COSTLESS SIGNALING WITH COSTLY SIGNALS

B.D. BERNHEIM AND A.L. BODOH-CREED

Abstract. We study signaling environments with two features that are common in practice: first, complete-information bliss points are heterogeneous across different types of senders; second, receivers observe many choices by each sender, rather than a single decision. We prove that, ironically, a sufficiently large increase in the weight attached to signaling costs allows senders to signal their true types at arbitrarily low overall cost. As an application, it follows that, when senders take a sufficient number of observable actions, their private information is revealed almost costlessly. Instead of becoming ubiquitous, costly signaling becomes essentially irrelevant.

1. INTRODUCTION

The Spence signaling model (Spence [15]) seeks to explain how information can be credibly conveyed when reputational incentives are either not present or not sufficient to insure honesty. The private information of a sender (she) is conveyed through the inferences made by a receiver (he) based on an action (e.g., length of education) that the sender chooses. In order to credibly convey information, the sender must take an action that is more costly to her than her bliss point action, the choice the sender would make if all agents had complete information (i.e., in the absence of the signaling incentive). This means, of course, that credible signaling requires welfare losses relative to the first-best.

Given the importance of credibly conveying information, the insights derived from the Spence model may lead one to believe that virtually all our decisions are distorted by the incentive to signal our types, and that the resulting pervasive waste must significantly erode the benefits of our social interactions. Our analysis suggests otherwise. In particular, we exhibit reasonably general conditions under which a profusion of observable actions enables people to signal their private information at negligible overall cost relative to the benefits of signaling. Ironically, the proliferation of potential signals can therefore render signaling irrelevant rather than ubiquitous, and lead people to behave as if private information were publicly observable. As an application, we point out that greater transparency, interpreted as an ability to observe a larger set of the sender's actions, can alleviate aggregate signaling distortions.

We first consider a general setting in which different types of senders have different bliss points,[1] and deviations from bliss points entail costs. Intuitively, as the number of observable decisions increases, so should the weight attached to those costs. We prove that, as that weight increases, the ratio of the total costs of signaling to the total benefits declines to zero. Thus, in the limit, senders reveal their information costlessly.

The notion that the total weight attached to signaling costs should increase with the number of observable actions, though intuitive, requires proof. In particular, it is not obvious a priori that senders will choose to send the same signal multiple times, rather than sending extreme signals on occasion. When the payoff for each action enters the sender's utility additively, we provide weak conditions under which the welfare-optimal separating equilibrium is one in which the sender chooses the same action repeatedly. Under those conditions, the optimal equilibrium is isomorphic to that of a model with a single signal, where the weight attached to signaling costs is proportionately greater. Applying our primary result to this setting, we conclude that if agents of different types have different bliss points, then the the ratio of the costs of signaling to the benefits vanishes at a rate of $O\left(\frac{1}{N^{1-\alpha}}\right)$ for any $\alpha > 0$, where N is the number of observable actions. We also consider more general utility functions that allow for non-linear aggregation across the sender's actions, and we provide sufficient conditions under which our main conclusion generalizes.

We prove that heterogeneous bliss points are necessary for our result to hold by studying an example in which all of the sender-types share the same bliss point. In

[1]Signaling models with heterogeneous bliss points are widely employed in the literature. Examples include Spence [16] on signaling with productive educational investments; Mailath [11] on price signaling; Banks [2] on political competition; Miller and Rock [13] on dividend signaling; Bernheim [4] on conformity; Bagwell and Bernheim [3], Ireland [10], and Corneo and Jeanne [8] on conspicuous consumption; Bernheim and Severinov [6] on bequests; Bernheim and Andreoni [1] on fairness; or Bernheim and Bodoh-Creed [5] on decisive leadership.

order for the sender to credibly signal her type, the signal must entail a non-negligible cost for any type that would benefit from mimicking the sender. In a situation with heterogeneous bliss points, these costs are non-negligible only when they are assessed according to the bliss points of the mimicking types: we prove that the signal of a sender of type t entails negligible costs when assessed according to her own bliss point. Of course, if all of the agents share the same bliss point, then the signal must also have a non-negligible cost even for the type t sender. This means that the total cost of signaling cannot vanish as the number of actions increases if agents of different types share the same bliss point,

We are unaware of any other work that has pointed out the relationship between heterogeneous bliss points, the number of observed signals, and the total cost of signaling. However, there is a rich literature studying the properties of models wherein either (or both) the signal or the underlying private information are multi-dimensional. Most of this theoretical literature is dedicated to providing conditions under which a separating equilibrium exists (e.g., [9], [14], and [18]). Our model is more specialized than those considered in this literature, but its restrictive features allow us to cleanly demonstrate our main points.

Section 2 introduces our benchmark model in which the sender's marginal cost of deviating from her bliss point is parameterized abstractly. Section 3 provides several examples of the benchmark model in order to build intuition for the main result, which Section 4 proves. Section 5 provides results supporting our interpretation of the marginal cost parameter as the number of signals observed by the receiver. We close in Section 6. All proofs appear in the appendix.

2. MODEL

A receiver observes the action of a sender. The sender's private information is represented by her type $t \in [\underline{t}, \overline{t}] = T \subset \mathbb{R}$. For our benchmark model we assume the sender chooses an action $a \in \mathbb{R}_+$ that will serve as the signal used by the receiver to make inferences about the sender's type. Action a yields direct utility $\lambda\pi(a,t)$ where π represents the costs and benefits of the action in the absence of any signaling incentive (i.e., with complete information) and $\lambda > 0$ is a decision weight. We assume throughout that $\pi(a,t)$ is continuous. The *bliss point* for an agent of type t is:

$$a_{BP}(t) = \underset{a}{arg\,max}\ \pi(a,t) \tag{1}$$

$a_{BP}(t)$ is the action the sender would choose if her type were publicly observed.

Having observed a, the receiver uses Bayes's rule to form a belief about the sender's type. The belief of the receiver following an observation of action a is denoted $\delta(a) \in \Delta(T)$ where $\Delta(T)$ is the set of Borel measures over T, and we refer to $\delta(a)$ as the receiver's *perception* of the sender. When we focus our analysis on fully separating equilibria, the receiver's equilibrium beliefs place probability 1 on the sender having the type $\widehat{t}(a)$. When convenient we will suppress the arguments of δ and $\widehat{t}$ and refer to the sender as "choosing" the receiver's perception.

Given the receiver's perception, the sender receives benefits $B\left(t, \delta(a)\right)$.[2] The sender's total utility is given by the function:

$$U(a,t;\lambda) = B\left(t, \delta\left(a\right)\right) + \lambda\pi(a,t) \tag{2}$$

In a fully separating equilibrium of Spence's [16] job market model, a represents a level of investment in human capital, $\pi\left(a,t\right)$ represents the payoff from these productive investments, and $B\left(t, \widehat{t}(a)\right)$ represents the wage the agent receives given her type t when she is perceived to be of type $\widehat{t}\left(a\right)$.

We make several assumptions that are easily verified in applications. Throughout we assume the existence of all referenced derivatives. For ease of notation, we first assume that each agent has a single best choice with complete information.

Assumption 1. $a_{BP}(t)$ *is unique and continuous.*

The crux of our argument is an analysis of the costs of signaling when the equilibrium strategy approaches $a_{BP}(t)$. To provide a general analysis, we use polynomial expansions of $\pi(a,t)$ around $(a_{BP}(t),t)$. Assumption 2 part (1) (below) allows us to focus on second order effects and ignore 3^{rd} and higher order effects without loss of

[2]It is straightforward to allow the action a to influence B. If we were to include a in B, then we would require that the partial derivatives $B_a(t,\delta,a)$, $B_{aa}(t,\delta,a)$, and $B_{aaa}(t,\delta,a)$ be defined for all (t,δ,a) and that $B_{aaa}(t,\delta,a)$ be uniformly bounded from above.

generality.[3] Part (2) of Assumption 2 extends the assumption of part (1) to cases where (for example) $\pi_{aa} = 0$ and $\pi_{aaaa} < 0$ — in other words, we can focus on the lowest relevant non-zero derivative and ignore higher order effects.[4]

Assumption 2. *One of the following holds:*

(1) $\pi_{aa}(a,t) < 0$ *at* $a = a_{BP}(t)$ *and there is* $C < \infty$ *such that* $\|\pi_{aaa}(a,t)\| \leq C$ *for all pairs* $(a_{BP}(t), t)$.
(2) *There is* $C < \infty$ *such that for all pairs* $(a_{BP}(t), t)$ *we have* $\frac{\partial^i \pi(a,t)}{\partial a^i} = 0$ *for* $i \in \{2, .., k < \infty\}$, $\frac{\partial^{k+1}\pi(a,t)}{\partial a^{k+1}} \neq 0$, *and* $\left\|\frac{\partial^{k+2}\pi(a,t)}{\partial a^{k+2}}\right\| \leq C$.

