Derivation of equations (8) through (12)

In this note, we derive equations (8) through (12). Equation (8) concerns the optimal allocation of a given level of overall lending, $\overline{L_i}$, between countries *a* and *b*. Given overall lending, by (3) it can be seen that C_{i1} is invariant to the allocation decision. As such, the optimal allocation satisfies

$$\frac{\partial E(C_2)}{\partial L_a} = \frac{\partial E(C_2)}{\partial L_b}$$

By (4) and (5):

$$E(C_{i2}) = Y_{i2} + \int_{\underline{e}}^{e_{ia}^*} (1-q) g\left[E(T_{ia}) + e\right] f(e)\, de + \int_{e_{ia}^*}^{\overline{e}} \left\{ g\left[E(T_{ia}) + e\right] - D_{ia} \right\} f(e)\, de$$
$$+ \int_{\underline{e}}^{e_{ib}^*} (1-q) g\left[E(T_{ib}) + e\right] f(e)\, de + \int_{e_{ib}^*}^{\overline{e}} \left\{ g\left[E(T_{ib}) + e\right] - D_{ib} \right\} f(e)\, de$$

We take overall lending as given

$$L_{ia} + L_{ib} = \overline{L}$$

By the creditor zero profit conditions in (7)

$$\left[1 - F\left(e_{ib}^*\right)\right] D_{ib} = r\overline{L} - \left[1 - F\left(e_{ia}^*\right)\right] D_{ia}$$

Substituting and simplifying

$$E(C_{i2}) = Y_{i2} + g\left[E(T_{ia}) + E(T_{ib})\right] - r\overline{L} - \int_{\underline{e}}^{e_{ia}^*} qg\left[E(T_{ia}) + e\right] f(e)\, de - \int_{\underline{e}}^{e_{ib}^*} qg\left[E(T_{ib}) + e\right] f(e)\, de$$

Differentiating $E(C_{i2})$ with respect to L_{ia} and L_{ib} yields

$$\frac{\partial E(C_{i2})}{\partial L_{ia}} = -qg\left[E(T_{ia}) + e_{ia}^*\right] f\left(e_{ia}^*\right) \frac{\partial e_{ia}^*}{\partial L_{ia}}$$

and

$$\frac{\partial E(C_{i2})}{\partial L_{ib}} = -qg\left[E(T_{ib}) + e_{ib}^*\right] f\left(e_{ib}^*\right) \frac{\partial e_{ib}^*}{\partial L_{ib}}$$

Combining (6) and (7) yields

$$\left[1 - F\left(e_{ij}^*\right)\right] qg\left[E(T_{ij}) + e_{ij}^*\right] = rL_{ij}$$

Totally differentiating yields the first-order Taylor approximation

$$\frac{de_{ij}^*}{dL_{ij}} = \frac{r}{qg\left\{\left[1-F\left(e_{ij}^*\right)\right] - f\left(e_{ij}^*\right)\left[E\left(T_{ij}\right)+e_{ij}^*\right]\right\}}$$

for $j = a,b$. This is equation (11).

In the relevant range this term will be positive, implying that the probability of default is increasing in borrowing levels from that country.

Subsitituting

$$\frac{\partial E\left(C_{i2}\right)}{\partial L_{ia}} = -\frac{r\left[E\left(T_{ia}\right)+e_{ia}^*\right] f\left(e_{ia}^*\right)}{\left[1-F\left(e_{ia}^*\right)\right] - f\left(e_{ia}^*\right)\left[E\left(T_{ia}\right)+e_{ia}^*\right]}$$

and

$$\frac{\partial E\left(C_{i2}\right)}{\partial L_{ib}} = -\frac{r\left[E\left(T_{ib}\right)+e_{ib}^*\right] f\left(e_{ib}^*\right)}{\left[1-F\left(e_{ib}^*\right)\right] - f\left(e_{ib}^*\right)\left[E\left(T_{ib}\right)+e_{ib}^*\right]}$$

So the first-order condition satisfies

$$\left[E\left(T_{ia}\right)+e_{ia}^*\right] f\left(e_{ia}^*\right)\left[1-F\left(e_{ib}^*\right)\right] = \left[E\left(T_{ib}\right)+e_{ib}^*\right] f\left(e_{ib}^*\right)\left[1-F\left(e_{ia}^*\right)\right]$$

From above

$$\left[E\left(T_{ij}\right)+e_{ij}^*\right] qg\left[1-F\left(e_{ij}^*\right)\right] = rL_{ij}$$

Substituting

$$\frac{L_{ia}}{L_{ib}} = \left(\frac{f\left(e_{ib}^*\right)}{f\left(e_{ia}^*\right)}\right)\left[\frac{1-F\left(e_{ia}^*\right)}{1-F\left(e_{ib}^*\right)}\right]^2$$

This is equation (8). We next turn to the impact of an increase in $E\left(T_{ia}\right)$ (equation (9)). Holding overall lending constant

$$\frac{\partial E\left(C_{i2}\right)}{\partial L_{ia}} = -r\left\{\frac{\left[E\left(T_{ia}\right)+e_{ia}^*\right] f\left(e_{ia}^*\right)}{\left[1-F\left(e_{ia}^*\right)\right] - f\left(e_{ia}^*\right)\left[E\left(T_{ia}\right)+e_{ia}^*\right]} + \frac{\left[E\left(T_{ib}\right)+e_{ib}^*\right] f\left(e_{ib}^*\right)}{\left[1-F\left(e_{ib}^*\right)\right] - f\left(e_{ib}^*\right)\left[E\left(T_{ib}\right)+e_{ib}^*\right]}\right\}$$

