A Note on the CAPM with Assumptions on Preferences and Distributions Alone [1]

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[1] A copy of this note, in addition to any other working paper by the author, is available on the author's WWW home page which can be accessed from: "http://finance.commerce.ubc.ca/".

In this note I will use the results in Berk[1] to derive more explicit (and potentially more useful) restrictions on primitives that give the CAPM. I focus attention on the classes of restrictions on preferences and distributions alone that provide the CAPM. Restrictions are not imposed on endowments because they are difficult to support empirically. For one thing, since endowments are unobservable, any restriction is untestable. For another, since endowments vary through time, any restriction that holds in one time period is unlikely to hold in another.

I begin by defining a class of distribution functions of J mean-zero random variables that are elements of $L^N(\pi)$, the vector space of all random variables whose first N moments are finite. The joint distribution of $J \geq 2$ mean-zero random variables $\tilde{y}_1, \ldots, \tilde{y}_J \in L^N(\pi)$ is *pseudo-elliptical of order* N if for every positive integer $m \leq N$ and $l, k \leq J$ and every non-negative integer $\{n_j\}_{j=1}^J$ such that $\sum_{j=1}^J n_j = m$,

$$E\left[\prod_{j=1}^{J} \tilde{y}_j^{n_j}\right] = \begin{cases} 0 & \text{if any } n_j \text{ is odd,} \\ (n_k - 1)E\left[\tilde{y}_l^2 \tilde{y}_k^{n_k-2} \prod_{j\neq k,l} \tilde{y}_j^{n_j}\right] & \text{if } n_l = 0 \text{ and every } n_j \text{ is not odd,} \\ \left(\frac{n_l-1}{n_k+1}\right) E\left[\tilde{y}_l^{n_l-2} \tilde{y}_k^{n_k+2} \prod_{j\neq k,l} \tilde{y}_j^{n_j}\right] & \text{o.w.} \end{cases}$$

Hence, if the joint distribution of J mean-zero random variables is pseudo-elliptical of order N then all odd central moments less than and equal to N are zero and the even moments of each order N or less are multiples of each other. I will assume that the univariate distribution of all single mean-zero random variables is pseudo-elliptical of order ∞. This assumption implies that the factor distribution of any two-fund separating distribution (i.e., $J = 1$) is pseudo-elliptical of order ∞. The name *pseudo-elliptical* is used because the central moments of all uncorrelated elliptically distributed random variables satisfy these properties, that is, Chamberlain's[2] and Owen and Rabinovitch's[3] elliptical distributions are pseudo-elliptical of order ∞.

The next proposition establishes a class of joint restrictions on preferences and distributions that yield the CAPM.

Proposition 1 *The CAPM holds on a choice space in $L^N(\pi)$ whenever every agent's utility function is a polynomial of order N (i.e., $U_\kappa(x) = \sum_{n=0}^N a_n^\kappa x^n$) or less, the joint distribution of the factors is pseudo-elliptical of order N and the equilibrium consumption allocation is admissible.*

Proof: I only prove the case when $J = 2$. The proof for arbitrary J is algebraically more complex but follows identical logic. It is therefore left for the interested reader. The strategy of the proof is to show that if all agents' utility functions are an N^{th} order polynomial or less and the joint distribution of the factors is pseudo-elliptical of order N then the conditions in Corollary 3.2 (in Berk[1]) will be satisfied and so the CAPM holds.

Take a consumption allocation such that every agent κ's consumption, α^κ, is given by a levered position in the market portfolio, i.e.,

$$\tilde{x}_\kappa = W_\kappa \sum_{i=0}^{I} \alpha_i^\kappa \tilde{R}_i \equiv \theta_\kappa + \beta_\kappa(b_{m1}\ \tilde{\delta}_1 + b_{m2}\ \tilde{\delta}_2),\ \text{where } \alpha^\kappa, \beta_\kappa \in \Re, \tag{1}$$

and is admissible. Now, by Corollary 3.2, the CAPM holds in this economy so long as (13), in Berk[1], is satisfied at this allocation. By cross multiplying (13), from Berk[1]), I get:

$$E[U_\kappa'(\tilde{x}_\kappa)\tilde{\delta}_1]b_{m2} = E[U_\kappa'(\tilde{x}_\kappa)\tilde{\delta}_2]b_{m1}.$$

This implies that,

$$E[U_\kappa'(\tilde{x}_\kappa)\ (b_{m1}\tilde{\delta}_1 - b_{m2}\tilde{\delta}_2)] = 0. \tag{2}$$

Finally, substituting in the fact that all agents have polynomial utility functions of order N or less leaves,

$$\sum_{n=0}^{N-1} E\left[a_n^\kappa(\theta_\kappa + \beta_\kappa(b_{m1}\ \tilde{\delta}_1 + b_{m2}\ \tilde{\delta}_2))^n\ (b_{m2}\tilde{\delta}_1 - b_{m1}\tilde{\delta}_2)\right] = 0. \tag{3}$$