The next two assumptions pertain to the benefit function.

Assumption 3. *There is* $\gamma > 0$ *such that* $0 \leq \frac{\partial B(t,\hat{t})}{\partial \hat{t}} \leq \gamma$.

Assumption 4. $B(t,\delta) \in [\underline{B}, \overline{B}]$ *for all* $\delta(a) \in \Delta(T)$ *where* $-\infty < \underline{B} \leq \overline{B} < \infty$. *If the support of* $\delta(a)$ *is* $\mathcal{S}$, *then* $B(t, \delta(a)) \geq \min_{\widehat{t} \in S} B(t, \widehat{t})$.

Assumption 3 bounds the rate at which the sender's benefit can change with her perceived type in a fully separating equilibrium. Assumption 4 imposes a bound on the benefit of signaling. The second half of this assumption bounds the sender's benefit when the receiver is not confident about the sender's type, which may occur following the observation of an off-path action by the sender.

Our next assumption requires that first order changes in type result in first order changes in bliss points. This assumption plays a critical role in characterizing the range of types that would be willing to choose $a_{BP}(t)$ in equilibrium. If Assumption 5 holds then this set will shrink quickly.

Assumption 5. *There exists* $\beta > 0$ *such that for any* $t > t'$, *we have* $a_{BP}(t) - a_{BP}(t') \geq \beta(t - t')$.

Our main result establishes that the standard separating equilibrium exhibits vanishing total signaling costs as $\lambda \to \infty$. Denote the strategy used in the separating

[3] Since we are analyzing actions in the neighborhood of the bliss point, first order effects are absent as $\pi_a(a_{BP}(t), t) = 0$.
[4] $\pi_{aa}(a_{BP}(t), t) = 0$ implies $\pi_{aaa}(a_{BP}(t), t) = 0$, since otherwise $a_{BP}(t)$ would not be a local maximum of $\pi(a, t)$.

equilibrium as $a_{SEP}(t)$. We can write the equilibrium utility for an agent of type t who mimics the action of an agent of type $\widehat{t}$ as:

$$V(\widehat{t}, t; \lambda) = B\left(t, \widehat{t}\right) + \lambda \pi\left(a_{SEP}\left(\widehat{t}; \lambda\right), t\right)$$

By standard arguments $a_{SEP}(t; \lambda)$ is the solution to the following differential equation:

$$\left.\frac{\partial a_{SEP}(\circ; \lambda)}{\partial \widehat{t}}\right|_{\widehat{t}=t} = \frac{-1}{\lambda} \left.\frac{\partial B}{\partial \widehat{t}}\right|_{(t,\widehat{t})=(t,t)} \left.\frac{1}{\pi_a(a, t)}\right|_{(a,t)=(a_{SEP}(t),t)}$$

with the initial condition $a_{SEP}(\underline{t}; \lambda) = a_{BP}(\underline{t})$. The single-crossing property is sufficient to guarantee that this solution is in fact a separating equilibrium, but is stronger than required for our results. To allow for greater generality, we directly assume that the solution is incentive compatible.

Assumption 6. $a_{SEP}(\circ; \lambda)$ *forms a Bayes-Nash equilibrium.*

3. EXAMPLES

We illustrate our framework through two examples. The first has has heterogeneous bliss points. We will show that the difference between the bliss-point and the equilibrium action vanishes at the rate $O\left(\lambda^{-1}\right)$. The high speed of convergence of the equilibrium actions to the bliss points is crucial for the total cost of signaling to vanish.

Example 1. *Suppose* $B(t, \widehat{t}) = \widehat{t}$, $\pi(a, t) = -(a-t)^2$, *and* $T = [0, 1]$. *The bliss points are (obviously)* $a_{BP}(t) = t$. *The ODE defining the fully separating equilibrium is:*

$$\left.\frac{\partial a_{SEP}}{\partial \widehat{t}}\right|_{\widehat{t}=t} = \frac{1}{2\lambda(a_{SEP}(t) - t)}$$

We use the change of variables $z(t) = a_{SEP}(t) - t$. *Solving the inverse ODE yields:*

$$t = -\left[z + \frac{1}{2\lambda} \ln\left(\frac{1}{2\lambda} - z\right)\right] + C$$

The initial condition $z(0) = 0$ *implies:*

$$t = -\left[z + \frac{1}{2\lambda} \ln\left(1 - 2\lambda z\right)\right]$$

Reversing our change of variables and rearranging, we find:

$$z(t) = \frac{1 - e^{2\lambda a_{SEP}(t)}}{2\lambda}$$

The total cost of signaling is then:

$$\lambda z(t)^2 \leq \lambda \left(\frac{1}{2\lambda}\right)^2 = \frac{1}{4\lambda}$$

which is $O\left(\lambda^{-1}\right)$ *as claimed.*

In the second example, the agents share the bliss point of $a_{BP} = 0$, $a_{SEP}(t)$ converges to $a_{BP}(t)$ at the rate $O\left(\lambda^{-0.5}\right)$, and the slow convergence causes the total cost of signaling to be bounded away from 0. Intuitively, the progressively greater bunching of low t types around 0 puts pressure on higher t types to choose higher actions. This pressure is absent in Example 1 since the different types are attracted to different bliss points. In fact, we show that while $a_{SEP}(t;\lambda) \to t$ as $\lambda \to \infty$, the total cost of signaling is invariant to λ.

Example 2. *Suppose* $B(t,\widehat{t}) = \widehat{t}$, $\pi(a,t) = \frac{-a^2}{t+\gamma}, \lambda > 0$, *and* $T = [0,1]$. *The bliss points are all* $a_{BP}(t) = 0$, *which means they are homogenous. The ODE defining the fully separating equilibrium is:*

$$\left.\frac{\partial a_{SEP}}{\partial \widehat{t}}\right|_{\widehat{t}=t} = \frac{t+\gamma}{2\lambda a_{SEP}(t)}$$

We can write this in a more convenient form as:

$$2\lambda a_{SEP}(t) \left.\frac{\partial a_{SEP}}{\partial \widehat{t}}\right|_{\widehat{t}=t} = t + \gamma$$

Integrating both sides and using our initial condition yields:

$$\lambda a_{SEP}(t)^2 = \frac{1}{2}(t+\gamma)^2 - \frac{\gamma^2}{2}$$

The total cost of signaling, $\lambda a_{SEP}(t)^2$, *is invariant with respect to* λ.

4. MAIN RESULT

Our main result proves that the utility obtained by the sender in a fully separating equilibrium approaches the utility she receives with complete information when she chooses her bliss point. If one thinks of $\lambda[\pi(a_{SEP}(t;\lambda),t) - \pi(a_{BP}(t),t)]$ as the total cost of signaling, then our main result implies that the total cost vanishes in the limit as $\lambda \to \infty$. Because our normalization of the utility holds the benefits of signaling fixed as the number of signals increases, one can interpret this result as indicating that the total costs of signaling decline to zero when expressed as a fraction of the benefit as the relative weight attached to signaling costs grows without bound.

Before proceeding to our main result, we prove the following lemma. In addition to serving as the first step of the proof of our main result, the lemma is of additional interest since it characterizes *all* of the equilibria of our signaling model, not just the fully separating equilibrium referenced in our main result. Let $a(t;\lambda)$ denote an equilibrium of the signaling game, and let $\{a(t;\lambda_i)\}_{i=1}^{\infty}$, $\lambda_i \to \infty$, be a convergent sequence of equilibrium strategies. Lemma 1 implies that the limit must be $a_{BP}(t)$.

Lemma 1. *Let Assumptions 1, 2, and 4 hold. Then* $\|a(t;\lambda_i) - a_{BP}(t)\| = O\left(\frac{1}{\sqrt{\lambda_i}}\right)$ *as* $\lambda_i \to \infty$.

Lemma 1 distills a somewhat obvious truth - if the cost of deviating from the bliss-point increases while the benefits remain fixed, then the deviations must shrink as their costs grow. (The lemma also specifies the rate, which is critical for our analysis.) From this result, it follows that pools must vanish as λ_i increases.

For the remainder of the paper we focus on fully separating equilibrium. Before discussing our main theorem, we present a lemma showing that each agent takes an action above her bliss point. The logic is that as λ grows, agents must signal using actions close to their bliss points. If $a_{SEP}(t) < a_{BP}(t)$, then for large enough λ the separating action will equal the bliss point of some type $t' < t$, which gives the agent of type t' incentive to deviate to $a_{BP}(t') = a_{SEP}(t)$.

