HOW EFFICIENT ARE DECENTRALIZED AUCTION PLATFORMS?

AARON L. BODOH-CREED - UNIVERSITY OF CALIFORNIA, BERKELEY
JÖRN BOEHNKE - HARVARD UNIVERSITY
BRENT HICKMAN - QUEEN’S UNIVERSITY

Abstract. We model a decentralized, dynamic auction market platform in which a continuum of buyers and sellers participate in simultaneous, single-unit auctions each period. Our model accounts for the endogenous entry of agents and the impact of intertemporal optimization on bids. We estimate the structural primitives of our model using Kindle sales on eBay. We find that just over one third of Kindle auctions on eBay result in an inefficient allocation with deadweight loss amounting to 14% of total possible market surplus. We also find that partial centralization—for example, running half as many 2-unit, uniform-price auctions each day—would eliminate a large fraction of the inefficiency, but yield lower seller revenues. Our results also highlight the importance of understanding platform composition effects—selection of agents into the market—in assessing the implications of market redesign. We also prove that the equilibrium of our model with a continuum of buyers and sellers is an approximate equilibrium of the analogous model with a finite number of agents.

Keywords: Dynamic Auctions, Approximate Equilibrium, Internet Markets.

JEL Codes: C73, D4, L1

Date: August 2018.

We thank (in no particular order) Lanier Benkard, John van den Berghe, Michael Dinerstein, Dan Quint, and Alex Torgovitsky for helpful comments and feedback. We also thank seminar participants at the University of Wisconsin–Madison and University of Chicago, as well as participants of the 2014 IIOC Conference (Northwestern University), the 2014 Conference on Auctions, Competition, Regulation, and Public Policy (Lancaster University), 2015 Econometric Society World Congress (Montreal), 2015 INFORMS Conference (Philadelphia), and the 2016 Econometric Society North American Summer Meetings (University of Pennsylvania). We are also greatly indebted to Vincent Johnson and Steven Mohr (UChicago SSCS), without whose dedicated technical support this project—from web scraping to estimation—would not have been possible. Jörn Boehnke gratefully acknowledges the support of the Center of Mathematical Sciences and Applications at Harvard University during the course of this work.
1. INTRODUCTION

Online market platforms are increasingly important in today’s economy. For example, in 2014 eBay reported USD$82.95 billion in sales volume and 8.5% annual growth after nearly two decades in business. Other platforms with hundreds of millions or billions of dollars of transactions include StubHub (event tickets) and Upwork (contract workers). Since many participants are exchanging a broad array of products on these platforms, each platform has sophisticated search tools to help users find partners to transact with.

Given the power of modern search algorithms and the thickness of the markets, one might conjecture that these platforms would do an excellent job of matching buyers and sellers, eliminating market frictions, and generating efficient trade. This conjecture is particularly compelling in cases where the products are homogenous and buyer and seller reputation are not significant barriers to trade. Our goal is to test this conjecture by estimating a novel model of the eBay auction platform using data on sales of new Amazon Kindle Fire tablets.

On the eBay platform a large number of participants compete in a large number of auctions each day, and buyers and sellers can participate across successive days. In this paper we provide a rich model of such an auction platform in which a continuum of buyers is matched to a continuum of seller auctions each period. After matching has occurred, each single-unit auction is executed independently, auction winners (and the associated sellers) exit the market, losing bidders move on to the next period, and new bidders enter at the end of each period. We include a costly per-period entry decision to capture the time and effort costs of participation. We use an extensive dataset on new Amazon Kindle Fire tablets to estimate the structural model primitives such as the matching process that allocates potential buyers to auction listings, the monetized cost of participation, and the steady-state distributions of buyer valuations and seller reserve prices. While the participation cost we find is low, on the order of $0.10, it turns out to be an important regulator of the number and types of buyers in the market.

On the empirical front, we make several contributions to the literature on identifying auction models. A key feature of our estimator is that it requires only observables that are readily available on many platform websites. In particular, we are able to identify the average number of buyers per auction without assuming that we observe all of the bidders in each auction. If bid submission times are randomly ordered, then some auction participants with an intent to bid may be prematurely priced out of the spot-market before they get a chance to submit their bid. Therefore, the total number of unique bidders within a given eBay auction constitutes a lower bound on the actual number of competitors. Our nonparametric identification argument for the dynamic structural model requires only that

\footnote{Information downloaded from \url{https://investors.ebayinc.com/secfiling.cfm?filingID=1065088-15-54&CIK=1065088} on 11/17/2015.}
we observe this random lower bound on the number of competitors, the seller reserve price, and the highest losing bid within each auction.

Our identification strategy lets us separately identify bid shading (i.e., bidding strictly below one’s private valuation) due to the use of a nontruthful pricing rule (e.g., a first-price auction) and bid shading due to intertemporal incentives. From a buyer’s perspective, failure to win an auction today is no tragedy since he can return tomorrow and bid again, which implies there is an opportunity cost to winning today. We show that when the spot-market pricing rule is non-second-price—so that the winner’s bid may directly affect the sale price—then the additional, static demand shading incentive is layered on top of the dynamic demand shading incentive in an intuitive way that allows for straightforward econometric identification. This represents what we refer to as a “plug-and-play” property: identification of the dynamic structural model is obtained under any auction format where identification is known for the static, one-shot game. This is important since many electronic auction pricing rules (including eBay’s) deviate from the second-price form in empirically relevant ways. Given a value for the time discount factor, we show that the degree of demand shading is nonparametrically identified.

To make this concrete, Figure 1 plots the equilibrium bidding strategies under first and second-price auction rules in a dynamic auction market given the economic primitives we estimate. The 45-degree line can be interpreted as the equilibrium of a static second-price auction that omits the dynamic opportunity cost. The bid shading caused by the intertemporal opportunity cost is represented by the difference between the bidding strategy under the second-price auction (SPA) mechanism in a dynamic setting and the 45-degree line. The additional demand shading caused by switching to a nontruthful mechanism such as a first-price auction (FPA) rule is represented by the gap between the first and second-price bidding strategies in our dynamic setting. We also include the probability of winning in

\[^{2}\]We chose to plot bidding strategies under first- and second-price spot-markets because they represent the polar extremes of static demand shading incentives among common pricing mechanisms.
the conclusion we draw from the plot is that dynamic incentives tied to opportunity costs play a clearly dominant role in shaping behavior: for all bidder types with non-trivial win probabilities, the demand shading caused by intertemporal opportunity costs is an order of magnitude larger than the static demand shading. In other words, it is more important for bidders to understand intertemporal opportunity costs than how to strategically bid under a nontruthful pricing rule.

Having estimated the structural model, we move on to investigate the efficiency of the market. Platform markets like eBay exist for the purpose of reducing frictions that impede trade and converting some of the resulting efficiency gains into profits for the platform’s owners. With this in mind, it is natural to assess how closely eBay approaches the ideal of fully efficient trade. To fix ideas, suppose there are exactly two listings and four bidders, two bidders with high valuations and two bidders with low valuations. The social planner’s preferred outcome is one where each auction attracts one high value bidder as this would guarantee that the high value bidders win in any monotone bidding equilibrium. However, when there is randomness in the bidder-listing match process, there will be a positive probability that one auction listing will not have a high-value participant, meaning a low value bidder wins at a low price and a high-value bidder loses. Another way of putting it is that the matching frictions mean some auctions have too much competition and others too little relative to an efficient allocation.

We begin our counterfactual analysis by using two separate methods to measure inefficiency under the current market conditions. Our first method relies on the raw data. Using our estimated buyer-seller ratio we can count the number of times in our data that a bidder with an inefficiently low value (i.e., low bid) won an auction and prevented a high value bidder from receiving that item. This method gives a lower bound on the prevalence of inefficient allocations because, for example, it cannot detect scenarios where multiple high-value losers attended the same auction. We find that within the listings for new Kindles, at least 27.6% of all auctions allocate goods to buyers with inefficiently low valuations.

In our second method, we use the structural estimates to get a precise value for the fraction of auction listings that award an object to a buyer whose private value is inefficiently low. This method also allows us to quantify the deadweight loss, which is defined as the average difference in value between high-value losers and low-value winners. We calculate that 36% of new Kindle listings result in inefficient allocations and that the total deadweight loss amounts to roughly 14% of potential market surplus.

Next, we explore the implications of alternative spot-market mechanisms eBay could use to improve efficiency. Specifically, we consider the welfare cost of eBay’s choice to use single-unit auctions, which we refer to as decentralization. We use our estimates to analyze outcomes of alternative markets where, instead of single-unit auctions, eBay runs \( u \)-unit, uniform-price auctions, and we use \( u \) as a measure of the market’s centralization. The most
extreme version of this counterfactual would be a single multi-unit, uniform-price auction each day. We find that aggregating auctions together so that eBay runs half as many auction listings for 2-units each day recovers 35% of the welfare loss by improving the efficiency of the allocation, while running a quarter as many 4-unit auctions recovers over 57% of the welfare loss. However, centralization reduces the expected sale price, which in turn reduces eBay’s revenues. In addition to being a vehicle for analyzing the welfare losses, we believe that centralizing auctions is a practical design strategy in settings wherein the goods are homogenous (e.g., new Kindles).

While we conduct our estimates within the particular eBay context, we believe that the degree of welfare losses we find should temper optimistic expectations that online platform markets can eliminate all market frictions. In addition, the market centralization solution we propose is far more broadly applicable than a single, isolated eBay market, and we believe it could be worthwhile considering similar centralization-oriented designs in other platform market contexts. For example, centralization may be possible for standardized back-office tasks that are bought and sold on Upwork.

Our last counterfactual explores the importance of controlling for platform composition when computing counterfactuals or redesigning platform markets. We repeat our welfare and revenue exercises, but we hold the distribution and number of buyers fixed. Under this mis-specified model, one would predict that welfare gains from redesign are larger and that centralizing the market increases seller revenue. These results are both primarily driven by the effect that the composition of the pool of buyers has on bidder behavior.

Finally, our paper also makes a contribution to the theory underlying the large market models we use. The notion of a large market approximation, sometimes referred to as an Oblivious Equilibrium, is not novel to this paper. However, proving a formal relationship between a model with a continuum of players and the finite markets that exist in reality is difficult when the market mechanism admits discontinuities, and an auction setting provides several points where such discontinuities can arise. In Appendix C, we prove that despite these issues, one can view an equilibrium of the model with a continuum of buyers and sellers as an ε-equilibrium of the model with a finite number of buyers and sellers. We view this result as a justification for our use of the continuum model in our estimation and counterfactual exercises.

The remainder of this paper has the following structure. In Section 2 we develop a theory of bidding within a dynamic platform based on a model with a continuum of buyers and sellers. In Section 3 we use this model to specify a parsimonious structural model of eBay, which we show is identified from observables. We also propose a semi-nonparametric estimator based on B-splines. In Section 4 we present our model estimates, and Section 5 contains our counterfactual analyses. Proofs for the technical claims in the main text are relegated to Appendix A. Appendix B provides an algorithm for computing counterfactuals.
and testing for uniqueness. In Appendix C, we prove that our model with a continuum of agents approximates an analogous model with a large, but finite, number of participants. Appendix D provides details on our estimation techniques and algorithms. Appendix E empirically investigates our random matching assumption. Appendix F provides a variety of additional counterfactuals on revenue, participation costs, and seller incentives.

1.1. Related Literature. The most closely related predecessor to our paper is Backus and Lewis [2016], which studies a model of eBay where bidders participate in a sequence of single-unit, second-price auctions. Backus and Lewis [2016] focuses on identifying a buyer demand with flexible substitution patterns across different goods and the possibility that individual bidder demand evolves over time. Their model could be used to compute welfare counterfactuals like ours in a setting with homogenous goods, but this (obviously) excludes substitution between products. However, since Backus and Lewis [2016] does not include participation costs or a procedure for estimating the measure and type distribution of buyers entering the market, the model remains silent on how market structure influences entry and exit through the interaction of changing auction format and participation costs. Our structure also decomposes static and dynamic demand shading incentives, which allows us to compare the relative strengths of these forces. This decomposition also links existing results on estimating static auction models with the literature on dynamic auction markets. Finally, we prove that our large market model approximates a more realistic model with a finite number of agents. Due to the different foci and contributions of each work, we view our papers as complementary.

Another related paper is Hickman [2010], which shows that the pricing rule on eBay is actually a hybrid of the first-price and a second-price auction formats. This is because minimum bid increments imply there is a positive probability that the winner will pay her own bid. Hickman, Hubbard, and Paarsch [2016] explore empirical implications of the non-standard pricing rule on eBay within a static auction model and show that estimates may become biased in an economically significant way if it is ignored. We build on these two papers in the following ways. First, our model incorporates both dynamic demand shading incentives and static demand shading incentives. Second, we extend the estimator of Hickman et al. [2016] to allow for binding reserve prices, which affects identification of the bidder arrival process and the private value distribution in complicated ways.

Our methodology analyzes approximate equilibria played by a large number of agents, which has been a prominent theme in the microeconomics and industrial organization literatures. Due to the broad scope of this literature, we provide a brief survey and a sample of the important papers related to the topic. Early papers focused on conditions under which underlying game-theoretic models could be used as strategic microfoundations for general equilibrium models (e.g., Hildenbrand [1974], Roberts and Postlewaite [1976], Otani and Siciliano [1990], Jackson and Manelli [1997]). Other papers focused on conditions under which
a generic game played by a finite number of agents approaches some limit game played by a continuum of agents (e.g., Green [1980], Green [1984], and Sabourian [1990]). A recent branch of this literature applies these ideas to simplify the analysis of large markets with an eye to real-world applications (e.g., Fudenberg, Levine, and Pesendorfer [1998]; MacLean and Postlewaite [2002]; Budish [2008]; Weintraub, Benkard, and Roy [2008]; Krishnamurthy, Johari, and Sundararajan [2014]; and Azevedo and Leshno [2016]).

Nekipelov [2007] and Hopenhayn and Saeedi [2016] develop models of intra-auction price dynamics with repeated bidding in a single auction. Their goal is to rationalize common empirical patterns concerning the timing of bids. In our model, we abstract away from intra-auction dynamics, and instead we concentrate on inter-auction dynamics and how future periods shape bidding incentives today. Peters and Severinov [2006] develop a model of a multi-unit auction environment similar to eBay with the goal of studying the sorting of buyers into sellers’ auctions in a static setting.

The first paper we are aware of that attempts to estimate a model of a sequence of auctions is Jofre-Bonet and Pesendorfer [2003], which applies the pioneering identification and estimation strategy of Guerre, Perrigne, and Vuong [2000] to a dynamic setting. Building on Jofre-Bonet and Pesendorfer [2003], Balat [2013] and Groeger [2014] also focus on capacity constraints and/or learning-by-doing in procurement settings (highway construction contracts). In this line of research, allocative efficiency within a given spot market auction is a concern because with a relatively small number of bidders, past auction outcomes induce bidder asymmetry through capacity constraints and/or learning by doing. In contrast, we study a large market setting with single-unit demands, where consumer types remain constant over time. While a given spot market auction is guaranteed to allocate efficiently within the set of bidders that show up, the set of auctions that run within a period will be collectively inefficient due to the role of randomness in matching of bidders to auctions.

Donald, Paarsch, and Robert [2006] estimated a model of sequential timber auctions where a small number of firms with multi-unit demand compete over time.

2. A MODEL OF PLATFORM MARKETS

Before entering the eBay market, the buyer considers her own value for the good, makes an assessment of the opportunity cost of winning, and formulates her bid. We assume that the time a bidder chooses to enter the market is driven by factors exogenous to eBay (e.g., the schedule of work breaks), which means the buyer only considers bidding on a small and randomly selected fraction of the auctions that close during that day. If a buyer wins the spot-market auction, then she does not participate in future days. Remaining buyers return to eBay the next day to place a bid. We summarize the timing in Figure 2.

2.1. Model Primitives. The market evolves in discrete time with periods indexed $t \in \{0, 1, 2, \ldots\}$. We treat sellers as a source of exogenous supply. This modeling choice is driven
by the fact that sellers face very weak incentives to set the optimal starting price, which makes estimates based on a model of seller behavior less credible. In each period there is a measure of sellers with starting prices described by cumulative density function (CDF) $G_R$ with support $[0, r]$. $G_R$ may have a mass point, but only at the lowest possible reserve price, $r = 0$, and has a probability density function (PDF) $g_R(R | R > 0)$ that is strictly bounded away from zero over the rest of its support.

We refer to the set of buyers present at the start of period $t$ as potential entrants; at the beginning of each period they make decisions based on the observed number and type distribution of the other potential entrants and their own types. Each period, the first choice a potential entrant must make is whether or not to enter the market and participate in the platform. We denote the choice to participate as Enter and refer to the agents that make this choice as entrants. The choice to not participate is denoted Out, and any agent choosing Out leaves the game permanently and earns a payoff of 0 in every future period.

Throughout we assume that the goods for sale are homogenous and that buyers have demand for a single unit. Each buyer’s value for the good is her private information, which we denote as $v$. A buyer that wins a good on the eBay platform and pays a price of $p$ receives a payoff in that period of $v - p - \kappa$, where $\kappa$ is a per-period participation cost paid by entrants regardless of whether they win. We assume $\kappa > 0$; this may reflect the opportunity cost of time spent searching for a listing and participating in the market, or it may reflect an actual monetary participation fee that the platform designer imposes. If an entrant does not engage in trade, her payoff is simply $-\kappa$.

A continuum of potential entrants with measure $\mu$ is added to the economy at the end of each period, and each of the newly added agents has a value drawn independently from CDF $T_V(\cdot)$ with PDF $t_V(\cdot)$. These private values are persistent and do not change with time. We assume that $t_V$ is strictly positive over the support $[0, 1]$. The total set of potential entrants therefore includes both the newly added agents and those buyers that failed to

---

3In Appendix F, we show that the sellers earn less than $1 in increased revenue by moving from a starting price of $0 to the revenue maximizing starting price.

4The lowest starting price on eBay is $0.99, but this does not affect our theoretical results. eBay also allows sellers to choose reservation prices that are hidden from buyers, but this is done so infrequently that we ignore it in our modeling.
Win in the previous period and are carried over to the current period. A generic, measure 0 buyer is denoted using the subscript $i$.

After choosing Enter, each entrant formulates a strategic bid in a simultaneous-move spot market without knowing either the number or identity of the other agents participating in the particular auction to which she is matched. The form of the random matching process, the distribution of entrant types, and the exogenous distribution of starting prices is known to agents at the point when they choose their bids. If a measure $C$ of buyers chooses to enter the auction market—including some new potential entrants and holdovers from previous periods—they are randomly assigned to auctions, with each auction receiving a random number of bidders $K \sim \pi(k; C) = Pr\{K = k\}$. We refer to $C$ as the market tightness parameter since it is the buyer-seller ratio.

Assumption 2.1. We require that $\pi$ satisfy the following conditions:

2. $\pi(1; C) > 0$.
3. $\pi$ is continuous in $C$.

Part (1) requires that all of the buyers be matched into auctions. If we allowed $E[K] > C$, then in aggregate the auctions would be assigned more bidders than entrant buyers that exist in the market. If we allowed $E[K] < C$, then a positive measure of bidders would not be assigned to any auction. This is why $C$ appears as an argument of the bidder arrival process $\pi$, because the latter must be consistent with the former in equilibrium. However, our theory requires no functional form assumptions on $\pi$, as there are many such distributions that could be consistent with a given value of $C$. Part (2) ensures that an entrant with the lowest value amongst all entrants can win an auction with positive probability. Part (3) is a continuity assumption that we use in the proof that an equilibrium exists.

From the perspective of a single bidder, let $M$ denote a random variable representing the number of competitors she faces, and let $\pi_M(M; C)$ denote its probability mass function (PMF). Her beliefs over $M$ are given by (see [Myerson 1998])

$$\pi_M(m; C) = Pr\{m \text{ opponents}|C\} = \pi(m + 1; C) \frac{(m + 1)}{E[K]}.$$

The random matching process results in the stochastic independence of the valuations of the bidders assigned to any particular auction. To complete the description of the game, we need to characterize the matching of bidders to auctions when the measure of entering bidders vanishes. In this “no participation by buyers” limit (i.e., $C \to 0$), each bidder that does choose to enter is the sole bidder in the auction with arbitrarily high probability.

---

5See Appendix A for details on topologies used in our analysis.

6Such an assumption is necessary to evaluate deviations by a buyer from a candidate equilibrium wherein no buyers enter the market.
Assumption 2.2. $\pi_M(0; C) \to 1$ as $C \to 0$.

Examples that satisfy Assumptions 2.1 and 2.2 include a Poisson distribution with parameter $C$, generalized Poisson, and a geometric distribution with parameter $C^{-1}$.

Finally, note that our continuum model is “large” in the sense that the actions of individual bidders have no effect on the aggregate distribution of auction outcomes. However, the actions of individual bidders have a large effect on the auction to which they have been assigned.

We now address a few potential objections to our framing of the bidding model. First, we assume that the buyer formulates her bid before entering the market, rather than updating her bidding strategy in real time as the observed price path within the current auction unfolds. As we discuss in the introduction (see Figure 1) and in Section 4, intertemporal incentives that are independent of any single auction play a far more important role in determining optimal bids than the competitive environment of today’s auction. In equilibrium, the bidder takes into account the expected static bid-shading incentive, but she has relatively weak incentives to update her bid based on the particular details of the auction she is matched into. The weak incentives to update her bid suggest it is not unreasonable to assume she does not tailor her bid to the observed price path.

Second, we assume a bidder does not repeatedly bid within the same auction. As we discuss in Section 4.1, we only consider bids arriving in the last 60 minutes of the auction. One reason we do this is to avoid the question of how to handle dynamic behavior within an auction. As it stands, we see few instances in the data of a bidder returning to increase her bid if she has already bid within the narrow window we consider at the close of each auction.

Third, by assuming a buyer participates in an auction closing near her time of entry, we rule out strategic selection into particular auctions. This assumption would be violated if, for example, at the time of entry, bidders could predict which auctions were likely to close with a low price. To investigate this possibility, we refer to the raw data and define the relative price (RP) as the median price for auctions ending within 24 hours before/after a given auction minus the final sale price for that same auction. When the RP is large and positive, the bidder chose well; otherwise, she did not. To test whether bidders can predict an auction’s RP, we first restricted attention to the set of bidders whose bidding affected the price path during the final 60 minutes of the auction. We regressed the RP on the current price when those bidders first entered the auction, and found a small $R^2$ value of 0.109. Because many of those bidders who shape the terminal price path enter prior to the final 60 minutes of the auction, we also regressed RP on the current price at various points in time. We found that the $R^2$ of these regressions with 60 minutes remaining was 0.34, meaning that price predictability was low even for auctions closing in the near future. Moreover, at only 60 seconds prior to close of bidding, over a third of the variation in
relative price cannot be captured by current price. Unless a bidder is willing to return to eBay dozens of times each day, price predictability is minimal. We conclude that bidders have limited ability to select into auctions due to the low predictability of relative prices, and therefore, randomness plays a significant role in determining who is matched to each auction. We present further details of this analysis and other metrics on price predictability in Appendix E.

Fourth, one might also worry that buyers can see features of the auction that are unobservable to us that might be proxied for by, for example, the starting price of the auction \cite{Roberts2013}. If this were a significant issue, one might find correlation between the starting price and the closing price of the auction. We find in our data that the correlation coefficient between the starting and the closing price is -0.015 and statistically indistinguishable from zero. A lack of significance, of course, does not prove that there is no selection at work. We discuss this issue in more depth in Section 4.

2.2. Equilibrium. Our analysis focuses on stationary equilibria where the aggregate market states are constant across periods. We let $C$ denote the measure of the set of all potential entrants (newly added and carried over from the prior period), and we denote the distribution of all potential entrant types by $F_V$. Thus, $\mu$ and $T_V$ are primitives, while $C$ and $F_V$ are equilibrium objects. The state variables describing the economy in a given period are a vector $(C, F_V, G_R)$. A symmetric bidding strategy is a function $\beta : [0, 1] \rightarrow [0, 1]$ with a typical bid denoted $\beta(v)$. We do not consider asymmetric bidding strategies.

The entry decision is a function $\zeta : [0, 1] \rightarrow \{0, 1\}$, where $\zeta(v) = 1$ indicates the agent enters the market and $\zeta(v) = 0$ that the agent stays out of the market. $\sigma = (\beta, \zeta)$ denotes the full strategy. These strategies explicitly condition on the agent’s private information and implicitly condition on the equilibrium values of the (stationary) aggregate state variables.

The spot-market auction mechanism is defined by an allocation rule and a pricing rule that are functions of the reservation price and bids. The allocation rule $x(b, C, F_V, G_R | \sigma)$ is a binary random variable taking on a value of 1 (0) in the event that the buyer wins (loses) an auction with a bid of $b$, given the aggregate state $(C, F_V, G_R)$ and a common strategy $\sigma$. The pricing rule $p(b, C, F_V, G_R | \sigma)$ is a random variable describing the transfer from the buyer to the seller conditional on a bid of $b$, given $(C, F_V, G_R)$ and $\sigma$. We assume throughout that $p(b, C, F_V, G_R | \sigma) \leq b$. Both $x$ and $p$ are defined by the auction mechanism used on the platform. We develop theoretical results in terms that apply for arbitrary spot-market.

\footnote{In Appendix C we discuss how to extend our model to the nonstationary case and what is lost by assuming stationarity.}

\footnote{The character roman-C here is the measure of the set of potential entrants, whereas calligraphic-C (used in Section 2.1) is the measure of the set of actual entrants. The difference between the two is that C includes new potential entrants arriving in the current period who will choose not to enter, while $C$ does not.}

\footnote{Proposition 2.3, part 3, proves that only a countable set of buyer types have multiple best responses, which in turn means that asymmetric bidding strategies would have to agree for almost all types.}
auction formats—e.g., first-price, second-price, etc.—so that our model may serve as a general framework for platform markets in a variety of settings.

To simplify notation we also define

\[ \chi(b,C,F_V,G_R | \sigma) = E \left[ x(b,C,F_V,G_R | \sigma) \right] \quad \text{and} \quad \rho(b,C,F_V,G_R | \sigma) = E \left[ p(b,C,F_V,G_R | \sigma) \right]. \]

Hereafter we suppress the notation for the aggregate state variables \( C, F_V, \) and \( G_R \) and strategy \( \sigma \) in the functions \( \chi \) and \( \rho \). Note that \( \rho \) represents expected transfers that are not conditional on sale. That is, each entering bidder has an ex-ante expectation of paying \( \rho(b) \) in the spot-market, although under any winner-pay pricing rule only one bidder will pay a positive amount, ex-post.