Differentiating with respect to $E\left(T_{ia}\right)$ yields

$$\frac{\partial E\left(C_{i2}\right)}{\partial L_{ia}\partial E\left(T_{ia}\right)} = \frac{r\left[E\left(T_{ia}\right)+e_{ia}^*\right]\left\{f'\left(e_{ia}^*\right)\left[1-F\left(e_{ia}^*\right)\right] + f\left(e_{ia}^*\right)^2\right\}}{\left\{\left[1-F\left(e_{ia}^*\right)\right] - f\left(e_{ia}^*\right)\left[E\left(T_{ia}\right)+e_{ia}^*\right]\right\}^2} > 0$$

Totally differentiating the first-order condition with respect to L_{ia} and $E\left(T_{ia}\right)$

then yields (9)

$$\frac{\partial L_{ia}}{\partial E\left(T_{ia}\right)} = -\frac{r\left[E\left(T_{ia}\right)+e_{ia}^{*}\right]\left\{\left[1-F\left(e_{ia}^{*}\right)\right]f'\left(e_{ia}^{*}\right)+f\left(e_{ia}^{*}\right)^{2}\right\}}{\frac{\partial^{2}E\left(C_{i2}\right)}{\partial L_{ia}^{\;2}}\left\{\left[1-F\left(e_{ia}^{*}\right)\right]-f\left(e_{ia}^{*}\right)\left[E\left(T_{ia}\right)+e_{ia}^{*}\right]\right\}^{2}} > 0 .$$

Where the denominator can be signed as negative by the debtor's second-order condition.

We next turn to the overall borrowing decision. Differentiating (2) with respect to $\overline{L}$ yields

$$\frac{\partial E\left(U_{i2}\right)}{\partial \overline{L}} = U' + b\,\frac{\partial E\left(C_{i2}\right)}{\partial \overline{L}}$$

The debtor's first-order condition satisfies

$$U' + b\,\frac{\partial E\left(C_{i2}\right)}{\partial \overline{L}} = 0$$

Totally differentiating with respect to $\overline{L}$ and $E\left(T_{ia}\right)$ yields

$$\frac{\partial \overline{L}}{\partial E\left(T_{ia}\right)} = -\frac{b\,\frac{\partial^{2}E\left(C_{i2}\right)}{\partial \overline{L}\,\partial E\left(T_{ia}\right)}}{U'' + b\,\frac{\partial^{2}E\left(C_{i2}\right)}{\partial \overline{L}^{2}}}$$

Since the denominator can be signed as negative from the debtor's second-order condition, the sign will be that of the numerator. Differentiating $E\left(C_{i2}\right)$ with respect to $\overline{L}$ yields

$$\frac{\partial E\left(C_{i2}\right)}{\partial \overline{L}} = -r - qg\left\{\left[E\left(T_{ia}\right)+e_{ia}^{*}\right]f\left(e_{ia}^{*}\right)\frac{\partial e_{ia}^{*}}{\partial L_{ia}}\frac{\partial L_{ia}}{\partial \overline{L}} + \left[E\left(T_{ib}\right)+e_{ib}^{*}\right]f\left(e_{ib}^{*}\right)\frac{\partial e_{ib}^{*}}{\partial L_{ib}}\frac{\partial L_{ib}}{\partial \overline{L}}\right\}$$

From the first-order condition above

$$\frac{\partial E\left(C_{i2}\right)}{\partial \overline{L}} = -r - qg\left\{\left[E\left(T_{ia}\right)+e_{ia}^{*}\right]f\left(e_{ia}^{*}\right)\frac{\partial e_{ia}^{*}}{\partial L_{ia}}\right\}$$

The first-order condition then satisfies

$$U' - b\left\{r + qg\left\{\left[E\left(T_{ia}\right)+e_{ia}^{*}\right]f\left(e_{ia}^{*}\right)\frac{\partial e_{ia}^{*}}{\partial L_{ia}}\right\}\right\} = 0$$

Differentiating with respect to $E\left(T_{ia}\right)$ then yields

$$\frac{\partial^2 E(C_{i2})}{\partial \overline{L} \partial E(T_{ia})} = -qg\left[E(T_{ia}) + e_{ia}^*\right]\left[-f'(e_{ia}^*)\frac{\partial e_{ia}^*}{\partial L_{ia}} + f(e_{ia}^*)\frac{\partial^2 e_{ia}^*}{\partial L_{ia}\partial E(T_{ia})}\right]$$

From the analysis above $\partial e_{ia}^* / \partial L_{ia} > 0$ in the relevant range. Differentiating this term with respect to $E(T_{ia})$ yields

$$\frac{\partial^2 e_{ia}^*}{\partial L_{ia}\partial E(T_{ia})} = -\frac{r\left\{f(e_{ia}^*) + f'(e_{ia}^*)\left[E(T_{ia}) + e_{ia}^*\right]\right\}}{qg\left\{\left[1 - F(e_{ia}^*)\right] - f(e_{ia}^*)\left[E(T_{ia}) + e_{ia}^*\right]\right\}^2} < 0$$

It follows that $\partial^2 e_{ia}^* / \partial L_{ia}\partial E(T_{ia}) < 0$.

Substituting

$$\frac{\partial \overline{L}}{\partial E(T_{ia})} = -\frac{b\,r\left[E(T_{ia}) + e_{ia}^*\right]\left[f'(e_{ia}^*)\left[1 - F(e_{ia}^*)\right] + f(e_{ia}^*)^2\right]}{\left[U'' + b\,\frac{\partial^2 E(C_{i2})}{\partial \overline{L}^2}\right]\left\{\left[1 - F(e_{ia}^*)\right] - f(e_{ia}^*)\left[E(T_{ia}) + e_{ia}^*\right]\right\}^2} > 0$$

since the denominator can be signed as negative by the debtor’s second-order condition. This is equation (12).