Lemma 2. *Let Assumptions 1 and 3 hold. Then in any fully separating equilibrium we have for all* t *that* $a(t;\lambda_i) \geq a_{BP}(t)$ *for* λ_i *sufficiently large.*

While Lemma 1 has the virtue of generality, it is not strong enough for us to prove that the cost of signaling declines as λ grows. Consider a cost function of the form $\pi(a,t) = (a-t)^2$, which we studied in Example 2. If the convergence of $a(t)$ to $a_{BP}(t)$ proceeds at a rate of $O\left(\frac{1}{\sqrt{\lambda}}\right)$, the cost of the signal would remain $O(1)$ in the limit as $\lambda \to \infty$. In order to prove that the total cost of the signal vanishes, we need to prove that the convergence of $a(t)$ to $a_{BP}(t)$ is faster.

We provide two versions of our main result. We state both in terms of the convergence of $\pi(a_{SEP}(t), t)$ to $\pi(a_{BP}(t), t)$, but one can also interpret them as implying that the ratio of the benefits to the costs of signaling vanishes at the rate $O\left(\frac{1}{\lambda^{1-\alpha}}\right)$. Theorem 1 applies a *dominance refinement* to limit the beliefs of the receiver following a deviation. The refinement requires that the receiver place 0 probability on the sender having a type t if that type of sender would find the deviation suboptimal relative to her bliss point for any inferences the receiver might hold.[5]

Definition 1. *The receiver's beliefs satisfy the **dominance refinement** if following any off-path action a the receiver places 0 probability on the sender having a type drawn from the set* $\{t : \underline{B} + \lambda\pi(a_{BP}(t), t) \geq \overline{B} + \lambda\pi(a, t)\}$.

The dominance refinement allows us to tighten the convergence rate of Lemma 1, and the faster convergence rate implies that the costs of signaling vanish as $\lambda \to \infty$.

Theorem 1. *Let Assumptions 1, 2, 4, and 5 hold and assume the receiver's beliefs satisfy the dominance refinement. Then* $\lambda\left[\pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t)\right] = O\left(\frac{1}{\sqrt{\lambda}}\right)$.

The second result assumes that both the bliss points and the fully separating equilibrium are continuous, which allows us to ignore the issue of off-path beliefs when making arguments similar to those used in the proof of Theorem 1. With this assumption, we can prove that the rate at which the total cost of signaling vanishes is at least $O\left(\frac{1}{\lambda^{1-\alpha}}\right)$.

Theorem 2. *Let Assumptions 1 - 5 hold and suppose that $a_{BP}(t)$ and $a_{SEP}(t)$ are continuous for all λ. Then* $\lambda\left[\pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t)\right] = O\left(\frac{1}{\lambda^{1-\alpha}}\right)$ *for any* $\alpha > 0$.

It is often possible to build intuition concerning the properties of continuous signaling models by considering discrete analogs. In the current setting, these analogs are less useful than they initially appear. Consider a model with discrete sets of types and actions in which each type has a distinct bliss point. There is plainly some value

[5]Our dominance refinement is similar to the dominance refinement studied in Cho and Kreps [7]. Since the receiver's response to the sender's signal is modeled through B, we provide our formulation of the refinement for clarity.

of λ sufficiently large that no agent would find it optimal to make a discrete deviation from her bliss point even to elicit the best possible reaction from the sender:

$$\underset{\widehat{t}}{arg\,max}\ B\left(t,\widehat{t}\right)$$

A discrete version of our main result follows immediately. Yet this line of intuition is potentially misleading. By similar reasoning, a related conclusion follows for discrete models with homogeneous bliss points (e.g., the discrete analog of Example 2). In particular, as $\lambda \to \infty$ the agents pool on the common bliss point - signaling costs vanish and no information is transmitted - whereas in the continuous setting separation continues to occur as $\lambda \to \infty$ and the total cost of signaling remains nontrivial. This observation suggests that there are subtle differences between the discrete and continuous cases, and that intuition built on the discrete models does not reliably transfer to the continuous models.

The logic behind our theorem is more subtle than the discrete analog suggests. As a first step, Lemma 1 shows that $a_{SEP}(t) - a_{BP}(t) = O\left(\frac{1}{\sqrt{\lambda}}\right)$. When combined with our lower bound on the rate of change of $a_{BP}(t)$ (Assumption 5), we can bound the equilibrium inferences made in response to any deviation. The bounds on the equilibrium inferences combined with our bound on the rate of change of $B(t,\circ)$ (Assumption 3) yields an upper bound on the cost of deviating from $a_{SEP}(t)$ to $a_{BP}(t)$ in terms of the benefit of signaling, $B\left(t,\widehat{t}\right)$. Repeated applications of this bound allows us to prove that $a_{SEP}(t) - a_{BP}(t)$ converges to 0 at a rate of $O\left(\frac{1}{\lambda^{1-\alpha}}\right)$. Combined with the fact that deviations from $a_{BP}(t)$ cause only second order losses in utility for agents of type t, we find that the total cost of the signal in our separating equilibrium vanishes as $O\left(\frac{1}{\lambda^{1-\alpha}}\right)$.

5. MULTIPLE SIGNALS

We now apply Theorem 2 to a setting where the receiver observes multiple signals from the sender simultaneously. We consider a sequence of models indexed by $N \in \mathbb{N}$, where the N^{th} model features a sender who takes N simultaneous actions $a \in \mathbb{R}$ yielding an action vector $\boldsymbol{a}^N \in \mathbb{R}^N$ with the i^{th} component denoted a_i^N. The utility for the sender from these N actions is $\boldsymbol{\pi}_{Agg}(\boldsymbol{a}^N, t)$. We consider three separate cases for the aggregator function $\boldsymbol{\pi}_{Agg}(\boldsymbol{a}^N, t)$. We start with the simplest case wherein $\boldsymbol{\pi}_{Agg}(\boldsymbol{a}^N, t)$ is additively separable and symmetric across actions. This formulation is restrictive, but allows us to characterize the optimal equilibrium. We then consider nonadditive aggregators that satisfy an intuitive separability property, and we identify conditions under which our results continue to hold for separating equilibria that are symmetric in the sense that $a_i^N = a_j^N$ for all i, j. Finally, we prove a version of our main result for general aggregator functions. In the second and third cases we cannot characterize the optimal equilibrium, but our results imply nevertheless that, under the specified conditions, the signaling costs must vanish as the number of signals grows without bound.

5.1. **Additive Aggregators.** Having observed $\boldsymbol{a}^N$, the receiver uses Bayes's rule to form beliefs about the sender's type. Since we focus on fully separating equilibria, in equilibrium the receiver has a degenerate belief placing probability 1 on the sender's type being $\widehat{t}\left(\boldsymbol{a}^N\right)$. Given the receiver's perception, the sender receives utility of $B\left(t, \widehat{t}\left(\boldsymbol{a}^N\right)\right)$. We can therefore write the total utility of the sender in the N^{th} model as:

$$U_N(\boldsymbol{a}^N, t) = B\left(t, \widehat{t}\left(\boldsymbol{a}^N\right)\right) + \sum_{i=1}^{N} \pi\left(a_i^N, t\right)$$

Let $\boldsymbol{a}_{BP}(t) = (a_{BP}(t), a_{BP}(t), ..., a_{BP}(t)) \in \mathbb{R}_+^N$ where $a_{BP}(t) = \underset{a}{arg\,max}\ \pi\left(a, t\right)$ denotes the bliss point of the type t agent in the N-action model.

Consider the symmetric signaling equilibrium where $a_1(t) = ... = a_N(t) = a_{SEP}(t; N)$. We can write the equilibrium utility of an agent of type t that mimics the action of an agent of type $\widehat{t}$ as

$$V_N(\widehat{t}, t) = B\left(t, \widehat{t}\right) + N\pi\left(a_{SEP}\left(\widehat{t}\right), t\right)$$

The function $a_{SEP}(t)$ is defined by the following differential equation

$$\left.\frac{\partial a_{SEP}}{\partial \widehat{t}}\right|_{\widehat{t}=t} = \frac{-1}{N} \left.\frac{\partial B}{\partial \widehat{t}}\right|_{(t,\widehat{t})=(t,t)} \left.\frac{1}{\pi_a(a, t)}\right|_{(a,t)=(a_{SEP}(t),t)}$$

with the initial condition $a_{SEP}(\underline{t}) = a_{BP}(\underline{t})$. The full vector of actions taken in the N^{th} economy is $\boldsymbol{a}_{SEP}^N(t) = (a_{SEP}(t), a_{SEP}(t), ..., a_{SEP}(t)) \in \mathbb{R}^N$.