All agents discount future payoffs using a per-period discount factor \( \delta \in (0,1) \). The value function given a (symmetric) strategy vector \( \sigma = (\beta, \zeta) \) played by all agents is:

\[
V(v,C,F_V,G_R | \sigma) = \zeta(v) \left[ \chi[\beta(v)]v - \rho[\beta(v)] - \kappa + (1 - \chi[\beta(v)])\delta V(v,C,F_V,G_R | \sigma) \right] \\
+ (1 - \zeta(v))\delta V(v,C,F_V,G_R | \sigma) \\
= \zeta(v) \left[ \chi[\beta(v)](v - \delta V(v,C,F_V,G_R | \sigma)) - \rho[\beta(v)] - \kappa) \right] + \delta V(v,C,F_V,G_R | \sigma) 
\]

Where confusion will not result, we suppress the notation for the aggregate state variables \( C, F_V, \) and \( G_R \) and strategy \( \sigma \) in the value function \( V \). When describing agent behavior, we refer to “agents best-responding to the aggregate state,” which we think better captures the economic intuition than the more common “agents best responding to the actions of the other players.” This is analogous to describing an agent in a general equilibrium economy as best responding to prices rather than the actions that generate the prices.

Suppose that the distribution of types \( F_V \) and the strategy \( \sigma \) generate a distribution of bids \( G_B \) in the auctions on the platform. We let \( \beta^s \) denote the best-response in the static version (i.e., when \( \delta = 0 \)) of the spot-market mechanism given a distribution of bids \( G_B \) and starting prices \( G_R \). One useful property of our model is that we can describe the best-response to the aggregate state variables in terms of \( \beta^s \). To see this, we first define a bidder’s private value minus her opportunity cost as her dynamic value, denoted \( \bar{v}_v \equiv v - \delta V(v) \). If a buyer enters the market, her optimal bid is defined by:

\[
\argmax_b \chi[b](v - \delta V(v)) - p[b] = \argmax_b \chi[b]\bar{v}_v - p[b] 
\]

But this is the problem faced by a buyer with value \( \bar{v}_v \) in the static form of the spot-market. Thus, \( \beta(v) = \beta^s(v - \delta V(v)) \) is a best-response bidding strategy to the aggregate state \( (C,F_V,G_R) \) and a strategy vector \( (\beta, \zeta) \) played by the other buyers.

\(^{10}\)Proposition 2.6 proves an equilibrium exists, and agents must have a weakly positive expected utility in equilibrium since they would otherwise exit. Recall that private values \( v \in [0,1] \) are bounded and therefore \( p \in [0,1] \) in equilibrium. Since an agent can only win one auction and an infinite discounted sum of the entry fees is finite, expected equilibrium payoffs must be finite. This implies the existence of a value function.
In most static spot-market mechanisms, one’s equilibrium bid is chosen to balance out opposing forces: a higher bid will increase the chance of winning, but it may also raise the price one pays as well. Whenever the second force is present, bidders shade their demand; henceforth, we refer to this as static incentives. Intertemporal dynamics introduce an additional demand shading incentive: if the spot-market price is sufficiently high today, then a bidder would prefer to wait in expectation of lower prices tomorrow. Therefore, even when the spot-market game follows a second-price rule, rational bidders in equilibrium engage in demand shading; we refer to this as dynamic incentives.

Proposition 2.3 summarizes four useful properties of the best-response, \( \tilde{\beta}, \tilde{\zeta} \), to \( (C, F_V, G_R) \).

Proposition 2.3. The best-response strategy \( (\tilde{\beta}, \tilde{\zeta}) \) satisfies:

1. \( \tilde{\beta}(v) = \beta^s(v - \delta V(v)) \) is a best-response bidding strategy.
2. \( \tilde{\beta}(v) \) is increasing in \( v \) if \( \chi(b) \) is increasing.
3. \( \tilde{\beta}(v) \) is strictly increasing in \( v \) if \( \chi(b) \) is differentiable and strictly increasing in \( b \). In addition, \( \tilde{\beta}(v) \) is uniquely defined for a set of \( v \) of Lebesgue measure 1.
4. There exists a cutoff \( \tilde{e} \) such that \( \tilde{\zeta}(v) = 1 \) if and only if \( v \geq \tilde{e} \).

Part (1) summarizes the relationship between the static and dynamic auctions. Part (2) uses the supermodularity of the buyer’s decision problem to show that best-responses by the bidders must be monotonically increasing. Part (3), which holds in our application, proves that if \( \chi(b) \) is strictly increasing, then the best-response function is strictly increasing. The strict monotonicity implies that almost all buyer-types have a unique best response, which also implies that the best-response function is symmetric across agents. Part (4) implies that we can describe the best-response strategy as \( (\tilde{\beta}, \tilde{e}) \), where \( \tilde{\zeta} \) has been replaced with the cutoff \( \tilde{e} \).

With these results in hand, we can define our notion of stationary equilibrium.

Definition 2.4. The strategy vector \( \sigma = (\beta, e) \) and the states \( C \) and \( F_V \) are a Stationary Competitive Equilibrium (SCE) if for all bidder values \( v \) we have:

1. Optimal Bids: \( \beta(v) = \arg \max_b \chi(b)(v - \delta V(v)) - \rho(b) \)
2. Optimal Entry: The entry cutoff is determined by the equation:
   \[
   \chi[\beta(e)]e - \rho[\beta(e)] = \kappa
   \]
3. Stationarity: The measure and distribution of agents entering the market equals those of the exiting agents:
   \[
   \text{For all } v \geq e, \quad \mu_{TV}(v) = \chi(\beta(v))f_V(v)C
   \]

Part (1) of Definition 2.4 implies the bidding strategy of a buyer is optimal given that she enters. Part (2) requires that the lowest valuation buyer that chooses to enter must be
indifferent between entering and staying out. Since the equilibrium is stationary, buyers with a value \( v < e \) prefer to stay out in every period (and hence buyers with these values exit the game immediately), while buyers with a value \( v > e \) strictly prefer to enter in every period. The marginal buyer, \( v = e \), is indifferent between entering and staying out and earning a payoff of 0, which means this buyer’s continuation value is 0 (and the buyer faces no dynamic incentives). Therefore, the marginal agent’s utility in any given period is:

\[
V(e) = \chi[\beta(e)]e - \rho[\beta(e)] - \kappa = 0
\]

which yields the equation in part (2) of Definition 2.4. Part (3) of Definition 2.4 requires that the state variables \( C \) and \( F_V \) be consistent with the laws of motion of the game. For an economy to be stationary, the distribution and measure of buyers that win auctions and exit the game must be replaced by an identical distribution of new entrants. Recall that \( \mu \) is the measure of buyers entering each period with distribution \( t \), so the left-hand side of Equation 6 describes the mass of agents of each type entering the game each period. \( \chi(\beta(v)) \) describes the probability an agent of type \( v \) wins and exits the economy, so \( \chi(\beta(v))f_V(v)C \) is the mass of agents of type \( v \) who win an auction and exit the game each period.

To get some insight into the forces driving agent behavior in our model, consider the SPA spot-market mechanism. Since it is an equilibrium in weakly undominated strategies for a bidder to bid her value in a static SPA, the SCE strategy is \( \beta(v) = v - \delta V(v) \).\(^{11}\) In the static, one-shot setting, the opportunity cost (i.e., \( V(v) \)) is 0 as outside options are assumed not to exist. In our dynamic model, the opportunity cost of winning today is the continuation value the bidder receives if she returns to the market to bid again in a future period.

The following assumption restricts the best-respondes of the bidders. It is stated in terms of the (static) spot-market best-responses since the properties of the static models have been characterized in the past. For example, Assumption 2.5 is trivially satisfied in the symmetric, weakly-undominated equilibrium of the SPA.\(^{12}\) Note that Assumption 2.5 places restrictions on the best-reply correspondence (not the equilibrium).

**Assumption 2.5.** Let \( Q[0, \bar{\eta}] \) denote the space of measures over \([0,1]\) that admit pdfs bounded from above by \( 0 < \bar{\eta} < \infty \). For any \( \bar{\eta} \), the best-response bidding strategy of the spot-market mechanism, \( \tilde{\beta}^s \), is continuous with respect to any \( G_B \in Q[0, \bar{\eta}] \); any distribution of starting prices, \( G_R \), that has full-support with atoms only at 0; and the market tightness parameter, \( C \).


\(^{12}\)Lizzeri and Persico [2000] established it for the first-price auction and the all-pay auction; and Hickman [2010] proved Assumption 2.5 for the auction mechanism on eBay.
We can choose $\varphi \in (0, 1)$ such that for any static best response $\tilde{\beta}^s$ to $(\lambda, G_B, G_R)$ where $G_B \in \mathcal{Q} [0, q], q < \infty$, and any $v > v'$ we have:

\[
\tilde{\beta}^s(v) - \tilde{\beta}^s(v') \in \left[ \varphi(v - v'), \frac{v - v'}{\varphi} \right]
\]  

(8)

Assumption 2.5 provides continuity properties that are essential for our proof of equilibrium existence. Our proof relies on a fixed point argument that requires the best-responses of the buyers and the market aggregates be continuous with respect to each other, which is the crucial contribution of the first half of Assumption 2.5. The remainder of the assumption mandates bounds on the slope of the bidding function. The lower bound insures that there will not be atoms in the bid distribution, which would destroy the model’s continuity. The upper bound insures the set of bidding strategies we need to consider is Lipschitz continuous (and hence equicontinuous and compact). Together continuity and compactness allow us to apply Schauder’s fixed point theorem to prove an equilibrium exists. We discuss the issue of uniqueness of the equilibrium in Section 5.

**Proposition 2.6.** A stationary competitive equilibrium exists, and a positive mass of buyers choose to enter the market if $\kappa$ is not too large.

Finally, one must acknowledge that our model assumes a continuum of agents, whereas the actual eBay market is only used by a finite number of buyers and sellers. The size of the state-space of an analogous finite model grows exponentially in the number of agents, making it impossible to solve the model or generate counterfactuals. In Appendix C we prove that an SCE is an “approximate equilibrium” of a finite model with sufficiently many agents, meaning that any individual agent has at most a small incentive to deviate from the stationary strategy when all agents play the SCE strategy. Intuitively, the stationary state of the continuum model is a good approximation of the aggregate state of a finite model with many agents, so the SCE bidding strategy will be approximately optimal for the state realized in the finite model. If there is little to be gained by costly monitoring of and optimizing with respect to the state realized in the finite-agent model, then bidders would find it optimal to act “as if” the state is fixed at the SCE value.

Proving such a result forces us to state concretely the finite-agent market structures that we believe our model approximates. In our case, we assume that as the market grows, a large number of auctions occur each day and the agents can only participate in a randomly chosen auction each day. In addition, approximation results typically rely on the continuity of the model, and our auction setting includes a large number of potential discontinuities that make it non-obvious that such an approximation result holds.
3. AN EMPIRICAL MODEL OF DYNAMIC PLATFORM MARKET BIDDING

We now shift to developing a structural model based on the SCE concept. Letting $L$ denote the number of sampled auctions, the observables, $\{\tilde{k}_l, r_l, y_l\}_{l=1}^L$, include $\tilde{k}_l$, the observed number of bidders within the $l$th auction; $r_l$, the reserve price; and $y_l$, the highest losing bid. We assume only that the highest losing bid is available for reasons discussed in Section 4.1. We assume throughout that $(\tilde{k}_l, r_l, y_l)$-triples are independent across auctions $l = 1, \ldots, L$ and that the data-generating process (DGP) is a SCE of our model. Our DGP corresponds to only a single realization of the market tightness parameter $C$, but we are able to nonparametrically identify the bidder matching process $\pi(\cdot; C)$ present in the DGP. For that reason we suppress the second argument during our discussion on identification. However, our counterfactual exercises will induce exit by low-value bidders, thus resulting in values of $C$ that are not present in the data. To facilitate pinning down counterfactual market structures later on, our estimator will use a finite-dimensional parametric assumption about $\pi(\cdot; C)$.

For simplicity of discussion, consider the decision problem of a bidder who has decided to enter and finds herself competing within a spot-market auction; we will refer to her as bidder 1. As before, the number of opponents she faces is $M \equiv K - 1 \geq 0$, which follows distribution $\pi_M(\cdot)$. Prior to bidding, bidder 1 observes her private valuation $v$ and she views her opponents’ private values as independent realizations of a random variable $V \sim F_V$ having strictly positive density $f_V$ on support $[v, \bar{v}]$ with $\bar{v} \geq 0$. The theory from the previous section depicted a set of potential buyers, some of whom choose to enter the bidding market and some of whom don’t, with Equation 5 determining the relevant entry cutoff. Since we are unable to collect real-world observations on non-entrants, we adopt the convention that $F_V$ is the steady-state distribution of buyer types who choose Enter in the DGP. Let $\underline{v} = \bar{v}$ denote the type that is just indifferent to entering, leading to the following formula that we refer to as the “zero surplus condition:”

**Assumption 3.1.** $V(\underline{v}) = 0^{14}$

Bidder 1 views the bids of her opponents as a random variable $B = \beta(V)$, which follows distribution $G_B(B) = F_V[\beta^{-1}(B)]$ with support $[\underline{B}, \bar{B}]$. Let $B_M$ denote the maximal bid among all of 1’s opponents, and we adopt the convention that $B_M = 0$ if there are no other opponent bidders. The distribution of $B_M$ is then $G_{B_M}(B_M) = \pi_M(0) + \sum_{m=1}^{\infty} \pi_M(m) G_B(B_M)^m$. $R$ is the starting price of the auction, which is randomly drawn from CDF $G_R$. In order to win, player 1’s bid must exceed the realized value of the random variable $Z \equiv \max\{R, B_M\}$. Note

---

13 Our identification results would be similar if we used the winning bid instead.
14 See discussion of Definition 2.4 for intuition on why this must be true.
that the distribution of $Z$ is the same as the win probability function:

$$\chi(b) = G_R(b) \sum_{m=0}^{\infty} \pi_M(m) G_B(b)^m = G_Z(b).$$  \tag{9}$$

3.1. Model Identification. The structural primitives to identify are $\mu$, $\kappa$, $\pi$, $T_V$, and $G_R$. Throughout we assume that the bids within each auction and across auctions are independent. The model is identified if there is a unique set of structural primitives that rationalizes the joint distribution of observables, $\{\bar{k}_l, r_l, y_l\}_{l=1}^L$. The primary econometric challenge is correcting for two forms of sample selection in order to identify $\pi(\cdot)$ and $G_B(\cdot)$, which are required to identify the structural primitives. We first establish nonparametric identification of these objects in Section 3.1.1. Section 3.1.2 proves that this initial result implies nonparametric identification of the structural primitives when the spot market is a SPA. This result is straightforward, but it lacks the generality needed for a viable empirical framework since many platform markets (including eBay) use non-SPA formats. In 3.1.3 we prove a second, more novel result extending our method to platforms with broad varieties of selling mechanisms, which formalizes the “plug-and-play” property. Since static and dynamic bid-shading incentives can be decomposed in a parsimonious way, identification of the dynamic structural model is obtained under any auction format where identification is known for the static, one-shot game. We relegate all identification proofs to Appendix A.3.

3.1.1. Identification of the Bid Distribution and Bidder Arrival Process. If the econometrician knew exactly how many bidders attended each auction, then the empirical problem here would be simple. For example, if the number of bidders was known to be $k$, then the CDF of the (observable) highest losing bid, call it $H_k(b)$, is related to its (unobserved) parent distribution through the following bijective mapping:

$$H_k(b) = G_B(b)^k + kG_B(b)^{k-1} [1 - G_B(b)]$$

$$= k(k-1) \int_0^{G_B(b)} u^{k-2} (1-u) du \equiv \phi_k^{-1}(G_B(b)).$$

From the above integral it is easy to see that $\phi_k$ is a bijection mapping quantile ranks of $H_k(b)$ into quantile ranks of $G_B(b)$: the upper limit of integration ranges from zero to one (onto), and since the integrand is always positive, it is monotone in $G_B$ (one-to-one). Our setting is more complex though: the number of bidders $K$ is a random variable, following PMF $\pi(K)$, so the observed distribution of the highest losing bid, $H(\cdot)$, takes the form

$$H(b) = \sum_{k=2}^{\infty} \frac{\pi(k)}{1 - \pi(0) - \pi(1)} (H_k(b)) \equiv \phi^{-1}(G_B(b)).$$  \tag{10}$$

\footnote{Our exposition might appear to treat $F_V$ as a primitive, but in reality it is endogenously pinned down by $T_V$ (and other variables) through equation (6).}
*H*(*b*) is a weighted average of the distributions of second order statistics from samples of varying *k*, where the weights are the probability that a given *k* will occur as the number of bidders matched to a particular listing. Note that since *φ*−1(*GB*(*b*)) is a convex combination of monotone bijective transformations *φ*−1(*k*), it must also be a bijection.

A more pressing challenge to empirical work is that the observed number of bidders in each auction, *˜K*, is only a lower bound on the actual number of bidders matched to the auction, *K*. Due to random ordering of bid submission times across all bidders who watch an item with intent to compete, some may find that their planned bid was surpassed before they had a chance to submit it to the online server. These bidders will never be visible to the econometrician, even though they were matched to the auction and competing to win.

To solve this problem we incorporate an explicit model of the sample selection process into our identification strategy. We adopt an approach similar to that of [Hickman et al. 2016](#) who proposed a model of a filter process executed by Nature that randomly withhold some bidders from the econometrician’s view. For a given auction with *k* total matched bidders, this filter process first randomly assigns each bidder an index {1, 2, …, *k*}, where one’s position in the list determines the ordering of bid submission times. Nature then visits each bidder in the order of her index within the list, keeping a running record of the current lead bidder and current price as she goes. As Nature visits each bidder in the list, she only records bids that cause the price to update because they exceed the second-highest previous bid. Otherwise, Nature skips bidder *i*’s submission as if it never happened and reports to the econometrician only the record of price path updates, which reveals *˜k* ≤ *k* observed bidder identities. This filter process is meant to depict the way information is recorded on real-world platform markets, where some bidders will not appear to have participated even though they had intent to bid. This view of intra-auction dynamics assumes that the ordering of bidders’ submission times is random, but we are agnostic as to how the bidders choose the moments in calendar time to place their bids.

Note that the distribution of *˜K* conditional on a given *k* can be characterized without knowing the form of *π*(·) or *GB*. Since a bidder’s visibility to the econometrician only depends on whether her bid exceeds the second-highest preceding bid, the researcher can easily simulate the filter process based on quantile ranks to compute conditional probabilities Pr[*˜k*|*k*] for various (*˜k*, *k*) pairs. We adopt a special notation for this function, *P*0(*˜k*, *k*) ≡ Pr[*˜k*|*k*], and treat it as known since it can be computed without data on bidding. Since *˜k* is observable, we can use this information to express its PMF, denoted *˜π*(*˜k*), as a function of the primitive distribution: *˜π*(*˜k*) = ∑*k*∞*P*0(*˜k*, *k*)π(*k*).

16In a similar setting, Platt [2015] explored parametric inference assuming that *K* is Poisson distributed. However, our empirical estimates with a more flexible model strongly reject the Poisson assumption in favor of a model with greater variance of bidder participation in each auction.
However, this equation will not suffice in our case: unlike [Hickman et al.], our data exhibit binding reserve prices, which introduce a further random wedge between actual participation $k$ and observed participation, $\tilde{k}$. Not only do some bidders go unobserved because the filter process withholds them from view, but an additional fraction of bidders, who would have otherwise been reported by Nature, go unobserved because their bids fall below the reserve price. This second layer of selection produces substantial complications since $G_B$ and $G_R$ now determine how the second source of selection influences the relation between the distribution of observed $\tilde{K}$ and the underlying distribution of actual $K$.

We propose an adjusted filter process. First, for each auction Nature randomly draws $k$ from $\pi(k)$ and $r$ from $G_R$. The $k$ randomly chosen bidders formulate their strategic bids without knowing the realization of $k$ or $r$. Nature then compiles a reported list of bidders for the econometrician in two steps. First, she dismisses any bidders whose bids do not exceed the reservation price. Second, she randomly assigns a timing index $i \in \{1, \ldots, k'\}$ to the remaining $k' \leq k$ bidders and then executes the standard filter process on that subset. Finally, Nature reports $\tilde{k}$ and $r$ to the econometrician.

In order to characterize the conditional distribution of $\tilde{K}$ given $r$, first note that if there are $K = k$ total bidders, the probability that $j$ of them are screened out by $r$ is $\binom{k}{j} G_B(r)^j [1 - G_B(r)]^{k-j}$. Now suppose there are $\tilde{k}$ observed bidders in an auction with $K$ total bidders. We combine the two levels of selection in the adjusted filter process with the following equation:

$$\Pr[\tilde{K} = \tilde{k}|K = k, r] = \sum_{j=0}^{k-\tilde{k}} \binom{k}{j} G_B(r)^j [1 - G_B(r)]^{k-j} P_0(\tilde{k}, k-j).$$

The sum is to account for the fact that any number of bidders between $0$ and $k - \tilde{k}$ could be screened out by selection on reserve prices. The trailing term involving $P_0$ is to account for the standard filter process running its course with the surviving set of bidders. Equation (11) characterizes the distribution of $\tilde{K}$, conditional on reserve price $r$, as

$$\pi(\tilde{k}|r) = \sum_{k=\tilde{k}}^{\infty} \Pr[\tilde{K} = k|k, r] \pi(k).$$

Lemma 3.2. Suppose that there exists a set $k_1 < k_2 < \cdots < k_I$ where $\sum_{i=1}^{I} \pi(k_i) = 1$. Then for any finite $I$ it follows that there is a unique $(\pi(\cdot), G_B(\cdot))$ pair that is consistent with the joint distribution of the observables $(\tilde{K}, R, Y)$.

Lemma 3.2 does not require any shape restrictions on $G_B$, and it does not depend on the specific value of $I$ in any way (aside from its finiteness). In other words, the identification result holds for models of the bidder arrival process that become arbitrarily flexible as $I$ grows. In that sense, the preceding result may be considered as nonparametric. However, the proof of the lemma is non-constructive in that it does not immediately suggest a method for estimating the parameters. For a more intuitive understanding, first note that bijectivity
of the mapping $\phi$ implies that fixing $\pi$ pins down a unique $G_B^n(b) = \phi(H(b); \pi)$ through equation (10). Therefore, if we condense equations (10)–(12) we can write

$$\hat{\pi}(k|\tau = \sum_{k=0}^{\infty} \pi(k) \sum_{j=0}^{k-1} \binom{k}{j} \phi(H(b); \pi)^j [1 - \phi(H(b); \pi)]^{k-j} p_0(\hat{\pi}, k - j),$$

(13)

which depends only on the function $\pi(\cdot)$ and the observables. In principle then, for any finite, $I$-dimensional, restriction of the bidder arrival process, equation (13) allows the researcher to choose from a continuum of moment conditions, with each possible moment condition depending on a distinct $(\hat{\pi}, \tau)$ pair.

3.1.2. Baseline Model: Second-Price, Sealed-Bid Spot-Market Auctions. Using Lemma 3.2 we are now ready to develop our main identification results concerning the structural primitives. A SPA spot-market mechanism implies a specific form for the expected payment function,

$$\rho(b) \equiv \mathbb{E}[p_B(b)] = \int_0^b t g_Z(t) dt$$

(14)

In steady-state, we can express the Bellman equation and bidding strategy as

$$\mathcal{V}(v) = \max_{b \in \mathbb{R}_+} \left\{ \chi(b)v - \rho(b) - \kappa + [1 - \chi(b)] \delta \mathcal{V}(v) \right\}$$

(15)

$$\beta(v) = v - \delta \mathcal{V}(v)$$

(16)

The demand shading factor given by bidder 1’s continuation value, $\delta \mathcal{V}(v)$, is uniquely characterized by four things: the per-period entry cost, $\kappa$; the distribution of bids, $G_B(b)$; the distribution of starting prices, $G_R$; and the bidder arrival process $\pi(\cdot)$. Note that (16) implies $\mathcal{V}(v) = \frac{v - \beta(v)}{\delta}$. By substituting this expression into equation (15) and using the shorthand notation $b^* = \beta(v)$, we can rearrange terms to get

$$v = b^* \frac{1 - \delta (1 - \chi(b^*))}{1 - \delta} = \frac{\delta}{1 - \delta} \left( \rho(b^*) + \kappa \right) = \beta^{-1}(b^*)$$

(17)

**Proposition 3.3.** For a given discount factor $\delta$, suppose that there exists a set $k_1 < k_2 < \cdots < k_\mathcal{I}$ where $\sum_{i=1}^{\mathcal{I}} \pi(k_i) = 1$. Then when the spot-market mechanism is a sealed-bid, second-price auction, for any finite $\mathcal{I}$ there is a unique configuration of model parameters $\Theta = (\mu, \kappa, \pi, T_V, G_R)$ that is consistent with the joint distribution of the observables $(\hat{\pi}, R, Y)$.

3.1.3. Model Identification Under Alternative Spot Market Mechanisms. We now extend our identification result to cover platforms that use general spot-market formats, which may cause both static and dynamic demand shading incentives. Recall that a buyer with valuation $v$ has a dynamic value equal to $\tilde{\mathcal{V}} = v - \delta \mathcal{V}(v)$. Proposition 2.3 implies that each buyer bids as if they are playing a static auction with their dynamic bid shading incentives collapsed into their dynamic value. We show that if the right set of observables are available to identify the mapping $\beta^*$ that would arise in a static auction with allocation rule $\chi$ and pricing rule $\rho$, then the value function $\mathcal{V}$ and the private value $v$ from the dynamic
20 HOW EFFICIENT ARE DECENTRALIZED AUCTION PLATFORMS?

auction market are also identified. We refer to this as our “plug-and-play” identification result since it demonstrates how existing identification results for static auction models can be reused in the context of a dynamic platform market. Note that plugging $\beta^s$ into equation (15) yields:

$$V(v) = \chi[\beta^s(\sigma_v)]v - \rho[\beta^s(\sigma_v)] - \kappa,$$

Using the shorthand $b^* = \beta^s(\sigma_v) = \beta(v)$ and substituting in the definition of $\sigma_v$, we can rearrange terms further to get:

$$v = \sigma_v \left(1 - \delta \frac{1 - \chi(b^*)}{1 - \delta} \right) - \delta \left(\rho(b^*) + \kappa\right) = \beta^{-1}(b^*).$$  \hspace{1cm} (18)

In the case of a SPA spot-market, $b^* = \beta^s(\sigma_v) = \sigma_v$, so (18) reduces to (17) above.

**Proposition 3.4. Plug and Play Identification.** For a given discount factor $\delta$, suppose that there exists a set $k_1 < k_2 < \cdots < k_I$ where $\sum_{i=1}^I \pi(k_i) = 1$. Then for any finite $I$ the model parameters $\Theta \equiv (\mu, \kappa, \pi, T_V, G_R)$ are uniquely identified under any spot-market mechanism for which the optimizer of equation 4, $\beta^s(v)$, could be identified from the available observables $(\tilde{K}, R, Y)$ if they were generated from a sample of static, one-shot auction games.

Once again, Proposition 3.4 may also be considered as nonparametric in the sense that the logic of its proof need not involve shape restrictions on $G_B, F_V$ or $T_V$, and it holds for models of the bidder arrival process that become arbitrarily flexible as $I$ grows. Proposition 3.4 is also useful because it broadens the applicability of our model and methodology to allow for empirical work for any spot-market mechanism that admits a monotone equilibrium in the static setting, and for which the pricing and allocation rules can be expressed in terms of $\pi(\cdot)$, $G_R$, and $G_B$. For example, any platform model where the spot-market is a first-price auction will still be identified from commonly available observables (see Guerre et al. [2000]). In our empirical application, the eBay pricing rule is a non-standard hybrid of the first-price and second-price formats, which causes bidders to engage in additional demand shading due to static, strategic incentives.

