and F_N are strictly monotone. But this continuity implies that the first bracketed term of equation 21 approaches 0 as $s_T \rightarrow s$, and from the compactness of \mathcal{S} this convergence is uniform over s .

Now consider the second bracketed term. From the uniform convergence of σ_j^T to σ_j^* , for any $d > 0$ we can choose T sufficiently large that for all agents $j \neq i$ and types θ we have

$$\sigma_j^T(\theta) - d < \sigma_j^*(\theta) < \sigma_j^T(\theta) + d$$

For any strategy vector σ_{-i} let $\sigma_{-i} + d$ refer to a vector of strategies $\tilde{\sigma}_{-i}$ where for all

agents $j \neq i$ we have $\tilde{\sigma}_j(\theta) = \sigma_j(\theta) + d$. Then we can write

$$\left\| \Pi_i(s_T, \sigma_{-i}^T, \theta) - \Pi_i(s_T, \sigma_{-i}^*, \theta) \right\| \leq \Pi_i(s_T, \sigma_{-i}^* - d, \theta) - \Pi_i(s_T, \sigma_{-i}^* + d, \theta) \quad (22)$$

From the strict monotonicity of σ_j^* , we know that the right side of equation 22 vanishes in the limit as $d \rightarrow 0$, which means that the second bracketed term of equation 21 vanishes in the limit as $T \rightarrow \infty$. \square

Now we use lemma 2 to close our argument regarding existence. Since σ^T represents an equilibrium of the T -gridded game, we know that for all players and all types θ and all actions $s \in \mathcal{S}_T$

$$\Pi_i(\sigma_i^T(\theta), \sigma_{-i}^T, \theta) \geq \Pi_i(s, \sigma_{-i}^T, \theta) \quad (23)$$

For any $s \in \mathcal{S}$ we can construct a sequence of $(s_T)_{T=1}^\infty$, $s_T \in \mathcal{S}_T$, such that $s_T \rightarrow s$. Lemma 2 allows us to conclude that $\Pi_i(s_T, \sigma_{-i}^T, \theta) \rightarrow \Pi_i(s, \sigma_{-i}^*, \theta)$. Since $\sigma_i^T(\theta) \rightarrow \sigma_i^*(\theta)$ we also have $\Pi_i(\sigma_i^T(\theta), \sigma_{-i}^T, \theta) \rightarrow \Pi_i(\sigma_i^*(\theta), \sigma_{-i}^*, \theta)$. Together with equation 23 we have

$$\Pi_i(\sigma_i^*(\theta), \sigma_{-i}^*, \theta) \geq \Pi_i(s, \sigma_{-i}^*, \theta)$$

which implies $\sigma_i^*(\theta)$ is a best response to σ_{-i}^* for player i . Therefore σ^* is a pure strategy, strictly monotone Bayesian-Nash equilibrium. \square

Theorem 2. *The equilibrium of any of our limit game affirmative action systems is essentially unique, almost all of the agents take pure actions, and the strategies are group-symmetric.*

Proof. In any equilibrium of any of our contests, $P_i^j(s)$ must be monotone increasing, which implies that the function is almost everywhere (under the Lebesgue measure) continuous and differentiable. From supermodularity, we have that the correspondence $s(\theta)$ defined by

$$\mathbf{s}(\theta) = \arg \max_s U(P_i^j(s), s, \theta) - C(s, \theta)$$

is decreasing in the strong set order. Since the objective function is strictly supermodular, $\mathbf{s}(\theta)$ is single valued for almost all θ (under either F_M or F_N). For an agent of type θ to violate group symmetry in equilibrium, it must be that agents of that type have two optimal actions. Similarly, for an agent of type θ to take a mixed action in equilibrium, he must have at least two optimal actions. Since $\mathbf{s}(\theta)$ is almost everywhere single valued, this implies that almost all agents take a pure action in equilibrium and that any equilibrium is group symmetric.

Consider a quota game and assume that σ_i^q is defined by equation 8 over the intervals where Q_i has support. Note that the boundary condition is uniquely defined for any

quota scheme as follows

$$\begin{aligned}\sigma_i^q(\bar{\theta}) &= \arg \max_s U(\underline{p}_i, s, \bar{\theta}) - C(s, \bar{\theta}) \\ \underline{p}_i &= \inf\{p : Q_i(p) > 0\}\end{aligned}$$

$\sigma_i^q(\bar{\theta})$ must be single valued since the problem is strictly concave. By standard results in differential equations, if $[p, p']$ is an interval where Q_i has support and type θ is assigned to college p with human capital level $\sigma_i^q(\theta) = s$, then the strategy is uniquely defined by equation 8 for all types that are (in equilibrium) assigned a seat at a college in $[p, p']$. The only way there could exist two equilibria σ_i^q and $\tilde{\sigma}_i^q$ is if there is some discontinuity at some θ , which could only occur at a gap in the support of Q_i , such that

$$\lim_{\varepsilon \rightarrow 0^+} \sigma_i^q(\theta + \varepsilon) \neq \lim_{\varepsilon \rightarrow 0^+} \tilde{\sigma}_i^q(\theta + \varepsilon)$$

In other words, the jump over an interval in which Q_i lacks support is not uniquely defined. We prove that the first such jump must be uniquely defined. An (omitted) induction step using essentially the same argument can be used to prove that all of the jumps must be uniquely defined.

Suppose that Q_i lacks support over the interval (p, p') and Q_i has full support over both $[p, p]$ and $[p', p' + \delta]$ for some $\delta > 0$. Let θ satisfy $P_i^q(\sigma_i^q(\theta)) = p$ - in other words, θ is where the first jump must occur. In equilibrium it must be the case that θ is indifferent about whether to make this jump, so

$$U(p, s, \theta) - C(s, \theta) = U(p', s', \theta) - C(s', \theta)$$

where $s = \sigma_i^q(\theta)$ and $s' = \lim_{\varepsilon \rightarrow 0^+} \sigma_i^q(\theta + \varepsilon)$. Suppose there was a second equilibrium $\tilde{\sigma}_i^q$ starting at type θ such that $s'' = \lim_{\varepsilon \rightarrow 0^+} \tilde{\sigma}_i^q(\theta + \varepsilon) > s'$. Then it must be the case that

$$U(p, s, \theta) - C(s, \theta) = U(p', s', \theta) - C(s', \theta) = U(p', s'', \theta) - C(s'', \theta)$$

But since U_{ss} is strictly concave and C_{ss} is strictly convex, this cannot be true. Therefore s' is uniquely defined.

Since any two equilibria of a quota scheme σ_i^q and $\tilde{\sigma}_i^q$ must be identical except for the countable points θ where a discontinuity occurs, the measure of $\{\theta : \sigma_i^q(\theta) \neq \tilde{\sigma}_i^q(\theta)\}$ is measure 0, which implies the equilibrium is essentially unique.

Now consider an admissions preference game where σ_i^{ap} is defined by equation 11 and is therefore continuous. Standard results on differential equations imply that σ_i^{ap} is uniquely defined.

In all of the affirmative action systems used in the limit game, the equilibria have been defined by ODEs that enforce the local incentive compatibility conditions, but it remains to prove that global incentive compatibility holds. We provide a proof for the admissions preference case, but analogous arguments hold for the quota game. Suppose $\tilde{\theta} < \theta$ and let $P_i^{ap}(\tilde{\theta}) = \tilde{p} > p = P_i^{ap}(\theta)$ and $\sigma_i^{ap}(\tilde{\theta}) = \tilde{s} > s = \sigma_i^{ap}(\theta)$. We then have from local incentive compatibility that

$$U_p(p, s, \theta) \left. \frac{dP_i^{ap}(\hat{\theta})}{d\hat{\theta}} \right|_{\hat{\theta}=\theta} + U_s(p, s, \theta) \left. \frac{d\sigma_i^{ap}(\hat{\theta})}{d\hat{\theta}} \right|_{\hat{\theta}=\theta} = C_s(s, \theta) \left. \frac{d\sigma_i^{ap}(\hat{\theta})}{d\hat{\theta}} \right|_{\hat{\theta}=\theta}$$

Since $\tilde{\theta} < \theta$, we have from assumption 5 that $U_p(p, s, \tilde{\theta}) > U_p(p, s, \theta)$, and from assumption 6 that $U_s(p, s, \tilde{\theta}) > U_s(p, s, \theta)$ and $C_s(s, \tilde{\theta}) < C_s(s, \theta)$. The value of the first order condition that results if type $\tilde{\theta}$ deviated from truthfulness upwards by declaring $\hat{\theta} = \theta$ is

$$U_p(p, s, \tilde{\theta}) \left. \frac{dP_i^{ap}(\hat{\theta})}{d\hat{\theta}} \right|_{\hat{\theta}=\theta} + U_s(p, s, \tilde{\theta}) \left. \frac{d\sigma_i^{ap}(\hat{\theta})}{d\hat{\theta}} \right|_{\hat{\theta}=\theta} > C_s(s, \tilde{\theta}) \left. \frac{d\sigma_i^{ap}(\hat{\theta})}{d\hat{\theta}} \right|_{\hat{\theta}=\theta}$$

which, since there is an inequality, implies declaring $\hat{\theta} = \theta$ cannot be optimal if the true type is $\tilde{\theta}$. Similar arguments imply that deviating downward from truthfulness also cannot be optimal.