We now show that the symmetric separating equilibrium is the welfare optimal separating equilibrium for all types, which motivates our focus.

Theorem 3. *Assume that $\pi(a,t)$ is supermodular in (a,t) and for all t and $a > a_{BP}(t)$ we have $\pi_a(a,t) \leq 0$ and $\frac{\pi_{at}(a,t)}{\pi_a(a,t)}$ is weakly increasing in a. For any fixed N, $\boldsymbol{a}_{SEP}^N$ maximizes the payoff of each type of sender relative to any other separating equilibrium.*

The first requirement (supermodularity) is standard. The final two requirements of Theorem 3 are satisfied for standard functional specifications, such as $\pi(a,t) = -(a-t)^2$.

It follows from Theorem 3 that the N-signal setting reduces to a special case of the model of Section 4 where N plays the role of λ. The following corollary to Theorem 2 implies that the ratio of the costs of signaling to the benefits converges to 0 as $N \to \infty$ in the symmetric separating equilibrium.[6]

Corollary 1. *Let Assumptions 1 - 5 hold and suppose that $a_{BP}(t)$ and $a_{SEP}(t)$ are continuous for all N. Then $N\left[\pi(a_{BP}(t),t) - \pi(a_{SEP}(t),t)\right] = O\left(\frac{1}{N^{1-\alpha}}\right)$ for any $\alpha > 0$*

As an application, our analysis has intriguing implications concerning the effects of *transparency.* In a setting where an individual takes many actions, greater transparency implies that a larger fraction of their actions is publicly observable. Trivially, in a symmetric equilibrium, greater transparency reduces the degree to which signaling distorts any single action. Our analysis implies a much stronger property: transparency also reduces the overall signaling distortion and, in the limit, yields the full-information outcome. Of course, in settings where signaling offsets other inefficiencies, greater transparency can be welfare-reducing; see Bernheim and Bodoh-Creed [6] for an application involving competition among politicians who attempt to signal decisiveness.

An astute reader may have observed by this point that one could interpret the one-dimensional signal of Section 2 as the sum of many signals, $a = \sum_{i=1}^N a_i$. Our formulation rules out this version of the multiple-signal problem, because we assume that aggregate signaling costs are given by $\sum_{i=1}^N \pi(a_i,t)$, rather than $\pi(\sum_{i=1}^N a_i, t)$. Moreover, our results obviously do not extend to this alternative. If only the sum of

[6]Theorem 1 yields an analogous corollary, which we omit for the sake of brevity.

the actions matters, breaking a single signal into an increasing number of additive components cannot affect the signaling equilibrium in any material way.

5.2. **Nonadditive Aggregators.** So far in this section, we have assumed that senders aggregate the costs of multiple actions additively. We now turn our attention to non-linear aggregators.. Consider the following utility function:

$$U_N(\boldsymbol{a}^N, t) = B\left(t, \widehat{t}\left(\boldsymbol{a}^N\right)\right) + \boldsymbol{\pi}_{Agg}\left(a_1, .., a_N, t\right)$$

As an example, suppose the cost function has the following form:

$$\boldsymbol{\pi}_{Agg}\left(a_1, .., a_N, t\right) = \sqrt{\sum_{m=1}^{N} \pi(a_m, t)}$$

where $\sqrt{\pi(a,t)}$ satisfies Assumptions 2 and 5.[7] Then for a separating equilibrium with symmetric actions we can write:

$$U_N(\boldsymbol{a}^N, t) = B\left(t, \widehat{t}\left(\boldsymbol{a}^N\right)\right) + \sqrt{N}\sqrt{\pi(a_{SEP}(t), t)}$$

Theorem 2 implies $U_N(a_{BP}(t), t; \lambda) - U_N(a_{SEP}(t; \lambda), t; \lambda) = O\left(\frac{1}{N^{0.5-\alpha}}\right)$ for any $\alpha > 0$, which means that our conclusions generalize.

More generally, suppose one can write the equilibrium utility for a symmetric separating equilibrium, $\boldsymbol{a}_{SEP}(t) = (a_{SEP}(t), a_{SEP}(t), ..., a_{SEP}(t))$, as:

$$U_N(\boldsymbol{a}_{SEP}(t), t) = B\left(t, t\right) + \lambda(N) g\left(a_{SEP}(t), t\right)$$

Assume the functions g and λ satisfy the following two assumptions:

Assumption 7. *$g_{aa}(a,t) < 0$ and there exists $C < \infty$ such that $\|g_{aaa}\| \le C$.*

Assumption 8. *$\lambda(N) \to \infty$ as $N \to \infty$.*

Assumption 7 is analogous to Assumption 2 and allows us to focus our analysis on the second order expansion of the function $g(a,t)$. Assumption 8 requires that $\lambda(N)$, which is analogous to the decision weight of Section 4, diverges to infinity as $N \to \infty$. The additively separable model of Subsection 5.1 and the example leading this subsection satisfy these requirements.

[7]Theorem 3 applies in any situation where $\boldsymbol{\pi}_{Agg}$ is a function of $\sum_{m=1}^{N} \pi(a_m, t)$, which would justify our focus on symmetric separating equilibria.

Under these assumptions, an argument analogous to the proof of Theorem 2 shows that aggregate signaling costs vanish as N grows.[8] For completeness, we state this result as the following corollary, where $a_{BP}(t) = \underset{a}{arg\,max}\ g\left(a,t\right)$.[9]

Corollary 2. *Let Assumptions 1, 3 - 5, 7, and 8 hold and assume that $a_{SEP}(t)$ and $a_{BP}(t)$ are continuous. Then*

$$\lambda(N)\left[g(a_{BP}(t),t) - g(a_{SEP}(t),t)\right] = O\left(\frac{1}{\lambda(N)^{1-\alpha}}\right)$$

for any $\alpha > 0$.

5.3. **General Aggregator Functions.** The previous section generalized our main result to a restricted class of nonadditive aggregators. It is natural to wonder whether our conclusions hinge on those restrictions, or instead follow from modest technical assumptions within a much broader class of environments. Accordingly, in this section we explore further generalizations. As before, we consider a sequence of models, the N^{th} of which allows the sender to choose N actions. The utility of the receiver remains

$$U_N(\boldsymbol{a}^N,t) = B\left(t,\widehat{t}\left(\boldsymbol{a}^N\right)\right) + \boldsymbol{\pi}_{Agg}\left(\boldsymbol{a}^N,t\right)$$

We define $\boldsymbol{a}_{BP}(t)$ as

$$\boldsymbol{a}_{BP}^N(t) = \underset{\boldsymbol{a}^N \in R^N}{arg\,max}\ \boldsymbol{\pi}_{Agg}(\boldsymbol{a}^N,t) \tag{3}$$

We make the following assumptions, which are satisfied by the aggregators studied in Sections 5.1 and 5.2.

Assumption 9. *$\boldsymbol{a}_{BP}^N(t)$ is the unique solution to (3). There exists a scalar $\beta > 0$ such that for all N and any $t > t'$, we have*

$$\boldsymbol{a}_{BP}^N(t) - \boldsymbol{a}_{BP}^N(t') \geq \beta(t-t')\mathbf{1}^N$$

where $\mathbf{1}^N = (1,1,...,1) \in \mathbb{R}^N$.

[8] To see this point, if one replaces λ with $\lambda(N)$ throughout the proof of Theorem 2, the same technical argument applies.

[9] Theorem 1 admits an analogous corollary, which we omit for the sake of brevity.

Assumption 9 requires that an increase in t produces a non-trivial increase in all bliss-point actions, and it resembles Assumption 5 from the one-dimensional case. The lower bound on the rate of change helps determine how the receiver's inferences change if the sender deviates from $\boldsymbol{a}^N_{SEP}(t)$ to $\boldsymbol{a}^N_{BP}(t)$.

Assumption 10. *$\boldsymbol{\pi}_{Agg}(\boldsymbol{a},t)$ is continuous. If $\widetilde{\boldsymbol{a}} > \widehat{\boldsymbol{a}} > \boldsymbol{a}_{BP}(t)$ or $\widetilde{\boldsymbol{a}} < \widehat{\boldsymbol{a}} < \boldsymbol{a}_{BP}(t)$, then $\boldsymbol{\pi}_{Agg}(\boldsymbol{a}_{BP},t) > \boldsymbol{\pi}_{Agg}(\widehat{\boldsymbol{a}},t) > \boldsymbol{\pi}_{Agg}(\widetilde{\boldsymbol{a}},t)$.*

Assumption 10 insures that $\boldsymbol{\pi}_{Agg}$ is continuous, and also provides a partial ordering of $\boldsymbol{\pi}_{Agg}(\circ,t)$. This partial ordering is used in our proof to compare the payoff from deviating to another sender-type's bliss point.