3.2. A Semi-Parametric Estimator. Following our identification argument, we recover the structural primitives using a strategy inspired by the two-stage approach pioneered by Guerre et al. [2000] and Jofre-Bonet and Pesendorfer [2003]. In the first stage we flexibly estimate $\pi(\cdot)$, $G_B$, and $G_R$, and in the second stage we construct the remaining objects $\chi(\cdot)$, $\rho(\cdot)$, $\kappa$, $(\beta^s)^{-1}(\cdot)$, $\beta^{-1}(\cdot)$, $V(\cdot)$, $F_V(\cdot)$, $\mu$, and $T_V$ as functions of first-stage parameter estimates. However, Stage I is an estimation step, while Stage II is a purely computational step based on the outputs from Stage I.

---

17 Alternatively, if the optimizer of equation 4 is unique and the allocation rule $\chi(b)$ and pricing rule $\rho(b)$ can be identified from the available observables $(\tilde{K}, R, Y)$, then $\beta^s(v)$ is identified.
Thus far we have left the bidder arrival process \( \pi(k) \) unrestricted in order to demonstrate that the observables and the theoretical model on their own are sufficient to identify the structural primitives. In this section we develop an estimator to implement our identification strategy, but for the purpose of computing counterfactuals we now assume \( K \) follows a generalized Poisson (GP) distribution (Consul and Jain [1973]) with PMF:

\[
\pi(k; \lambda) = \lambda_1 (\lambda_1 + k \lambda_2)^{k-1} e^{-\lambda_1 - k \lambda_2} / k!, \quad \lambda_1 > 0, \quad |\lambda_2| < 1.
\]

The GP reduces to a regular Poisson distribution when \( \lambda_2 = 0 \), with fatter tails when \( \lambda_2 > 0 \) and thinner tails when \( \lambda_2 < 0 \). Thus, we refer to \( \lambda_1 \) as the size parameter and \( \lambda_2 \) as the dispersion parameter. The GP model implies that \( \pi_M(m, \lambda) = \pi(m + 1; \lambda)(m + 1)(1 - \lambda_2)/\lambda_1 \).

Following our identification argument, \( G_B \) and \( \lambda \) must be jointly estimated, which rules out many common methods such as kernel smoothing. We choose to specify \( G_B \) as a B-spline, which is a linear combination of globally defined basis functions that mimic the behavior of piecewise, local splines (the name “B-splines” is short for basis splines). By the Stone–Weierstrass theorem, B-splines can be used to approximate any continuous function to arbitrary precision given sufficiently many basis functions. B-splines provide a combination of flexibility and numerical convenience that is ideally suited to our application.

Let \( n_b = \{ n_{b1} < n_{b2} < \cdots < n_{b, I_b + 1} \} \) be a set of knots on bid domain \([\hat{b}, \tilde{b}] = [\min_i \{y_i\}, \max_i \{y_i\}]\) that create a partition of \( I_b \) subintervals. This need not be a uniform partition, but we do require that \( n_{b1} = \hat{b} \) and \( n_{b, I_b + 1} = \tilde{b} \) so that the partition spans \([\hat{b}, \tilde{b}]\). The knot vector, in combination with the Cox-de Boor recursion formula, defines a set of \( I_b + 3 \) cubic B-spline basis functions \( F_{bi} : [\hat{b}, \tilde{b}] \to \mathbb{R}, i = 1, \ldots, I_b + 3 \) that parameterizes the bid distribution:

\[
\hat{G}_B(b; \alpha_b) = \sum_{i=1}^{I_b+3} \alpha_{bi} F_{bi}(b)
\]

We also use this approach to estimate \( G_R \) and \( F_V \). Let \( n_r = \{ n_{r1} < \cdots < n_{r, I_r + 1} \} \) and \( n_v = \{ n_{v1} < \cdots < n_{v, I_v + 1} \} \) denote knot vectors that span the reserve prices, \([r_1, \hat{r}] = [0.99, \max_i \{r_i\}]\), and the space of buyer values, \([\hat{v}, \tilde{v}]\) (with the bounds to be estimated). These knot vectors determine our other basis functions \( F_{ri} : [r_i, \hat{r}] \to \mathbb{R}, i = 1, \ldots, I_r + 3 \) and \( F_{vj} : [\hat{v}, \tilde{v}] \to \mathbb{R}, i = 1, \ldots, I_v + 3 \) which then parameterize \( \hat{G}_R(r; \alpha_r) = \sum_{i=1}^{I_v+3} \alpha_{ri} F_{ri}(r) \) and \( \hat{F}_V(v; \alpha_v) = \sum_{i=1}^{I_v+3} \alpha_{vi} F_{vi}(v) \).

3.2.1. Stage I: \( \lambda \), \( G_B \), and \( G_R \). Recalling that the matrix of conditional probabilities \( P_0(\tilde{k}, k) \) is known beforehand, we now define the model-generated conditional PMF of \( \tilde{k} \) given \( r \) as

\[
\tilde{\pi}(\tilde{k}|r; \lambda, \alpha_b) = \sum_{k=\hat{k}}^\tilde{k} \left\{ \sum_{j=0}^{k} \binom{k}{j} G_B(r; \alpha_b)^j \left[ 1 - G_B(r; \alpha_b) \right]^{k-j} P_0(\tilde{k}, k - j) \right\} \pi(k; \lambda).
\]

\[\text{A standard text on B-splines is de Boor [2001]. See also Hickman et al. [2016, Online Appendix].}\]

\[\text{For discussion on choice of knot locations used in our empirical implementation, see Appendix D.1.}\]
where \( \bar{K} \) is an upper bound on the auction sizes we consider. We also adopt the following as the empirical analog of the conditional PMF:

\[
\hat{\pi}(\tilde{k}|r) = \frac{\sum_{l=1}^{L} I(\tilde{k}_l = \tilde{k}) \cdot \bar{K}(\frac{r-r_l}{h_R})}{\sum_{l=1}^{L} \bar{K}(\frac{r-r_l}{h_R})}
\]

(21)

where \( I(\cdot) \) is an indicator function, \( \bar{K} \) is a boundary-corrected kernel function, and \( h_R \) is an appropriately chosen bandwidth. The model-generated highest loser bid distribution is

\[
H(b; \lambda, \alpha_b) = \sum_{k=2}^{\infty} \frac{\pi(k; \lambda)}{1 - \pi(0; \lambda) - \pi(1; \lambda)} \left( G_B(b; \alpha_b)^{k-1} - G_B(b; \alpha_b) \right)
\]

and its empirical analog is \( \hat{H}(b) = \sum_{i=1}^{L} I(y_i \leq b) / L \). Using these separate pieces we can define a method of moments estimator as

\[
(\hat{\lambda}, \hat{\alpha}_b) = \arg \min_{(\lambda, \alpha_b) \in \mathbb{R}^{b+3}} \sum_{l=1}^{L} \left\{ \left[ \hat{\pi}(\tilde{k}_l|y_l; \lambda, \alpha_b) - \hat{\pi}(\tilde{k}_l|r_l) \right]^2 + \left[ H(y_l; \lambda, \alpha_b) - \hat{H}(y_l) \right]^2 \right\}
\]

subject to

\[
\alpha_{b,1} = 0, \quad \alpha_{b,l+3} = 1,
\]

\[
\alpha_{b,i} \leq \alpha_{b,i+1}, \quad i = 1, \ldots, I_b + 2.
\]

(23)

The estimate \((\hat{\lambda}, \hat{\alpha}_b)\) is chosen to make the model-generated conditional distribution of \( \bar{K} \) match its empirical analog as closely possible. The constraints on the empirical objective function enforce boundary conditions and monotonicity of our parameterization for \( \hat{G}_B \).

Finally, we separately estimate \( \hat{G}_R \) by a simpler method of moments procedure as

\[
\hat{\alpha}_r = \arg \min_{\alpha_r \in \mathbb{R}^{b+3}} \sum_{l=1}^{L} \left\{ \left[ \hat{G}_R(r_l; \alpha_r) - \hat{G}_R(r_l) \right]^2 \right\}
\]

subject to

\[
\alpha_{r,1} = \hat{G}_R(r), \quad \alpha_{r,l+3} = 1,
\]

\[
\alpha_{r,i} \leq \alpha_{r,i+1}, \quad i = 1, \ldots, I_r + 2,
\]

(24)

where \( \hat{G}_R(r) = \sum_{i=1}^{L} I(r_i \leq r) / L \) is the empirical CDF of reserve prices.

---

20 The GP distribution has unbounded support, but for implementation we impose a finite bound \( \bar{K} \). In practice, \( \bar{K} \) is chosen so that there is essentially 0 probability it would be exceeded in the set of 1,732 auctions we analyze.

21 The boundary-corrected kernel we use follows Karunamuni and Zhang [2008]. See Hickman and Hubbard [2015] for an in-depth discussion of its advantages and uses in structural auctions models.

22 A fully nonparametric estimator for \( \pi \) is possible, but with additional complications. The main challenge is that only finitely many \( \Pr[K = k] \) can be estimated with finite sample size \( L \). Thus, one could choose an upper bound \( \bar{K}_L < \infty \) and restrict \( \Pr[K = k] = 0 \) for each \( k > \bar{K}_L \). The estimator must also specify the rate at which \( \bar{K}_L \) should grow with the sample size. In a simpler setting than ours—a static bidding model of eBay laptop computer auctions with no binding reserve prices—Hickman et al. [2015] found strong evidence that a GP model produced estimates that could not be improved upon by relaxations of its parametric form in a sample size of roughly 750 auctions.
3.2.2. Stage II: Having these estimates in hand, we are able to directly re-construct the remaining structural primitives. Some Stage II objects will depend on the time discount factor, and where this is the case we so note by including $\delta$ as a parameter argument for the relevant functional. Another difficulty is that eBay employs a hybrid of the usual first- and second-price auction formats that uses a bid increment denoted $\Delta > 0$. Typically, the price is set equal to the second highest bid plus the increment. However, if the top two bids are within $\Delta$ of each other, then the second-price rule yields a price above the highest bid. In that case, the price is set equal to the high bidder’s own bid as in a first-price auction. Thus, eBay’s pricing rule is $p(b) = \min\{Z + \Delta, b\}$.

Hickman [2010] proved existence and uniqueness of a monotone Bayes-Nash equilibrium under this pricing rule in a static auction where the number of bidders is known. This equilibrium involves demand shading because there is positive probability that the winner’s own bid will determine the price she pays. Hickman et al. [2016] showed, in a static bidding game with stochastic participation and no binding reserve prices, that a bidder’s private value is identified from the distribution of bids through the equation:

$$v = b + \frac{G_Z(b) - G_Z[\tau(b)]}{\hat{g}_Z(b)} \quad \text{where} \quad \tau(b) = \begin{cases} b & \text{if } b \leq b + \Delta \\ b - \Delta & \text{otherwise}, \end{cases} \quad (25)$$

Using Stage I estimates we can construct the allocation rule:

$$\chi(b; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) = \hat{G}_R(b; \hat{\alpha}_r) \sum_{m=0}^{\infty} \pi_M(m; \hat{\lambda}) \hat{G}_B(b; \hat{\alpha}_b)^m.$$

Using the zero surplus condition, we can recover the per-period entry cost as:

$$\hat{\kappa} = \chi(\hat{v}_b; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) \hat{v}_b - \rho(\hat{v}_b; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) \quad (29)$$
as well as the dynamic inverse bid function and value function which are:

\[
\hat{\vartheta} = \hat{\beta}^{-1}(b; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r, \delta) = \hat{\vartheta}_b - \frac{1 - \delta}{1 - \delta} \left[ 1 - \chi(b; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) \right] - \frac{\delta}{1 - \delta} \left[ \rho(b; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) + \hat{\kappa} \right]
\] (30)

\[
\hat{\mathcal{V}}(v; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r, \delta) = \frac{v - \hat{\vartheta}_b}{\delta}.
\] (31)

We parameterize the private value distribution \(F_v\) using B-spline functions to obtain a flexible representation that is convenient for computing counterfactuals. We begin by specifying a grid of \(j = I_v + 1\) points spanning the bid support, \(b_j = \{b_1, \ldots, b_I\}\), and a knot vector \(n_v\) that spans \([\hat{b}_1, \hat{b}]: = \left[ \beta^{-1}(\hat{b}; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r, \delta), \bar{\beta}^{-1}(\hat{b}; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r, \delta) \right]\). This defines our basis functions \(F_v: [\hat{b}_1, \hat{b}^\infty] \rightarrow \mathbb{R}, i = 1, \ldots, I_v + 3\), from which we can compute \(\hat{\alpha}_v:\)

\[
\hat{\alpha}_v = \arg\min_{\alpha_v \in \mathbb{R}^{I_v + 3}} \sum_{j=1}^{I_v + 3} \left\{ \mathcal{G}_b(b_j; \hat{\alpha}_b) - \sum_{i=1}^{I_v + 3} \alpha_v(b_i) \mathcal{F}_v \left[ \beta^{-1}(b_i; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r, \delta) \right] \right\}^2
\]

subject to

\[
\alpha_{v,1} = 0, \quad \alpha_{v, I_v + 3} = 1, \\
\alpha_{v,i} \leq \alpha_{v,i+1}, \quad i = 1, \ldots, I_v + 2.
\] (32)

Finally, the steady-state measure and distribution of new agents each period are:

\[
\hat{\mu} = \left[ 1 - \pi\left(0; \hat{\lambda}\right) \right] G_R(b; \hat{\alpha}_r) + \int_b^\infty g_R(r; \hat{\alpha}_r) \left( \sum_{k=1}^{\infty} \pi\left(k; \hat{\lambda}\right) \left[ 1 - G_B(b; \hat{\alpha}_b)^k \right] \right) dr
\] (33)

\[
t_V(v; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r, \delta) = \frac{\chi\left(\beta\left(v; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r, \delta\right); \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r\right) f_V(v; \hat{\alpha}_b, \delta) \hat{\lambda}_1}{\hat{\mu} \hat{\lambda}_2}.
\] (34)

3.2.3. Asymptotics and Standard Errors. In Appendix D.1 we give recommendations for B-spline tuning parameters (i.e., choice of number and locations of knots) that achieve a reasonable mix of utility and tractability in our empirical application. A fixed choice of tuning parameters can be viewed as defining a flexible, though finite-dimensional, parametric model. In Appendix D.2 we show why, under this view, our Stage I parameter estimators \(\hat{\lambda}, \hat{\alpha}_b\), and \(\hat{\alpha}_r\) fall within the class of GMM estimators, meaning that standard asymptotic theory applies to establish sampling distributions for all Stage I and Stage II parameters.\(^{23}\)

In our implementation we use the standard nonparametric bootstrap as a computationally convenient method of accounting for the role of sampling variability. We re-sample from our auction-level observations (with replacement) to construct 1000 bootstrap samples of size \(L\). Then we execute Stages I and II of our estimator as described above to obtain confidence bounds on the parameters.

\(^{23}\)In Appendix D.2 we also sketch how one might extend our analysis using semi-parametric sieve methods by (for example) increasing the number of B-spline knots as more data become available to increase the flexibility of the parametric model. However, proving the requisite estimator properties that yield point-wise asymptotic normality of the underlying functionals is nontrivial and beyond the scope of this paper.
4. DATA AND RESULTS

We use a unique dataset on Amazon Kindle Fire tablet devices that we scraped from eBay during March through July 2013. Our scraping algorithm captured all Kindle listings on eBay during that period, and for each one we downloaded and stored various .html files including the item listing page and the bid history page. During the sample period we observed a total of 1,732 Kindle Fires listed as “new” (i.e., unused in a factory sealed box) or “new other” (i.e., unused in an unsealed box) for an average of 11.25 per day.

Each Kindle tablet had eight gigabytes of internal storage and a seven-inch screen with standard-definition resolution of 1024x600. The Kindle Fire tablets came pre-loaded with Amazon’s proprietary version of the Android-based operating system that prevents the user from accessing the full Android app market. This makes the Kindle Fire a poor substitute for a standard tablet (e.g., Apple iPad) that can serve a dual role as a productivity tool. The Kindle Fire is designed to exclusively access Amazon’s electronic media market, which includes e-books, periodicals, audiobooks, music, and movies. All transactions were covered by the eBay Money Back Guarantee to insure consumers against potential unscrupulous sellers.

In order to further probe the homogeneity of our Kindle auctions sample, we manually examined a 10% sample of the raw .html files. Unlike many other tablet devices, accessories are rarely coupled with Kindle Fires: of these listings, only one mentioned an accessory (a Kindle case) that the seller had bundled into the sale. The majority of the listings with a condition of “new” had been opened. A common explanation was that the seller was checking that all parts (e.g., charging cord) were present. We conclude that the “new” listings are best interpreted as items that are like new and essentially unused.

Because listings with a low closing price identify the participation cost, \( \kappa \), we manually scrutinized all of these items. We identify $66 as the minimal observed sale price, but we examined all listings with a closing price of less than $80. Of these we removed listings that (for example) were selling Kindle accessories (e.g., cases) rather than the actual device or were offering a Kindle running a user-modified version of the Android OS. These atypical listings were largely isolated to the lower tail of the price distribution.

One final concern is that there may be residual auction-specific variation which our manual survey missed, and which is not included in our econometric model. Unobserved

---

24 It requires specialized knowledge to uninstall the proprietary operating system, and doing so is costly since it invalidates all product guarantees issued by Amazon.

25 Amazon also maintains its own limited app market—primarily dedicated to entertainment and online shopping, but in June 2013 it contained less than one tenth the number of apps available in Apple’s App Store for iPhones or Google Play for Android devices. See https://en.wikipedia.org/wiki/App_Store_(iOS); https://en.wikipedia.org/wiki/Google_Play; and https://en.wikipedia.org/wiki/Amazon_Appstore; information retrieved on 7/15/2016.

26 As of 7/15/2016, details on eBay’s consumer protection program were available at http://pages.ebay.com/ebay-money-back-guarantee/questions.html.
heterogeneity (UH)—auction characteristic that bidders see but the econometrician does not—is a common problem, and various approaches have been developed to deal with it (e.g., see Krasnokutskaya [2011], Roberts [2013], Balat [2013]). Each approach assumes bidder valuations are separable in the UH and the idiosyncratic component, which makes it possible to deconvolve UH from agent-specific variation in bids. Roberts [2013] proposed a method to correct for UH when only one bid is observed per auction. Since we can only be confident that the highest losing bid in each of our auctions is fully reflective of equilibrium strategies (see discussion below), Roberts [2013] is the most relevant approach to the current context. Assuming reserve prices are a separable function of the UH variable, he shows that one can use joint movement in reserve prices and bids to deconvolve the UH and identify private valuations.

In our data we observe non-trivial variation in sellers’ reserve prices with roughly one third of them being binding for a positive fraction of the bidder population. Therefore, one might reasonably suspect that if UH is present then higher values of the unobserved characteristic prompt sellers to increase reserve prices. A necessary condition for UH in the Roberts [2013] model is co-movement of bids and reserve prices, which is testable. We find in our data that the correlation between seller reserve price and the highest losing bid is statistically indistinguishable from zero and very small in a practical sense, at -0.015. We conclude that there is not unobserved heterogeneity as captured by Roberts [2013], but we cannot rule out other forms of unobserved heterogeneity as per Krasnokutskaya [2011] or Balat [2013]. The combination of market features (e.g., our manual survey of .html pages, uniform buyer insurance, proprietary operating system and limited app market, and uniform characteristics of the Kindle Fire tablets) and the lack of evidence from our UH test is consistent with our assumption of a homogeneous goods market. These characteristics of the eBay Kindle data allow us to avoid significant complications covered by other work, such as identifying UH or complex substitution patterns (see Backus and Lewis [2016]), and instead focus on questions of bidding behavior, allocative efficiency, and market design.

4.1. Intra-Auction Dynamics. For each auction listing, we observe the timing and amount of each bid submission as well as an anonymized hash of the bidder identity. As previous empirical work has recognized, one challenge for interpreting eBay data is a large number of bids that fall too far below realistic transaction prices to be taken seriously. Many bidders place repeated bids, often within a few dollars or cents of each other, and then become inactive long before the price approaches a reasonable level. The question of intra-auction dynamics is broad, complicated, and beyond the scope of this work. In our case, inter-auction dynamics are the primary concern for answering our research questions on

Roberts [2013] assumes only that reserves are a monotone separable function of UH. For example, it does not matter whether reserves are chosen to optimize projected revenues or whether they are chosen to hedge against the risk of selling at an unacceptably low price, since both scenarios would satisfy monotonicity.
allocative efficiency and market design. To deal with observed early low bids, we adopt the approach of Bajari and Hortacsu [2003] by partitioning individual auctions into two stages. During the first phase bidders may submit cheap-talk bids that are viewed as uninformative of the final bids or the final sale price. The second stage is treated as a sealed-bid auction as per our model of Sections 2 and 3. Finally, consistent with the previous section, the ordering of bidders’ submission times is assumed to be random.

This requires us to take a stand on differentiating between bids that are a meaningful part of competition and those that are superfluous. We define a serious bid as one that affects the price path within the second stage. Note that our definition of serious bidding also counts the top two submissions from the first stage of the auction as these bids fix the price at the start of the second stage. This allows us to avoid drawing too sharp a distinction between the two stages of the auction, since some serious bidders’ submission times may still occur early in the life of the auction. Likewise, a serious bidder is one who is observed to submit at least one serious bid. Of course, the possibility always exists that some bidders who are determined to be non-serious by the above criterion had intent to compete for the item, but were priced out before submitting their planned, serious bid during the terminal stage. This is, however, part of the problem that our model of the adjusted filter process solves (i.e., observed participation by serious bidders is a lower bound on actual participation).

We specify the terminal period as the last 60 minutes of an auction, during which we see an average of 4.01 observed serious bidders per auction. Figure 3 shows the empirical distribution for time remaining when the winning bid was submitted, which occurs within the final 60 minutes in over 95% of auctions in the sample. The figure also shows the empirical distribution for time remaining across all serious bid submissions in the sample. These figures are not sensitive to alternate specifications of the terminal period cutoff. If it is chosen as 80 minutes the mean number of serious bidders becomes 4.25, and if it is chosen as 40 minutes the mean number of serious bidders becomes 3.67.

---

28The vast majority of eBay auctions in our data are won by bidders who bid in the final moments and the behavior of the price path near the end of the auction is independent of auction duration.
Table 1. Descriptive Statistics

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Median</th>
<th>St. Dev.</th>
<th>Min</th>
<th>Max</th>
<th># Obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Remaining (minutes)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Winning Bid Submission:</td>
<td>6.69</td>
<td>0.11</td>
<td>38.31</td>
<td>0.00</td>
<td>593.30</td>
<td>1,460</td>
</tr>
<tr>
<td>High Loser Bid Submission:</td>
<td>12.49</td>
<td>0.56</td>
<td>52.85</td>
<td>0.00</td>
<td>604.35</td>
<td>1,397</td>
</tr>
<tr>
<td>Observed Participation</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ᶜ (serious bidders only):</td>
<td>4.01</td>
<td>4</td>
<td>1.82</td>
<td>0</td>
<td>12</td>
<td>1,462</td>
</tr>
<tr>
<td>Monetary Outcomes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sale Price:</td>
<td>$124.96</td>
<td>$125.00</td>
<td>$17.74</td>
<td>$67.00</td>
<td>$190.00</td>
<td>1,460</td>
</tr>
<tr>
<td>Highest Losing Bid:</td>
<td>$123.84</td>
<td>$124.50</td>
<td>$17.34</td>
<td>$66.00</td>
<td>$189.50</td>
<td>1,397</td>
</tr>
<tr>
<td>Seller Reserve Price:</td>
<td>$33.56</td>
<td>$0.99</td>
<td>$45.27</td>
<td>$0.99</td>
<td>$175.00</td>
<td>1,462</td>
</tr>
</tbody>
</table>

One challenge remains: bidders may choose to submit their strategic bid to the server and make use of eBay’s automated proxy bidding or incrementally raise their bid manually to the level of their strategic bid. Roughly one third of serious bidders are observed to engage in incremental bidding. Because it is difficult to interpret gradually changing bids from a particular buyer, we assume only that the highest losing bid is fully reflective of equilibrium play. This leaves us with the three data points from each auction needed for identification: \( \hat{k}_t \), the observed number of serious bidders; \( r_t \), the seller’s reserve price; and \( y_t \), the highest loser bid from auctions with at least two bids. After dropping .html pages for which our software was unable to extract data because of formatting problems, we have 1,462 total auctions, 2 of which logged no bids, and 1,397 of which had 2 or more observed bidders (so that we observed a highest losing bid). Table I displays descriptive statistics on bid timing, observed participation, sale prices, and highest losing bids.

4.2. Choosing \( \delta \). The final free parameter is the time discount factor, \( \delta \). As in many other empirical contexts, this part poses a difficult challenge. Luckily, \( \delta \) does not enter Stage I estimation, so all of the necessary building blocks to compute the final structural primitives will be unaffected. Several Stage II objects are also unaffected, including the win probability \( \chi(\cdot) \), the expected payment function \( \rho(\cdot) \), the per-period participation cost \( \hat{\kappa} \), and the exogenous, per-period measure of new agents flowing into the market \( \hat{\mu} \). However, the remaining objects depend on \( \delta \). There is an intuitive reason why: \( \beta(\cdot), \mathcal{V}(\cdot), \mathcal{F}_V(\cdot), \text{ and } T_V(\cdot) \) are influenced by the opportunity cost of losing today, and \( \delta \) plays a pivotal role in shaping agents’ attitude toward present versus future consumption.