As a final step we must rule out cases where it might be optimal for the agent to choose a human capital level outside of the range of σ_i^{ap} . Since σ_i^{ap} is continuous, this implies that the deviation satisfies $\tilde{s} > s = \sigma_i^{ap}(\theta)$. Let $p = P_i^{ap}(\theta)$. We know from our first order conditions that¹⁸

$$U_p(p, s, \theta) \left. \frac{dP_i^{ap}(\hat{\theta})}{d\hat{\theta}} \right|_{\hat{\theta}=\theta} + U_s(p, s, \theta) \left. \frac{d\sigma_i^{ap}(\hat{\theta})}{d\hat{\theta}} \right|_{\hat{\theta}=\theta} = C_s(s, \theta) \left. \frac{d\sigma_i^{ap}(\hat{\theta})}{d\hat{\theta}} \right|_{\hat{\theta}=\theta}$$

Increasing from s to \tilde{s} does not change the school p , and so it can only be optimal if

$$U_s(p, \tilde{s}, \theta) = C_s(\tilde{s}, \theta) \tag{24}$$

Since $U_{ss} < 0$, $C_{ss} > 0$, and $U_s(p, s, \theta) \leq C_s(s, \theta)$, equation 24 cannot hold, which means deviating to \tilde{s} cannot be optimal. \square

Theorem 4. Consider some $P_i(s) : \mathcal{S} \rightarrow \mathcal{P}$, $i \in \mathcal{M}, \mathcal{N}$. $P_i(s) : \mathcal{S} \rightarrow \mathcal{P}$, $i \in \mathcal{M}, \mathcal{N}$ is the result of an equilibrium of some quota system if and only if there is an equilibrium of an admissions preference system that also yields these assignment functions and admits the same equilibrium

¹⁸We have written the first order condition with the derivatives defined using limits from the right (i.e., using sequences contained in Θ).

strategies.

Proof. Suppose $P_i^q : \mathcal{S} \rightarrow \mathcal{P}$, $i \in \mathcal{M}, \mathcal{N}$, is the result of an equilibrium of some quota system and denote the equilibrium strategies $\sigma_i^q : \Theta \rightarrow \mathcal{S}$. Since P_i^q and σ_i^q are strictly monotone, the functions are invertible. Let $P^{ap}(s)$, the assignment function under admissions preferences, be $P^{ap}(s) = P_{\mathcal{N}}^q$. Since the assignment functions are the same for the non-minority students, P^{ap} and $P_{\mathcal{N}}^q$ generate identical decision problems for the non-minorities. Therefore, if $\sigma_{\mathcal{N}}^q$ was an equilibrium for non-minority students under $P_{\mathcal{N}}^q$, then $\sigma_{\mathcal{N}}^{ap}(\theta) = \sigma_{\mathcal{N}}^q(\theta)$ will be an equilibrium for the non-minority students under $P^{ap}(s)$.

To construct the outcome equivalent score function that generates $P^{ap}(s)$, let

$$\tilde{S}(s) = (P_{\mathcal{N}}^q)^{-1}(P_{\mathcal{M}}(s))$$

A minority student who chooses human capital level s will then be assigned to college

$$P^{ap}(\tilde{S}(s)) = P_{\mathcal{N}}^q(\tilde{S}(s)) = P_{\mathcal{N}}^q((P_{\mathcal{N}}^q)^{-1}(P_{\mathcal{M}}(s))) = P_{\mathcal{M}}^q(s)$$

Since the assignment functions are the same for the minority students, P^{ap} and $P_{\mathcal{M}}^q$ generate identical decision problems for the minorities. Therefore, if $\sigma_{\mathcal{M}}^q$ was an equilibrium for minority students under $P_{\mathcal{M}}^q$, then $\sigma_{\mathcal{M}}^{ap}(\theta) = \sigma_{\mathcal{M}}^q(\theta)$ will be an equilibrium for the non-minority students under $P^{ap}(s)$.

Now suppose $P^{ap} : \mathcal{S} \rightarrow \mathcal{P}$ with score function \tilde{S} is the result of an equilibrium of some admissions preference system and denote the equilibrium strategies $\sigma_i^{ap} : \Theta \rightarrow \mathcal{S}$, $i \in \mathcal{M}, \mathcal{N}$. To define the equivalent quota system, we need to define allocations of seats to each group. Let these distributions be denoted Q_i , $i \in \mathcal{M}, \mathcal{N}$, and define them as

$$\begin{aligned} \text{For all } p \text{ let } Q_{\mathcal{M}}(p) &= 1 - F_i \left[\psi_i^{ap} \left(\tilde{S}^{-1} \left((P^{ap})^{-1}(p) \right) \right) \right] \\ \text{For all } p \text{ let } Q_{\mathcal{N}}(p) &= 1 - F_i \left[\psi_i^{ap} \left((P^{ap})^{-1}(p) \right) \right] \end{aligned}$$

Given these definitions, $P_{\mathcal{N}}^q(s) = P^{ap}(s)$ and $P_{\mathcal{M}}^q(s) = P^{ap}(\tilde{S}(s))$. Note that the total measure of non-minority students choosing s and minority students choosing $\tilde{S}^{-1}(s)$ under P^{ap} (in equilibrium is)

$$\begin{aligned} 1 - \mu F_{\mathcal{M}} \left[\psi_{\mathcal{M}}^{ap} \left(\tilde{S}^{-1} \left((P^{ap})^{-1}(p) \right) \right) \right] - (1 - \mu) F_{\mathcal{N}} \left[\psi_{\mathcal{N}}^{ap} \left((P^{ap})^{-1}(p) \right) \right] = \\ \mu Q_{\mathcal{M}}(p) + (1 - \mu) Q_{\mathcal{N}}(p) = F_{\mathcal{P}}(p) \end{aligned}$$

which implies $Q_{\mathcal{M}}$ and $Q_{\mathcal{N}}$ are feasible quotas. As argued above, since the decision problems for the agents are the same, the equilibrium strategies in the original admissions

preference scheme and the constructed quota are the same. \square

Theorem 5. *HC investment for all types in the color-blind admissions scheme exceeds that in the first-best outcome.*

Proof. Let σ^{cb} be the equilibrium strategy under color-blind admissions and σ^{FB} be the ex post investment strategy in the first-best benchmark. The boundary condition for both problems is the same:

$$\sigma^{cb}(\bar{\theta}) = \sigma^{FB}(\bar{\theta}) = \underline{s}$$

Since the first order condition must hold at $\theta = \bar{\theta}$ in the color-blind scheme, we have

$$\frac{\partial U(P(\underline{s}), \underline{s}, \bar{\theta})}{\partial p} \cdot \frac{dP(\underline{s})}{ds} + \frac{\partial U(P(\underline{s}), \underline{s}, \bar{\theta})}{\partial s} = \frac{dC(\underline{s}; \bar{\theta})}{ds} \geq \frac{\partial U(P(\underline{s}), \underline{s}, \bar{\theta})}{\partial s} \quad (25)$$

We can only have equality across equation 25 if $\frac{dP(\underline{s})}{ds} = 0$, which requires $\frac{\partial \sigma^{cb}(\theta)}{\partial \theta} \rightarrow -\infty$ as $\theta \rightarrow \bar{\theta}$. In this case, we know for an interval of θ in the neighborhood of $\bar{\theta}$ that $\sigma^{cb}(\theta) \geq \sigma^{FB}(\theta)$ where the inequality is strict for $\theta \neq \bar{\theta}$ in that neighborhood.