Assumption 11. *$D_{\boldsymbol{a}}\boldsymbol{\pi}_{Agg}(\boldsymbol{a},t) = \left(\frac{\partial\boldsymbol{\pi}_{Agg}(\boldsymbol{a},t)}{\partial\boldsymbol{a}_1},\dots,\frac{\partial\boldsymbol{\pi}_{Agg}(\boldsymbol{a},t)}{\partial\boldsymbol{a}_N}\right)$ exists for all pairs $(a_{BP}(t),t)$ and all of the following hold:*

1. *There exists a sequence $\{\phi_N\}_{N=1}^{\infty}$ such that $\phi_N < 0$, $\sum_{i,j=1}^{N}\frac{\partial^2\boldsymbol{\pi}_{Agg}(\boldsymbol{a}^N_{BP}(t),t)}{\partial\boldsymbol{a}_i\partial\boldsymbol{a}_j} < \phi_N$, and $\phi_N \to -\infty$.*
2. *The ratio of third to second order effects is bounded in the sense that there exists $C < \infty$ such that in an open neighborhood of the pairs $(a_{BP}(t),t)$ we have:*

$$\left\| \frac{\sum_{i,j,k=1}^{N}\frac{\partial^3\boldsymbol{\pi}_{Agg}(\boldsymbol{a}^N,t)}{\partial\boldsymbol{a}_i\partial\boldsymbol{a}_j\partial\boldsymbol{a}_k}}{\sum_{i,j=1}^{N}\frac{\partial^2\boldsymbol{\pi}_{Agg}(\boldsymbol{a}^N,t)}{\partial\boldsymbol{a}_i\partial\boldsymbol{a}_j}} \right\| < C$$

Assumption 11 includes several properties that are necessary for our Taylor expansion argument. Part (1) ensures that a small standardized departure from the agent's bliss point ("one unit" in every dimension) becomes increasingly costly, without bound, as N grows as a result of the second-order terms.[10] In particular, when combined with Assumption 9, it implies that the costs of selecting the bliss point of an agent of a given higher type grows without bound. Part (2) implies that we can neglect the third and higher order effects in the Taylor expansion as $a_{SEP}(t) \to a_{BP}(t)$.

We now prove that the overall signaling costs vanish at a rate of $O(-S_N(t))$ where $S_N(t) = \sum_{i,j=1}^{N}\frac{\partial^2\boldsymbol{\pi}_{Agg}(\boldsymbol{a}^N_{BP}(t),t)}{\partial a_i\partial a_j}(< 0)$. The logic of our argument is similar to that of Theorem 1 in that we constrain the receiver's off-path beliefs using a dominance

[10]Alternatively, we could provide an assumption along the lines of Assumption 2, Part (2), but it would be notationally intensive.

refinement, and then use these restricted beliefs to provide a tight bound on the rate at which the costs of signaling vanish. The formal statement of the dominance refinement for the multidimensional case is the same as for the single dimensional case, except that one interprets the action a as a vector.

Since $\boldsymbol{a}_{BP}^N(t)$ is strictly increasing by Assumption 9, the dominance refinement sharply pins down receiver beliefs following a deviation by type t to $\boldsymbol{a}_{BP}^N(t)$. Since such a deviation is always an option for type t, the restricted beliefs of the receiver bound the cost of signaling. Formally:

Theorem 4. *Let Assumptions 3, 4, and 9 - 11 hold, and assume the receiver's beliefs satisfy the dominance refinement. Then* $\boldsymbol{\pi}_{Agg}\left(\boldsymbol{a}_{BP}^N(t), t\right) - \boldsymbol{\pi}_{Agg}\left(\boldsymbol{a}_{SEP}^N(t), t\right) = O\left(\frac{1}{\sqrt{-\phi_N}}\right)$.

Assumption 11, part 1, insures that $-S_N(t) \leq -\phi_N \to \infty$ as $N \to \infty$, implying that the cost of signaling vanishes in this limit.

The assumptions cited in Theorem 4 are sufficient for our main conclusions, rather than necessary. However, it is important to emphasize that our main conclusions do not hold universally in settings with multidimensional signals and heterogeneous bliss points. We demonstrate this point through a counterexample that violates Assumption 11, part 1, since the second order terms vanish as the sender's action approaches her bliss point.

Example 3. *Consider a utility function of the form:*

$$U_N(\boldsymbol{a}^N, t) = B\left(t, \widehat{t}\left(\boldsymbol{a}^N\right)\right) - \prod_{m=1}^{N} (a_m - t)$$

where $t \in [0,1]$ *and* $a_i \geq 0$. *If we let* $B\left(t, \widehat{t}\right) = \widehat{t}$, *then the exact solution to the ODE defining the fully separating equilibrium is:*

$$a_{SEP}(t) = t^{1/N} + t$$

This implies that the cost of signaling is:

$$\left(a_{SEP}(t) - t\right)^N = t$$

which is invariant with respect to N *(and hence nonvanishing).*

6. CONCLUSION

A common message of most signaling analyses is that credibly conveying information is costly. This note establishes that if agents' bliss points are heterogeneous, then, ironically, a sufficiently large increase in the weight attached to signaling costs allows senders to signal their true types at arbitrarily low cost. As an application, it follows that, when senders take a sufficient number of observable actions, their private information is revealed almost "for free." Instead of becoming ubiquitous, signaling becomes essentially irrelevant. Because inferences about an actor are usually drawn from many actions rather than from a single act, these observations may have broad implications. In some instances, they may imply that signaling effects are unimportant. For example, the multidimensionality of conspicuous consumption raises the possibility that concerns about status may not lead to wasteful signaling, contrary to the theories of Ireland [10], Bagwell and Bernheim [3], Corneo and Jeanne [8], and others. In other instances, our analysis suggests that increases in transparency, defined as an increase in the fraction of decisions that are publicly observable, may ameliorate wasteful signaling.[11]

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Appendix A. PROOF APPENDIX

Lemma 1. *Let Assumptions 1, 2 and 4 hold. Then* $\|a(t;\lambda_i) - a_{BP}(t)\| = O\left(\frac{1}{\sqrt{\lambda_i}}\right)$ *as* $\lambda_i \to \infty$*.*

Proof. The bound on $a(t;\lambda_i) - a_{BP}(t)$ will be derived from the following inequality, which must hold in equilibrium:

$$\lambda_i \left[\pi(a_{BP}(t), t) - \pi(a(t;\lambda_i), t)\right] \leq \overline{B} - \underline{B} \tag{4}$$

We first prove our result for the case where $\pi_{aa}(a_{BP}(t), t) < 0$, and then consider what occurs when $\pi_{aa}(a_{BP}(t), t) = 0$.

The uniqueness of the bliss point from for type t (Assumption 1) implies that $\pi(a_{BP}(t), t) - \pi(a(t;\lambda_i), t) > 0$, and the continuity of $\pi(\circ, t)$ implies $a(t;\lambda_i) - a_{BP}(t) \to$

0 as $\lambda \to \infty$. The Taylor expansion of $\pi(a,t)$ around $(a_{BP}(t),t)$ is:

$$\pi(a_{BP}(t),t) - \pi(a(t;\lambda_i),t) = \frac{-1}{2}\pi_{aa}(a_{BP}(t),t)\left(a(t;\lambda_i) - a_{BP}(t)\right)^2 - \frac{\pi_{aaa}(\xi)}{6}\left(a(t;\lambda_i) - a_{BP}(t)\right)^3 \tag{5}$$

where $\xi \in [a_{BP}(t), a(t;\lambda_i)]$. Suppose $\pi_{aaa}(\xi)\left(a(t;\lambda_i) - a_{BP}(t)\right) < 0$. Then

$$\pi(a_{BP}(t),t) - \pi(a(t;\lambda_i),t) \geq \frac{-1}{2}\pi_{aa}(a_{BP}(t),t)\left(a(t;\lambda_i) - a_{BP}(t)\right)^2$$

Suppose $\pi_{aaa}(\xi)\left(a(t;\lambda_i) - a_{BP}(t)\right) > 0$. Then since $\|\pi_{aaa}(a,t)\| \leq C$ (Assumption 2) and $a(t;\lambda_i) - a_{BP}(t) \to 0$ as $\lambda_i \to \infty$, we can choose λ_i^* such that for all $\lambda_i > \lambda_i^*$

$$\left\|\frac{\pi_{aaa}(\xi)}{6}\left(a(t;\lambda_i) - a_{BP}(t)\right)\right\| \leq \frac{-1}{4}\pi_{aa}(a_{BP}(t),t)$$

which means we have

$$\pi(a_{BP}(t),t) - \pi(a(t;\lambda_i),t) \geq \frac{-1}{4}\pi_{aa}(a_{BP}(t),t)\left(a(t;\lambda_i) - a_{BP}(t)\right)^2 \tag{6}$$