In lieu of taking a stand on the particular value of \( \delta \) applicable to our study, we present results both here and in our counterfactual section for a range of values of \( \delta \). Where possible, we provide statistics that are stable across choices of \( \delta \). For example, instead of providing a dollar value for deadweight loss, which is sensitive to \( \delta \), we present deadweight loss as a percentage of the buyer values, which is stable across different choices of \( \delta \).
Table 2. Estimation Results

<table>
<thead>
<tr>
<th>Variable:</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\kappa$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Point Estimate:</td>
<td>5.9100</td>
<td>0.2579</td>
<td>0.0654</td>
<td>0.9649</td>
</tr>
<tr>
<td>Standard Error:</td>
<td>(0.384)</td>
<td>(0.058)</td>
<td>(0.0174)</td>
<td>(0.0261)</td>
</tr>
</tbody>
</table>

Figure 4. Matching Process Estimates

Figure 5. Inverse Bid Function Estimates Given Various Values of $\delta$

4.3. Estimates. Table 2 displays point estimates and standard errors for the market tightness parameters, the per-period participation cost, and the per-period measure of new agents. Figure 4 depicts point estimates for the empirical CDF of observed bidders per auction (thick, dashed line), the estimated distribution of total auction-level participation $K$ (thick, solid line), and 95% point-wise confidence bounds (vertical box plots). As the figure demonstrates, failing to account for unobserved bidders would lead to a very different view of the distribution of auction participation. This substantial difference shows up in both the mean—4.07 for observed bidders per auction versus 7.96 for actual bidders per auction—and also in the variance—3.19 for observed bidders and 14.46 for actual bidders. Note also $\lambda_2 > 0$ and highly significant, rejecting an assumption that $\pi$ is Poisson. Appendix D.1 contains similar plots of point estimates for $\hat{G}_B(b; \alpha_b)$ and $\hat{G}_R(r; \alpha_r)$. 
Table 3. Mean Private Values and Information Rents For Various $\delta$

<table>
<thead>
<tr>
<th>Discount Factor $\delta$:</th>
<th>0.75</th>
<th>0.81</th>
<th>0.87</th>
<th>0.8871</th>
<th>0.93</th>
<th>0.95</th>
<th>0.98</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Private Value:</td>
<td>$48.57$</td>
<td>$51.29$</td>
<td>$56.32$</td>
<td>$59.63$</td>
<td>$68.83$</td>
<td>$86.15$</td>
<td>$153.24$</td>
</tr>
<tr>
<td>Mean Winner Private Value:</td>
<td>$208.39$</td>
<td>$230.98$</td>
<td>$269.26$</td>
<td>$293.58$</td>
<td>$358.55$</td>
<td>$474.05$</td>
<td>$875.56$</td>
</tr>
<tr>
<td>Mean Winner Information Rent:</td>
<td>$54.66$</td>
<td>$69.11$</td>
<td>$94.08$</td>
<td>$111.09$</td>
<td>$157.44$</td>
<td>$245.84$</td>
<td>$583.91$</td>
</tr>
<tr>
<td>Mean Information Rent Percentage:</td>
<td>26.23%</td>
<td>29.92%</td>
<td>34.94%</td>
<td>37.84%</td>
<td>43.91%</td>
<td>51.86%</td>
<td>66.69%</td>
</tr>
</tbody>
</table>

Figure 5 presents the dynamic inverse bid functions $\hat{\beta}^{-1}(b; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r, \delta)$ which we estimate for a uniform grid of values of the time discount factor $\delta$ between 0.75 and 0.98. We also include an additional value at 0.8871 taken from an experimental study by Augenblick, Niederle, and Sprenger [2016] where they elicited hyperbolic time discount parameters at the daily level from college students.\footnote{Recall from Figure 1 that the vast majority of demand shading is driven by the option value of returning to the market in future periods if one does not win today.\footnote{To our knowledge ours is the first paper to make such a comparison in the context of platform markets. Procurement models also include static and dynamic incentives, although the differences in the setting (e.g., time-varying cost types) make a direct comparison difficult. Jofre-Bonet and Pesendorfer [2003] estimate that just over half of bid shading is due to intertemporal incentives.}} Figure 5 depicts the important role of this third piece.

Table 3 displays various descriptive statistics derived from Stage II estimates, including average private values, average private values of winners, and information rents (i.e., the difference between the winner’s private value and the spot-market price). The last row of the table shows information rents as a fraction of the winner’s private value, on average.

Figure 6 presents other Stage II estimates related to the distribution of buyer values. The upper pane displays the PDF of the distribution of market participants’ private values in steady state under our preferred specification, $f_V(v_\delta; \alpha_v, \delta = 0.8871)$ (dash-dot line), as well as the type distribution for new market entrants each period, $i_V(v; \hat{\lambda}, \hat{\alpha}_b; \hat{\alpha}_r, \delta = 0.8871)$ (solid line), with 95% point-wise confidence bounds (vertical box plots). The PDFs $t_V$ and $f_V$ are tied together by the win probability, $\chi$. Since $f_V$ depicts the type distribution for all market participants—including players remaining from previous periods—it represents a measure $\lambda_1/(1 - \lambda_2) = 7.96$ of agents (recall that sellers are assumed to have measure 1). On the
other hand, $t_V$ describes the type distribution of the measure $\mu = 0.9649$ of new agents that enter the market each period in order to maintain the steady state. Under $f_V$ there is a larger mass of low-value bidders who are less likely to win, and therefore they pile up in the market and remain for many periods until finally winning. $t_V$ is a selected set of buyers who move in and out of the market at much higher frequency: they have higher private values and are much more likely to win in the spot-market in a given period.

5. WELFARE COUNTERFACTUALS

We now investigate the welfare implications of our model. First we explore market efficiency and revenue. We then analyze the importance of controlling for buyer type distribution when conducting these exercises, which would be neglected in a static analysis of the market. The structural primitives of our model being held fixed in the counterfactual scenarios are $\mu$, $t_V$, $\kappa$, $G_R$, $\lambda_2$, and the space of types $[v, \bar{v}]$. We describe how the counterfactual equilibria are computed and how we test for uniqueness in Appendix B.

We adopt the usual notion of efficiency as the tendency for goods to be allocated to those who value them most within a given period. Even when the spot-market mechanism is efficient within an auction, auction platforms with search frictions still exhibit two related sources of inefficiency. First, there is the chance that a high-value buyer that ought to receive the good in an efficient allocation is competing against another high-value buyer, so one of them cannot receive the good. Second, an auction may fail to attract any high-value buyers, which means a low-value buyer will receive the good when she would not under
an efficient outcome. The first case is one in which there is “too much” competition within
the auction, while the second case is one in which there is “too little” competition.

5.1. “Model Anemic” Inefficiency Calculations. In this section we use our Stage I esti-
mates to bound the percentage of auctions resulting in an inefficient sale. We refer to these
calculations as “model anemic” since they do not rely on our equilibrium bidding model
and thereby employ the fewest possible assumptions. These calculations rely only on our
filter process model to correct for sample selection in the observed number of bidders.

To proceed, we first find the cutoff $v_{eff}$ that separates high-value buyers that ought
to receive the good in an efficient allocation from lower-value buyers that ought not. Since the buyer-seller ratio is $\lambda_1/(1 - \lambda_2)$, the efficient cutoff in private value space is

$$v_{eff} \equiv \frac{1}{F_Y} \left(1 - \frac{1 - \lambda_2}{\lambda_1}\right).$$

However, since quantile orderings are invariant to monotone trans-
formations, we can re-define this cutoff in bid space (where the raw data live) as

$$b_{eff} \equiv \frac{1}{G_B} \left(1 - \frac{1 - \lambda_2}{\lambda_1}\right).$$

If the highest losing bid in a given auction exceeds $b_{eff}$, then the corre-
sponding bidder would receive the good in an efficient allocation but does not win the
item this period. We find that 28.47% of the auctions in our sample satisfy this criterion.
For each high-value bidder who loses an auction there is a low-value bidder in some other
auction who inefficiently wins, so high-value buyers losing and low-value buyers winning
are two sides of the same coin. This measure is a lower bound on the frequency of inef-
ficiency because we cannot account for auctions where two or more losing bids surpassed
$b_{eff}$ without observing all of the bids. Another disadvantage of the model-anemic approach
is that it offers no way of measuring the magnitude of unrealized gains from trade.

5.2. Structural Welfare Calculations. Our full Stage II structural estimates allow us to get
a more complete idea of the frequency and magnitude of market inefficiency. Using equa-
tion (6) we can compute the the fraction of all inefficient transactions:

$$\Pr[v_{winner} < v_{eff}] = \frac{C \int_{v_{eff}}^{v_{s}} \chi(\beta(s)) f_Y(s) ds}{C \int_{v_{eff}}^{v_{s}} f_Y(s) ds}.$$ 

We find that 35.89% of Kindle auctions end with an inefficient out-
come. Deadweight loss calculations in levels will be sensitive to the choice of $\delta$. To address
this problem, we adopt the following measure, which we refer to as the efficiency ratio:

$$E_{u,\delta} = \frac{\frac{C \int_{v_{eff}}^{v_{s}} s \chi_u(\beta_u(s)) f_Y(s) ds}{C \int_{v_{eff}}^{v_{s}} f_Y(s) ds}}{\frac{C \int_{v_{eff}}^{v_{s}} f_Y(s) ds}{C \int_{v_{eff}}^{v_{s}} f_Y(s) ds}}$$

The numerator is the realized gains from trade in our market (within a given period), and
the denominator represents gains from trade generated by a fully efficient allocation. The
$u$ subscript denotes number of units involved in each auction listing for our counterfactual
centralization analysis below; for now we fix $u = 1$. By expressing surplus as a fraction
of total possible surplus, the separate influences of $\delta$ in the numerator and denominator
largely cancel out and we get a measure that is stable with respect to different choices of
δ (see Table 4). We also compute the efficiency ratio under a hypothetical lottery system, denoted $E_{\text{lot},\delta}$, as the minimum efficiency benchmark (see the last row of Table 4).

Our point estimates imply that the fraction of total deadweight loss is $1 - E_{1,0.8871} = 0.135$ under our preferred specification. To put this number into context, deadweight loss under a lottery system is estimated to be $1 - E_{\text{lot},0.8871} = 0.53$, meaning that eBay’s auction market platform achieves 76% of total gains from trade above the lottery benchmark. Note, however, that this is only a “partial equilibrium” assessment; were a social planner with complete knowledge of the bidder values to implement the efficient allocation each period, then the steady-state distribution of buyers’ values and the buyer-seller ratio would change. However, we believe our figures have the benefit of giving a sense of the welfare losses while imposing minimal structural assumptions on the estimates.

5.3. **Counterfactual Market Centralization.** We now consider the extent to which inefficiencies can be mitigated by changing the market structure to one in which the same number of Kindles are allocated each period, but using fewer $u$-unit, uniform-price auctions with $u \geq 2$. Since new Kindles are relatively homogeneous, we think it is reasonable to assume that buyers view them as near perfect substitutes; therefore, considering multi-unit auctions is reasonable. For products that are not perfect substitutes (e.g., used cars), the implications of selling disparate products in a multi-unit auction become much more difficult to formalize. However, our estimates provide a sense of the efficiency loss due to search frictions when selling items through decentralized, single-unit auctions.

Several aspects of our model need to be adjusted in the multi-unit auction setting. We subscript the endogenous quantities with $u, \delta$ to denote the degree of centralization and the time discount factor. First, each $u$-unit auction attracts a number of bidders $K_u$ distributed as a generalized Poisson random variable where

$$E[K_u] = \frac{\lambda_1 u}{1 - \lambda_2} = uC. \quad (35)$$

Equation (35) mandates that the (endogenous) measure $C$ of bidders be exactly assigned to the $1/u$ measure of $u$-unit auctions, much as in part 1 of Assumption 2.1. We assume $\lambda_2$, the dispersion parameter, is fixed at the estimated value and allow $\lambda_1,u$, the size parameter, to adjust so equation (35) is satisfied in our counterfactuals.

In our status-quo model, we assume each seller draws an independent starting price from $G_R$. For comparability to the status quo, we assume that a single starting price is drawn from $G_R$ for each $u$-unit auction, and that starting price applies to all $u$ units being allocated in that auction. Each bidder submits a bid to the auction to which she is matched, and the $u$ highest bids that are larger than the auction’s starting price win an item. Each winning bidder then pays a sum equal to the larger of the $(u+1)^{th}$ highest bid and the starting price.
Table 4. Counterfactual Efficiency Ratios $\mathcal{E}_{u,\delta}$

<table>
<thead>
<tr>
<th># Units Per Listing</th>
<th>Discount Factor $\delta =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75</td>
<td>0.80</td>
</tr>
<tr>
<td>1</td>
<td>0.89</td>
</tr>
<tr>
<td>2</td>
<td>0.92</td>
</tr>
<tr>
<td>4</td>
<td>0.94</td>
</tr>
<tr>
<td>8</td>
<td>0.95</td>
</tr>
<tr>
<td>Lottery</td>
<td>0.58</td>
</tr>
</tbody>
</table>

If $v = \beta^{-1}(b)$, the probability of winning in a $u$-unit auction is:

$$
\chi_u(b) = G_R(b) \left[ \sum_{m=0}^{u-1} \pi_M(m) + \sum_{m=u}^{\infty} \pi_M(m) \sum_{i=0}^{u-2} \binom{m}{m-i} F_V(v)^{m-i}(1 - F_V(v))^i \right].
$$

(36)

One of the general takeaways from our research is that understanding the impact of platform market design on participation decisions is crucial. The social planner’s welfare calculus will be strongly influenced by changes in entry behavior (e.g., how many low-value buyers leave the market?) and the steady state-distribution of private values for market participants (e.g., how many low-value bidders accumulate in the market when they are less likely to win an item?). We find that $e$ increases as the market becomes more centralized (i.e., as $u$ grows) since player types with very low values will see a decrease in their probability of winning as market allocations become more efficient.

Table 4 provides results for counterfactual efficiency ratio statistics for $u \in \{1, 2, 4, 8\}$. Recall from above that the efficiency ratio compares gains from trade in a single period of a $u$-unit model with the welfare generated by the efficient allocation from clearing the market once per period with a single, large, multi-unit, uniform-price auction. Note that the efficiency ratios are remarkably stable across different specifications of the time discount factor $\delta$. Also, the majority of possible gains from centralization can be realized by 2- or 4-unit uniform-price auctions, so there is little need to shift to a fully centralized market.

One might expect that if eBay could re-design their platform market to increase allocative efficiency, then it should be able to benefit by capturing some of the increased gains from trade. From a static perspective of a single auction, the revenue equivalence theorem implies that the choice of payment rule has no effect on the revenues of the seller. If the revenues change with the altered auction format it must be due to a combination of (1) the new allocation rule and (2) the change in the measure and value distribution of the bidders. An examination of the moving parts within the model indicates that the sign of these effects on revenue is ambiguous. On the one hand, a bidder with a value above the new participation cutoff $e_u$ faces fewer competitors in the market. On the other hand, her

---

31 There are examples where this general intuition does not hold; for example, the literature on optimal auctions suggests that efficiency-reducing reservation prices can increase revenue.
How efficient are decentralized auction platforms?  

Table 5. Counterfactual Mean Auction Revenues

<table>
<thead>
<tr>
<th># Units Per Listing</th>
<th>Discount Factor $\delta$ = 0.75</th>
<th>0.80</th>
<th>0.86</th>
<th>0.88</th>
<th>0.92</th>
<th>0.95</th>
<th>0.98</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$115.05$</td>
<td>$114.90$</td>
<td>$114.71$</td>
<td>$114.59$</td>
<td>$114.37$</td>
<td>$114.10$</td>
<td>$113.65$</td>
</tr>
<tr>
<td>2</td>
<td>$112.79$</td>
<td>$112.33$</td>
<td>$112.97$</td>
<td>$112.72$</td>
<td>$112.26$</td>
<td>$111.96$</td>
<td>$112.28$</td>
</tr>
<tr>
<td>4</td>
<td>$111.73$</td>
<td>$111.21$</td>
<td>$111.09$</td>
<td>$110.84$</td>
<td>$110.40$</td>
<td>$110.31$</td>
<td>$111.30$</td>
</tr>
<tr>
<td>8</td>
<td>$110.05$</td>
<td>$109.54$</td>
<td>$109.12$</td>
<td>$108.94$</td>
<td>$108.64$</td>
<td>$108.81$</td>
<td>$109.75$</td>
</tr>
</tbody>
</table>

Figure 7. The Efficiency-Revenue Link

remaining competitors also value the object more highly on average. This combination of effects make it difficult to derive a priori predictions on bidding behavior and revenue. Table 5 demonstrates that the average sale price actually falls as $u$ increases.

To explain why revenue drops as efficiency rises, Figure 7 plots the probability of winning and equilibrium bids for each agent type in the $u = 1$ (solid line) and $u = 8$ (dashed line) cases. The left panel reveals that for most agents (especially those most likely to win), greater efficiency raises the probability that they will win in a given period. This raises their continuation values, and therefore the opportunity cost of winning an auction today. This in turn promotes further demand shading as shown in the second panel of Figure 7. Reduced bids translate into decreased lower for both sellers and eBay, which currently charges sellers a percentage commission on auction revenue. This highlights an interesting point: what is good for market welfare is not necessarily good for platform market designers like eBay.

In Appendix F.2 we analyze the effect of centralization on the participation costs paid by the agents. Centralizing from single-unit auctions to 8-unit auctions reduces the participation costs paid by roughly 60%.

5.4. The Importance of Platform Composition. One of the main findings of Section 5.3 is that the revenue of the sellers decreases as the efficiency of the within-period allocation increases. In this section we try to understand whether these effects are due more to the change in the spot market mechanism or to the change in the composition of the pool of
buyers using the market. In order to tease apart these effects, we recompute the welfare and revenue statistics holding the measure and type distribution of the buyers fixed at the status-quo values.\footnote{All of the analysis in this section is out-of-equilibrium since we are not accounting for how altering the auction format changes the steady-state measure and type distribution of the buyers.} The most natural interpretation of our exercise is that we are studying how our predictions would differ had we used a mis-specified model that assumed the measure and type-distribution of the agents are exogenous objects. As we will see, assuming the population of buyers is exogenous will lead us to over-estimate the efficiency gains from centralization. More surprisingly, we will find the mis-specified model would predict a sharp increase in auction revenues, which is the opposite of what the true model predicts. The primary take-away from our exercise is that it is crucial to account for the endogenous changes to the population of buyers when producing counterfactuals.

We now step through the changes to the endogenous objects (i.e., $\chi_u(b)$, $V(v)$, and $\beta(v)$) when evaluating the centralization counterfactual holding $F_V$ and $C$ fixed at the status-quo values. The left pane of Figure 8 plots $\chi_1(b)$ and $\chi_8(b)$ for $\delta = 0.88$.\footnote{The results are qualitatively similar for other choices of $\delta$.} The central pane of Figure 8 displays the value function, which is uniformly lower when $u = 8$ than when $u = 1$, although the difference is small for the highest value buyers. This contrasts with the results of Section 5.3 wherein the value function for the agents that remain in the market increases as $u$ rises. The decrease in the value function causes the bids, which are displayed in the right pane of Figure 8, to increase as $u$ increases.

If we had accounted for the changes in $F_V$ and $C$ that centralization causes (as in Section 5.3), we would predict that the number of buyers (i.e., $C$) drops and, in particular, the high-value buyers leave the platform quickly. The first effect reduces competition for all

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure8}
\caption{Out-of-Equilibrium Outcomes}
\end{figure}
Table 6. Mis-specified Counterfactual Results

<table>
<thead>
<tr>
<th>Efficiency Ratios, $\varepsilon_{u,\delta}$</th>
<th>Discount Factor $\delta$ =</th>
<th>0.75</th>
<th>0.80</th>
<th>0.86</th>
<th>0.88</th>
<th>0.92</th>
<th>0.95</th>
<th>0.98</th>
</tr>
</thead>
<tbody>
<tr>
<td># Units Per Listing</td>
<td>1</td>
<td>0.89</td>
<td>0.88</td>
<td>0.87</td>
<td>0.87</td>
<td>0.85</td>
<td>0.84</td>
<td>0.82</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.94</td>
<td>0.94</td>
<td>0.93</td>
<td>0.93</td>
<td>0.92</td>
<td>0.92</td>
<td>0.91</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.97</td>
<td>0.97</td>
<td>0.96</td>
<td>0.96</td>
<td>0.96</td>
<td>0.96</td>
<td>0.95</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
<td>0.98</td>
</tr>
<tr>
<td>Lottery</td>
<td>0.58</td>
<td>0.55</td>
<td>0.50</td>
<td>0.47</td>
<td>0.41</td>
<td>0.35</td>
<td>0.25</td>
<td></td>
</tr>
</tbody>
</table>

| Revenue |
|---|---|---|---|---|---|---|---|---|
| # Units Per Listing | 0.75 | 0.80 | 0.86 | 0.88 | 0.92 | 0.95 | 0.98 |
| 1 | $120.83$ | $120.77$ | $120.68$ | $120.63$ | $120.52$ | $120.37$ | $120.06$ |
| 2 | $129.33$ | $130.00$ | $131.17$ | $131.91$ | $133.80$ | $136.90$ | $145.95$ |
| 4 | $134.11$ | $135.52$ | $138.03$ | $139.59$ | $143.67$ | $150.49$ | $171.54$ |
| 8 | $137.11$ | $139.23$ | $143.00$ | $145.36$ | $151.60$ | $162.29$ | $197.46$ |

of the buyers, and the second effect reduces the number of high-value buyers and mutes competition between them. In contrast, there is a large population of high-value buyers in the status quo $F_V$. Even for buyers with very high values, they are less likely to win the good and, when they do win, the price is typically set by the bid of another high-value buyer. In other words, the high-value buyers win less frequently and pay higher prices. These effects depress the value function of all of the agents.

The out-of-equilibrium efficiency ratios and revenues (Table 6) are both higher than in equilibrium due to two effects. First, the status quo type distribution has a large population of high-value buyers relative to the equilibrium type distribution for $u = 8$. The mis-specified model predicts large welfare gains by more efficiently allocating goods to this group of buyers. Second, the higher bids have the effect of increasing revenue, whereas the lower bids in the $u = 8$ SCE depress revenue. In Appendix F.1 we explore further the role of platform market composition and its relation to bidding incentives. There we find that in terms of revenue and efficiency, incentives driving selection into the platform are at least as important as the incentives driving behavior once a buyer participates. We believe these counterfactuals provide practical guidance to eBay and other online market designers regarding what issues are of most importance when considering changes to a platform.

6. CONCLUSION

Our goal has been to provide a model of a dynamic auction platform that is both rich enough to capture the salient features of the market (e.g., the large number of auctions concluding each day, the cost of participation) and yet remain tractable enough to facilitate empirical analysis. To accomplish this, we have developed a model with a continuum
of buyers and sellers that is easy to estimate and solve, and we show that this model approximates the more realistic setting with a finite number of agents in Appendix C. We have also demonstrated that the structural components of this model can be identified from observables that are commonly available from real-world platform markets. In constructing these identification results we have overcome several important problems including sample selection in the number of observed spot-market competitors and allowing for pricing rules that give rise to static demand shading incentives. Finally, we have also proposed a flexible GMM estimator for the structural primitives.

Most platform markets exist in order to eliminate barriers to trade and allow for buyers and sellers to interact in a relatively low-friction environment. However, the sheer size of the markets may give rise to search frictions which prevent market outcomes from attaining the social optimum. We have estimated our model within the context of the market for Kindle Fire tablets, and we use these estimates both to compute the welfare loss under the present design and to suggest novel designs to mitigate these welfare losses. We begin by providing a “model-anemic” analysis that relies only on the bid distribution. We find a lower bound of at least 28% of auctions that close with a highest losing bidder whose private value exceeds that of some winner from another auction on that same day.

We then use our structural estimates to compute the deadweight loss and to study alternative spot-market mechanisms that reduce the welfare loss due to search frictions. We find that over 36% of the auctions within a day allocate goods to winners with inefficiently low private values, which causes a 14% welfare loss due to the decentralized nature of the platform. This implies that the single-unit auction market attains three quarters of total possible welfare improvement over a pure lottery system. By taking small steps toward a more centralized market structure—such as running multi-unit, uniform-price auctions with as few as 4 units each—2/3 of the remaining welfare loss can be recovered. However, centralization causes seller revenue to fall.

Our final counterfactual explores the importance of controlling for the composition of the buyers on the platform, which is shaped both by the flow of new buyers into the market, the participation cost, and the choice of spot-market auction format. We recompute our centralization statistics holding the market composition, \( F_V \) and \( C \), fixed at the status quo levels. We find that the value function decreases as the market becomes more centralized. Therefore, if we fail to account for endogenous changes to \( F_V \) and \( C \), we would overestimate the welfare improvements yielded by centralization and we would predict that auctioneer revenues would increase.

References


Appendix A. ONLINE SUPPLEMENTAL MATERIALS FOR
How Efficient are Decentralized Auction Platforms?
by Aaron L. Bodoh-Creed, Jörn Boehnke, and Brent R. Hickman

THEORY: TECHNICAL PROOFS APPENDIX

We break our proof appendix into parts that correspond to the different sections of the main text. Subsection A.1 presents some background results on the topologies we employ. Subsection A.2 proves our claim that an equilibrium of our model exists. Subsection A.3 provides the proofs of our identification propositions.

A.1. Preliminary Results. We now define the topologies used in our analysis. For real-valued parameters, we use the standard topology on $\mathbb{R}$. Our random variables are elements of the space of probability measures over $[0,1]$, denoted $\Delta([0,1])$. Unless stated otherwise, $\Delta([0,1])$ is endowed with the Kolmogorov topology, which is metrized by the sup-norm over the space of CDFs associated with the measures in $\Delta([0,1])$:

$$d_K(F,G) = \sup_x \|F(x) - G(x)\|$$

When either $G$ or $F$ is continuous (i.e., the underlying measure is atomless), then $d_K$ metrizes the weak-* topology (Petrov [1995], p. 43). We use the natural product topology where relevant.

Remark 1. Consider a random variable $X : \Omega \rightarrow \mathbb{R}$ with CDF $F(x)$. For $N$ i.i.d. realizations, $\{x_1, ..., x_N\}$, drawn from $F$, denote the $N$ realization empirical CDF as $F_N(x)$. Then from the Glivenko-Cantelli theorem we have

$$\sup_{x \in \mathbb{R}} \|F_N(x) - F(x)\| \rightarrow 0 \text{ almost surely as } N \rightarrow \infty$$

Unless noted otherwise, we refer to almost sure convergence under $d_K$ when making statements about the convergence of measures or their respective CDFs.

A.2. Proofs from Section 2. The highlight of this section is an equilibrium existence result using a fixed point argument. Before offering our proof, we first prove Proposition 2.3, which provides several essential preliminary results.

Proposition 2.3. The best-response strategy $(\bar{\beta}, \bar{\zeta})$ satisfies:

1. $\bar{\beta}(v) = \beta^b(v - \delta \mathcal{V}(v))$ is a best-response bidding strategy.
2. $\bar{\beta}(v)$ is increasing in $v$ if $\chi(b)$ is increasing.
3. $\bar{\beta}(v)$ is strictly increasing in $v$ if $\chi(b)$ is differentiable and strictly increasing in $b$. In addition, $\bar{\beta}(v)$ is uniquely defined for a set of $v$ of Lebesgue measure 1.
4. There exists a cutoff $\bar{v}$ such that $\bar{\zeta}(v) = 1$ if and only if $v \geq \bar{v}$.
Proof. Part (1) summarizes the discussion of the relationship between the static and dynamic auctions given in Section 2.2. To prove part (2), note that if a buyer enters the market, her optimal bid is defined by:

\[
\arg\max_b \chi[b](v - \delta V(v)) - \rho[b]
\]  

(1)

Lemma A.5 proves that \( \frac{\partial V(v, C, G_B, G_R|\tilde{\sigma})}{\partial v} \in [0, 1) \) and strictly positive for types with a best-response choice of Enter. Since the probability of being allocated the good is increasing in \( b \), equation 1 is supermodular in \((v, b)\). This implies that the best response function, \( \tilde{\beta}(v) \), is increasing in \( v \) (Milgrom and Shannon [1994]). If \( \chi(b) \) is strictly increasing and differentiable, then equation 1 is strictly supermodular and differentiable in \((v, b)\). This implies \( \tilde{\beta}(v) \) is strictly increasing (Edlin and Shannon [1998]). In this case \( \tilde{\beta}(v) \) can admit only a countable set of discontinuities, which represent points where the bidder is indifferent between two bids. Since the set of such \( v \) is countable, the set has Lebesgue measure 0. Therefore the set of \( v \) where the best-response is uniquely defined is of Lebesgue measure 1.