Now assume that $\sigma^{cb}(\theta^*) = \sigma^{FB}(\theta^*)$ for some $\theta < \bar{\theta}$. In this case, we have, as per the argument above that $\frac{dP(\underline{s})}{ds} = 0$, which implies $\sigma^{cb}(\theta) > \sigma^{FI}(\theta)$ for θ sufficiently close to, but less than, θ^* . Therefore, it cannot be the case that σ^{cb} and σ^{FB} ever cross - σ^{cb} is always weakly greater than σ^{FB} . \square

A.2 Proving Theorem 3

Since the equilibrium strategies are strictly decreasing (Proposition 1), we know immediately that the equilibrium strategy must be almost everywhere differentiable. We now prove that there is a lower bound on the derivative of the equilibrium strategy, which implies that the distribution of human capital in any equilibrium must be nonatomic. Moreover, it implies that if we look at sequences of equilibrium strategies, the resulting limit strategy generates a nonatomic distribution of human capital.

Lemma 3. *There exists $\omega < 0$ such that for any equilibrium strategy, $\sigma(\theta)$, of either a finite game or the limit game, we have $\frac{d}{d\theta}\sigma(\theta) < \omega$ at points where the strategy is differentiable.*

Proof. We prove our lemma for the color-blind game, but the proof extends directly to the quota game if we by treat each group separately. Finally, the proof technique easily extends to the score function game, although the notation becomes cumbersome since one must accommodate both groups of students and account for the variation in \tilde{S} .

Suppose there is no such upper bound on the derivative, which means that for any $\omega < 0$ there exists θ such that $\sigma'(\theta) > \omega$. From the a.e. differentiability of σ , there exists

an interval $[\theta_L, \theta_U]$ such that $\sigma'(\theta) > \omega$ for all $\theta \in [\theta_L, \theta_U]$ where $\sigma'(\theta)$ exists. Without loss of generality, we assume $\sigma'(\theta)$ exists at θ_L . Let $s_L = \sigma(\theta_L)$ and $s_U = \sigma(\theta_U)$, and note that $0 < s_L - s_U < \omega(\theta_L - \theta_U)$. Since σ must be decreasing, we have

$$\Pr\{\sigma(\theta) \in [s_U, s_L]\} = F(\theta_U) - F(\theta_L)$$

Rearranging this we find

$$\frac{\Pr\{\sigma(\theta) \in [s_U, s_L]\}}{s_L - s_U} = \frac{F(\theta_U) - F(\theta_L)}{\sigma(\theta_U) - \sigma(\theta_L)} > \frac{-1}{\omega} \frac{F(\theta_U) - F(\theta_L)}{\theta_U - \theta_L}$$

Let $\eta_\theta = \inf_{\theta} f(\theta) > 0$. Taking limits we find

$$\lim_{\theta_U \rightarrow \theta_L} \frac{\Pr\{\sigma(\theta) \in [\sigma(\theta_U), \sigma(\theta_L)]\}}{\sigma(\theta_U) - \sigma(\theta_L)} > \frac{-1}{\omega} \lim_{\theta_U \rightarrow \theta_L} \frac{F(\theta_U) - F(\theta_L)}{\theta_U - \theta_L} = \frac{-1}{\omega} f(\theta_L) > \frac{-1}{\omega} \eta_\theta > 0$$

This means that in intervals where $\sigma'(\theta)$ is close to 0, the “density” of individuals making the associated human capital choices is arbitrarily large. We call such a point of high density a pseudo-atom.

Let $\delta_p = \sup_p f_p(p) < \infty$. Increasing the human capital choice from s_L to s_U yields a minimal benefit of increasing the rank of one’s school in the limit game by

$$-\frac{1}{\omega} \frac{\eta_\theta}{\delta_p} (s_L - s_U)$$

If we let $\eta_U = \min_{s,p,\theta} U_p(p, s, \theta) > 0$, then the utility benefit must be at least

$$-\frac{1}{\omega} \frac{\eta_\theta}{\delta_p} \eta_U (s_L - s_U)$$

Let the maximum marginal cost of human capital that we can observe in any equilibrium be denoted

$$\delta_C = \max_{s \in \mathcal{S}, \theta \in \Theta} C_s(s, \theta) < \infty$$

This means the cost of deviating from s_U to s_L is bounded from above by

$$(s_L - s_U) \delta_C$$

For such a deviation to be suboptimal, we must have

$$(s_L - s_U) \delta_C \geq -\frac{\eta_U \eta_\theta}{\omega \delta_p} (s_L - s_U)$$

which requires

$$\sigma'(\theta) \leq \omega \leq -\frac{\eta_\theta \eta_U}{\delta_p \delta_C}$$

In the game with K students, the formation of pseudo-atoms is probabilistic. Consider an agent with type θ_U who in equilibrium chooses $s_U = \sigma^K(\theta)$. Suppose such a student considers increasing her human capital choice to s_L . For each student she leap frogs, her school placement improves by at least $(\delta_p)^{-1} \frac{1}{K}$, which generates a utility benefit of at least

$$\frac{\eta_U}{\delta_p} \frac{1}{K}$$

For each student, there is a probability of at least $-\frac{\eta_\theta}{\omega} (s_L - s_U)$ of observing a human capital choice in $[s_U, s_L]$, which yields a lower bound on the expected benefit of the deviation equal to

$$\frac{\eta_U}{\delta_p} \frac{1}{K} E[i]$$

where i is distributed binomially with K draws using a parameter equal to $-\frac{\eta_\theta}{\omega} (s_L - s_U)$. Given the distribution of i , we can write

$$\frac{\eta_U}{\delta_p} \frac{1}{K} E[i] = \frac{\eta_U}{\delta_p} \frac{1}{K} \left[-\frac{\eta_\theta}{\omega} (s_L - s_U) K \right] = -\frac{\eta_U \eta_\theta}{\delta_p \omega} (s_L - s_U)$$

The remainder of the argument proceeds as above. □

Before proving our approximation results, we provide a few background results from the theory of the weak convergence of empirical processes. Below we restate the Glivenko-Cantelli theorem for reference.

Theorem 9. *Consider a random variable $X : \Omega \rightarrow \mathbb{R}$ with CDF $F(y) = \int_{\Omega} 1\{x \leq y\} * \pi_0(dx)$. For N i.i.d. realizations, $\{X_1, \dots, X_N\}$ denote the N realization empirical CDF as $F_N(y)$. Then we have*

$$\sup_{y \in \mathbb{R}^d} \|F_N(y) - F(y)\| \rightarrow 0 \text{ almost surely as } N \rightarrow \infty$$

The topology over the space of measures generated by the sup-norm over the space of CDFs is referred to as the *Kolmogorov topology*. It is straightforward to show that the Kolmogorov topology and the weak-* topology¹⁹ are identical over any space of nonatomic measures.

With this background in hand, we can now proceed with the proof of our approximation results.

¹⁹The Prokhorov-Levy metric that metricizes the weak-* topology and the sup-norm over the space of CDFs are identical when one of the measures is nonatomic.

Theorem 3. Consider any admissions preference game or a quota game where $Q_{\mathcal{M}}$ and $Q_{\mathcal{N}}$ admit strictly positive PDFs over a connected support. Let σ_i^j , $i \in \mathcal{M}, \mathcal{N}$ and $j \in \{cb, ap, q\}$ denote an equilibrium of the limit game. Under assumptions 1 -9 and given $\varepsilon, \delta > 0$, there exists $K^* \in \mathbb{N}$ such that for any $K \geq K^*$ we have that σ_i^j is a ε -approximate equilibrium and a δ -approximate equilibrium of the K -agent game.

Proof. We prove our theorem through a series of lemmas. These lemmas are necessary for the application of the theorems in Bodoh-Creed [7].

Since our proofs rely on results on the convergence of empirical processes to their true distributions, we will need to define the spaces in which these measures live. The true distributions of the student types are $F_{\mathcal{M}}(\theta)$ or $F_{\mathcal{N}}(\theta)$. We have assumed that these distributions have strictly positive densities, which means that their respective PDFs are bounded from below. This means that we can define a compact set Δ^{Θ} of measures over Θ such that $F_{\mathcal{M}}(\theta), F_{\mathcal{N}}(\theta) \in \Delta^{\Theta}$ and all of the measures in Δ^{Θ} have PDFs that are uniformly and strictly bounded above 0. We endow the space Δ^{Θ} with the Kolmogorov topology.