In either case, using equation 4 we can write

$$\left(a(t;\lambda_i) - a_{BP}(t)\right)^2\left(\frac{-1}{4}\pi_{aa}(a_{BP}(t),t)\right) \leq \frac{\overline{B} - \underline{B}}{\lambda}$$

which in turn yields

$$\|a(t;\lambda_i) - a_{BP}(t)\| \leq \frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4(\overline{B} - \underline{B})}{\pi_{aa}(a_{BP}(t),t)}} \tag{7}$$

In the case where $\pi_{aa}(a_{BP}(t),t) = 0$, the first higher-order partial derivative that is nonzero must be of an even-numbered order.[12] If the order of the first nonzero derivative with respect to a is k, we can use Assumption 2, Part (2) to make an

[12]To see this more formally, if $\pi_{aa}(a_{BP}(t),t) = 0$, then Equation 5 has the form

$$\pi(a_{BP}(t),t) - \pi(a(t;\lambda_i),t) = \frac{-1}{6}\pi_{aaa}(a_{BP}(t),t)\left(a_{BP}(t) - a(t;\lambda_i)\right)^3 - \frac{\pi_{aaaa}(\xi)}{24}\left(a_{BP}(t) - a(t;\lambda_i)\right)^4$$

The fourth order term is negligible relative to the third order term for λ sufficiently large (i.e., $a_{BP}(t)$ and $a(t;\lambda_i)$ sufficiently close), so if $\pi_{aaa}(a_{BP}(t),t) \neq 0$ utility would be increased by either a slight increase or decrease in a from $a_{BP}(t)$. But then $a_{BP}(t)$ cannot be optimal, and from this contradiction we conclude that $\pi_{aa}(a_{BP}(t),t) = 0$ entails $\pi_{aaa}(a_{BP}(t),t) = 0$. We conclude that if $\pi_{aa}(a_{BP}(t),t) = 0$, then the first higher-order partial derivative that is nonzero must be of an even-numbered order.

argument analogous to that provided above to show convergence at the rate

$$\|a(t;\lambda_i) - a_{BP}(t)\| = O\left(\frac{1}{\lambda_i^{k/2}}\right)$$

□

Lemma 2. *Let Assumptions 1 and 3 hold. Then in any fully separating equilibrium we have for all t that $a(t;\lambda_i) \geq a_{BP}(t)$ for λ_i sufficiently large.*

Proof. First note that the claim holds for $\underline{t}$ since $a_{SEP}(\underline{t}) = a_{BP}(t)$ in any fully separating equilibrium. Now consider an arbitrary $t > \underline{t}$ and suppose our claim fails to hold. Then for any $\lambda^* > 0$ there exists $\lambda > \lambda^*$ such that $a_{SEP}(t) < a_{BP}(t)$. Equation 7 implies that for such a choice of λ sufficiently large that $a_{SEP}(\underline{t}) \leq a_{SEP}(t) < a_{BP}(t)$. But from the continuity of $a_{BP}(t)$ (Assumption 1), there exists $t' \in [\underline{t}, t)$ such that $a_{BP}(t') = a_{SEP}(t)$. Since the equilibrium is fully separating, it must be that $a_{SEP}(t') \neq a_{BP}(t')$ or types t and t' would pool. Since $B(t, \widehat{t})$ is increasing in $\widehat{t}$ (Assumption 3), t' can profitably deviate to $a_{BP}(t')$ and have the receiver infer her type to be $\widehat{t} = t > t'$. This contradiction proves our claim. □

Theorem 1. *Let Assumptions 1, 2, 4, and 5 hold and assume the receiver's beliefs satisfy the dominance refinement. Then $\lambda\left[\pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t)\right] = O\left(\frac{1}{\sqrt{\lambda}}\right)$ for any $\alpha > 0$.*

Proof. For the duration of the proof we assume that λ is sufficiently large that $a_{SEP}(t) \geq a_{BP}(t)$ as per Lemma 2. The goal of this proof is to tighten the bound provided by Lemma 1. To that end, suppose agent t deviates from $a_{SEP}(t)$ to $a_{BP}(t)$. The proof of Lemma 1 showed that even if a deviation from her bliss point could shift $B(t, \delta(a))$ from $\underline{B}$ to $\overline{B}$, the largest she would be willing to deviate is of $O(\lambda^{-0.5})$. If there exists t' such that $a_{SEP}(t') = a_{BP}(t)$, then the receiver infers (incorrectly) that the sender is of type t'. Using Equation 7 we can write:

$$a_{BP}(t) = a_{SEP}(t') \leq a_{BP}(t') + \frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4\left(\overline{B} - \underline{B}\right)}{\pi_{aa}(a_{BP}(t), t)}}$$

From Assumption 5 we have

$$\beta(t-t') \leq a_{BP}(t) - a_{BP}(t') \leq \frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}$$

$$t' \geq t - \frac{1}{\beta}\frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}} \tag{8}$$

If there is no t' such that $a_{SEP}(t') = a_{BP}(t)$, then Lemma 1 combined with the dominance refinement implies that the receiver must believe that the sender has some type t' that satisfies $\|a_{BP}(t') - a_{BP}(t)\| = O(\lambda^{-0.5})$. Using Equation 7 we can write this formally as:

$$a_{BP}(t) \leq a_{BP}(t') + \frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}$$

As before, Assumption 5 implies

$$t' \geq t - \frac{1}{\beta}\frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}} \tag{9}$$

We now use Equation 8 or 9 as appropriate to bound the effect on the signaling incentive $B(t,\widehat{t})$ more tightly. The core idea is that the the cost of signaling can be no larger than the benefit received by having the receiver infer that the sender has type t instead of type t'. Using Assumption 3 we have:

$$\begin{aligned}\pi(a_{BP}(t),t) - \pi(a_{SEP}(t),t) &\leq \frac{1}{\lambda}\left[B(t,t) - B\left(t, t - \frac{1}{\beta}\frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}\right)\right] \\ &\leq \frac{\gamma}{\lambda^{1.5}}\left[\frac{1}{\beta}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}\right]\end{aligned}$$

This then yields:

$$\begin{aligned}\lambda\left[\pi(a_{BP}(t),t) - \pi(a_{SEP}(t),t)\right] &\leq \frac{\gamma}{\sqrt{\lambda}}\left[\frac{1}{\beta}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}\right] \\ &= O\left(\frac{1}{\sqrt{\lambda}}\right)\end{aligned}$$

□

Theorem 2. *Let Assumptions 1 - 5 hold and suppose that $a_{BP}(t)$ and $a_{SEP}(t)$ are continuous for all λ. Then* $\lambda\left[\pi(a_{BP}(t),t)-\pi(a_{SEP}(t),t)\right]=O\left(\frac{1}{\lambda^{1-\alpha}}\right)$ *for any* $\alpha>0$.

Proof. The goal of this proof is also to tighten the bound provided by Lemma 1. The key difference with the prior argument is that due to the continuity of $a_{BP}(t)$ and $a_{SEP}(t)$, if an agent deviates from $a_{SEP}(t)$ to $a_{BP}(t)$, then there exists a t' such that the receiver believes that sender has type $t'<t$ — namely t' such that $a_{SEP}(t')=a_{BP}(t)$. This allows us to avoid the issue of off-path beliefs.