To see that \( \zeta \) has the claimed cutoff form, suppose that it is optimal for a buyer with value \( v \) to enter and bid \( b \). Since such a buyer could receive 0 by staying out, it must be that \( V(v, C, G_B, G_R|\tilde{\sigma}) \geq 0 \). Since \( \frac{\partial V(v, C, G_B, G_R|\tilde{\sigma})}{\partial v} \in [0, 1) \), it must be that a buyer of type \( v' > v \) also finds it optimal to enter. Therefore \( \zeta \) has the claimed cutoff form. \( \square \)

Our equilibrium existence argument consists of several parts. First we argue that the state variables, \((C, F_V, G_R)\), live in a compact space that we denote \( \Gamma \). We must show that the best responses to the state variables and the transition operator for the state variables are continuous with respect to \( \Gamma \). Recall that we impose the Kolmogorov topology on the space \( \Delta([0,1]) \), which is metrized by \( d_K \) (described above). When considering continuity with respect to multiple variables, we impose the product topology on the Cartesian product of the relevant spaces. We use the notation \( V(v, C, G_B, G_R|\sigma'_i, \sigma_{-i}) \) when buyer \( i \) uses strategy \( \sigma'_i \) and all other agents follow \( \sigma \).

As part of our argument we must prove that the bid strategies we consider have both a lower and an upper bound on their derivatives so that the induced distributions of bids do not admit atoms. Stated formally, we need to show that the best response function is continuous mapping from the space \( C_M[0,1|\varphi] \), \( \varphi \in (0,1) \), into itself, where \( C_M[0,1|\varphi] \) contains all continuous, strictly increasing mappings from \([0,1]\) into \([0,1]\) that satisfy:

\[
f(v) - f(v') \in \left[ \varphi(v - v'), \frac{v - v'}{\varphi} \right], \quad \forall v > v'.
\]

Since the functions in \( C_M[0,1|\varphi] \) are Lipschitz continuous with modulus of continuity \( 1/\varphi \), this set of functions is equicontinuous and bounded over a compact domain. From the Arzelá-Ascoli theorem, \( C_M[0,1|\varphi] \) is compact when \( C_M[0,1|\varphi] \) is endowed with the sup-norm topology. The strategy space we need to consider can be defined as \( \sigma = (e, \beta) \in \Xi = \)
which implies that for any \( y \). Once we show that our best response and state transition operators are continuous, a straightforward fixed point argument applies.

Our proof characterizes two functions. First, we define the best response dynamics,

\[
BR (C, F_V, G_R, \sigma) = \tilde{\sigma} = \left( \tilde{e}, \tilde{\beta} \right),
\]

that describe how each type of buyer responds given the belief that \((C, F_V, G_R)\) is stationary and all other agents use the strategy \( \sigma \). We show that \( BR \) is continuous in \((C, F_V, G_R, \sigma)\) and maps into \( \Xi \).

The function \( T(C, F_V, G_R | \tilde{\sigma}) = (\tilde{C}, \tilde{F}_V, \tilde{G}_R) \) describes aggregate state transitions. We show that \( T \) is continuous in \((C, F_V, G_R, \tilde{\sigma})\) for \((C, F_V, G_R) \in \Gamma \) and \( \tilde{\sigma} \in \Xi \), and \( T (\cdot | \tilde{\sigma}) : \Gamma \rightarrow \Gamma \) as long as \( \tilde{\beta} \in \mathcal{C}_M[0,1] \). After proving \( T \) and \( BR \) are continuous operators over compact spaces (and hence have compact images), we define the total operator \( \mathcal{L} : \Gamma \times \Xi \rightarrow \Gamma \times \Xi \) where

\[
\mathcal{L}(C, F_V, G_R, \sigma) = (\tilde{C}, \tilde{F}_V, \tilde{G}_R, \tilde{\sigma})
\]

\[
BR (\sigma, C, F_V, G_R, \sigma) = \tilde{\sigma}
\]

\[
T(C, F_V, G_R | \tilde{\sigma}) = (\tilde{C}, \tilde{F}_V, \tilde{G}_R)
\]

Since \( G_R \) is constant across periods, \( G_R = \tilde{G}_R \). A straightforward application of Schauder’s fixed point theorem to the operator \( \mathcal{L} \) gives us existence of a stationary equilibrium. However, proving \( T \) and \( BR \) are continuous operators over compact spaces requires many small steps. Several of our results require that \( F_V \) and \( G_R \) admit bounded PDFs, \( f_V \) and \( g_r \), except at \( r = 0 \). For notational purposes, we let \( \mathcal{Q}[0,\bar{\eta}] \) denote the space of measures over \([0,1]\) that admit pdfs bounded from above by \( \bar{\eta} > 0 \).

**A.2.1. Continuity of Best Responses.** In this section we prove that the best responses are continuous as required and lie in \( \mathcal{C}_M[0,1|\varphi] \) for some choice of \( \varphi \in (0,1) \). Our first result is that the measure of entering agents and the distribution of bids are continuous in the underlying economy. To prove this, we need the following intermediate result. We add the “open neighborhood” qualifier to the statement of Lemma [A.1] so that our result applies to empirical measures that are close to a nonatomic measure with a bounded PDF. We exclusively work with measures with bounded PDFs in Section [A.2] but Section [C.3] requires us to work with empirical measures near the SCE steady-state distributions.

**Lemma A.1.** Consider any increasing function \( f \in \mathcal{C}_M[0,1|\varphi], \varphi \in (0,1) \), and let \( Z \) be an atomless CDF over \( \mathbb{R} \) that admits a pdf that is bounded from above by \( M \). Then \( Y(s) = Z(f^{-1}(s)) \) is uniformly continuous in \( f \in \mathcal{C}_M[0,1|\varphi], \varphi \in (0,1) \) and an open neighborhood of \( Z \).

**Proof.** Consider \( f, \tilde{f} \in \mathcal{C}_M[0,1|\varphi] \) where \( \| f - \tilde{f} \| \leq \gamma \). We can then write:

\[
f(x + \gamma / \varphi) > \tilde{f}(x) > f(x - \gamma / \varphi)
\]

which implies that for any \( y \) we have \( \tilde{f}^{-1}(y) \in [f^{-1}(y) - \gamma / \varphi, f^{-1}(y) + \gamma / \varphi] \), and in turn,

\[
Z \left( \tilde{f}^{-1}(y) \right) \in \left[ Z \left( f^{-1}(y) - \gamma / \varphi \right), Z \left( f^{-1}(y) + \gamma / \varphi \right) \right]
\]
Since the PDF of $Z$ is bounded by $M$, $Z$ can increase by at most $M\gamma/\varphi$ over the intervals $[f^{-1}(y) - \gamma/\varphi, f^{-1}(y)]$ or $[f^{-1}(y), f^{-1}(y) + \gamma/\varphi]$, and we then have:

$$\left\| Z\left(\tilde{f}^{-1}(y)\right) - Z\left(f^{-1}(y)\right) \right\| < \frac{M\gamma}{\varphi}$$

Since this bound holds uniformly over $x$, our result regarding continuity with respect to $f$ is proven. Continuity with respect to $Z$ is immediate given our use of the metric $d_K$ over the space of measures. □

**Lemma A.2.** The distribution of bids, $G_B$, and the measure of entering buyers, $\mathcal{C}$, are continuous in $C, F_V$, and $\sigma = (e, \beta)$ provided that the entry cutoff point $e < 1$, $F_V$ admits a bounded pdf, and $\beta \in \mathbb{C}_M[0,1|\varphi], \varphi \in (0,1)$.

**Proof.** The distribution of entering buyers is

$$F^E_V(x) = \frac{F_V(x) - F_V(e)}{1 - F_V(e)} \quad \text{for } x \geq e, 0 \text{ otherwise}$$

and the measure of entering buyers is $\mathcal{C} = C_1 - F_V(e)$. Note that $F^E_V$ and $C$ are continuous in $(C, F_V)$. The distributions of bids can be described as

$$G_B(b) = F^E_V(\beta^{-1}(b))$$

Using Lemma A.1, we find that $G_B$ is continuous in $\beta$. Finally, if $F_V$ is atomless, then $F_V(e)$ varies continuously in $e$, so $(C, G_B)$ is also continuous in $e$. □

Since we can state many of our lemmas most concisely in terms of continuity with respect to $\chi$, we now prove that $\chi$ is continuous with respect to the underlying aggregate states and the agent’s bid.

**Lemma A.3.** Assume $G_B$ is atomless. Then $\chi$ is continuous in $C, G_R, G_B$, and $b$. $\chi$ is strictly increasing in $b$.

**Proof.** A bidder’s beliefs about the number of other agents assigned to her auction are

$$\pi_M(k; C) = \frac{\pi(k; C)(k + 1)}{E[K]} \quad \text{with } E[K] = C$$

Since $C$ is continuous in $C, F_V$, and $\sigma, \pi_M(k; C)$ is continuous with respect to these variables as well. The probability of winning the good (from a bidder’s perspective) is

$$\chi(b) = G_R(b) \sum_{k=0}^{\infty} \pi_M(k; C) G_B(b)^k$$

Recall that we have imposed the Kolmogorov topology over the spaces of measures, which means that two measures are close if their respective CDFs are close in the sup-norm. Under this topology, $\chi(b)$ is continuous in $C, G_R,$ and $G_B$. $\chi(\cdot)$ is clearly continuous in $b$ if $G_B$ is atomless. To see that $\chi$ is strictly increasing, it suffices to note that $G_R$ has full support, which means $\chi$ is strictly increasing even if $b$ is not in the support of $G_B$. □
Lemma [A.3] would not be true if we had imposed the weak-* topology over the bid or starting price distributions. To see this, suppose that \( G_B \) had an atom of measure 1 at \( b = 0.5 \), \( G^*_B \) has an atom of measure 1 at \( b = 0.5 + \varepsilon \), and \( G_R(0) = 1 \). Under the Levy-Prokhorov metrization of the weak-* topology, \( G_B \) and \( G^*_B \) differ by \( \varepsilon \). However, a bid of \( b = 0.5 + \varepsilon/2 \) would win with certainty under \( G_B \) and lose with certainty under \( G^*_B \). In other words, \( \chi \) fails to be continuous with respect to \( G_B \). In contrast, under the Kolmogorov metric \( G_B \) and \( G^*_B \) differ by 1, so continuity is not threatened.

We required the assumption that \( \varepsilon < 1 \) in order to prove Lemma [A.2]. Our next lemma proves that we can restrict attention to \( e \in [0, \bar{e}] \) where \( \bar{e} < 1 \). The basic idea is that for large \( e \), any entering buyer effectively bids against the distribution of starting prices. If \( \kappa \) is small, then buyers with values close to \( v = 1 \) find it strictly optimal to enter the market. This in turn implies that the best response \( e \) is always strictly less than 1. For the duration, we assume that any entry cutoff, \( e \), is drawn from the set \([0, \bar{e}]\) where \( \bar{e} < 1 \).

**Lemma A.4.** Fix \( G_R \). Then as \( e \to 1 \), a buyer with type \( v = 1 \) strictly prefers to enter for \( \kappa > 0 \) sufficiently small.

**Proof.** As \( e \to 1 \), the measure of entering buyers approaches 0. Assumption 2.2 implies that as \( C \to 0 \) we have \( \pi_M(0; \mathcal{C}) \to 1 \). Thus, the payoff for a bidder that chooses a bid of \( b \) is:

\[
G_R(b)v - \rho(b) \geq G_R(b)(v - b)
\]

Since \( G_R \) has full support over \([0,1]\), \( G_R(b) > 0 \) for all \( b > 0 \), which means \( G_R(b)(v - b) > 0 \) for \( b < v \). Therefore, the best response for buyers with \( v \) sufficiently large is to enter and bid \( b < v \) as \( e \to 1 \). \( \square \)

This result on the continuity of the bids combined with Assumption 2.5 yields the following result.

**Lemma A.5.** \( BR(C, F_V, G_R, \sigma) = \bar{\sigma} = (\bar{\sigma}, \bar{\beta}) \) is continuous in \((C, F_V, G_R)\) and \( \sigma \) if \( F_V \) admits a bounded PDF. Moreover, \( \bar{\sigma} \in \mathcal{C}_M[0,1|\bar{\varphi}], \bar{\varphi} \in (0,1) \).

**Proof.** Let \( \mathcal{V}(v, C, G_B, G_R|\sigma) \) denote the value function generated by best responding to \((C, G_B, G_R)\) given a value of \( v \). Theorem 1 of Pavan, Segal, and Toikka [2014] implies that \( \mathcal{V} \) is almost everywhere differentiable and the derivative, where it exists, takes the value:

\[
\frac{\partial \mathcal{V}(v, C, G_B, G_R|\sigma)}{\partial v} = \sum_{t=1}^{\infty} \delta^{t-t}(1 - \chi(\beta(v)))^{t-t} \chi(\beta(v)) \text{ if } v \geq e
\]

and \( \partial \mathcal{V}(v, C, G_B, G_R|\sigma) / \partial v = 0 \) if \( v < e \). Since the strategies are best responses, the probability of sale, \( \chi(b) \), must be positive for all types that enter. Since equation 2 is the sum of the probability of disjoint events with the sum exponentially discounted by \( \delta \), equation 2 is
strictly bounded between 0 and 1. From the continuity of \( \chi \), we know that \( V \) is continuous in \((C, G_B, G_R, \tilde{\sigma})\)

Given the continuity of \( G_B \) and \( \mathcal{C} \) with respect to \((C, F_V, G_R)\) and \( \sigma \), Assumption 2.5 implies that the spot-market strategy that is a best response to \((C, F_V, G_R), \tilde{\beta}^s(v)\), is continuous with respect to \((C, F_V, G_R)\) and \( v \). Since \( V \) is continuous in these variables, the bidding strategy in our dynamic market, \( \tilde{\beta}(v) = \tilde{\beta}^s(v - \delta V(v, C, G_B, G_R | \tilde{\sigma})) \), is continuous in these variables as well.

Recall from Assumption 2.5 that \( \tilde{\beta}^s \in \mathcal{C} \) \([0, 1|\tilde{\phi}]\), \( \tilde{\phi} \in (0, 1) \), which implies that \( \tilde{\beta}^s \) is almost everywhere differentiable. Where the derivative of \( \tilde{\beta}^s \) exists we can write:

\[
\frac{d\tilde{\beta}(v)}{dv} = \frac{d\tilde{\beta}^s(v)}{dv} \left(1 - \delta \frac{\partial V(v, C, G_B, G_R | \tilde{\sigma})}{\partial v}\right) \in \left(\phi(1 - \delta), \frac{1}{\phi}\right)
\]

Therefore \( \tilde{\beta} \in \mathcal{C} \) \([0, 1|\tilde{\phi}]\), \( \tilde{\phi} \) where \( \tilde{\phi} = \phi(1 - \delta) \).

Since we know that the lowest valuation buyer willing to enter (i.e., \( v = \epsilon \)) has a continuation value of 0, it must be the case that:

\[
V(\epsilon, C, G_B, G_R | \tilde{\sigma}) = 0
\]

Since \( V(, C, G_B, G_R | \tilde{\sigma}) \) is continuous in \((C, F_V, G_R)\) and \( \sigma \) and strictly increasing in \( v \), the cutoff \( \epsilon \) is continuous in \((C, F_V, G_R)\) and \( \sigma \).

A.2.2. Continuity of Aggregate States. Now we turn to proving that the evolution of the aggregate states is continuous. Before doing so, let us describe the operator \( T(C, F_V, G_R | \tilde{\sigma}) = (\tilde{\mathcal{C}}, \tilde{F}_V, \tilde{G}_R) \) in more depth. We iterate the states by using steady-state relations that, in equilibrium, are consistent with the aggregate state. We will need to use the following function, which the reader is encouraged to think of as an unnormalized version of \( \tilde{f}_V \)

\[
\tilde{J}(v) = \frac{\mu t_V(v)}{\chi(\tilde{\beta}(v))} \text{ if } \tilde{\epsilon} \leq v
\]

where \( \tilde{\epsilon} \) is the best-response entry threshold and \( \tilde{J}(v) = 0 \) if \( \tilde{\epsilon} > v \). We can define \( \tilde{C} \) as

\[
\tilde{C} = \int_0^1 \tilde{J}(s) ds
\]

The distributions of buyer types and starting prices are

\[
\tilde{f}_V(v) = \frac{\tilde{J}(v)}{\tilde{C}}
\]

\[
\tilde{G}_R = G_R
\]

There is, at this point, no assurance that \( \tilde{f}_V \) is well-defined since \( \chi \) could have arbitrarily low values for some \( b \). We rule this problem out with the following lemma.

**Lemma A.6.** \( \chi(b) \geq \kappa > 0 \) if the agent is best-responding.
Proof. For a buyer to find it optimal to enter given a positive entry cost, his expected payoff from entering must at least cover the cost of entry. This implies:

\[ \chi v - \rho \geq \kappa \]

Therefore

\[ \chi = \Pr\{\text{Transaction}\} \geq \frac{\kappa}{v} \geq \kappa \]

where the last inequality follows from the fact that \( v \in [0,1] \).

Lemma [A.6] implies that

\[ \tilde{f}(v) \leq \frac{\mu t_V(v)}{\kappa} \]

for entrants. These relations give us

\[ \tilde{C} \in [\mu, \mu/\kappa] \]

\[ \tilde{f}_V(v) \in \left[0, \frac{t_V(v)}{\kappa}\right] \]

Therefore we can restrict attention to \( \tilde{f}_V \in Q[0,\bar{q}] \) where \( \bar{q} \) is the maximum of \( t_V(\cdot)/\kappa \). In other words, \( f_V \) admits a bounded PDF. Furthermore, \( f_V \) is continuous in \( \chi \) as required. Finally, \( \tilde{C}, \tilde{f}_V \) and \( \tilde{G}_R \) inherit continuity with respect to \( C, G_R, G_B, \) and \( \sigma \) from the continuity of \( \chi \).

A.2.3. Existence Result. We can now prove the existence of a stationary equilibrium of the continuum model.

Proposition 2.6. A stationary competitive equilibrium exists, and a positive mass of buyers choosing to enter the market if \( \kappa \) is not too large.

Proof. Lemmas [A.1] through [A.6] prove that \( \mathcal{L} \) is a continuous mapping from \( \Xi \times \Gamma \) into itself. Given the continuity of the mapping and the compactness of the spaces, we know that \( \mathcal{L} \) has a compact image. Schauder’s fixed point theorem implies that there exists a fixed point that defines a stationary equilibrium of our model. Lemma [A.4] establishes that \( e < 1 \) (i.e., some buyers enter the market) for \( \kappa > 0 \) sufficiently small.

A.3. Identification Proofs of Section 3.1. Now we present the proofs of Lemma 3.2 and Proposition 3.3, which prove that our model is identified.

Lemma 3.2. Suppose that there exists a set \( k_1 < k_2 < \cdots < k_I \) where \( \sum_{i=1}^{I} \pi(k_i) = 1 \). Then for any finite \( I \) it follows that there is a unique \((\pi(\cdot), G_B(\cdot))\) pair that is consistent with the joint distribution of the observables \((\tilde{K}, R, Y)\).
Proof. If we condition on the event $\bar{k} = k_I$ (i.e., no participants were filtered out) we then have $k = \bar{k} = k_I$. Therefore, we can write the distribution of the highest-losing-bid conditional on this event and reservation price $r$ as:

$$H(b|r,k = k_I) = k_I G_B(b|b \geq r)^{k_I - 1} [1 - G(b|b \geq r)] + G_B(b|b \geq r)^{k_I}.$$ 

Since the above relationship is bijective (see explanation in Section 3.1.1) and since $H(b|r,k = k_I)$ is known, we can invert it to obtain $G_B(\cdot|b \geq r)$ for any $r$. Since the theoretical model implies buyers only enter the market if their bid might win an auction, the infimum of the support of $r$ must be weakly lower than the infimum of the bid support (i.e., $G_B(r) = 0$). Therefore,

$$G_B(b|b \geq r) = \frac{G_B(b) - G_B(r)}{1 - G_B(r)} = G_B(b)$$

and we recover the parent distribution of bids $G_B(b)$ on its entire domain.

Having identified $G_B$ also uniquely pins down a vector of probabilities $(\pi(k_1), \ldots, \pi(k_I))$. To see why, consider the following equation:

$$\Pr[\bar{k} = 0|r] = \sum_{i=1}^{I} G_B(r)^{k_i} \pi(k_i).$$

(5)

For any choice of $(r_1, ..., r_I)$ where $G_B(r_i) \neq G_B(r_j), i \neq j$, the resulting $I$ equations are linearly independent, which means only a single configuration of probabilities $(\pi(k_1), \ldots, \pi(k_I))$ can be consistent with $G_B$ and the joint distribution of the observables. Since Assumption 2.5 ensures that $\beta$ is strictly increasing and we have assumed $F_V$ is continuous, then for any $r_j > r_i$ where $G_B(r_i) \in (0,1)$ implies $G_B(r_i) < G_B(r_j)$. Therefore we can choose from an uncountable set of conditions of the form of equation 5 to pin down our finite-dimensional parameter vector $(\pi(k_1), \ldots, \pi(k_I))$. Finally, since the logic of the proof does not depend on the value of $I$, then it must be true for all finite $I$.

\[ \square \]

Proposition 3.3. For a given discount factor $\delta$, suppose that there exists a set $k_1 < k_2 < \cdots < k_I$ where $\sum_{i=1}^{I} \pi(k_i) = 1$. Then when the spot-market mechanism is a sealed-bid, second-price auction, for any finite $I$ there is a unique configuration of model parameters $\Theta \equiv (\mu, \kappa, \pi, T_V, G_R)$ that is consistent with the joint distribution of the observables $(\bar{K}, R, Y)$.

Proof. First note that Lemma 3.2 establishes that for any finite $I$ there is a unique $(\pi(\cdot), G_B(\cdot))$ pair that is consistent with the joint distribution of the observables, and $G_R(r)$ is known. Given these three pieces, it also follows that the win probability $\chi(b)$ and the expected winner payment $\rho(b)$ are identified through equations (9) and (14). To identify the participation

\[1\]To see this, let $r$ be the infimum of the support of $G_B$. Conditional on the event $r = r$ we have $G(b|b \geq r) = \frac{G_B(b) - 0}{1} = G_B(b)$.\[1\]
cost, combine equation (15) with the zero surplus condition (3.1) to find the following relation:

\[ \chi(v) v - \rho(v) = \kappa \]  

(6)

In other words, the marginal market participant reaps just enough benefit in expectation to offset the cost of participation.

With \( \chi(b), \rho(b), \) and \( \kappa \) known, equation (17) shows that \( \beta^{-1} \) is also identified if the discount factor \( \delta \) is known, and in turn, the private value distribution is identified through the relationship \( F_V(v) = G_B[\beta(v)] \). With \( F_V \) known, \( \mu \) is identified through either of the following two equivalent expressions which determine the mass of transactions each period, and therefore the total mass of buyers exiting the market:

\[
\mu = C \int_v^\pi \chi[\beta(v)] f_V(v) dv = C \left[ 1 - \pi(0) \right] G_R(b) + C \int_b^\Omega g_R(r) \left( \sum_{k=1}^\infty \pi(k) \left[ 1 - G_B(r)^k \right] \right) dr.
\]

(7)

Finally, once \( \mu \) is known \( T_V \) is identified through equation (6).

\[ \square \]

**Proposition 3.4. Plug and Play Identification.** For a given discount factor \( \delta \), suppose that there exists a set \( k_1 < k_2 < \cdots < k_I \) where \( \sum_{i=1}^I \pi(k_i) = 1 \). Then for any finite \( I \) the model parameters \( \Theta = (\mu, \kappa, \pi, T_V, G_R) \) are uniquely identified under any spot-market mechanism for which the optimizer of equation 4, \( \beta^*(v) \), could be identified from the available observables \( (\tilde{K}, R, Y) \) if they were generated from a sample of static, one-shot auction games.

**Proof.** The argument for \( \pi, G_B, \) and \( G_R \) is the same as in Lemma 3.2. Now consider a hypothetical world where the same set of observables were actually generated from a sample of static, one-shot auctions, based on underlying private valuations \( \tilde{v}_v \). If the observables (including \( \pi(\cdot), G_B, \) and \( G_R \)) are known to identify the inverse bid mapping in that static world, then once again we can treat \( \beta^*(\cdot) \) and \( \tilde{v}_v \) as known. The value \( \kappa \) is then pinned down by the optimal entry condition (equation 5) or alternately Assumption 3.1. Finally, equation (18) maps each observed bid \( b \) into a private value \( v \) that rationalizes \( b \) as a best response to market conditions both within-period and future. This implies that \( F_V \) is identified, after which equation (6) and the steady-state identities \( \mu = C \int_v^\pi \chi[\beta(v)] f_V(v) dv \) and \( \mu t_V(v) = C \chi[\beta(v)] f_V(v) \) identify \( \mu \) and \( T_V \).

\[ \square \]

**Appendix B. ONLINE SUPPLEMENTAL MATERIALS FOR How Efficient are Decentralized Auction Platforms?**

2Alternatively, if the optimizer of equation (4) is unique and the allocation rule \( \chi(b) \) and pricing rule \( \rho(b) \) can be identified from the available observables \( (K, R, Y) \), then \( \beta^*(v) \) is identified.

3Note that the marginal agent only wins when he is alone in the auction and his bid is greater than the reservation price. The probability of winning and expected transfer can be computed from \( \pi(1) \) and \( G_R \).
We now describe the algorithm used to compute the counterfactuals. For expositional clarity, we focus on the SPA pricing rule. The structural primitives of our model are \( \mu, t_V, \kappa, G_R, \lambda_2 \), and the space of types \([v, \bar{v}]\). These structural parameters remain fixed when performing our counterfactual exercises unless otherwise noted. Recall from the theory and identification sections that the bidder arrival process we estimate, \( \pi(\cdot; \lambda_1, \lambda_2) \), must be consistent with the value of \( C \) present in the DGP. However, our counterfactual scenarios will induce exit from the market, giving rise to other values \( C' \) not present in the DGP. Since the bidder arrival process is 2-dimensional, we hold the dispersion parameter fixed and adopt the convention that \( \lambda_1 = C'(1 - \lambda_2) \), which means we could write: \( \pi(k; C', \lambda_2) = C'(1 - \lambda_2)(C'(1 - \lambda_2) + k\lambda_2)^{k-1}e^{-\left[C'(1-\lambda_2)+k\lambda_2\right]}. \) Thus, we need to pin down the endogenous variables \( e, \beta, C, \lambda_1, \) and \( F_V \).