First we establish some initial properties of the objects we are working with. Lemma 3 establishes that the equilibrium strategies have an upper bound on their derivative, which implies that pseudo-atoms cannot exist. Let Σ^R be the set of strategies that adhere to the bound prescribed by lemma 3, and we endow this space with the L^1 norm. Since these are functions of bounded variation, Σ^R is compact by the Helly Selection Theorem. From Egorov's theorem, the convergence is also "almost" uniform in the sense that it can be metricized by the norm:

$$d_{AU}(\sigma, \tilde{\sigma}) < \delta \text{ if } \|\sigma(\theta) - \tilde{\sigma}(\theta)\| < \delta \text{ for a set of } \Theta_{AU} \text{ of measure } 1 - \delta \text{ w.r.t. } F_j$$

The monotonicity of the members of Σ^R implies that $\Sigma^R - \Theta_{AU}$ can be described as a countable set of closed intervals.²⁰

Let $\Delta^R(\mathcal{S})$ denote the space of pushforward measures generated by a strategy $\sigma \in \Sigma^R$ and a distribution of student types, $F_i(\theta) \in \Delta^{\Theta}$. The set $\Delta^R(\mathcal{S})$ is a tight family of measures and is compact (Theorem 15.22 of Aliprantis and Border [1]). We endow the space $\Delta^R(\mathcal{S})$ with the weak-* topology. Since the measures in $\Delta^R(\mathcal{S})$ are all nonatomic, the weak-* topology is equivalent to the Kolmogorov topology generated by the sup-norm applies to the space of CDFs.

Let $\Delta_K(X)$ denote the set of empirical measures generated K draws from the set X . We will let $G_{\mathcal{N}}^{K_{\mathcal{N}}}$ denote the CDF of the empirical measure of human capital choices for

²⁰These intervals may be required to include points where the uniform closeness condition holds, although the measure of Θ_{AU} will still meet the requirements of the almost uniform convergence.

the non-minority students when there are K_N such students in the economy, and we use the notation $\Delta_{K_N}(\mathcal{S})$ to refer to the set of such CDFs. $G_{\mathcal{M}}^{K_M}$ denotes the empirical measure of human capital choices for the minority students when there are K_M such students in the economy, and we use $\Delta_{K_M}(\mathcal{S})$ to refer to the set of such CDFs. We endow $\Delta_{K_N}(\mathcal{S})$ and $\Delta_{K_M}(\mathcal{S})$ with the weak-* topology.

We now prove two useful lemmas about inverse functions that, for some reason, we cannot find in the existing literature.

Lemma 4. *Let $G_j, j \in \{\mathcal{M}, \mathcal{N}\}$ be the CDF of the pushforward measure generated by $\sigma_j \in \Sigma^R$ and the type distribution $F_j \in \Delta^\Theta$. Then G_j is uniformly continuous in σ_j when the space of CDFs is endowed with the sup-norm.*

Proof. Consider σ_j with the associated pushforward measure G_j . Then

$$G_j = F_j \left(\sigma_j^{-1}(s) \right) = F_j \left(\{ \theta : \sigma_j(\theta) \leq s \} \right)$$

Choose any $\epsilon > 0$. Consider an arbitrary $\tilde{\sigma}$ such that $d_{AU}(\tilde{\sigma}_j, \sigma_j) < \delta$, and denote the associated pushforward measure $\tilde{G}_j(s)$. For any $T \in \mathbb{N}$ consider the set $\{0, \frac{1}{T}, \frac{2}{T}, \dots, 1\}$ that grids quantile space. Let Θ_{Cut} include all θ such that $\|\tilde{\sigma}_j(\theta) - \sigma_j(\theta)\| > \delta$, and (as noted above) there exists a Θ_{Cut} that is a countable set of closed intervals. This means that for δ sufficiently small we can choose a set $\Theta_{Grid} = \{\theta_1, \dots, \theta_T\}, \theta_i \in [\underline{\theta}, \bar{\theta}]$, where $\theta_i \in \left[F_j^{-1}\left(\frac{i}{T}\right), F_j^{-1}\left(\frac{i+1}{T}\right) \right]$ and $\theta_i \notin \Theta_{Cut}$.

For each θ_i we have for $\delta > 0$ sufficiently small $\|\tilde{\sigma}_j^{-1}(\sigma_j(\theta_i)) - \theta_i\| < \epsilon$ for all $\theta_i \in \Theta_{Grid}$.²¹ This combined with the continuity and monotonicity of F_j implies that for all $\theta \in \Theta_{Grid}$ we have $\|G_j(\sigma_j(\theta)) - \tilde{G}_j(\sigma_j(\theta))\| < \epsilon + \frac{2}{T}$. Since G and \tilde{G}_j are monotone, for any $\epsilon > 0$ we can choose $\delta > 0$ sufficiently small and T sufficiently large that

$$\sup_{s \in [\underline{s}, \bar{s}]} \|G_j(s) - \tilde{G}_j(s)\| < \epsilon$$

□

Lemma 5. *Let K be any increasing function where for some $\gamma > 0$ and any $t > t'$ we have $K(t) - K(t') \geq \gamma(t - t')$. Then $K^{-1}(\cdot)$ is uniformly continuous.*

²¹A quick proof of this claim starts by noting that for any θ

$$\begin{aligned} \sigma_j(\theta - \epsilon) &< \sigma_j(\theta) - \omega\epsilon \\ \sigma_j(\theta + \epsilon) &< \sigma_j(\theta) + \omega\epsilon \end{aligned}$$

For any ϵ choose $\delta < \omega\epsilon$ and choose $\tilde{\sigma}_j$ such that $\|\sigma_j(\theta) - \tilde{\sigma}_j(\theta)\| < \delta$, which implies $\tilde{\sigma}_j(\theta) \in [\sigma_j(\theta) - \delta, \sigma_j(\theta) + \delta] \subset (\sigma_j(\theta) - \omega\epsilon, \sigma_j(\theta) + \omega\epsilon)$ which implies $\tilde{\sigma}_j^{-1}(s) \in (\theta - \epsilon, \theta + \epsilon)$ as required.

Proof. Consider k, k' where $K(t) = k$ and $K(t') = k'$. Then we have

$$\begin{aligned}\|k - k'\| &> \gamma \|t - t'\| \\ \|t - t'\| &< \frac{1}{\gamma} \|k - k'\|\end{aligned}$$

which implies that $K^{-1}(\cdot)$ is uniformly continuous. \square

Our next lemma proves that the limit assignment mapping must be continuous in both s and the distributions of agent actions, $G_j, j \in \{\mathcal{M}, \mathcal{N}\}$. To this end, let $P_j^r(\cdot; G_{\mathcal{N}}, G_{\mathcal{M}})$ denote the assignment mapping generated if the agent actions are distributed as per $(G_{\mathcal{N}}, G_{\mathcal{M}})$.

Lemma 6. $P_j^r(s; G_{\mathcal{N}}, G_{\mathcal{M}}), j \in \{\mathcal{M}, \mathcal{N}\}$ and $r \in \{cb, q, ap\}$, is uniformly continuous in $(s, G_{\mathcal{N}}, G_{\mathcal{M}})$.