To that end, suppose agent t deviates from $a_{SEP}(t)$ to $a_{BP}(t)$. The type t' such that $a_{SEP}(t')=a_{BP}(t)$ defines the inference made by the receiver following the deviation by type t. Equation 7 yields:

$$a_{SEP}(t')=a_{BP}(t)\leq a_{BP}(t')+\frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}$$

From Assumption 5 we have

$$\begin{aligned}\beta(t-t') &\leq a_{BP}(t)-a_{BP}(t')\leq\frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}\\ t' &\geq t-\frac{1}{\beta}\frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}\end{aligned}$$

Therefore deviating from $a_{SEP}(t)$ to $a_{BP}(t)$ leads to an inference that the sender's type is at least $t-\frac{1}{\beta}\frac{1}{\sqrt{\lambda}}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}$. This allows us to bound the effect on the signaling incentive $B(t,\widehat{t})$ more tightly. Using Assumption 3 we have:

$$\begin{aligned}\pi(a_{BP}(t),t)-\pi(a_{SEP}(t),t) &\leq \frac{1}{\lambda}\left[B(t,t)-B\left(t,t-\frac{1}{\beta}\frac{1}{\lambda}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}\right)\right]\\ &\leq \frac{\gamma}{\lambda^{1.5}}\left[\frac{1}{\beta}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}\right]\end{aligned}$$

If Assumption 2, Part (1) holds, we can then write

$$\begin{aligned}\frac{-1}{4}\pi_{aa}(a_{BP}(t),t)\left[a_{SEP}(t)-a_{BP}(t)\right]^2 &\leq \pi(a_{BP}(t),t)-\pi(a_{SEP}(t),t)\\ &\leq \frac{1}{\lambda^{1.5}}\frac{\gamma}{\beta}\sqrt{\frac{-4\left(\overline{B}-\underline{B}\right)}{\pi_{aa}(a_{BP}(t),t)}}\end{aligned}$$

where the first inequality can be derived using an argument essentially identical to that used to derive Equation 6 from the proof of Lemma 1. Simplifying we have:

$$a_{SEP}(t)-a_{BP}(t)\leq \frac{1}{\lambda^{3/4}}\sqrt{\frac{\gamma}{\beta}}\left(\frac{-4}{\pi_{aa}(a_{BP}(t),t)}\right)^{3/4}\left(\overline{B}-\underline{B}\right)^{1/4}$$

If Assumption 2, Part (2) applies, we can make an analogous argument that yields an even tighter bound on $a_{SEP}(t)-a_{BP}(t)$ as in the proof for Lemma 1.

Iterating this process K times yields

$$a_{SEP}(t)-a_{BP}(t)\leq \frac{C_K}{\lambda^{1-0.5^K}}$$

When we use this in our Taylor expansion we get

$$\begin{aligned}\pi(a_{BP}(t),t)-\pi(a_{SEP}(t),t) &= \left(a_{SEP}(t)-a_{BP}(t)\right)^2\left(\frac{-1}{2}\pi_{aa}(a_{BP}(t),t)-\right.\\ &\qquad\left.\frac{\pi_{aaa}(\xi)}{6}\left(a_{SEP}(t)-a_{BP}(t)\right)\right)\\ &= \frac{C_K^2}{\lambda^{2-0.5^{K-1}}}\left(\frac{-1}{2}\pi_{aa}(a_{BP}(t),t)-\frac{\pi_{aaa}(\xi)}{6}\left(\frac{C_K}{\lambda^{1-0.5^K}}\right)\right)\end{aligned}$$

Using the negligibility of the third order terms, we find

$$\begin{aligned}\lambda\left[\pi(a_{BP}(t),t)-\pi(a_{SEP}(t),t)\right] &\leq \frac{C_K^2}{\lambda^{1-0.5^{K-1}}}\left(\frac{-1}{2}\pi_{aa}(a_{BP}(t),t)-\frac{\pi_{aaa}(\xi)}{6}\left(\frac{C_K}{\lambda^{1-0.5^K}}\right)\right)\\ &= O\left(\frac{1}{\lambda^{1-0.5^{K-1}}}\right)\end{aligned}$$

as desired. □

Theorem 3. *Assume that $\pi(a,t)$ is supermodular in (a,t) and for all t and $a > a_{BP}(t)$ we have $\pi_a(a,t)\leq 0$ and $\frac{\pi_{at}(a,t)}{\pi_a(a,t)}$ is weakly increasing in a. For any fixed N,*

$\boldsymbol{a}_{SEP}^{N}$ *maximizes the payoff of each type of sender relative to any other separating equilibrium.*

Proof. Suppose we have a separating equilibrium with action functions $\mathbf{a}(t) = (a_1(t), ..., a_N(t))$. Defining

$$\Gamma(\mathbf{a}, \mathbf{t}) \equiv \sum_{i=1}^{N} \pi\left(a_i, t\right) \tag{10}$$

we can write the first-order condition for type t's optimal choice as:

$$\left.\frac{\partial B(t,\widehat{t})}{\partial \widehat{t}}\right|_{\widehat{t}=t} + \sum_{i=1}^{N} \frac{\partial \pi\left(a_i, t\right)}{\partial a_i}\frac{da_i(t)}{dt} = 0 \tag{11}$$

We are interested in determining type t's total payoff in equilibrium. If we let $V\left(t, \widehat{t}\right)$ denote the payoff of a type t sender having chosen the action of type $\widehat{t}$, we have by definition:

$$V\left(t, \widehat{t}\right) = B\left(t, \widehat{t}\right) + \Gamma(\mathbf{a}(\widehat{t}), t) \tag{12}$$

and the Envelope Theorem yields:

$$\frac{dV\left(t,t\right)}{dt} = \frac{dB\left(t,t\right)}{dt} + \frac{\partial \Gamma(\mathbf{a}(t), t)}{\partial t}$$

Notice that only the final term depends on the particular separating equilibrium. Let $\mathbf{a}^0$ denote the symmetric separating equilibrium with payoffs V^0, and $\mathbf{a}^A$ denote an asymmetric separating equilibrium with payoffs V^A. To demonstrate that payoffs in the symmetric separating equilibrium are strictly higher than in the asymmetric separating equilibrium, we will establish the following Property (capitalized for clarity of subsequent references): if it were the case for some t that either (i) $V^0(t,t) = V^A(t,t)$ and $\mathbf{a}^0(t) \neq \mathbf{a}^A(t)$, or (ii) $V^0(t,t) < V^A(t,t)$, then we would have $\frac{dV^0(t,t)}{dt} > \frac{dV^A(t,t)}{dt}$.[13]

To understand why this Property delivers the desired conclusion, note that $V^A(t',t') - V^0(t',t')$ would shrink as t' rises over $[\underline{t}, t]$ if the property holds. But then we would have a violation of the boundary condition $V^0(\underline{t},\underline{t}) = V^A(\underline{t},\underline{t}) = B(\underline{t},\underline{t}) - \Gamma(\boldsymbol{a}_{BP}(\underline{t}), \underline{t})$

[13]Suppose our claim is true. Then if either condition (i) or (ii) holds for t, then condition (ii) must hold for all $t' \in (\underline{t}, t)$.

where $\boldsymbol{a}_{BP}(t) = (a_{BP}(t), ..., a_{BP}(t)) \in \mathbb{R}^N$. In light of Equation 12, we can rewrite the Property as follows: if it were the case for some t that either (i)$'$ $\Gamma(\mathbf{a}^0(t), t) = \Gamma(\mathbf{a}^A(t), t)$ and $\mathbf{a}^0(t) \neq \mathbf{a}^A(t)$, or (ii)$'$ $\Gamma(\mathbf{a}^0(t), t) > \Gamma(\mathbf{a}^A(t), t)$, then we would have $\frac{\partial \Gamma(\mathbf{a}^0(t),t)}{\partial t} > \frac{\partial \Gamma(\mathbf{a}^A(t),t)}{\partial t}$.

We now establish the Property. Supposing condition (i)$'$ were satisfied for some $t > \underline{t}$, we would begin by defining:[14]

$$\overline{a}_m = \begin{cases} a_m^A(t) \text{ if } a_m^A(t) \geq a_{BP}(t) \\ a \geq a_{BP}(t) \text{ s.t. } \pi(a,t) = \pi(a_m^A(t), t) \text{ otherwise} \end{cases}$$

Let $Q \equiv \left\{ m \mid a_m^A(t) < a_{BP}(t) \right\}$. Then:

$$\frac{\partial \Gamma(\overline{\boldsymbol{a}}, t)}{\partial t} - \frac{\partial \Gamma(\boldsymbol{a}^A(t), t)}{\partial t} = \sum_{m \in Q} \pi_t(\overline{a}_m, t) - \pi_t(a_m^A, t) \geq 0$$

with strict inequality if Q is non-empty.