Recall that in our dataset \( e = v \), but in the counterfactuals we consider we often find \( e > v \). Any equilibrium of a market with a SPA spot market must satisfy the following conditions:

\[
\begin{align*}
\mu t_V(v) &= \chi(\beta(v)) f_V(v) C \\
\beta(v) &= \beta_s(v - \delta V(v)) = v - \delta V(v) \\
V(v) &= \sum_{t=0}^{\infty} \delta^t \int_v^\infty (1 - \chi(\beta(s)))t \chi(\beta(s)) ds \\
\chi(\beta(e))e - \rho(\beta(e)) &= \kappa \\
F_V(\bar{v}) &= 1
\end{align*}
\]

Equation 8 requires that the distribution and measure of buyers exiting after winning an auction equals the distribution and measure of buyers flowing in, which is the stationarity condition of Definition 2.4. Equation 9 pins down the bidding strategy, while equation 10 uses the envelope theorem to describe the value function. Equation 9 would use an ordinary differential equation in non-SPA spot markets to describe \( \beta_s \), but this is a relatively trivial modification to our structure. Equation 11 requires that the lowest value buyer that enters be indifferent to entering, which is the optimal entry condition of Definition 2.4. Finally, Equation 12 requires that the steady-state distribution of types be properly normalized.

In practice, our software uses a bisection algorithm to search for the equilibrium value of \( e \). Given \( e \), we let the auction size parameter, \( \lambda_1 \), adjust so that Equation 11 holds. Since \( C = E[K] = \frac{\lambda_1}{1-\lambda_2} \) by assumption, we immediately have \( C \). Given \( e, \lambda, \) and \( C \) we can solve Equations 8 - 10 to obtain candidate values for \( f_V, \beta, \) and \( V \). If equation 12 fails to hold for
the candidate \( f_V \) (i.e., if the steady-state distribution of types is not properly normalized), then we adjust our guess for \( e \) and repeat the process.\(^4\)

**Remark 2** (Uniqueness of the Stationary Competitive Equilibrium). *It is difficult to provide general sufficient conditions for a unique SCE in our setting (even for an SPA spot market) because analytically characterizing the influence of \( e \) on \( F_V \) is hard due to the presence of participation costs. However, equations (8)–(12) provide a simple numerical test for uniqueness. Since the only unknown endogenous variable is \( e \), let \( F^\tilde{e}_V(\tilde{v}) \equiv \int_{\tilde{v}}^{\tilde{e}} f_V(v) dv \) be defined as the un-normalized integral over bidder types that is attained by solving equations (8)–(11) for some candidate cutoff, \( \tilde{e} \), without imposing the normalization in (12). Then one can compute this quantity for a grid of \( \tilde{e} \) ranging over the baseline type support \([\underline{v}, \bar{v}]\) to see whether \( F^\tilde{e}_V(\tilde{v}) \) is strictly monotone in \( \tilde{e} \). When this condition is satisfied—and it is for the counterfactuals we compute—there is a unique value of \( \tilde{e} \) such that \( F^\tilde{e}_V(\bar{v}) = 1 \) and equations (8)–(12) are satisfied, which in turn means the equilibrium is unique.\(^5\) This numerical test is easy to implement, so verifying uniqueness is quite tractable.

**Appendix C. ONLINE SUPPLEMENTAL MATERIALS FOR How Efficient are Decentralized Auction Platforms? by Aaron L. Bodoh-Creed, Jörn Boehnke, and Brent R. Hickman**

**THEORY: APPROXIMATING A FINITE MODEL**

We refer to a model with a finite number of buyers and sellers as a *finite model*. Since the real world is clearly finite, we ideally would have estimated and computed counterfactuals using a finite model. The goal of this section is to justify our use of the continuum model as an approximation of the more realistic finite setting. We first lay out the primitives of the finite model analog to the continuum model described in Section 2. We then prove that an SCE of the continuum model is an approximate equilibrium of the finite model. This approximation result is our justification for the use of a continuum model as a proxy for the intractable, but more realistic, finite model.

**C.1. Primitives of the Finite Model.** We consider a sequence of games indexed by \( N \) where \( N \) refers to the number of sellers that list goods for sale in each period. All variables pertaining to the \( N \)-agent game are superscripted with \( N \). Each seller has a starting price \( R \) that is drawn randomly from the distribution \( G_R \), which yields an empirical distribution \( G^N_R \).

---

\(^4\) Since we do not have any data on the distribution of buyer values below the \( v \) observed in the data, we can only perform counterfactuals that yield a value of \( e \geq v \).

\(^5\) In the case where \( \kappa = 0 \) as in Backus and Lewis [2016], \( e = 0 \) in equilibrium and equations (8)–(12) pin down the remaining endogenous variables, which implies the equilibrium of our model is unique.

\(^6\) It is not difficult to allow for a random number of sellers in the finite game. If we denote the (potentially stochastic) number of sellers entering in period \( t \) of the \( N \)-agent game as \( S^N_t \), we require that:

\[
\frac{S^N_t}{N} \to 1 \text{ almost surely as } N \to \infty
\]

We do not consider this extension due to the considerable number of notational aggravations it causes.
of starting prices equal to \( G_{R_{i,t}} \) in period \( t \). The numbers of potential entrant buyers at \( t = 0 \) is denoted \( C_0^N \). We assume that as \( N \to \infty \)

\[
\frac{C_0^N}{N} \to C \in \mathbb{R}_{++}.
\]

The population of potential entrants in period \( t \) is \( C_t^N \). Nature generates \([N \mu]\) new buyers at the end of each period and adds them to the set of potential entrants. Each time Nature generates a new potential entrant buyer, her private value \( v \) is drawn from \( T_V \). The measure \( F^N_{V,t} \) describes the distribution of potential entrant buyer values in period \( t \) of the \( N \)-agent game. As in the continuum game, buyers observe their own value for the good, the participation cost, and the number and value distribution of the other potential entrant buyers in the game prior to choosing whether to enter. A bidder makes her choice of a bid without knowing either the number or identity of the other agents participating in the particular auction to which she is matched.

We now describe the stochastic matching process that assigns bidders to auctions in the finite setting. We denote the number of buyers that enter the market in period \( t \) as \( C_t^N \). The buyers and sellers are randomly ordered into queues with the ordering independent across periods. Nature sequentially matches each seller in the respective queue with the next \( k \in \{0, 1, \ldots\} \) buyers from the buyer queue where \( k \) is a realization of random variable \( K \) that is distributed according to probability mass function \( \pi(K; C) \).

Intuitively, if we consider a limit where the number of entering buyers and sellers grows without bound, then in the limit all of the entrants are matched into auctions. In the finite model, if the supply of entrants is not completely assigned to auctions, the unassigned buyers are referred to as \textit{unmatched buyers}. Unmatched buyers proceed to the next period without transacting. We refer to a seller as being \textit{seller rationed} if the supply of buyers is not sufficient to provide that seller with the number of buyers they are allocated by the matching process. Conditional on being matched, a particular bidder wins her auction if her bid is larger than the maximum of all competing bids and the seller’s starting price. Ties between highest bidders are resolved by assigning the item to the tied bidders with equal probability, but if the highest bid is tied with the starting price, then we assume the bidder wins the item.

We require the following assumption on the distribution of auction sizes in the finite game:

\textbf{Assumption C.1.} A \textit{local limit theorem} applies, meaning that for the sequence \((K_1, K_2, \ldots)\) with \( Z_N = \sum_{i=1}^N K_i \) and \( \psi \) denoting the density function of the normal distribution we have:

\[
\sqrt{N \text{Var}[K]} \Pr\{Z_N = k\} \to \psi \left( \frac{k - NE[K]}{\sqrt{N \text{Var}[K]}} \right) \text{ uniformly over } k \in \mathbb{Z}
\]
We use this assumption to approximate the probability mass function of sums of realizations of $K$ using the probability density function of the normal distribution. Local limit theorems apply to most distributions of interest to economists including the generalized Poisson distribution used in our estimator. For the interested reader, Theorem 3.1 and Lemma 3.1 of McDonald [2005] provide conditions for the application of a local limit theorem, and these conditions amount to assuming that:

$$\sup_k N |PR\{Z_N = k + 1\} - PR\{Z_N = k\}|$$

has a finite limit as $N \to \infty$.

Bidding strategies can be written as functions $O : [0, 1] \times \mathbb{R}_+ \times \Delta([0, 1]) \times \Delta([0, 1]) \to [0, 1]$ with a typical bid denoted $O(v_i, C^N_{t'}, F^N_{V,t'}, G^N_{R,t'})$. The entry decision for participating buyers is a function of the form $\theta : [0, 1] \times \mathbb{R}_+ \times \Delta([0, 1]) \times \Delta([0, 1]) \to \{\text{Enter, Out}\}$ with a typical realization $\theta(v_i, C^N_{t'}, F^N_{V,t'}, G^N_{R,t'})$. We let $\Sigma$ denote the buyers’ strategy space. We let the bid and entry decision functions condition on the realized aggregate state, $(C^N_{t'}, F^N_{V,t'}, G^N_{R,t'})$, to account for fluctuations in the aggregate state or nonstationarities in the equilibrium. We continue to assume, as we did in the main text, that an agent that chooses $\text{Out}$ exits the game immediately.

It is relatively easy to extend the notation used in the main text to nonstationary equilibria as we have for the finite model, and the working paper version of this project did so. Such an extension would be useful if one wished to, for example, study the effect of a firm’s announcement of a release date for a new version of a consumer electronics product. In general, it is difficult to prove existence results for nonstationary models, but there are some cases which can be handled by simple extensions of our results. For example, suppose a company announces that a new product will be introduced $T$ periods in the future. It is relatively easy to characterize a stationary equilibrium that applies after the introduction and characterize the equilibrium in the $T$ periods leading up to the introduction using backwards induction. Another extension would be to assume that product introductions occur with a fixed probability in each period and are not announced in advance. To handle this case, one could include a state variable in the model that indicates the generation of product that exists in the market in the current period. If one assumes that innovation in the product ends after a fixed number of versions are released, then one could characterize the final stationary-state after the innovations have been exhausted and then use backwards induction to solve for the entire equilibrium.

We use the notation $x^N(b, C^N_{t'}, F^N_{V,t'}, G^N_{R,t'}) = 1$ ($0$) to denote the random event that a buyer wins (loses) an auction with a bid of $b$, and $p^N(b, C^N_{t'}, F^N_{V,t'}, G^N_{R,t'})$ denotes the random transfer

---

\[\text{One could alternatively allow innovation to continue indefinitely and simply ignore innovations after some period } T. \text{ This would result in an approximate equilibrium since the predictions of agent activity today would not account for innovations after period } T. \text{ The upshot is that the equilibrium would be tractable to compute.}\]
from the buyer to the seller/eBay conditional on a bid of $b$. We also define:

$$
\chi^N(b, C^N_t, F^N_{V,t}, G^N_{R,t}) = E^N_t \left[ x(b, C^N_t, F^N_{V,t}, G^N_{R,t}) \right]
$$

$$
\rho^N(b, C^N_t, F^N_{V,t}, G^N_{R,t}) = E^N_t \left[ p(b, C^N_t, F^N_{V,t}, G^N_{R,t}) \right]
$$

We superscript the expectation operator to emphasize that we are referring to the $N$-seller economy.

All agents discount future payoffs using a per-period discount factor $\delta \in (0,1)$. The value function given a (symmetric) SCE $\sigma = (\epsilon, \beta)$ for an agent with value $v$ that bids $b = \beta(v)$ is:

$$
\mathcal{V}^N(v, C^N_t, F^N_{V,t}, G^N_{R,t} | \sigma)) = \mathbb{I}\{v \geq \epsilon\} \left( \chi^N[v] \left( v - \delta E^N_t \left[ \mathcal{V}^N(v, C^N_{t+1}, F^N_{V,t+1}, G^N_{R,t+1} | \sigma) \right] \right) - \rho[b] - \kappa \right)
$$

$$
+ \delta E^N_t \left[ \mathcal{V}^N(v, C^N_{t+1}, F^N_{V,t+1}, G^N_{R,t+1} | \sigma) \right]
$$

We use the notation $\mathcal{V}^N(v, C^N_t, F^N_{V,t}, G^N_{R,t} | \sigma'_i, \sigma_{-i})$ when buyer $i$ uses strategy $\sigma'_i$ and all other agents follow $\sigma$.

We use the following definition of an equilibrium in our finite games.

**Definition C.2.** The strategy vector $\sigma$ and the initial state $C^N_0 \in \mathbb{R}_{++}$ and $F^N_{V,0}, G^N_{R,0} \in \Delta_N([0,1])$ is an $\epsilon$-Bayes-Nash Equilibrium ($\epsilon$-BNE) of the $N$-agent game if for all bidder values $v$ we have

For all $\sigma'_i \in \Sigma_C$, $\mathcal{V}^N(v, C^N_0, F^N_{V,0}, G^N_{R,0} | \sigma) + \epsilon \geq \mathcal{V}^N(v, C^N_0, F^N_{V,0}, G^N_{R,0} | \sigma'_i, \sigma_{-i})$

Given the dynamic nature of our game, a solution concept that incorporates some notion of perfection might be expected. Consider the two ways in which an $\epsilon$-BNE can yield an $\epsilon > 0$. First, it may be that the agent does not exactly optimize with respect to high probability events, which results in a small loss with high probability. Second, the strategy may not optimize with respect to very rare events. Failing to optimize with respect to rare events can be approximately optimal but severely violate perfection. A stationary strategy can be an $\epsilon$-BNE even though perfection may not even be approximately satisfied at the histories of the finite game in which the market aggregates differ significantly from the stationary state.

**C.2. Approximating the Large Finite Model.** It is not difficult to see why the model becomes too computationally complex to solve precisely as $N \to \infty$. In the $N$-agent game, the bidder’s strategy must condition on all possible values of $C^N_t$, $F^N_{V,t}$, and $G^N_{R,t}$, which means the complexity of the strategies grows exponentially with $N$. The bidding strategy in the continuum need only condition on the values of $C_t$, $F_{V,t}$, and $G_{R,t}$, which evolve deterministically in equilibrium.

Our goal is to prove that the limit model approximates the large finite model. The foundation of our proof is a mean field result that proves that the evolution of the continuum game and the economy of a finite game with sufficiently many players are approximately
the same over finite horizons. Mean field results usually require strong continuity conditions on the evolution of the economic primitives and on the strategies adopted by the agents, conditions that we need to prove hold despite auction models admitting a wide array of possible discontinuities. In addition, since the within-period matching process of the finite game samples without replacement from a finite set of buyers, auction outcomes are correlated. We prove that as the market grows, the auctions become independent of one another.

With our mean field result in hand, we demonstrate that the expected buyer utility in the large finite game and the limit game are approximately the same. This insight translates into our approximation result, which proves that any exact SCE strategy of the limit game is an \( \varepsilon \)-BNE of the finite game with sufficiently many players.

**Proposition C.3.** Consider a SCE \((\sigma, C, F_V, G_R)\). For any \( \varepsilon > 0 \) we can choose \( N^* \) and \( \eta > 0 \) such that for all \( N > N^* \), \( \sigma \) is an \( \varepsilon \)-BNE strategy if \((\omega^N_0, F^V_N, G^R_N)\) satisfies

\[
\|C^N_0 - C\| + \|F^N_{V,0} - F_V\| + \|G^N_{R,0} - G_R\| < \eta
\]  

Equation (15)

Proposition C.3 may be seen as providing an approximation to the actual equilibrium being played within the data-generating process, but it admits an alternative interpretation as a behavioral strategy. If one assumes that agents are subject to small computational costs, then in large markets it may be that they follow SCE behavioral predictions in lieu of solving a complex optimization problem for a vanishing benefit. Finally, note that while our result requires that the aggregate state in \( t = 0 \) be close to the SCE state, if we assume that seller and bidder types are drawn from \( F_V \) and \( G_R \) with numbers close to \( N_C \) and \( N \), then \((C^N_0, F^N_{V,0}, G^N_{R,0}) \to (C, F_V, G_R) \) almost surely as \( N \to \infty \). In other words, Equation 15 above is very likely to hold in large markets, and becomes increasingly so as \( N \to \infty \).

C.3. **Proofs from Section C.2.** Our approximation result requires two steps. First, we must show that the limit game has a utility structure that is close to the utility structure of a sufficiently large finite game. In particular, we must show that the correlation across auctions vanishes as \( N \to \infty \). Second, we must show that these facts imply that with high probability there are no deviations that yield a significant improvement in the utility of any agent. We conduct each task in separate sections. Throughout the sections we focus on an SCE strategy of the limit game \((\sigma, C, F_V, G_R)\).

C.3.1. **Convergence of Utility.** First note that if an agent chooses \( \text{Out} \), then his utility is 0 regardless of the number of other agents. For the remainder of the section we will assume the agent in question chooses \( \text{Enter} \). The utility of a bidder in the current period of the N-agent game given the bidder enters and bids \( b \) is

\[
\chi^N(b)v_i - \rho^N(b) - \kappa
\]
If we can show that
\[ \chi^N(b) \to \chi(b) \quad \rho^N(b) \to \rho(b) \tag{16} \]
uniformly over \( b \) when we hold \( b, C, F_V, \) and \( G_R \) fixed, then we will have shown that the utility function in the \( N \) agent game converges to the utility functions of the limit game.

We first show that the probability of buyers being unmatched vanishes as \( C_N \) increases.

**Lemma C.4.** \( \Pr(\text{A particular buyer is unmatched}) = O\left(\frac{1}{\sqrt{N}}\right) + \omega_N \) where \( \omega_N \to 0 \) as \( N \to \infty \).

**Proof.** Let \( D_l \) denote the number of bidders matched to auction \( l \). For any buyers to be unmatched, the total “demand” for bidders from sellers must fall short of the supply, \( C_N \), which means that \( i \) buyers are not matched if and only if
\[ \sum_{l=1}^{S_N} D_l = C_N - i \]
Since any buyer is equally likely to be amongst the unmatched buyers, conditional on \( i \) buyers being unmatched, the probability that a particular buyer is unmatched is \( i/C_N \). The total probability a particular buyer is unmatched is
\[ \Pr(\text{A particular buyer is unmatched}) = \sum_{i=1}^{C_N} \frac{i}{C_N} \Pr\left[ \sum_{l=1}^{C_N} D_l = C_N - i \right] \]
Using assumption 2.1, we can approximate the probability mass function of the sum of the \( D_l \) using a normal distribution PDF, and the error \( \omega_N \) vanishes as \( N \to \infty \). This lets us write:
\[ \sum_{i=1}^{C_N} \frac{i}{C_N} \Pr\left[ \sum_{l=1}^{C_N} D_l = C_N - i \right] = \frac{1}{C_N} \sum_{i=1}^{C_N} \frac{i}{\sqrt{N\text{Var}[K]}} \psi\left[ \frac{-i}{\sqrt{N\text{Var}[K]}} \right] + \omega_N \]
\[ \leq \frac{1}{C_N} \frac{C_N}{\sqrt{N\text{Var}[K]}} \psi[0] + \omega_N \]
\[ = \frac{\psi[0]}{\sqrt{N\text{Var}[K]}} + \omega_N = O\left(\frac{1}{\sqrt{N}}\right) + \omega_N \]
where \( \psi \) is the standard normal PDF. \( \square \)

We first show that the probability that a seller is rationed vanishes as \( N \) increases.

---

\[ ^8 \text{For any approximation error } \epsilon > 0 \text{ (i.e., the error in Equation 13), there exists } d < \infty \text{ independent of } N \text{ or } r \text{ such that if } N - r > d \text{ then the approximation error is less than } \epsilon. \text{ Since } \epsilon \text{ was chosen arbitrarily, the approximation error for these terms vanishes as } N \to \infty. \text{ The terms in equation 17 where } N - r \leq d \text{ which may have a nontrivial error when using the local limit approximation, are a vanishing fraction of the terms in the outer summation of equation 17. For large } N \text{ these also are the lowest probability terms in this summation since they represent the largest degree of seller rationing. Therefore, the total probability (and hence the total contribution to the approximation error) of the terms where } N - r \leq d \text{ vanishes as } N \to \infty. \text{ Combining the arguments for the case where } N - r > d \text{ and the case where } N - r \leq d \text{, we have that the total approximation error vanishes as } N \to \infty. \]
Lemma C.5. \( \Pr(\text{A particular seller is rationed}) = O\left(\frac{1}{\sqrt{N}}\right) + \omega_N \) where \( \omega_N \to 0 \) as \( N \to \infty \).

Proof. Let \( D_l \) denote the number of bidders matched to auction \( l \). For any sellers to be rationed, the total “demand” for bidders from sellers must exceed the supply, \( C_N \), which means that the probability that exactly \( r \) sellers are rationed can be written:

\[
\sum_{i=0}^{\infty} \Pr\left[ \sum_{l=1}^{N-r} D_l = C_N - i \right] \Pr[D_{N-r+1} \geq i]
\]

Since any seller is equally likely to be amongst the rationed sellers, the probability that a particular seller is rationed is:

\[
\sum_{r=1}^{N} \frac{r}{N} \sum_{i=0}^{\infty} \Pr\left[ \sum_{l=1}^{N-r} D_l = C_N - i \right] \Pr[D_{N-r+1} \geq i]
\]

Using assumption 2.1, we can approximate the probability mass function of the sum of the \( D_l \) using a normal distribution PDF, and the error \( \omega_N \to 0 \) as \( N \to \infty \). This lets us write:

\[
\sum_{r=1}^{N} \frac{r}{N} \sum_{i=0}^{\infty} \Pr\left[ \sum_{l=1}^{N-r} D_l = C_N - i \right] \Pr[D_{N-r+1} \geq i]
\]

Using a change of variables we have:

\[
\sum_{i=0}^{\infty} \sum_{x(i)=1}^{\pi(i)} \frac{i + x \sqrt{N \text{Var}[K]}}{N \text{E}[K]} \Pr[K \geq i] \psi\left(\frac{r \text{E}[K] - i}{\sqrt{(N-r)\text{Var}[K]}}\right) + \omega_N
\]

where

\[
\tilde{x}(i) = \frac{r \text{E}[K] - i}{\sqrt{(N-r)\text{Var}[K]}}
\]

\[
\tilde{x}(i) = \frac{(N-1) \text{E}[K] - i}{\sqrt{\text{Var}[K]}}
\]

We can decompose equation[18] into two terms. First consider:

\[
\sum_{i=0}^{\infty} \frac{i \Pr[K \geq i]}{N \text{E}[K]^2} \sum_{x(i)=1}^{\pi(i)} \psi\left(\frac{\text{E}[K]}{\sqrt{(N-r)\text{Var}[K]}}\right) \lesssim \sum_{i=0}^{\infty} \frac{i \Pr[K \geq i]}{N \text{E}[K]^2} \int_{-\infty}^{\infty} \psi(x) \, dx
\]

\[
= \sum_{i=0}^{\infty} \frac{i \Pr[K \geq i]}{N \text{E}[K]^2}
\]

\[
= \frac{1}{N}
\]
The second term is:

\[
\sum_{i=0}^{\infty} \sum_{\exists (i)=1}^{\pi(i)} x \sqrt{N \text{Var}[K]} \frac{\text{Pr}[K \geq i]}{E[K]} \psi(x) \frac{E[K]}{\sqrt{(N-r) \text{Var}[K]}} \leq \frac{\sqrt{\text{Var}[K]}}{\sqrt{N E[K]}} \sum_{i=0}^{\infty} \frac{\text{Pr}[K \geq i]}{E[K]} \int_{-\infty}^{\infty} ||x|| \psi(x) \, dx
\]

\[
= \frac{\sqrt{\text{Var}[K]}}{\sqrt{N E[K]}} \int_{-\infty}^{\infty} ||x|| \psi(x) \, dx
\]

\[
= O \left( \frac{1}{\sqrt{N}} \right)
\]

Combining these terms we find:

\[
\sum_{r=1}^{N} \frac{r}{N} \sum_{i=0}^{\infty} \text{Pr} \left[ \sum_{l=1}^{N-r} D_l = C^N - i \right] \text{Pr} [D_{N-r+1} \geq i] = O \left( \frac{1}{\sqrt{N}} \right) + \omega_N
\]

as required. □

The asymptotics results required to prove equation 16 holds are complicated by the fact that in the finite game the buyers are sampled without replacement when assigned to auctions. Intuition suggests that, much as in other settings with sampling without replacement, the covariance between two auctions ought to vanish as the number of auctions grows. We prove that this is indeed the case in Lemma C.6.

The proof of Lemma C.6 proceeds in two steps. First, we use the Chebyshev inequality to prove that the expected outcome of bidding \( b \) in the \( N \)-agent game, \( \chi^N(b) \) and \( \rho^N(b) \), approaches the average outcome a bid of \( b \) would have generated across the \( N \) auctions. The second step is to show that the expected outcome of the \( N \)-agent game approaches the expected outcome of the game with a continuum of agents. Both steps require grappling with the correlation of outcomes across auctions, which is the only difference in the game mechanics between the model with a finite set of buyers and the model with a continuum of buyers.

We analyze the covariance between auctions caused by a rejection sampling algorithm for generating samples without replacement. In the rejection algorithm, a sample of buyers for each auction is generated with replacement. If the sample violates the conditions of sampling without replacement, then the sample is rejected. This process is repeated until a sample is not rejected. We prove that the probability of rejection vanishes as \( N \) increases, which in turn means that the covariance between auctions under the sampling without replacement regime of the finite model converges to the 0 covariance of the sampling with replacement regime of the continuum model as \( N \) increases.

There are three events that cause a sample of buyers for two auctions to be rejected. First, it could be that one of the sellers is rationed because the demand for buyers outstrips the
Lemma C.5 proves that this effect vanishes at a rate of $N^{-0.5}$. Second, it could be that a particular buyer is unmatched. Lemma C.4 proves that this effect vanishes at a rate of $N^{-0.5}$. The third event that causes a sample to be rejected is that a bidder appears twice in the sample. We show that the probability of this event also vanishes at the rate of $N^{-0.5}$.

**Lemma C.6.** Suppose that all buyers follow some SCE of the limit game, $\sigma = (e, \beta)$. For any $\epsilon, \gamma > 0$ we can choose $N^*$ such that for any $N > N^*$ and any $(C_N, F^N, G_R)$ we have

$$\Pr \left[ \frac{1}{CN} \left| \sum_{i=1}^{CN} \left( x_i^N(b) - \chi^N(b) \right) \right| > \epsilon \right] < \gamma$$

(19)

$$\Pr \left[ \frac{1}{CN} \left| \sum_{i=1}^{CN} \left( p_i^N(b) - \rho^N(b) \right) \right| > \epsilon \right] < \gamma$$

$$\sup_{b \in [0,1]} \left| \chi^N(b) - \chi(b) \right| = O \left( \frac{1}{\sqrt{N}} \right)$$

(20)

$$\sup_{b \in [0,1]} \left| \rho^N(b) - \rho(b) \right| = O \left( \frac{1}{\sqrt{N}} \right)$$

The choice of $\epsilon$ and $\rho$ can be chosen uniformly over $(C_N, F^N, G_R)$.

**Proof.** We provide a proof for Equations 19 and 20, but essentially identical arguments suffice for the other results. For notational cleanliness, we provide a proof of the probability of large positive deviations, but the analogous result for large negative deviations is essentially identical.