Proof. We provide a proof for the admissions preference game since the other systems are special cases of an admissions preference scheme. In the admissions preference game, we have $P_{\mathcal{M}}^{ap}(s) = F_{\mathcal{P}}^{-1}\left((1 - \mu)G_{\mathcal{N}}(\tilde{S}(s)) + \mu G_{\mathcal{M}}(s)\right)$. Let

$$\begin{aligned}q(s) &= (1 - \mu)G_{\mathcal{N}}(\tilde{S}(s)) + \mu G_{\mathcal{M}}(s) \\ \tilde{q}(s') &= (1 - \mu)\tilde{G}_{\mathcal{N}}(\tilde{S}(s')) + \mu\tilde{G}_{\mathcal{M}}(s')\end{aligned}$$

$G_{\mathcal{N}}$ and $G_{\mathcal{M}}$ must be uniformly continuous with respect to s since the slopes of $\sigma_{\mathcal{N}}$ and $\sigma_{\mathcal{M}}$ are strictly bounded away from 0 and \tilde{S} is uniformly continuous from assumption 8. Therefore, for any $\gamma > 0$ we can choose $\delta > 0$ sufficiently small that for any $(s, G_{\mathcal{N}}, G_{\mathcal{M}})$ and $(s'; \tilde{G}_{\mathcal{N}}, \tilde{G}_{\mathcal{M}})$ that satisfy

$$\|G_{\mathcal{N}} - \tilde{G}_{\mathcal{N}}\| + \|G_{\mathcal{M}} - \tilde{G}_{\mathcal{M}}\| + \|s - s'\| < \delta$$

we have

$$\begin{aligned}\|\tilde{q}(s') - q(s)\| &< \left\|q(s) - \left[(1 - \mu)G_{\mathcal{N}}(\tilde{S}(s')) + \mu G_{\mathcal{M}}(s')\right]\right\| + \delta \\ &< \left\|q(s) - \left[(1 - \mu)G_{\mathcal{N}}(\tilde{S}(s)) + \mu G_{\mathcal{M}}(s)\right]\right\| + \delta + \frac{\gamma}{2} \\ &= \delta + \frac{\gamma}{2} < \gamma\end{aligned}$$

$f_{\mathcal{P}}$ has a lower bound, meaning $F_{\mathcal{P}}$ has a lower bound on its slope and from Lemma 5 $F_{\mathcal{P}}^{-1}$ is uniformly continuous. Therefore for any $\varepsilon > 0$ we can choose $\delta > 0$ sufficiently

small that for any $(s, G_{\mathcal{N}}, G_{\mathcal{M}})$ and $(s'; \tilde{G}_{\mathcal{N}}, \tilde{G}_{\mathcal{M}})$ that satisfy

$$\|G_{\mathcal{N}} - \tilde{G}_{\mathcal{N}}\| + \|G_{\mathcal{M}} - \tilde{G}_{\mathcal{M}}\| + \|s - s'\| < \delta$$

we have

$$\|P_j^r(s; G_{\mathcal{N}}, G_{\mathcal{M}}) - P_j^r(s'; \tilde{G}_{\mathcal{N}}, \tilde{G}_{\mathcal{M}})\| < \varepsilon$$

which establishes our claim. \square

We use the notation $\Pi_j^r(s, \theta; G_{\mathcal{N}}^{K_{\mathcal{N}}}, G_{\mathcal{M}}^{K_{\mathcal{M}}}, K)$ to refer to the expected utility in the K -agent game of an agent of type θ in demographic group $j \in \{\mathcal{M}, \mathcal{N}\}$ that chooses human capital level s given $G_{\mathcal{N}}^{K_{\mathcal{N}}}$ and $G_{\mathcal{M}}^{K_{\mathcal{M}}}$ and admissions system $r \in \{cb, q, ap\}$. Let $\Pi_j^r(s, \theta; G_{\mathcal{N}}^{K_{\mathcal{N}}}, G_{\mathcal{M}}^{K_{\mathcal{M}}})$ refer to the utility received by an agent of type θ that chooses human capital level s given $G_{\mathcal{N}}^{K_{\mathcal{N}}}$ and $G_{\mathcal{M}}^{K_{\mathcal{M}}}$ in the limit game. Finally we use our lemmas to prove that the agent utility is continuous, which is the linchpin of our ε -approximate equilibrium result.

Lemma 7. $\Pi_j^r(s, \theta, G_{\mathcal{N}}, G_{\mathcal{M}})$ is uniformly continuous in $(s, \theta, G_{\mathcal{N}}, G_{\mathcal{M}}) \in \mathcal{S} \times \Theta \times \Delta(\mathcal{S}) \times \Delta(\mathcal{S})$.

Proof. The continuity result follows from Lemma 6 and the continuity of U and C with respect (p, s, θ) . The uniform equicontinuity comes from the compactness of $\mathcal{S} \times \Theta \times \Delta(\mathcal{S}) \times \Delta(\mathcal{S})$. \square

We now establish our two approximation results in separate lemmas.

Lemma 8. Given the previous assumptions, if $(\sigma_{\mathcal{M}}^{ap}, \sigma_{\mathcal{N}}^{ap})$ is an exact equilibrium of the limit game, then for any $\varepsilon > 0$ we can choose K^* such that for any $K > K^*$ we have that $(\sigma_{\mathcal{M}}^{ap}, \sigma_{\mathcal{N}}^{ap})$ is an ε -approximate equilibrium of the K -agent game

Proof. First note that uniform convergence of the utility functions of the finite model to the limit model is straightforward since conditional on the distribution of human capital, distribution of school qualities, and a particular human capital choice, the school assignment is the same in the finite and the continuum model. Combining this with our result on uniform continuity of the limit game (Lemma 7), Theorem 8 of Bodoh-Creed [7] implies that if $(\sigma_{\mathcal{M}}^{ap}, \sigma_{\mathcal{N}}^{ap})$ is an exact equilibrium of the limit game, then for any $\varepsilon > 0$ we can choose K^* such that for any $K > K^*$ $(\sigma_{\mathcal{M}}^{ap}, \sigma_{\mathcal{N}}^{ap})$ is an ε -approximate equilibrium of the modified K -agent game. \square

Our next result deals with sequences of equilibrium strategies. We use the notation $(\sigma_{\mathcal{M}}^{ap}(K), \sigma_{\mathcal{N}}^{ap}(K))$ to denote an exact equilibrium of the limit game.

Lemma 9. Consider any sequence $\{(\sigma_{\mathcal{M}}^{ap}(K), \sigma_{\mathcal{N}}^{ap}(K))\}_{K=2}^{\infty}$ such that $(\sigma_{\mathcal{M}}^{ap}(K), \sigma_{\mathcal{N}}^{ap}(K)) \rightarrow (\sigma_{\mathcal{M}}^{ap}, \sigma_{\mathcal{N}}^{ap})$. Then under the previous assumptions, $(\sigma_{\mathcal{M}}^{ap}, \sigma_{\mathcal{N}}^{ap})$ is an exact equilibrium of the limit game.²²

Proof. Let $(G_{\mathcal{N}}^K, G_{\mathcal{M}}^K)$ refer to the pushforward measure generated by $(\sigma_{\mathcal{M}}^{ap}(K), \sigma_{\mathcal{N}}^{ap}(K))$ and the student type distributions. From Lemma 4 we have $(G_{\mathcal{N}}^K, G_{\mathcal{M}}^K) \rightarrow (G_{\mathcal{N}}, G_{\mathcal{M}})$ in the sup-norm. Let $(\widehat{G}_{\mathcal{N}}^K, \widehat{G}_{\mathcal{M}}^K)$ denote an empirical measure defined by random draws from $(G_{\mathcal{N}}^K, G_{\mathcal{M}}^K)$ in the K -agent game. From the Glivenko-Cantelli theorem, for any $\rho, \delta > 0$ we can choose K sufficiently large that with probability at least $1 - \rho$

$$\|G_{\mathcal{N}}^K - \widehat{G}_{\mathcal{N}}^K\| + \|G_{\mathcal{M}}^K - \widehat{G}_{\mathcal{M}}^K\| < \delta$$

Suppose $(\sigma_{\mathcal{M}}^{ap}, \sigma_{\mathcal{N}}^{ap})$ is not an equilibrium of the limit game. For concreteness, let us assume that minority students of some type θ where $\sigma_{\mathcal{M}}^{ap}(\theta) = s^*$ have a profitable deviation. Formally that means there exists s' and $\varepsilon > 0$ such that

$$\Pi_{\mathcal{M}}^r(s^*, \theta; G_{\mathcal{N}}, G_{\mathcal{M}}) + \varepsilon < \Pi_{\mathcal{M}}^r(s', \theta; G_{\mathcal{N}}, G_{\mathcal{M}})$$

We now translate this into a statement regarding payoffs in the K -agent game. For sufficiently large K we have with probability at least $1 - \rho$

$$\begin{aligned} \Pi_{\mathcal{M}}^r(s^*, \theta; \widehat{G}_{\mathcal{N}}^K, \widehat{G}_{\mathcal{M}}^K) + \frac{\varepsilon}{2} &< \Pi_{\mathcal{M}}^r(s', \theta; \widehat{G}_{\mathcal{N}}^K, \widehat{G}_{\mathcal{M}}^K) \\ \Pi_{\mathcal{M}}^r(\sigma_{\mathcal{M}}^{ap}(K), \theta; \widehat{G}_{\mathcal{N}}^K, \widehat{G}_{\mathcal{M}}^K) + \frac{\varepsilon}{4} &< \Pi_{\mathcal{M}}^r(s', \theta; \widehat{G}_{\mathcal{N}}^K, \widehat{G}_{\mathcal{M}}^K) \end{aligned} \quad (26)$$

The first line follows from our continuity result (Lemma 7). The second line follows from our continuity result (with respect to action) and the fact that $\sigma_{\mathcal{M}}^{ap}(K) \rightarrow s^*$ as $K \rightarrow \infty$. Equation 26 implies that for ρ sufficiently small s' represents a profitable deviation for an agent of type θ in the K -agent game. Since the utility function is bounded, allowing us to ignore whatever utility is realized in the residual probability $1 - \rho$ event, this means s' is a profitable deviation from $\sigma_{\mathcal{M}}^{ap}(K)$. From this contradiction we conclude that $(\sigma_{\mathcal{M}}^{ap}, \sigma_{\mathcal{N}}^{ap})$ is an exact equilibrium of the limit game \square

Since Lemma 9 holds for all such sequences of strategies, we have from Theorem 17.16 of Aliprantis and Border [1] that the equilibrium correspondence is upper hemicontinuous.

