Continuous Time, Closed Form Solution for the Case Without Learning and $\pi(t)$ Non-Constant

(Addendum to: "Valuation and Return Dynamics of New Ventures" by Berk, Green and Naik, forthcoming *Review of Financial Studies*)

When the success intensity is not constant but a known function of the stage alone, $\pi(t) = \pi_n$, the Bellman-Hamiliton-Jacobi equation becomes

$$\frac{1}{2}x^2\sigma^2\frac{\partial^2}{\partial x^2}V^n(x) + \hat{\mu}x\frac{\partial}{\partial x}V^n(x) - \hat{r}V^n(x) + u_n^*(x)\left(\pi_n\left[V^{n+1}(x) - V^n(x)\right] - a - bx\right) = 0$$

where $u_n^*(x)$ denotes the optimal investment rule. This recursive system of ordinary differential equations is solved by starting with the terminal value and working backwards. At each stage the solution involves several constant parameters, and a number of constants for stage n that are defined recursively given the constants obtained solving the previous stage and the boundary conditions in Corollary 2 in the main paper. The next proposition states this solution.

Proposition 1 *When the success intensity at each stage is known to be π_n the value of the R&D project, $V_n(x)$, is*

$$V_n(x) = \begin{cases} \sum_{i=n}^{N-1} F_n^i x^{\gamma_i} + B_n x + G_n & x \geq x_n^* \\ A_n x^{\beta} & x < x_n^* \end{cases} \tag{1}$$

where,

$$\beta = \frac{(\sigma^2 - 2\hat{\mu}) + \sqrt{8\hat{r}\sigma^2 + (\sigma^2 - 2\hat{\mu})^2}}{2\sigma^2}$$

$$\gamma_n = \frac{(\sigma^2 - 2\hat{\mu}) - \sqrt{8(\hat{r} + \pi_n)\sigma^2 + (\sigma^2 - 2\hat{\mu})^2}}{2\sigma^2}$$

$$G_n = \frac{G_{n+1} - \frac{a}{\pi_n}}{1 + \frac{\hat{r}}{\pi_n}}$$

$$B_n = \frac{B_{n+1} - \frac{b}{\pi_n}}{1 + \frac{\hat{r} - \hat{\mu}}{\pi_n}}$$

$$F_n^i = \begin{cases} 0 & i = N \\ \frac{\pi_n}{\pi_n - \pi_i} F_{n+1}^i & n < i < N \\ \frac{(x_n^*)^{-\gamma_n}}{\pi_n}\left((\hat{r} - \hat{\mu})B_n x_n^* + \hat{r}G_n - \sum_{i=n+1}^{N-1} \pi_i\ (x_n^*)^{\gamma_i}\ F_n^i\right) & i = n \end{cases}$$

$$A_n = \frac{(x_n^*)^{-\beta}}{\beta\pi_n}\left((\pi_n + (\hat{r}-\hat{\mu})\gamma_n)B_n x_n^* + \hat{r}\gamma_n G_n + \sum_{i=n+1}^{N-1}(\gamma_i\pi_n - \gamma_n\pi_i)\ (x_n^*)^{\gamma_i}\ F_n^i\right).$$

and x_n^* solves the following equation:

$$\left(1 - \frac{1}{\beta} + \frac{(\hat{r}-\hat{\mu})\left(1-\frac{\gamma_n}{\beta}\right)}{\pi_n}\right) B_n x_n^* + \left(1 + \frac{\hat{r}\left(1-\frac{\gamma_n}{\beta}\right)}{\pi_n}\right) G_n$$
$$= \sum_{i=n+1}^{N-1}\left(\frac{\gamma_i}{\beta} + \frac{\pi_i\left(1-\frac{\gamma_n}{\beta}\right)}{\pi_n} - 1\right)\ (x_n^*)^{\gamma_i}\ F_n^i.$$

The constants are solved for recursively, subject to the following two boundary conditions:

$$G_N = 0$$
$$B_N = \frac{1}{\hat{r}-\hat{\mu}}.$$

The proof of this proposition follows exactly the same logic as the proof of Proposition 7 in the main paper.