From Chebyshev’s inequality we have

$$\Pr \left[ \frac{1}{N} \sum_{i=1}^{N} \left( x_i^N(b) - \chi^N(b) \right) > \epsilon \right] \leq \frac{1}{\epsilon^2} \text{Var} \left( \frac{1}{N} \sum_{i=1}^{N} \left( x_i^N(b) - \chi^N(b) \right) \right)$$

(21)

$$= \frac{1}{\epsilon^2} \frac{1}{N^2} \left[ \sum_{i=1}^{N} \text{Var} \left( x_i^N(b) - \chi^N(b) \right) + 2 \sum_{i=2}^{N} \sum_{j=1}^{N} \text{cov} \left( x_i^N(b) - \chi^N(b), x_j^N(b) - \chi^N(b) \right) \right]$$

(22)

Since $x_i^N(b)$ is bounded, we know

$$\frac{1}{N^2} \sum_{i=1}^{N} \text{Var} \left( x_i^N(b) - \chi^N(b) \right) = O(N^{-1})$$

(23)

We now bound the covariance term by assessing the covariance that would be generated by using rejection sampling to generate a set of buyers for two auctions. A rejection sampling algorithm draws samples of buyers for the auctions using sampling with replacement (SWR) from the pool of buyers in the finite game, but the sample generated is rejected if it either (1) the buyer is unmatched, (2) one of the sellers is rationed, or (3) the sample of
buyers contains two "copies" of the same agent. This process is repeated until a sample is not rejected. Since the covariance between auctions is 0 under sampling with replacement, any covariance between auctions must be generated by events in which a sample is rejected. Since the price and probability of winning are bounded, we can bound the covariance due to these rejection events by bounding the probability that the sample is rejected. In the following argument we prove that the probability a sample is rejected vanishes, so the covariance terms vanish in the limit as $N \to \infty$.

Lemmas C.4 and C.5 provide a uniform upper bound on the probability of events (1) and (2). Let $E_S$ denote the event the auctions share a bidder in a SWR regime, which allows us to write
\[
\text{cov} \left( x_i^N(b) - \chi^N(b), x_j^N(b) - \chi^N(b) \right) \leq \text{Pr} [E_S] + O \left( \frac{1}{\sqrt{N}} \right) + \omega_N
\]
where the second term captures the probability of a buyer being unmatched or a seller being rationed and $\omega_N \to 0$ as $N \to \infty$.

Fix two auctions with $m$ and $n$ bidders. The probability the auctions do not share a bidder is:
\[
\left( 1 - \frac{1}{CN} \right) \left( 1 - \frac{2}{CN} \right) \ldots \left( 1 - \frac{m + n - 1}{CN} \right) > \left( 1 - \frac{m + n}{CN} \right)^{m+n}
\]
The probability the auctions share an agent is no more than
\[
\sum_{\{m,n:m+n \leq CN\}} \pi(m;CN)\pi(n;CN) \left[ 1 - \left( 1 - \frac{m + n}{CN} \right)^{m+n} \right]
\]
A binomial expansion yields:
\[
\left( 1 - \frac{m + n}{CN} \right)^{m+n} = 1 - \frac{(m + n)^2}{CN} + o \left( \frac{1}{CN} \right)
\]

\footnote{If we think of drawing the $m + n$ buyers in order, the second buyer chosen cannot have the same identity as the first (first term), the the third buyer chosen must not have the same identity as either of the first two (second term), etc.}
Therefore
\[
\sum_{\{m,n:m+n\leq C\}} \pi(m;C)\pi(n;C) \left[ 1 - \left( 1 - \frac{m+n}{C} \right)^{m+n} \right] = \sum_{\{m,n:m+n\leq C\}} \pi(m;C)\pi(n;C) \left( \frac{m+n}{C} \right)^2 + o \left( \frac{1}{C} \right)
\]
\[
\leq \frac{2E[K^2]}{C} + o \left( \frac{1}{C} \right)
\]
\[
= O \left( \frac{1}{C} \right)
\]

Putting all of these results together, we have that
\[
Pr \left[ E_S \right] = O \left( \frac{1}{C} \right) = O \left( \frac{1}{N} \right)
\]
where the final equality follows from the fact that $C = \Theta(N)$.

Pulling our argument together we have
\[
\frac{1}{\epsilon^2 N^2} \sum_{i=2}^{N-1} \sum_{j=1}^{i-1} \text{cov} \left( x^N_i(b) - \chi^N(b), x^N_j(b) - \chi^N(b) \right) = O \left( \frac{1}{\sqrt{N}} \right) + \omega_N, \omega_N \to 0 \text{ as } N \to \infty
\]

Referring back to equation 21, we then have
\[
Pr \left[ \frac{1}{N} \sum_{i=1}^{N} (x^N_i(b) - \chi^N(b)) > \epsilon \right] = O \left( \frac{1}{\sqrt{N}} \right) + \omega_N, \omega_N \to 0 \text{ as } N \to \infty
\]

The final step is proving that $\chi^N$ and $\chi$ are close to one another. Note that the difference between these two functions is generated by the fact that $\chi^N$ is generated by constructing the set of auctions with a sampling with replacement process, buyers can be unmatched and sellers can be rationed. Our argument regarding the correction required to account for these differences implies:
\[
\| \chi^N(b) - \chi(b) \| = O \left( \frac{1}{\sqrt{N}} \right)
\]

The uniformity with respect to $b$ follows from standard arguments based on the separability of the reals and both functions being monotone in $b$. The uniformity with respect to $(C^N, F^N, G_R)$ follows from the fact that the proof is completely independent of these variables.

The following lemma implies that the within-period utility of the game with a finite number of agents converges to the within-period utility function of the continuum limit game.
**Lemma C.7.** Suppose that all agents follow some SCE of the limit game, \((e, \beta)\). For any \(\varepsilon > 0\) we can choose \(N^*\) such that for any \(N > N^*\) and any \((C^N, F^N_V, G^N_R)\) we have

\[
\sup_{b,\varepsilon \in [0,1]} \| \chi^N(b)\sigma - \rho^N(b) - (\chi(b)\sigma - \rho(b)) \| < \varepsilon
\]

**(24)\)**

**Proof.** The result follows from Lemma C.6. \(\square\)

C.3.2. **Mean Field Lemma.** The goal of this subsection is to prove that the evolution of the finite game approaches the deterministic evolution of the limit game as \(N \to \infty\). We start by showing the initial distribution of agent types in the finite and limit game converges as \(N \to \infty\).

**Lemma C.8.** The empirical distribution of types and starting prices in period 0 of the \(N\)-agent game converges to \(F_V\) and \(G_R\) with a convergence rate of \(O(N^{-0.5})\).

**Proof.** Follows from Remark 1. \(\square\)

We use time indices for the variables in the next proposition to make the evolution of the aggregate variables clear. Let \(Q_N(C^N_t, F^N_{V,t}, F^N_{C,t+1} | \sigma) = (C^N_{t+1}, F^N_{V,t+1}, F^N_{C,t+1})\) denote the aggregate state iterator in the \(N\)-agent game, where \((C^N_{t+1}, F^N_{V,t+1}, F^N_{C,t+1})\) is a random variable.

**Lemma C.9.** Consider a stationary SCE strategy \(\sigma = (e, \beta)\) and aggregate state \((C, F_V, G_R)\). For any \(\eta, \gamma > 0\), we can choose \(N^*\) such that for all \(N > N^*\) and \((\hat{C}^N_t, \hat{F}^N_{V,t}, \hat{G}^N_{R,t})\) such that

\[
\| \hat{C}^N_t - C \| + \| \hat{F}^N_{V,t} - F_V \| + \| \hat{G}^N_{R,t} - G_R \| < \frac{\eta}{2}
\]

we have the following with probability at least \(1 - \gamma\)

\[
\| \hat{C}^N_{t+1} - C \| + \| \hat{F}^N_{V,t+1} - F_V \| + \| \hat{G}^N_{R,t+1} - G_R \| < \eta
\]

**(25)\)**

where \(Q_N((\hat{C}^N_t, \hat{F}^N_{V,t}, \hat{G}^N_{R,t}) | \sigma_i, \sigma_i^t) = (\hat{C}^N_{t+1}, \hat{F}^N_{V,t+1}, \hat{G}^N_{R,t+1})\).

**Proof.** As an initial note, when any single agent deviates from \(\sigma\) in the limit game, no change in the aggregate variables occurs. When a single agent deviates in the finite game, it causes a change of at most \((C^N)^{-1}\). Since \((C^N)^{-1} \to 0\) as \(N \to \infty\) in any equilibrium of the finite game, these deviations do not affect the convergence arguments presented below.

First, we have \(G^N_{R,t+1} \to G_R\) by Remark 1. What remains is to show that \(\frac{C^N_{t+1}}{N} \to C_{t+1}\) and \(F^N_{V,t+1} \to F_{V,t+1}\). Focusing on the new potential entrants, there are \([N\mu]\) agents added to the game with types \((\bar{\sigma}_1, ..., \bar{\sigma}_{[N\mu]}\) drawn from \(T_V\). \(\frac{[N\mu]}{N} \to \mu\) as \(N \to \infty\), so Remark 1 implies

\[
\frac{1}{[N\mu]} \sum_{i=1}^{[N\mu]} 1 \{ v \geq \bar{\sigma}_i \geq e \} \to T_V(v) \text{ uniformly over } v
\]

This means that the only thing we need to show is that the number and type distribution of agents continuing onto the next period in the finite game converges to the analogous
measure and distribution in the limit game as \( N \to \infty \). We now show that the measure and distribution of buyers that exit each period converges in probability and uniformly over \( v \) to the analogous quantities in the continuum model as \( N \to \infty \). Lemma \( \text{C.6} \) implies:

\[
\frac{1}{C_t} \sum_{i=1}^{C_t^N} \mathbf{1}\{v_i \geq e\} \left(1 - x_i(\beta(v))\right) \to \int_0^1 \mathbf{1}\{v \geq e\} \left(1 - \chi(\beta(v))\right) dF_{V,t}(s)
\]

\[
\frac{1}{C_t} \sum_{i=1}^{C_t^N} \mathbf{1}\{v \geq v_i \geq e\} \left(1 - x_i(\beta(v))\right) \to \int_0^1 \mathbf{1}\{v \geq v_i \geq e\} \left(1 - \chi(\beta(v))\right) dF_{V,t}(s)
\]

The first equation refers to the measure of buyers that continue to the next period, while the second equation refers to the distribution of the types of these buyers. Bringing these results together, we have \((C_{t+1}, F_{V,t}^N, G_{R,t+1}^N) \to (C_{t+1}, F_{V,t+1}, G_R)\) in probability as \( N \to \infty \), which is equivalent to our desired result.

Iterating Lemma \( \text{C.9} \) immediately gives us the following:

**Corollary C.10.** Consider a stationary SCE strategy \( \sigma \) and aggregate state \((C, F_V, G_R)\). For any \( \Delta, \gamma > 0 \), we can choose \( \eta > 0 \) and \( N^* \) such that for all \( N > N^* \) and \((C_t^N, F_V^N, G_R^N)\) such that

\[
\|C_t^N - C\| + \|F_V^N - F_V\| + \|G_R^N - G_R\| < \eta
\]

we have for all \( t' \in \{t, \ldots, t + \tau\} \) with probability at least \( 1 - \gamma \)

\[
\|C_{t'}^N - C\| + \|F_{V,t'}^N - F_V\| + \|G_{R,t'}^N - G_R\| < \Delta
\]

(26)

where for \( k \in \{0, \ldots, \tau - 1\} \) we define \( Q_N \left(C_{t+k}, F_{V,t+k}^N, G_{R,t+k}^N | \sigma_{-i}, \sigma'_{t+k+1}\right) = \left(C_{t+1+k}, F_{V,t+k+1}^N, G_{R,t+k+1}^N\right)\).

C.3.3. No Profitable Deviations. In this subsection, we finally prove our approximation result. We start by proving that the limit model has a continuous per-period utility function.

**Lemma C.11.** \( \chi(b)v - \rho(b) - \kappa \) is continuous with respect to \((C, F_V, G_R)\) and \( \beta \) when \( F_V \) admits a PDF that is bounded from above

**Proof.** \( \chi(b) \) was proven to be continuous in Lemma \( \text{A.3} \). We can write

\[
\rho(b) = \pi_M(1; C) \int_0^b u \ G_R(du) + \sum_{k=2}^{\infty} \pi(k; C) \int_0^b \int_0^b \max\{u, t\} G_R(du) G_B^{k-1}(dt)
\]

Lemma \( \text{A.2} \) implies that \( G_B \) is continuous in \((C, F_V, G_R)\) and \( (e, \beta) \). Since the integrands are continuous and \( G_B \) and \( G_R \) are continuous, the resulting integral is continuous.\( \blacksquare \)

These results, together with the convergence of the utility functions, yields the following result on the convergence of value functions.

\( ^{10} \)We have implicitly used the fact that the topology generated by \( d_K \) is finer than the weak* topology.
Lemma C.12. Consider a stationary SCE strategy \( \sigma \) and aggregate state \((C, F_V, G_R)\). For any \( \epsilon, \gamma > 0 \) we can choose \( \eta > 0 \) and \( N^* \) such that for all \( N > N^* \) and \((C_0^N, F_{V,0}^N, G_{R,0}^N)\) such that
\[
\|C_0 - C\| + \|F_{V,0}^N - F_V\| + \|G_{R,0}^N - G_R\| < \eta
\] (27)
we have with probability at least \( 1 - \gamma \)
\[
\text{For all } v, \left\| \mathcal{V}^N \left( v, C_0^N, F_{V,0}^N, G_{R,0}^N | \sigma \right) - \mathcal{V}(v, C, F_V, G_R | \sigma) \right\| < \epsilon
\]
Proof. First note that if \( v < \epsilon \), then we are done since the buyer never enters the market and receives the same payoff in either game. For the duration we assume that \( v \geq \epsilon \).

Let \( E_{0}^{N} [x_t] \), etc. refer to an agent’s expectation in period 0 about an event that occurs in period \( t \) of the finite game. We can write the value functions as
\[
\mathcal{V}^N \left( v, C_0^N, F_{V,0}^N, G_{R,0}^N | \sigma \right) = \sum_{t=0}^{\infty} \delta^t E_{0}^{N} \left[ x_t \right. v - p_t - \kappa | C_0^N, F_{V,0}^N, G_{R,0}^N, \sigma \]
\[
\mathcal{V}(v, C, F_V, G_R | \sigma) = \sum_{t=0}^{\infty} \delta^t \left( \chi v - \rho \right)
\]
Choose \( T \) such that \( \delta^T < \frac{\epsilon}{3} \), and note:
\[
\sum_{t=0}^{\infty} \delta^t \left[ (x_t v - p_t - \kappa) \right] < (1 - \delta) \delta^T v < \frac{\epsilon}{3}
\]
From hereon, we consider only the first \( T \) periods.

Lemma [C.7] implies that for any sample path of \( \left\{ \left( C_t^N, F_{V,t}^N, G_{R,t}^N \right) \right\}_{t=0}^{\infty} \) we can choose an \( N^* \) sufficiently large so that for all \( t \in \{0, \ldots, T\} \)
\[
\sup_{b, v \in [0,1]} \left\| E_{0}^{N} \left[ x_t^N (b) v - p_t^N (b) | C_t^N, F_{V,t}^N, G_{R,t}^N \right] - \left( \chi (b) v - \rho (b) \right) \right\| < \epsilon
\] (28)
where \( \chi \) and \( \rho \) are conditioned on \( (C_t^N, F_{V,t}^N, G_{R,t}^N) \). Lemma [C.10] implies that for any \( \Delta, \gamma > 0 \) we can choose \( \eta \) sufficiently small and \( N \) sufficiently large such that
\[
\| C_{t+\tau}^N - C \| + \| F_{V,t+\tau}^N - F_V \| + \| G_{R,t+\tau}^N - G_R \| \leq \Delta
\] (29)
for all \( \tau \leq T \) with probability at least \( 1 - \gamma \).

From Lemma [C.11], if Equation [29] holds and \( \Delta \) is sufficiently small, we have for all \( t \in \{0, \ldots, T\} \) we have:
\[
\left\| E_{0} \left[ x v - p - \kappa | C_t^N, F_{V,t}^N, G_{R,t}^N, \sigma \right] - \left( \chi (b) v - \rho (b) \right) \right\| < \frac{\epsilon}{3T}
\]
where \( \chi \) and \( \rho \) are conditioned on the steady-state aggregate variables, \((C, F_V, G_R)\). Note that this result holds uniformly over \( v \) and the closed neighborhood defined by Equation [29]. Finally, in the complementary event that the sample path of \( \left\{ \left( C_t^N, F_{V,t}^N, G_{R,t}^N \right) \right\}_{t=0}^{\infty} \) is not
close to \((C,F,V,G_R)\) (i.e., Equation \([29]\) fails to hold for \(\Delta\) sufficiently small) we have:

\[
\left\| E_0^N \left[ (xv - p - \kappa) |C_i^N,F_i^N,G_{R,i}^N| \right] - E_0 \left[ (xv - p - \kappa) |C,F,V,G_R| \right] \right\| < 1
\]

Therefore, for \(\Delta\) (and hence \(\eta\)) sufficiently small and \(\gamma < \frac{\epsilon}{3T+1}\), we have uniformly over \(v\)

\[
\left\| \mathcal{V}^N(v,C_i^N,F_i^N,G_{R,i}^N|\sigma) - \mathcal{V}(v,C,F,V,G_R|\sigma) \right\| < T \star \frac{\epsilon}{3T} + \frac{\epsilon}{3} + (T + 1)\gamma < \epsilon
\]

where the first error term refers to errors that occur in the approximation when equation \([29]\) holds, the second term includes errors accruing in periods after \(T\), and the final term is the expected error from the event when equation \([29]\) fails to hold.

Now we prove our main approximation result.

**Proposition C.13.** Consider a SCE \((\sigma,C,F,V,G_R)\) where \(e(C,F,V,G_R) < 1\). For any \(\epsilon > 0\) we can choose \(\eta > 0\) and \(N^*\) such that for all \(N > N^*\) and \(\gamma > 0\), \(\sigma\) is an \(\epsilon\)-BNE strategy if the state in the first period of the \(N\)-agent game, \(\left(C_0^N,F_{V,0},G_{R,0}^N\right)\), satisfies

\[
\left\| C_0^N - C \right\| + \left\| F_{V,0} - F_V \right\| + \left\| G_{R,0}^N - G_R \right\| < \eta
\]

**Proof.** From the one-step deviation principle, it suffices to consider a deviation by a single agent in a single period. Without loss of generality, let us assume the deviation occurs in period 0 by a bidder that chooses Enter. Lemma \([C.9]\) implies that for any \(\eta, \gamma > 0\) we can choose \(\eta\) sufficiently small that with probability \(1 - \gamma\)

\[
\left\| C_1^N - C \right\| + \left\| F_{V,1} - F_V \right\| + \left\| G_{R,1}^N - G_R \right\| < \eta
\]

(30)
even if a single agent deviates from the SCE strategy in period 1. Lemma \([C.12]\) implies that for any \(\epsilon > 0\) we can choose \(\eta, \gamma > 0\) sufficiently small and \(N\) sufficiently large that with probability \(1 - \gamma\)

\[
\left\| \mathcal{V}^N (v,C_1^N,F_{V,1}^N,G_{R,1}^N|\sigma) - \mathcal{V}(v,C,F,V,G_R|\sigma) \right\| < \frac{\epsilon}{4}
\]

(31)
for \(\left(C_1^N,F_{V,1}^N,G_{R,1}^N\right)\) that satisfy equation \([30]\). Equation \([31]\) implies that the effect of the current deviation on future periods is small.

Let \(\beta_{dev}(v)\) denote the optimal deviation for type \(v\) at \(\left(C_0^N,F_{V,0},G_{R,0}^N\right)\). From Lemma \([A.2]\) and Assumption 2.5 we have that for any \(\delta \geq 0\) and all \(v\) that we can choose \(\eta\) sufficiently small that

\[
\left\| \beta_{dev}(v) - \beta(v) \right\| < \delta v
\]

(32)
From Lemma \([C.7]\) we have for \(N\) sufficiently large:

\[
\sup_{b,v \in [0,1]} \left\| E_0^N \left[ x(b)v - p(b) \left| C_0^N,F_{V,0}^N,G_{R,0}^N \right| \right] - E_0 \left[ x(b)v - p(b) \left| C_0^N,F_{V,0}^N,G_{R,0}^N \right| \right] \right\| < \frac{\epsilon}{6}
\]
Combining this with Lemma C.11 yields for \( \eta \) sufficiently small and \( N \) sufficiently large

\[
\|E_0^N [x(\beta_{dev}(v) )v - p(\beta_{dev}(v))|C_1,F_{V,0}^N G_{R,0}^N ] - E_0 [x(b)v - p(b) |C,F,V,G_R]| \| < \frac{\varepsilon}{4} \tag{33} \]

Equations [31] and [33] and the fact that the SCE strategy is weakly optimal in the limit game given the SCE aggregate values of \( C \), \( F_V \), and \( G_R \) we get

\[
\mathcal{V}^N (v, C_0^N, F_{V,0}^N G_{R,0}^N |\sigma ) + \varepsilon \geq E_0^N [(x(\beta_{dev}(v))v - p(\beta_{dev}(v)) - \kappa ) + (1 - x(\beta_{dev}(v)))\delta \mathcal{V}^N (v_i, C_1^N, F_{V,1}^N G_{R,1}^N |\sigma ) |\sigma , C_0^N, F_{V,0}^N G_{R,0}^N ]
\]

which implies that following the SCE strategy is an \( \varepsilon \)-BNE for \( \eta > 0 \) sufficiently small. \( \square \)

APPENDIX D. ONLINE SUPPLEMENTAL MATERIALS FOR
How Efficient are Decentralized Auction Platforms?
by Aaron L. Bodoh-Creed, Jörn Boehnke, and Brent R. Hickman
EMPIRICS: MODEL TUNING PARAMETERS AND ASYMPTOTICS

D.1. Model Tuning Parameters and Additional Figures. Our estimator from Section 3.2 relied heavily on B-splines to achieve a high degree of flexibility in our functional forms, while maintaining good numerical behavior. One of the benefits of B-splines is their ease of incorporating shape restrictions, many of which can be imposed as simple linear constraints on the parameter values themselves. For example, under the Cox-de Boor recursion formula (with concurrent boundary knots), the only basis functions to attain a non-zero value at the boundaries are \( F_{b,1}(\cdot) \) and \( F_{b,b+3}(\cdot) \), which both equal one at the upper and lower endpoints, respectively. Therefore, enforcing boundary conditions is equivalent to setting the first and/or last parameter value equal to the known boundary value(s) of the B-spline function, which also cuts down on computational cost by reducing the number of free parameters. Monotonicity is also quite simple: [de Boor, 2001, p.115] showed that a B-spline function \( \hat{G}_B(b;a_b) \) will be monotone increasing (decreasing) if and only if the parameters themselves are ordered monotonically increasing (decreasing). This avoids the necessity of imposing a set of complicated, nonlinear (and potentially non-convex) constraints on the objective function values, as would be the case with global polynomials, in order to enforce appropriate shape restrictions which ensure our solution is a valid CDF.

However, before implementing the estimator there remain several free parameters to pin down, the most important of which are the knot vectors \( n_b, n_r, \) and \( n_v \) which in turn define the B-spline basis functions. We adopt the convention that knots will be uniformly spaced, which then reduces the problem to choosing values for \( I_r, I_v, \) and \( I_b \) that dictate the number of knots to use in the relevant B-spline function. For the first two we first choose a grid of uniform points in \([0,1]\) (quantile rank space), and then we map these back into \( R \) space (or \( V \) space) using the empirical quantile functions. This procedure ensures that the influence of the data is spread evenly among the various basis functions. For \( n_v \), we chose knots that
are uniform in bid space. The reason for this is that \( \alpha_b \) directly parameterizes the parent distribution \( \hat{G}_B \), but in our estimator we are matching the empirical moments of the order statistic distribution \( H \) without knowing the quantiles of \( G_B \) ex ante.

In Stage I we chose \( I_b = 10 \), and we partitioned the reserve price support by the quintiles of the empirical conditional distribution \( \hat{G}_R(r|R > \bar{r}) \), meaning \( I_r = 5 \). This gives us a total of 13 parameters for \( \hat{G}_B \) and 8 for \( \hat{G}_R \). We chose \( I_v = 15 \) knots at the quantiles of the distribution \( \hat{G}_B \circ \hat{\beta} \), which is known from Stage I. We chose \( I_v > I_b \) because \( \hat{F}_V \) must conform to the nuances induced by all first-stage parameters in order to accurately represent the implied private value distribution. We find that these choices provide a good fit to the data and that adding more parameters renders little benefit.

Figure 1 displays plots of the distribution of the highest loser bid and includes an extra plot for the model-driven \( H(y; \hat{\lambda}, \hat{\alpha}_b) \) distribution, which is derived from both the market tightness and parent bid distribution parameters. The lower panel depicts model fit for the seller reserve price distribution. Note that in both cases, the B-spline functions provide a very good fit to the underlying data. The difference between the two cases is that in the latter our B-splines parameterize the distribution \( \hat{G}_R \), which is directly matched to its empirical quantiles, whereas in the former, we parameterize \( \hat{G}_B \) and then indirectly match the moments of the implied order statistic distribution \( H \). Figure 1 also provides intuition about the nature of the B-spline functions we use to fit the variables of our model. Since both panels describe observable variables, we have included the empirical CDFs as well. A comparison between the empirical CDFs and the B-spline fit shows a very close correspondence. To illustrate the underlying components of our B-spline functions, we have included the locations of the knots and the basis functions in the plot as well. The knot locations are described by the thin, vertical, solid lines extending at the bottom of each panel. The families of basis functions are drawn at the bottom of each panel in thin, dashed lines.

D.2. **GMM Standard Errors.** Throughout this section we will view the rules discussed in the previous section for location and number of B-spline knots as fixed decisions which complete the definition of a flexible, though finite-dimensional, parametric model. Under this view, we relate our Stage 1 and 2 estimators to well-known GMM asymptotic theory, which validates our bootstrap calculations of standard errors. Let

\[
\alpha = (\lambda_1, \lambda_2, \alpha_{b_2}, \ldots, \alpha_{b_{I_b-1}}, \alpha_{r_1}, \ldots, \alpha_{r_{I_r-1}})^T
\]

11The conditioning is due to the mass point at the lower bound.

12A fully semi-nonparametric estimation routine based on B-splines would involve specifying a rule for optimal choice of \( I \) within finite samples and the rate at which \( I \) should increase as the sample size \( L \to \infty \). This is an interesting econometric question, but one which is beyond the scope of this paper.
denote a column vector of all stage-I free parameters, let $I_1 = 2 + I_b + 1 + I_r + 2$ denote its dimension, and let $w_l = (\tilde{k}_l, y_l, r_l)$ denote the observables for the $l$th auction, $l = 1, \ldots, L$.