Our claim regarding δ -approximate equilibria could still fail if there were equilibria of the limit game that were not close to any equilibrium of arbitrarily large finite game. In

²²One might have thought theorem 6 of Bodoh-Creed [7] would provide this result. Unfortunately, the proof of that theorem required the strategy space to be compact, which is not the case in our application.

other words, the argument might fail if the equilibrium correspondence were not lower hemicontinuous in K . Since there is a unique equilibrium of the limit game, it must be the case that the sequence $\{\sigma^K : \Theta \rightarrow \Delta(\mathcal{S})\}_{K=1}^\infty$ converges in the sup-norm to the unique equilibrium of the limit game. \square

A.3 Noisy Human Capital Proofs

In this section we use the Lévy metric d_L on CDFs, and we note that convergence under the Lévy metric is equivalent to convergence in the weak-* topology. Given $Y, Z \in \Delta(\mathbb{R})$, $d_L(Y, Z) < \varepsilon$ if for all $x \in \mathbb{R}$ we have

$$Y(x - \varepsilon) - \varepsilon < Z(x) < Y(x + \varepsilon) + \varepsilon$$

This definition in hand, we now prove some basic continuity properties of our model.

Lemma 10. $G_j(s; \sigma_j), j \in \{\mathcal{M}, \mathcal{N}\}$ is continuous in σ_j in the weak-* topology.

Proof. Consider monotone decreasing $\sigma_j, \tilde{\sigma}_j \in \Sigma$ where for a measure $1 - \delta$ of θ we have $\|\sigma_j - \tilde{\sigma}_j\| < \delta$. By definition, $G_j(s; \sigma_j) = F_j(\{\theta : \sigma_j(\theta) \leq s\})$. Using this definition

$$\begin{aligned} G_j(s; \tilde{\sigma}_j) &= F_j(\{\theta : \tilde{\sigma}_j(\theta) \leq s\}) \\ &> F_j(\{\theta : \sigma_j(\theta) \leq s - \delta\}) - \delta \\ &= G_j(s - \delta; \sigma_j) - \delta \end{aligned} \tag{27}$$

Similar algebra can be used to show $G_j(s; \tilde{\sigma}_j) < G_j(s + \delta; \sigma_j) + \delta$, which implies $d_L(G_j(s; \sigma_j), G_j(s; \tilde{\sigma}_j)) < \delta$ as required. \square

Now we turn to the continuity of the assignment mappings in the finite game. First consider the color-blind assignment mapping.

Lemma 11. In the finite agent game, $\Pr \{P^{cb}(t_i, \mathbf{t}_{-i}) = k | s_i, \mathbf{s}_{-i}\}$ is continuous in s_i and \mathbf{s}_{-i} .

Proof. In the K agent finite game, we can write the distribution of NHC from the other players as

$$H_K^{cb}(t) = \frac{1}{K-1} \sum_{j \neq i} F_\varepsilon(t - s_j)$$

Since f_ε is uniformly continuous, $H_K^{cb}(t)$ is uniformly continuous in t and \mathbf{s}_{-i} . The probability that $P^{cb}(t) = k$ is just the probability that the agent i 's signal is the k^{th} highest out of \mathcal{K} , which can be written

$$\Pr \{P^{cb}(t_i, \mathbf{t}_{-i}) = k | s_i, \mathbf{s}_{-i}\} = \binom{\mathcal{K}}{k} \int_{-\infty}^{+\infty} H_K^{cb}(s_i + x)^{\mathcal{K}-k} (1 - H_K^{cb}(s_i + x))^{k-1} f_\varepsilon(x) dx$$

Uniform continuity of $\Pr \{P^{cb}(t_i, \mathbf{t}_{-i}) = k | s_i, \mathbf{s}_{-i}\}$ follows from the uniform continuity of H_K^{cb} in t and \mathbf{s}_{-i} . \square

The arguments for the quota case are essentially identical to the color-blind case and are omitted. Now we turn to the admissions preference scheme.

Lemma 12. *In the finite agent game, $\Pr \{P_j^{ap}(t_i, \mathbf{t}_{-i}) = k | s_i, \mathbf{s}_{-i}\}$ is continuous in s_i and \mathbf{s}_{-i} as long as \tilde{T} is continuous and invertible*

Proof. As our first step, we compute the distribution of NHC after applying the markup function, which we denote $H_K^{ap}(t)$. We let $i = 1$ for notational clarity and assume that agent 1 is a nonminority student, but it is obvious that an essentially identical argument would apply if agent i were a minority student. Let $\mathbf{s}_{-i} = (s_2, \dots, s_{K_N}, s_{K_N+1}, \dots, s_{K_N+K_M})$. The

$$H_K^{ap}(t) = \frac{1}{K_N - 1} \sum_{j \neq i} F_\varepsilon(t - s_j) + \frac{1}{K_N - 1} \sum_{j=K_N+1}^{K_N+K_M} F_\varepsilon(\tilde{T}^{-1}(t) - s_j)$$

Since f_ε is uniformly continuous and $H_K^{ap}(t)$ is uniformly continuous in t and \mathbf{s}_{-i} . We can then write

$$\Pr \{P_N^{ap}(t_i, \mathbf{t}_{-i}) = k | s_i, \mathbf{s}_{-i}\} = \binom{K}{k} \int_{-\infty}^{+\infty} H_K^{ap}(t)(s_i + x)^{K-k} (1 - H_K^{ap}(t)(s_i + x))^{k-1} f_\varepsilon(x) dx$$

Uniform continuity of $\Pr \{P_N^{ap}(t_i, \mathbf{t}_{-i}) = k | s_i, \mathbf{s}_{-i}\}$ follows from the uniform continuity of H_K^{ap} in t and \mathbf{s}_{-i} . \square

Recall that $\Pi_j^r(s_i, \mathbf{s}_{-i}; \theta)$ denotes the utility of an agent of type θ in the finite game that chooses s_i while the other agents choose \mathbf{s}_{-i} . From Lemmas 11 and 12 we immediately have the following.

Corollary 10. $\Pi_j^r(s_i, \mathbf{s}_{-i}; \theta)$ is continuous in (s_i, \mathbf{s}_{-i}) for $j \in \{\mathcal{M}, \mathcal{N}\}$ and $r \in \{cb, q, ap\}$.

We now prove the analogous result for the limit model.

Lemma 13. $\Pi_j^r(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}; \theta)$ is continuous in $(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$ for $j \in \{\mathcal{M}, \mathcal{N}\}$ and $r \in \{cb, q, ap\}$.

Proof. We provide a proof for the admissions preference case, which implies the result for the color-blind and quota systems as are special cases of the admissions preference model. Recall that

$$\Pi_j^r(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}; \theta) = E_\varepsilon \left[U(P_j^{ap}(s + \varepsilon), s, \theta) | \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}} \right] - C(s, \theta)$$

We can write the distribution of marked-up NHC values as

$$H_j^{ap}(t; \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}) = (1 - \mu) \int_{\underline{s}}^{\bar{s}} F_\varepsilon(t - s) dG_{\mathcal{N}}(s; \sigma_{\mathcal{N}}) + \mu \int_{\underline{s}}^{\bar{s}} F_\varepsilon(\tilde{T}^{-1}(t) - s) dG_{\mathcal{M}}(s; \sigma_{\mathcal{M}})$$

and since ε is nonatomic, it must be that $H_j(t; \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$ is continuous in t and strictly increasing. Since G_j is continuous in σ_j under the weak-* topology (Lemma 10) and F_ε is continuous, we know that $H_j(t; \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$ is also continuous in $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$. Similar argument imply that the PDF of h_j^{ap} is also continuous in t and $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$.