If $\boldsymbol{a}^0(t) = \overline{\boldsymbol{a}}$, we are done. If not, then since $\Gamma(\boldsymbol{a}^A(t), t) = \Gamma(\overline{\boldsymbol{a}}, t)$ by construction, there must exist i and j such that $\overline{a}_i > a^0(t) > \overline{a}_j$. Define the function $\widetilde{\boldsymbol{a}}(a_i)$ as follows: $\widetilde{a}_i(a_i) = a_i$, $\widetilde{a}_k(a_i) = \overline{a}_k$ for $k \neq i, j$, and $\Gamma(\widetilde{a}(a_i), t) = \Gamma(\overline{\boldsymbol{a}}, t)$. In other words, $\widetilde{a}_j(a_i)$ indicates how a_j must vary in response to changes in a_i to keep the value of Γ constant at its equilibrium value. Implicit differentiation reveals that:

$$\left. \frac{d\widetilde{a}_j}{da_i} \right|_{a_i = \overline{a}_i} = -\frac{\pi_a(\overline{a}_i, t)}{\pi_a(\widetilde{a}_j(\overline{a}_i), t)} < 0$$

Plainly, there exists a unique value $a_i^e > a_{BP}(t)$ such that $\widetilde{a}_j(a_i^e) = a_i^e$. For $a_i \in [a_i^e, \overline{a}_i(t))$, so we have:

$$\begin{aligned} \frac{d}{da_i}\left(\frac{\partial \Gamma(\widetilde{\mathbf{a}}(a_i), t)}{\partial t}\right) &= \frac{d}{da_i}\left(\sum_{i=1}^{N} \pi_t(\widetilde{a}_i(a_i), t)\right)_{a_i = \overline{a}_i} \\ &= \pi_{at}(a_i, t) + \pi_{at}(\widetilde{a}_j(a_i), t) \left. \frac{d\widetilde{a}_j}{da_i} \right|_{a_i = a_i} \\ &= \pi_{at}(a_i, t) - \pi_{at}(\widetilde{a}_j(a_i), t) \frac{\pi_a(a_i, t)}{\pi_a(\widetilde{a}_j(a_i), t)} < 0 \end{aligned}$$

[14]This step sets computes a cost-equivalent signal to $\boldsymbol{a}^A$ that has the intuitive property that $a_m^A \geq a_{BP}(t)$.

where we have used the fact that since $\overline{a}_i \geq a_i \geq a_i^e \geq \widetilde{a}_j(a_i) > a_{BP}(t)$ (which implies $\pi_a(\overline{a}_i, t) < 0$) and our assumption that for $a > a_{BP}(t)$ we have $\pi_a(a,t) \leq 0$ and:

$$\frac{\pi_{at}(a_i,t)}{\pi_a(a_i,t)} > \frac{\pi_{at}(\widetilde{a}_j(\overline{a}_i),t)}{\pi_a(\widetilde{a}_j(\overline{a}_i),t)}$$

If follows that $\frac{\partial \Gamma(\widetilde{a}(a_i^e),t)}{\partial t} > \frac{\partial \Gamma(\overline{a},t)}{\partial t}$ since $\overline{a}_i > a^0(t)$ is being reduced in this equalization step. Through repeated application of this equalization argument, we conclude that $\frac{\partial \Gamma(a^0(c),t)}{\partial t} > \frac{\partial \Gamma(\overline{a},t)}{\partial t} \geq \frac{\partial \Gamma(a^A(t),t)}{\partial t}$, as desired.

Next, supposing condition (ii)$'$ were satisfied for some $t > \underline{t}$, we would begin by defining a' s.t. $a'_1 = a'_2 = ... = a'_N > a_{BP}(t)$ and $\Gamma(a',t) = \Gamma(a^A(t),t)$. By the same argument as for condition (i)$'$, we infer $\frac{\partial \Gamma(a',t)}{\partial t} \geq \frac{\partial \Gamma(a^A(t),t)}{\partial t}$.[15] Because $\Gamma(a^0(t),t) > \Gamma(a^A(t),t) = \Gamma(a',t)$ by assumption, we have $a_m^0(t) > a'_m$. From our assumption of supermodularity we conclude:

$$\frac{\partial \Gamma(a^0(c),t)}{\partial t} - \frac{\partial \Gamma(a',t)}{\partial t} = \sum_{m=1}^{N} \pi_t(a_m^0,t) - \pi_t(a'_m,t) \geq 0$$

It follows that $\frac{\partial \Gamma(a^0(c),t)}{\partial t} > \frac{\partial \Gamma(a^A(c),t)}{\partial t}$, as desired.

Having established that the Property holds, the Proposition follows for the reasons given above. □

Theorem 4. *Let Assumptions 3, 4, and 9 - 11 hold, and assume the receiver's beliefs satisfy the dominance refinement. Then* $\boldsymbol{\pi}_{Agg}\left(\boldsymbol{a}_{BP}^N(t),t\right) - \boldsymbol{\pi}_{Agg}\left(\boldsymbol{a}_{SEP}^N(t),t\right) = O\left(\frac{1}{\sqrt{-\phi_N}}\right)$.

Proof. For the duration of the proof let $S_N(t) = \sum_{i,j=1}^N \frac{\partial^2 \boldsymbol{\pi}_{Agg}(\boldsymbol{a}_{BP}^N(t),t)}{\partial a_i \partial a_j} (< 0)$. We begin by imposing a bound on the size of the possible cost of signaling. Consider a deviation by type t from $a_{BP}(t)$ to $a_{BP}(t) + \delta \mathbf{1}^N$ where $\delta \geq 0$. Using algebra similar to that employed in the proof of Lemma 1, it is straightforward to show using a Taylor series approximation of $\boldsymbol{\pi}_{Agg}$ that for the following to hold

$$\boldsymbol{\pi}_{Agg}\left(\boldsymbol{a}_{BP}^N(t),t\right) - \boldsymbol{\pi}_{Agg}\left(\boldsymbol{a}_{BP}^N(t) + \delta\mathbf{1}^N,t\right) \leq \overline{B} - \underline{B} \tag{13}$$

[15]The inequality is weak because we include the possibility that $a' = a^A(t)$.

we must have:

$$\delta \leq \sqrt{\frac{-4(\overline{B}-\underline{B})}{S_N(t)}} = \overline{\delta} \tag{14}$$

Equation 14 combined with Assumption 10 implies that it cannot be be the case that $\boldsymbol{a}_{SEP}^N(t) > \boldsymbol{a}_{BP}^N(t) + \overline{\delta}\mathbf{1}^N$. Although the Taylor series argument underlying Equation 14 applies for small δ, Assumption 10 insures that Equation 14 is a necessary condition for Equation 13 for all δ.

Suppose that an agent of type t deviates from $\boldsymbol{a}_{SEP}^N(t)$ to $\boldsymbol{a}_{BP}^N(t)$ and there exists a type t' such that $\boldsymbol{a}_{BP}^N(t) = \boldsymbol{a}_{SEP}^N(t')$. The receiver infers (incorrectly) that the sender is of type t'. Using Equation 14 we can write:

$$\boldsymbol{a}_{BP}^N(t) = \boldsymbol{a}_{SEP}^N(t') \leq \boldsymbol{a}_{BP}^N(t') + \sqrt{\frac{-4(\overline{B}-\underline{B})}{S_N(t')}}\mathbf{1}^N$$

From Assumption 9 we have

$$\begin{aligned}\beta(t-t')\mathbf{1}^N &\leq \boldsymbol{a}_{BP}^N(t) - \boldsymbol{a}_{BP}^N(t') \leq \sqrt{\frac{-4(\overline{B}-\underline{B})}{S_N(t')}}\mathbf{1}^N \\ t' &\geq t - \frac{1}{\beta}\sqrt{\frac{-4(\overline{B}-\underline{B})}{S_N(t')}} \qquad (15)\end{aligned}$$

If there is no t' such that $\boldsymbol{a}_{BP}^N(t) = \boldsymbol{a}_{SEP}^N(t')$, then Equation 14 combined with the dominance refinement implies that the receiver must believe that the sender has some type t' that satisfies:

$$\boldsymbol{a}_{BP}^N(t) \leq \boldsymbol{a}_{BP}^N(t') + \sqrt{\frac{-4(\overline{B}-\underline{B})}{S_N(t')}}\mathbf{1}^N$$

As before, Assumption 9 implies

$$t' \geq t - \frac{1}{\beta}\sqrt{\frac{-4(\overline{B}-\underline{B})}{S_N(t')}} \tag{16}$$

We now use Equation 15 or 16 as appropriate to bound the effect on the signaling incentive $B(t,\widehat{t})$ more tightly. The core idea is that the the cost of signaling can be no larger than the benefit received by having the receiver infer that the sender has type t instead of type t'. Using Assumption 3 we have:

$$\begin{aligned}
\boldsymbol{\pi}_{Agg}\left(\boldsymbol{a}^N_{BP}(t),t\right) - \boldsymbol{\pi}_{Agg}\left(\boldsymbol{a}^N_{SEP}(t),t\right) &\leq B(t,t) - B\left(t, t - \frac{1}{\beta}\sqrt{\frac{-4(\overline{B}-\underline{B})}{S_N(t')}}\right) \\
&\leq \frac{\gamma}{\beta}\sqrt{\frac{-4(\overline{B}-\underline{B})}{S_N(t')}} \\
&= O\left(\frac{1}{\sqrt{-\phi_N}}\right)
\end{aligned}$$

□