D.2.1. Stage 1 Asymptotic Theory. Let $m(w_l; \alpha)$ denote our $(3L \times 1)$-dimensional moment condition function, for which the $n$th component is

\[
m_n(w_l; \alpha) = \begin{cases} \pi(\tilde{k}_n | r_n; \lambda, \alpha_b) - L\mathbb{1}(\tilde{k}_n = \bar{k}_n) \frac{\sum_{u=1}^L K\left(\frac{r_u - r_n}{s_R}\right)}{\sum_{u=1}^L K\left(\frac{r_u - y_n}{s_R}\right)}, & n \in \{1, \ldots, L\} \\
H(y_n; \lambda, \alpha_b) - \mathbb{1}(y_l \leq y_{n-L}), & n \in \{L+1, \ldots, 2L\}, \text{ and} \\
\hat{G}_{R}(r_n; \alpha_r) - \mathbb{1}(r_l \leq r_{n-2L}), & n \in \{2L+1, \ldots, 3L\}, \end{cases}
\]

and satisfying $E[m(w_l; \alpha)] = 0$. Note that a consistent estimator of the asymptotic variance of $m(w_l; \alpha)$, denoted $\hat{\Sigma}$, a $(3L \times 3L)$-dimensional matrix, is given by $\hat{\Sigma} = L^{-1} \sum_{l=1}^L m(w_l; \hat{\alpha}) m(w_l; \hat{\alpha})^\top$.

\[\text{Figure 1. Stage I Estimates}\]
The analogous empirical moment conditions are \( m_L(\alpha) = L^{-1} \sum_{t=1}^L m(w_t; \alpha) \), and note that our estimators 23 and 24 are equivalent to finding the standard GMM estimator \( \hat{\alpha} = \arg\min_{\alpha} Lm_L(\alpha)^T m_L(\alpha) \). Finally, let \( \tilde{M} \) denote the \((3L \times I_1)\)-dimensional matrix of (estimated) first derivatives, where the \((i,j)\)th element is defined as
\[
\frac{\partial m_i(w_t; \hat{\alpha})}{\partial \alpha_j} = \begin{cases} 
\frac{\partial \hat{\pi}(k_j | r_i; \lambda, \alpha)}{\partial \alpha_j} & i = 1, \ldots, L, \ j = 1, 2, \\
\frac{\partial H(u_{i,j}; \lambda, \alpha)}{\partial \alpha_j} & i = L + 1, \ldots, 2L, \ j = 1, 2, \\
0 & i = 2L + 1, \ldots, 3L, \ j = 1, 2, \\
\frac{\partial \hat{\pi}(k_j | r_i; \lambda, \alpha)}{\partial \alpha_{b, r_{ij} - j}} & i = 1, \ldots, L, \ j = 3, \ldots, I_b + 5, \\
\frac{\partial H(u_{i,j}; \lambda, \alpha)}{\partial \alpha_{b, r_{ij} - j}} & i = L + 1, \ldots, 2L, \ j = 3, \ldots, I_b + 5, \\
0 & i = 2L + 1, \ldots, 3L, \ j = 3, \ldots, I_b + 5, \\
0 & i = 2L + 1, \ldots, 3L, \ j = I_b + 6, \ldots, I_b + I_r + 8, \\
\frac{\partial \tilde{C}_2(r_{ij} - 1; \alpha)}{\partial \alpha_{b, r_{ij} - j}} & i = 2L + 1, \ldots, 3L, \ j = I_b + 6, \ldots, I_b + I_r + 8.
\end{cases}
\]

With these definitions, it follows from well-known econometric theory (see [Hayashi, 2000, Chapter 7]) that the parameter vector estimator \( \hat{\alpha} \) is consistent and asymptotically normal, converging at rate \( \sqrt{L} \), and a consistent estimator of its \((I_1 \times I_1)\)-dimensional asymptotic variance-covariance matrix is
\[
\hat{a}var(\hat{\alpha}) = L^{-1} \left( \tilde{M}^T \tilde{M} \right)^{-1} \tilde{M}^T \hat{\Sigma} \tilde{M} \left( \tilde{M}^T \tilde{M} \right)^{-1}.
\]

D.2.2. Stage 2 Asymptotic Theory. Given this result, a straightforward approach to deriving standard errors for Stage 2 objects is the delta method. First note that the value of any of our Stage 2 objects—including \( \chi(b; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) \), \( \rho(b; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) \), \( \kappa \), \( \mathcal{V}(v; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) \), \( \hat{\beta}(\hat{\nu}; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) \), \( \beta(v; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) \), and \( t_\mathcal{V}(v; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) \)—at a point in their respective domains (e.g., \( \mathcal{V}(v; \hat{\lambda}, \hat{\alpha}_b, \hat{\alpha}_r) \) for a particular \( v \in [\underline{v}, \overline{v}] \)) are differentiable functions of \( \hat{\alpha} \), as can be seen in equations (26)–(34). Let \( v(\alpha) = (v_1(\alpha), \ldots, v_{I_2}(\alpha))^T \) denote an \( I_2 \)-dimensional, differentiable, vector-valued function of the Stage 1 free parameters, and let \( Y(\alpha) \) denote its \( I_2 \times I_1 \)-dimensional partial derivative matrix, where the \((i,j)\)th element is given by
\[
Y_{ij}(\alpha) = \frac{\partial v_j(\alpha)}{\partial \alpha_i}, \quad i = 1, \ldots, I_2, \ j = 1, \ldots, I_1 \tag{14}
\]

Since \( \hat{\alpha} \) is approximately distributed as multivariate normal with mean \( \alpha \) and variance \( avar(\hat{\alpha}) \), the delta method applies and it is approximately true that
\[
v(\hat{\alpha}) - v(\alpha) \sim \mathcal{N} \left( 0, Y(\alpha) avar(\hat{\alpha}) Y(\alpha)^T \right).
\]

\[\text{In brief, the quantity } I_2 \text{ is at the researcher’s discretion. Once a suitable number of grid points are chosen from the domains of the various Stage 2 functionals, } I_2 \text{ is the sum of the total number of domain points where functional values are to be computed.}\]
D.2.3. Method of Sieves Extension. One possible extension to our method would be to consider an alternative sieve form where shape restrictions are gradually lifted as the sample size $L$ grows. For example, one could allow $I_b$ and $I_r$ to grow, and one could replace the GP form by a more flexible specification where $\pi(0), \ldots, \pi(I)$ are estimated where $\sum_{k=0}^{I} \pi(k) = 1$ and $I$ grows with $L$ also. In that case, the estimators $\hat{\pi}(\cdot), \hat{G}_B(\cdot),$ and $\hat{G}_R(\cdot)$ described in the body of the paper would belong to a broad class of semiparametric M-estimators. Since B-splines are mathematically equivalent to piecewise splines with differentiability conditions imposed at the interior knots (see de Boor [2001]), our Stage I estimators also fit within a subset of this broad class, being spline series least squares estimators. Large-sample properties within this class have been explored by Chen [2007] and Huang [2003], though certain non-trivial technical hurdles remain in our context in order to guarantee that their asymptotic theory applies to the sieve extension of our estimator. Moreover, establishing asymptotics for the Stage 2 estimators is also involved since conditions such as Hadamard directional differentiability of the Stage 2 estimators with respect to the stage 1 functionals must be established, among other things.

Appendix E. ONLINE SUPPLEMENTAL MATERIALS FOR
How Efficient are Decentralized Auction Platforms?
by Aaron L. Bodoh-Creed, Jörn Boehnke, and Brent R. Hickman

PRICE PREDICTABILITY AND THE RANDOM MATCHING ASSUMPTION

Our model assumes that bidders are randomly matched into auctions, which is compatible with a setting in which bidders randomly arrive on the platform based on exogenous factors (e.g., breaks in the work day, time constraints, internet availability, etc.) and bid on auctions that close in the near future. For selection on price to occur, bidders must be able to predict the auction’s closing price based on the information that is currently available. The first question to address then is, at what point during the life of the auction do bidders typically enter?

Figure 2 depicts the CDF of the time remaining in the auction when we observe a serious bidder first enter the auction by placing a bid. As one can see, there is a flurry of entry within the last hour of the auction, though there is a large tail of serious bidders that first interact with the auction several hours before the close. The median time that a serious bidder first bids is 55 minutes before the close of the auction, with a mean of 6.58 hours. When interpreting the information in the figure it is important to keep a few things in mind. First, after a bidder places her initial bid, eBay’s automated system sends her email updates throughout the remaining life of the auction (e.g., letting her know when her bid was overtaken by a competitor or reminding her that the end is near). Second, the figure depicts only entry time information for serious bidders; i.e., those who demonstrated commitment to competition for the good near the end by submitting bids that affected
the price path during the final hour. Across roughly 1,700 auctions in our sample, we observed 6,948 such serious bidders. Although it is impossible to know what specific logistical constraints are like for a given bidder (and thus her capacity to devote time and effort to selectively choosing a particular auction to enter), the figure gives some idea of the time horizons over which the typical serious bidder participates in a single auction.

The next relevant question to address is how feasible it is for a bidder to predict likely outcomes of the auction, given current price information at points in time before the close of bidding. To that end, we define the relative price ($RP$) of each auction as the median price across auctions within 24 hours before/after the auction, minus the realized final price in that auction. From the perspective of the $i$th serious bidder in the $l$th auction at the point in time when they first enter, $RP_l$ is an uncertain future event. When it is large and positive, the bidder chose well and got a bargain price; otherwise, he did not. The standard deviation of $RP_l$ is similar to that of the raw sale price, at $16.59, with a 90–10 range of $[-20, 18]$. We also define price at entry ($PE_{il}$) as the current price in the auction just before each serious bidder enters by submitting his first bid.

We measure the predictive power of current price by running the following regression

$$RP_l = \gamma_0 + \gamma_1 PE_{il} + \epsilon_{il}$$

across all 6,948 serious-bidder-auction pairs. We find that the $R^2$ of this regression is only 0.1095, indicating that the current price has very little predictive power at the point in time when the typical serious bidder enters the auction. Perhaps this is not entirely surprising since many serious bidders enter prior to the final 60 minutes of the auction as shown in Figure 2. As a further probe of relative price predictability, we also defined current price at

![Figure 2. Time of First Bid](image)
end minus $X$ minutes ($CP_{Xl}$) for the $l$th auction as the current price with precisely $X$ minutes remaining before close of bidding. We computed this quantity for each auction in our data and each $X \in \{1, 5, 10, 15, 30, 60, 90, 180\}$, and then ran the auction-level regression

$$RP_l = \xi_0 + \xi_1 CP_{Xl} + \epsilon_l.$$  

Figure 3 depicts our findings on predictive power, which remains low even relatively close to the end of the auction. We found that the $R^2$ of these regressions with 60 minutes remaining, 30 minutes remaining, 15 minutes remaining, and 1 minute remaining were roughly 0.34, 0.41, 0.47, and 0.65, respectively. In other words, at only 60 seconds prior to close of bidding, over a third of the variation in relative price cannot be captured by current price.

All of these results suggest that serious bidders ought to find it difficult to select into auctions based on favorable expectations of the final price. In turn, this suggests that randomness is likely to play a significant role in matching bidders to auctions.

**Appendix F. ONLINE SUPPLEMENTAL MATERIALS FOR How Efficient are Decentralized Auction Platforms? by Aaron L. Bodo-Creed, Jörn Boehnke, and Brent R. Hickman**

**ADDITIONAL COUNTERFACTUAL RESULTS**

F.1. **Relative Importance of Platform Composition and Dynamic Incentives.** Our model has two novel features relative to most of the empirical auctions literature: platform composition effects and dynamic incentive effects. Our goal in this section is to measure the relative importance of these two. As an illustrative example, we consider changes to the per-period participation cost $\kappa$. In addition to informing questions of academic interest,
this counterfactual provides practical guidance to eBay and other online market designers regarding what issues are of most importance when considering changes to a platform.

There are two effects when participation costs increase. First, agents’ continuation values drop, which in turn reduces demand shading and increases bids. Holding the starting price distribution $G_R$ fixed, these dynamic incentive (DI) effects increase allocative efficiency since bids are now more likely to exceed the starting price $R$. Second, an increase of the participation cost drives low-value buyers out of the market, which reduces the buyer-seller ratio and strengthens the steady-state distribution of active bidder types. The consequences of buyer selection out of the market are referred to as platform composition (PC) effects.

We consider a range of participation costs from the estimated status-quo value, which we denote $\kappa = 0.0657$, through a maximum of $10$. Our goal is to separate the DI and PC effects, which are tied together in equilibrium. For each counterfactual we consider the status-quo equilibrium with $\kappa$ and replace either the bid and value functions (which drive the DI effect) or the buyer-seller ratio and bidder value distribution (which drive the PC effect) of an alternative equilibrium with $\kappa' > \kappa$. The reader should keep in mind that neither of these exercises result in equilibrium outcomes; rather, they are meant to serve as a decomposition of the PC and DI effects.

To formally define the comparative statistics of interest, let $V_\kappa(v)$ denote the value function for a bidder with value $v$ in an equilibrium with participation cost $\kappa$. Let $C_\kappa$ denote the ratio of (active) buyers to sellers and $\lambda_\kappa$ denote the matching parameter in an equilibrium with participation cost $\kappa$. We use $e_\kappa$ to denote the endogenous entry threshold given $\kappa$. Let $f_{V\kappa}$ and $F_{V\kappa}$ denote the analogous steady-state PDF and CDF of (active) bidder types and note that these live on support $[e_\kappa, \bar{v}]$, with $\bar{v} < e_\kappa'$ whenever $\kappa < \kappa'$. Finally, let $\beta_\kappa(v)$ denote the equilibrium bidding strategy given participation cost $\kappa$. The probability of a buyer winning is: $\chi_\kappa(v; V_\kappa, \lambda_\kappa, F_{V\kappa}, \beta_\kappa) = G_R(\beta_\kappa(v)) \sum_{m=0}^{\infty} \pi_M(m, \lambda_\kappa) [F_{V\kappa}(v)]^m$ If all of the $\kappa$ subscripts take on the same value, then $\chi_\kappa$ is generated by the SCE for that particular value of $\kappa$.

The allocative efficiency of the assignment within a given period, $W$, is a function of the endogenous variables considered:

$$W(V_\kappa, \lambda_\kappa, F_{V\kappa}, \beta_\kappa) \equiv C_\kappa \int_{\underline{v}}^{\bar{v}} s\chi_\kappa(s; V_\kappa, \lambda_\kappa, F_{V\kappa}, \beta_\kappa)f_{V\kappa}(s)ds,$$

where for convenience we simply define $F_{V\kappa}(v) = f_{V\kappa}(v) = 0$ for each $v \in [\underline{v}, e_\kappa]$.

Our metric for the role of DI effects in shaping welfare is the dynamic gap, defined by:

$$DG(\kappa, \kappa') \equiv W(V_{\kappa'}, \lambda_{\kappa'}, F_{V\kappa'}, \beta_{\kappa'}) - W(V_\kappa, \lambda_\kappa, F_{V\kappa}, \beta_\kappa).$$

The dynamic gap is computed by comparing equilibrium allocative efficiency generated by $\kappa$ to an out-of-equilibrium market that uses the same matching parameter and steady-state distributions, but the value function ($V_{\kappa'}$) and bidding strategy ($\beta_{\kappa'}$) from an SCE with a
higher participation cost $\kappa'$. We hold fixed the endogenous quantities that correspond to PC effects ($\lambda$ and $F_V$) while allowing DI effects ($\beta$ and $V$) to vary with $\kappa$.

The platform gap, $\mathcal{PG}$, captures the importance of PC effects in determining welfare:

$$\mathcal{PG}(\kappa, \kappa') = W(V_{\kappa'}, \lambda_{\kappa'}, F_{V, \kappa'}, \beta_{\kappa'}) - W(V_{\kappa}, \lambda_{\kappa}, F_{V_{\kappa}}, \beta_{\kappa}).$$

This gap is computed by comparing equilibrium allocative efficiency generated by $\kappa$ to an out-of-equilibrium market with the same value function ($V_{\kappa}$) and bidding strategy ($\beta_{\kappa}$) but matching parameters and steady-state distributions of an equilibrium with a higher cost $\kappa'$. Here we hold DI effects ($\beta$ and $V$) fixed and vary endogenous quantities that correspond to the PC effects ($\lambda$ and $F_V$).

In Figure 4 we plot the ratio of the platform gap to the dynamic gap. When participation costs are low, the platform gap is twice as large as the dynamic gap. However, as costs rise, the platform gap becomes as much as ten times larger than the dynamic gap. The primary take-away from Figure 4 is that understanding the platform composition effects of market changes is more important than understanding the dynamic incentive effects of the changes. In terms of efficiency, the incentives driving selection into the platform market are at least as important as the incentives driving behavior once a buyer participates.

**F.2. Participation Cost Counterfactuals.** We would like to highlight the effect of centralization on the participation costs of the buyers through two different metrics. The first metric is the flow of average lifetime participation costs (FALPC) paid by the agents. At the point an agent of type $v$ bids for the first time, the expected lifetime participation costs paid by the agent is equal to $\kappa/\chi_u(\beta(v))$. The FALPC measures the total participation costs paid by the set of buyers entering the game in the current period. A measure $\mu$ of such buyers with types distributed as per $t_V(v)$ enter each period, meaning that the FALPC can be written as:

$$FALPC = \mu \int_0^{\infty} \frac{\kappa}{\chi_u(\beta(v))} t_{V,\beta}(v) dv$$
Table 1. Effects of Centralization on FALPC

<table>
<thead>
<tr>
<th># Units Per Listing</th>
<th>Discount Factor $\delta = $</th>
<th>0.75</th>
<th>0.80</th>
<th>0.86</th>
<th>0.88</th>
<th>0.92</th>
<th>0.95</th>
<th>0.98</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>$0.52$</td>
<td>$0.52$</td>
<td>$0.53$</td>
<td>$0.53$</td>
<td>$0.53$</td>
<td>$0.54$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$0.37$</td>
<td>$0.37$</td>
<td>$0.37$</td>
<td>$0.38$</td>
<td>$0.38$</td>
<td>$0.38$</td>
<td>$0.38$</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$0.27$</td>
<td>$0.27$</td>
<td>$0.27$</td>
<td>$0.27$</td>
<td>$0.27$</td>
<td>$0.28$</td>
<td>$0.29$</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>$0.20$</td>
<td>$0.20$</td>
<td>$0.20$</td>
<td>$0.20$</td>
<td>$0.21$</td>
<td>$0.21$</td>
<td>$0.23$</td>
</tr>
</tbody>
</table>

By focusing on the newly entering bidders, we are measuring the increase in the total expected participation costs in the current period, hence our choice to refer to this statistic as a “flow.” Since the distribution of entering bidder types is a structural primitive, the only effect of changing the spot-market mechanism is to change $\chi_u(b)$. The FALPC values are summarized in Table 1. Because the FALPC is not based on agent value, $\delta$ has very little effect on the realized values. On the other hand, $u$ has a strong influence through its effect in $\chi_u(b)$.

Across all of the discount factors, centralizing from $u = 1$ to $u = 8$ results in a roughly 60% decline in FALPC. There are two channels for the reduction in participation costs. First, because the good is allocated more efficiently when $u = 8$, high-valuation buyers are matched to the good more quickly. Second, and more importantly, since low-valuation buyers are less likely to win the good each period, these agents leave the game and do not incur participation costs in the first place. Inducing these buyer-types to exit has a particularly strong effect on participation costs since these low-valuation buyers stay in the market for many periods before winning when $u = 1$.

Our second metric, the stock of average lifetime participation costs (SALPC), assesses the expected lifetime participation costs of the agents that are present in the “stock” of agents characterized by the steady-state type distribution, $f_V(v)$. The equation for the SALPC is identical to equation 34 once we replace the measure and type distribution of the “flow” of entering buyers with the steady-state measure and distribution of agent types that characterize the “stock” of agents present in each period:

$$S\text{ALPC}_{u,\delta} = C \int_0^\infty \frac{\kappa}{\chi_u(\beta(v))} f_{V,u,\delta}(v) dv$$

The SALPC is described in Table 2.

There are two effects at work. First, when markets centralize, the total participation costs paid by a participant before winning an item goes slightly up on average. For example, when $\delta = 0.88$ and $u = 1$, the participants paid on average $8.98 each over their lifetimes in

\[\text{Since the estimation of } \kappa \text{ is independent of } \delta, \text{ the figures discussed below are essentially identical for all choices of } \delta.\]
Table 2. Effects of Centralization on SALPC

<table>
<thead>
<tr>
<th># Units Per Listing</th>
<th>Discount Factor $\delta = $</th>
<th>0.75</th>
<th>0.80</th>
<th>0.86</th>
<th>0.88</th>
<th>0.92</th>
<th>0.95</th>
<th>0.98</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>$66.57$</td>
<td>$67.77$</td>
<td>$68.95$</td>
<td>$69.58$</td>
<td>$71.27$</td>
<td>$72.97$</td>
<td>$75.92$</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$47.80$</td>
<td>$50.03$</td>
<td>$52.33$</td>
<td>$54.30$</td>
<td>$58.08$</td>
<td>$60.26$</td>
<td>$61.33$</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$34.59$</td>
<td>$36.09$</td>
<td>$37.33$</td>
<td>$38.07$</td>
<td>$40.14$</td>
<td>$44.86$</td>
<td>$61.51$</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>$22.55$</td>
<td>$23.84$</td>
<td>$26.05$</td>
<td>$27.73$</td>
<td>$32.46$</td>
<td>$41.49$</td>
<td>$65.59$</td>
</tr>
</tbody>
</table>

the market. When $\delta = 0.88$ and $u = 8$, the participants paid on average $9.50 over the course of their participation in the market. The average participation cost per bidder is pushed up by the fact that lower value agents must wait even longer (on average) before winning an item and exiting the market.

The larger effect is that fewer buyers participate in the market when $u$ increases. The buyer to seller ratio is 7.75 when $\delta = 0.88$ and $u = 1$, while the ratio is only 2.92 when $\delta = 0.88$ and $u = 8$. The SALPC is the product of the average per-bidder cost and the ratio of buyers to sellers. As seen in Table 2, the SALPC drops by roughly 60% as $u$ moves from 1 to 8 for the $\delta = 0.88$ case.

Obviously the FALPC values are much lower than the corresponding SALPC values. The gap reflects the differences between the structural primitives describing the agents that enter each period and the steady-state distribution. In short, because the steady-state includes a larger population of low value buyers that must wait many periods to win, the SALPC will disproportionately reflect the high lifetime participation costs incurred by these low value buyers.

We view the FALPC and SALPC as complementary metrics. The SALPC metric better reflects the population of buyers present at a given time, which makes SALPC the better metric of the participation costs as perceived by the population of agents typically present on the platform. If one were interested in the net welfare effect of participation costs, one could use FALPC as a metric of per-period welfare loss and compute the net present value of future welfare losses. Given the daily rate at which FALPC is measured, the appropriate discount factor ought to be very close to 1, which would result in net welfare losses of the same order of magnitude as the SALPC calculations presented in Table 2.

To place these results in the context of our finite model, one needs to recall that we normalize the measure of sellers to 1, which means one can interpret the costs above in terms of the cost per auction. For example, centralizing from $u = 1$ to $u = 8$ causes the FALPC to drop from $0.53$ to $0.21$ per auction when $\delta = 0.95$. Computing the net present value using this same discount factor, we find that centralization reduces the net present value of the participation costs by $6.40$ per auction, per day. Again, since this is a daily
discount factor, we view $\delta = 0.95$ as an extremely low estimate of the appropriate time
discount factor for computing the net present value. To find the total cost incurred in the
finite setting each day, we need to “de-normalize” the measure of sellers by multiplying
the per-auction statistics by the 11.25 auctions per day that we observe in our data. Given
this average number of auctions per day, the total welfare gain from the reduction of the
participation costs per day is $72 per day.[16]

F.3. Seller Incentives. As has been regularly noted about the eBay marketplace, sellers
tend to choose very low starting prices. In our data, almost 60% of the starting prices are
set at the lowest possible value of $0.99. It is easy to see that such a price is not optimal - a single seller could improve his profits if he set a starting price equal to $v$, the lowest
possible bid in the auction.[17] From the perspective of a single seller, choosing the optimal
starting price involves the same reasoning as in the classic optimal auctions literature: a
high starting price can increase the final price paid by the winner, but it also risks that the
good will go unsold.

We solve this problem numerically to get a sense of the strength of the incentives of the
sellers to carefully choose a revenue maximizing starting price. For this exercise we assume
that the seller has a supply cost of $0. Since we are considering a deviation by a single seller
in our limit game, the seller’s deviation has no effect on market aggregates. As a result, we
fix $\lambda, F_V, (e, \beta)$ at their status quo values. The problem the seller solves is:

$$\max_{r \geq 0} \Pr [B^{(1)} \geq r] E [\max \{r, B_M\} | B^{(1)} \geq r]$$

where $B^{(1)}$ is the highest bid in the auction and $B_M$ is the highest competing bid.

Our results are remarkably stable across different choices of $\delta$. The optimal starting price
varies from a low of $84.90 to a high of $85.80. At the optimal starting price, the revenue
generated is either $122.30 or $122.31 across all of the possible $\delta$. This represents an increase
in profit of just $0.95 relative to the profit generated by a starting price of $0.99.

The benefits from optimally choosing the starting price are small because each seller is
matched with 7.96 bidders in expectation, which means that the competition between bid-
ers is intense. Bulow and Klemperer [1996] show that in a static auction setting choosing
the starting price optimally pales in comparison to adding a single extra bidder to the mar-
ket.[18] With almost 8 bidders on average already participating, it should not be surprising

[16]One might have compared the welfare losses from the participation costs with the welfare gains created by
the market. Unfortunately, we do not believe such an exercise is credible since the consumer surplus is highly
dependent on the choice of $\delta$ whereas the participation costs are insensitive to $\delta$.

[17]Such a starting price would insure that if a single buyer was matched to the auction, the seller could extract
some value from that buyer. It would have no effect if two or more buyers were matched to the auction as
one of these buyers would necessarily set the sale price.

[18]Since we do not provide a model of seller activity, we view Bulow and Klemperer [1996] as merely suggest-
tive of what occurs in our setting.
Table 3. Probability of Winning In Equilibrium

<table>
<thead>
<tr>
<th>Valuation Quantile</th>
<th>Prob. of Winning Per Period</th>
<th>Exp. Number of Auctions Until Win</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>0.643</td>
<td>1.56</td>
</tr>
<tr>
<td>0.9</td>
<td>0.418</td>
<td>2.39</td>
</tr>
<tr>
<td>0.75</td>
<td>0.129</td>
<td>7.77</td>
</tr>
<tr>
<td>0.5</td>
<td>0.024</td>
<td>42.3</td>
</tr>
<tr>
<td>0.25</td>
<td>0.006</td>
<td>168</td>
</tr>
</tbody>
</table>

that there is little room left for optimizing the starting price to have a significant effect on auction revenues.

F.4. Additional Table. Table 3 displays the status quo probability of winning for various quantiles of the buyer valuation distribution given our estimates. We also include the expected number of periods until winning, which is simply the inverse of the probability of winning each period. Since winning is a binomial random variable, the various quantiles of the number of auctions until a win for a fixed valuation can be calculated using standard techniques. For example, the interquartile range on the number of periods until a win for a buyer at the 0.75 quantile of the valuation distribution (i.e., a binomial win probability of 0.129) is 2.08 to 10.04.

References