Our goal will be to show that $P_j^{ap}(s + \varepsilon)$ is continuous in s and $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$ under the weak-* topology. Since U is continuous in p , this immediately yields that $\Pi_j^{ap}(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}; \theta)$ is continuous in $(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$. We focus on the non-minority students for notational ease, but it is obvious that almost identical arguments hold for the minority students.

Noting that $(H^{ap}(\circ; \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}))^{-1}(F_{\mathcal{P}}(p))$ is the NHC level needed to be admitted to school p , we can write the probability of being assigned to school p given $(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$ as

$$\Pr \{P_{\mathcal{N}}^{ap}(s + \varepsilon) = p | \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}\} = \frac{f_\varepsilon((H^{ap})^{-1}(F_{\mathcal{P}}(p)) - s)}{h_j^{ap}((H^{ap})^{-1}(F_{\mathcal{P}}(p)))} f_{\mathcal{P}}(p) \quad (28)$$

where we have suppressed the dependence of $(H^{ap})^{-1}$ and h^{ap} on $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$. Equation 28 implies that $\Pr \{P_{\mathcal{N}}^{ap}(s + \varepsilon) = \circ | \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}\}$ will be continuous as long as $h_j^{ap}((H^{ap})^{-1}(\circ))$ is continuous. First consider the density of $h_j^{ap}(\circ; \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$

$$h_j^{ap}(t; \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}) = (1 - \mu) \int_{\underline{s}}^{\bar{s}} f_\varepsilon(t - s) dG_{\mathcal{N}}(s; \sigma_{\mathcal{N}}) + \mu \int_{\underline{s}}^{\bar{s}} f_\varepsilon(\tilde{T}^{-1}(t) - s) dG_{\mathcal{M}}(s; \sigma_{\mathcal{M}})$$

As $t \rightarrow \pm\infty$, h_j^{ap} approaches 0, which implies $(H_j^{cb})^{-1}$ and $\Pr \{P_{\mathcal{N}}^{ap}(s + \varepsilon) = p | \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}\}$ become discontinuous. However, for sets where h_j^{ap} is bounded above 0, $(H_j^{cb})^{-1}$ and $\Pr \{P_{\mathcal{N}}^{ap}(s + \varepsilon) = p | \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}\}$ are continuous. Our proof strategy is to prove continuity for a central interval of p , and then argue that the outlier school qualities are of such small mass that continuity of $P_{\mathcal{N}}^{ap}(s + \varepsilon)$ under the weak-* topology holds.

To begin, let $(\underline{\sigma}_{\mathcal{M}}, \underline{\sigma}_{\mathcal{N}})$ be the strategy pair where all types chooses \underline{s} , and let $(\bar{\sigma}_{\mathcal{M}}, \bar{\sigma}_{\mathcal{N}})$ be the strategy pair where all types chooses \bar{s} . In turn let $\underline{H}(\circ) = H^{ap}(\circ; \underline{\sigma}_{\mathcal{M}}, \underline{\sigma}_{\mathcal{N}})$ and $\bar{H}(\circ) = H^{ap}(\circ; \bar{\sigma}_{\mathcal{M}}, \bar{\sigma}_{\mathcal{N}})$. For a given γ , we now define define p_1 and p_2 where there is no more than an $\frac{\gamma}{2}$ probability of being assigned a school outside of $[p_1, p_2]$ for any choice of

s or $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$. To this end

$$\begin{aligned} \Pr \{P_{\mathcal{N}}^{ap}(s + \varepsilon) = p_1 | \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}\} &= F_{\varepsilon} \left((H^{ap}(\circ; \underline{\sigma}_{\mathcal{M}}, \underline{\sigma}_{\mathcal{N}}))^{-1} (F_{\mathcal{P}}(p_1)) - s \right) \\ &\leq F_{\varepsilon} \left(\underline{H}^{-1} (F_{\mathcal{P}}(p_1)) - \underline{s} \right) = \frac{\gamma}{2} \end{aligned}$$

Solving for p_1 we find

$$p_1 = F_{\mathcal{P}}^{-1} \left[\underline{H} \left(F_{\varepsilon}^{-1} \left(\frac{\gamma}{2} \right) + \underline{s} \right) \right]$$

Using symmetric algebra, we find

$$p_2 = F_{\mathcal{P}}^{-1} \left[\overline{H} \left(F_{\varepsilon}^{-1} \left(1 - \frac{\gamma}{2} \right) + \overline{s} \right) \right]$$

If we can prove continuity of $\Pr \{P_{\mathcal{N}}^{ap}(s + \varepsilon) = p_1 | \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}\}$ over $[p_1, p_2]$ for any $\gamma > 0$, then we have continuity of $P_{\mathcal{N}}^{ap}(s + \varepsilon)$ in the weak-* topology as we can “ignore” the arbitrarily low probability event that $p \notin [p_1, p_2]$.

Since ε has full support, there exists $\underline{h} > 0$ such that $h_j^{ap}(H^{ap})^{-1}(F_{\mathcal{P}}(p)) \geq \underline{h}$ for all $p \in [p_1, p_2]$. This in turn implies that $(H^{ap})^{-1}$ is continuous over $[F_{\mathcal{P}}(p_1), F_{\mathcal{P}}(p_2)]$ and also continuous in $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$. Combined with the continuity of f_{ε} and $f_{\mathcal{P}}$, we know that $\Pr \{P_{\mathcal{N}}^{ap}(s + \varepsilon) = p | \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}\}$ is continuous in p for $p \in [p_1, p_2]$. This implies that $P_{\mathcal{N}}^{ap}(s + \varepsilon)$ is continuous in the weak-* topology. As mentioned at the beginning, we then have that $\Pi_j^{ap}(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}; \theta)$ is continuous in $(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$. □

We now prove the supermodularity condition required to apply the equilibrium existence theorems of Athey [2].

Lemma 14. $\Pi_j^r(s_i, \mathbf{s}_{-i}; \theta)$ and $\Pi_j^r(s_i, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}; \theta)$ are strictly supermodular in $(s_i, -\theta)$.

Proof. To show supermodularity in the finite game, we must prove that $\frac{\partial \Pi_j^r(s_i, \mathbf{s}_{-i}; \theta)}{\partial \theta}$ is decreasing in s_i . There are two effects to consider. First consider the direct effect of a change in s_i . Assumption 6 implies that $E \left[U_{s\theta}(P_j^r(s_i + \varepsilon, \mathbf{t}_{-i}), s, \theta) \right] - C_{s\theta}(s, \theta) < 0$, so the direct effect has the desired sign. The second, indirect effect is the influence of s_i on P_j^r . Note that increases in s increases the distribution of school assignments in the sense of first order stochastic dominance. Assumption 5 implies that the changes in the distribution of p as s increases also decreases $\frac{\partial \Pi_j^r(s_i, \mathbf{s}_{-i}; \theta)}{\partial \theta}$. Therefore $\Pi_j^r(s_i, \mathbf{s}_{-i}; \theta)$ is supermodular in $(s, -\theta)$. Essentially identical arguments suffice to prove $\Pi_j^r(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}; \theta)$ is supermodular in $(s, -\theta)$ and are omitted. □

We now prove our equilibrium existence result for the finite game.

Theorem 1. *In the college admissions game $\Gamma(K_{\mathcal{M}}, F_{\mathcal{M}}, K_{\mathcal{N}}, F_{\mathcal{N}}, \mathcal{P}_{\mathcal{K}})$ with $r \in \{cb, q, ap\}$, under assumptions 1–8 there exists a monotone pure-strategy equilibrium.*

Proof. Lemmas 14 and 10 allow us to use Corollary 2.1 of Athey [2]. \square

We cannot immediately reply the results of Athey to prove an equilibrium exists, but we can come very close.

Theorem 2. *There exists a pure strategy Nash equilibrium of our limit model in the color-blind or admissions preference systems. There exists a pure strategy Nash equilibrium of our limit model in the quota system if Q_j admits a connected support.*

Proof. Define the best response correspondence as

$$s^{BR}(\theta; \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}) = \arg \max_s \Pi_i^r(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}; \theta)$$

Lemma 14 combined with Lemma 1 of Athey [2] implies that $s^{BR}(\circ; \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$ is monotonically ordered in the strong set order. Lemma 2 of Athey [2] implies $s^{BR}(\theta; \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$ is convex. Finally, because $\Pi_i^r(s, \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}}; \theta)$ is continuous in s and $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$ (Lemma 13), we have from the Theorem of the Maximum that $s^{BR}(\theta; \sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$ is upper hemicontinuous in $(\sigma_{\mathcal{M}}, \sigma_{\mathcal{N}})$. It is straightforward to adapt Lemma 3 and Theorems 1 and 2 of Athey [2], which at their core are results about the existence of fixed points and the convergence of sequences of functions, to prove the existence of an equilibrium. \square

We now turn to our approximation result. Given the continuity results we have established, the approximation results follow from theorems in Bodoh-Creed [6].

Theorem 3. *Let σ_i^j , $i \in \mathcal{M}, \mathcal{N}$ and $j \in \{cb, q, ap\}$ denote an equilibrium of the limit game. We can choose K^* such that σ_i^j is a ε -approximate equilibrium of the K -agent game for any $K > K^*$ in the color-blind and admissions preference systems. If Q_j has a connected support, then we can choose K^* such that σ_i^j is a ε -approximate equilibrium of the K -agent game for any $K > K^*$ in the color-blind and admissions preference systems.*

Proof. We can strengthen all of our continuity claims to uniform continuity due to the compactness of the spaces over which U and C are defined. Theorem 7 of Bodoh-Creed [6] then implies σ_i^j is a ε -approximate equilibrium of the K -agent game for any K sufficiently large. \square

B Online Appendix: Identifying Discontinuities in the Equilibrium

In this appendix we briefly describe how to identify discontinuities in the equilibrium of the limit game. This section will be of practical interest primarily to practitioners who wish to use equations 8 and 11 to compute equilibria numerically.

First let us consider quota schemes, where jumps are caused by gaps in the support of Q_j . The size of the jump must make the types on the edge of the gap indifferent about making the jump. Formally written, suppose (p_L, p_U) is an interval such that $Q_j((p_L, p_U)) = 0$ and for all $\varepsilon > 0$ we have $Q_j([p_L - \varepsilon, p_U]) > 0$ and $Q_j([p_L, p_U + \varepsilon]) > 0$. Let θ be such that $P(\sigma_j^q(\theta)) = p_L$. Then it must be that for $s = \lim_{\varepsilon \rightarrow 0^+} \sigma_j^q(\theta + \varepsilon)$ (i.e., the human capital choice on the other side of the jump) we have

$$U(p_L, \sigma_j^q(\theta), \theta) - C(\sigma_j^q(\theta), \theta) = U(p_U, s, \theta) - C(s, \theta)$$

which identifies s .

Now we discuss how to identify gaps in an admissions preference scheme, which are caused by kinks or discontinuities in \tilde{S} . When these issues arise, the marginal incentives for both groups will change. We handle each issue in turn.

First consider a kink in \tilde{S} at s such that $\frac{d}{ds}\tilde{S}$ jumps at s . Without loss of generality, assume that the strategies are lower semicontinuous²³ and let $\theta_{\mathcal{N}} = \psi_{\mathcal{N}}^{ap}(s)$ and $\theta_{\mathcal{M}} = \psi_{\mathcal{M}}^{ap}(\tilde{S}^{-1}(s))$, and consider the first order conditions that would have to hold at s if the strategies are continuous

$$\text{For } i = \mathcal{N}, U_p(P^{ap}(s'), s', \theta_{\mathcal{N}}) \left. \frac{dP^{ap}(s)}{ds} \right|_{s=\tilde{S}(s')} + U_s(P^{ap}(s'), s', \theta) = C_s(s', \theta)$$

$$\text{For } i = \mathcal{M}, U_p(P^{ap}(\tilde{S}(s')), s', \theta) \left. \frac{dP^{ap}(s)}{ds} \right|_{s=\tilde{S}(s')} \left. \frac{d\tilde{S}(s)}{ds} \right|_{s=s'} + U_s(P^{ap}(\tilde{S}(s')), s, \theta) = C_s(s', \theta)$$

For both group's strategies to be continuous, we would need the first order conditions across the discontinuity in $\frac{d}{ds}\tilde{S}$ to be continuous. In other words, we would require

$$\frac{dP^{ap}(s)}{ds} = \frac{dP^{ap}(\tilde{S}(s))}{ds} \frac{d\tilde{S}(s)}{ds}$$

which is clearly impossible at the discontinuity in $\frac{d\tilde{S}(s)}{ds}$.

²³Since the edges of these jumps are described by an indifference condition, we could just as easily construct an upper semicontinuous equilibrium.

To resolve this problem, one of the groups must jump. We will construct an equilibrium where the minority students jump, but the construction and the test for the validity of the construction is symmetric in the case where the non-minority students jump. If the minority student strategy exhibits a jump, then it must be the case that the first order condition for the non-minority students is smooth across the discontinuity, so $\frac{dP^{ap}(s)}{ds}$ must be continuous. Although convoluted, we write the equation for $\frac{dP^{ap}(s)}{ds}$ below for clarity:

$$\frac{dP^{ap}(s)}{ds} = -\frac{\Phi}{f_{\mathcal{P}}(1-\Phi)} \text{ where } \Phi = \frac{(1-\mu)f_{\mathcal{N}}(\psi_{\mathcal{N}}^{ap}(s))}{(\sigma_{\mathcal{N}}^{ap})'(s)} + \frac{\mu f_{\mathcal{M}}(\psi_{\mathcal{M}}^{ap}(\tilde{S}^{-1}(s)))}{(\sigma_{\mathcal{M}}^{ap})'(s)\tilde{S}'(s)}$$

and we let $(\sigma_i^{ap})'$ be infinity if group i has jumped across that level of human capital. In other words, if the minority students jump, it must be that $(\sigma_{\mathcal{N}}^{ap})'(s)$ drops discontinuously to keep $\frac{dP^{ap}(s)}{ds}$ constant at s . For the duration of the minority student jump, we can use equation 11 to describe the non-minority student strategy. This construction is successful if in the gap we have for all s within discontinuity in the minority student strategy

$$U_p(P^{ap}(\tilde{S}(s)), s, \theta) \frac{dP^{ap}(s)}{ds} \frac{d\tilde{S}(s)}{ds} + U_s(P^{ap}(\tilde{S}(s)), s, \theta) \geq C_s(s, \theta)$$

If the inequality is reversed, then it must be the case that non-minority student jump and minority students do not. Finally, we need to define the size of the jump in the minority student strategy. Suppose the minority strategy is lower semicontinuous and we have $\sigma_{\mathcal{M}}^{ap}(\theta) = s$ (i.e., θ is the type of minority student that jumps). To define the jump we need to compute s' such that

$$U(P^{ap}(\tilde{S}(s)), s, \theta) - C(s, \theta) = U(P^{ap}(\tilde{S}(s')), s', \theta) - C(s', \theta)$$

and let the minority strategy jump to s' at θ . Again, it will often be the case that the first order conditions for the two groups will not align and

$$\frac{dP^{ap}(s')}{ds} \neq \frac{dP^{ap}(s')}{ds} \frac{d\tilde{S}(s')}{ds}$$

If this occurs, it is treated as noted above. It is easy to construct examples where the groups repeatedly jump and never compete at the same college. For example, if $\tilde{S}(s) = s + \Delta$, $\Delta > 0$, the equilibrium has this structure.

Second, assume that there is a discontinuity in \tilde{S} at human capital level s . Since we have assumed \tilde{S} is increasing, \tilde{S} must jump upwards. Let $\sigma_j^q(\theta) = s$ and assume σ_j^q

is lower semicontinuous at θ .²⁴ In this case, there must be a jump in the equilibrium strategy of the minority students that is defined by

$$U(P^{ap}(\tilde{S}(s)), s, \theta) - C(s, \theta) = U(P^{ap}(\tilde{S}(s')), s', \theta) - C(s', \theta)$$

and we let the value of the minority student strategy jump to s' at θ . Again, if the first order conditions cannot both line up, then we will have to allow one of the groups' strategies to jump again, which requires the construction techniques outlined above.

²⁴The construction would be essentially the same if we chose to let σ_j^q be upper semicontinuous at θ .