I must now prove that the above equation is always satisfied when the distribution of the factors is pseudo-elliptical of order N. Expanding the n^{th} term on the left hand side of (3) gives,

$$a_n^\kappa E\left[\left[\sum_{i=0}^{n}\sum_{j=0}^{i}\begin{pmatrix} n \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\tilde{\delta}_1^j\tilde{\delta}_2^{i-j}\theta_\kappa^{n-i}\beta_\kappa^i b_{m1}^j b_{m2}^{i-j}\right](b_{m2}\tilde{\delta}_1 - b_{m1}\tilde{\delta}_2)\right]$$

$$\begin{aligned}
&= a_n^\kappa E\left[\sum_{i=0}^{n}\sum_{j=0}^{i}\begin{pmatrix} n \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\tilde{\delta}_1^{j+1}\tilde{\delta}_2^{i-j}\theta_\kappa^{n-i}\beta_\kappa^i b_{m1}^j b_{m2}^{i-j+1}\right] \\
&\quad -a_n^\kappa E\left[\sum_{i=0}^{n}\sum_{j=0}^{i}\begin{pmatrix} n \\ i \end{pmatrix}\begin{pmatrix} i \\ j \end{pmatrix}\tilde{\delta}_1^{j}\tilde{\delta}_2^{i-j+1}\theta_\kappa^{n-i}\beta_\kappa^i b_{m1}^{j+1} b_{m2}^{i-j}\right]
\end{aligned}$$

$$\begin{aligned}
&= a_n^\kappa \sum_{i=2}^{n}\theta_\kappa^{n-i}\beta_\kappa^i\begin{pmatrix} n \\ i \end{pmatrix}\Bigg\{E\left[\tilde{\delta}_1\tilde{\delta}_2^i\right]b_{m2}^{i+1} \\
&\quad +\sum_{j=1}^{i}\begin{pmatrix} i \\ j \end{pmatrix}\left[E\left[\tilde{\delta}_1^{j+1}\tilde{\delta}_2^{i-j}\right] - \left(\frac{j}{i-j+1}\right)E\left[\tilde{\delta}_1^{j-1}\tilde{\delta}_2^{i-j+2}\right]\right]b_{m1}^j b_{m2}^{i-j+1} \\
&\quad -E\left[\tilde{\delta}_1^i\tilde{\delta}_2\right]b_{m1}^{i+1}\Bigg\}.
\end{aligned} \tag{4}$$

If the factors in a two-fund economy are pseudo-elliptical of order N, then for every positive integer $m \leq N$ and non-negative integer $l \leq m$,

$$E\left[\tilde{\delta}_1^l \tilde{\delta}_2^{m-l}\right] = \begin{cases} 0 & \text{if either } l \text{ or } (m-l) \text{ is odd,} \\ (m-1)E\left[\tilde{\delta}_1^2 \tilde{\delta}_2^{m-2}\right] & \text{if } l = 0 \text{ and } m \text{ is even,} \\ \left(\frac{l-1}{m-l+1}\right) E\left[\tilde{\delta}_1^{l-2} \tilde{\delta}_2^{m-l+2}\right] & \text{o.w.} \end{cases} \quad (5)$$

Substituting (5) into (4) shows that (4) is zero for any $n \leq N-1$. Thus, by Corollary 3.2 the CAPM holds. ■

The intuition why this proposition is true is easily explained. In order for the CAPM to hold, all agents in the economy must effectively be mean-variance maximizers, in other words, in equilibrium, they must not care about any other central moment of their portfolios. Now if agents have polynomial utility functions there are two ways to reduce the number of central moments they care about. One can either reduce the order of the agents' utility functions, or one can specify return distributions whose central moments are related in such a way that agents are indifferent to changes in them. The two conditions that are known to provide the CAPM represent the extreme position of only using one of these strategies at a time. A trade off clearly exists between these two strategies. Intuitively, if agents' utility functions are polynomials of order N or less, then they obviously will not care about any moment above the N^{th}. Therefore, for the CAPM to hold one need only restrict the first N central moments. The proposition gives the precise restriction that is required on the distribution of asset returns to effect this, that is, that the distribution of the factors must be pseudo-elliptical of order N.

As I have already pointed out, the well known conditions that provide the CAPM are special cases of the above proposition. If the non-zero coefficients of the polynomial expansions are limited to just the first two (i.e., quadratic utility), then in order for the CAPM to hold, the distribution of the factors must be pseudo-elliptical of order 2. However, from the definition of the factor structure in Berk[1] it is clear that all factor distributions are pseudo-elliptical of order 2. Thus, the CAPM holds for any return distribution. On the other hand, if the joint distribution of the factors is pseudo-elliptical of order ∞, the CAPM will hold for any (including an infinite order) polynomial utility function. This set includes all analytic utility functions since any analytic function can be expanded (uniquely) in a power series (i.e. a polynomial of possibly infinite order). The elliptical distributions are pseudo-elliptical of order ∞, so I have Chamberlain's [2] condition, that is, that the CAPM holds whenever asset returns are elliptically distributed.[1]

[1] Of course, any joint distribution of the factors that is elliptical can be replaced with a two-fund separating distribution.

Besides the well known conditions, the above proposition provides many other distribution-preference pairs that will give the CAPM in equilibrium. For instance, since the odd moments of any symmetric distribution are zero, any symmetric distribution will be pseudo-elliptical of order 3. Therefore, any symmetric distribution of the factors with cubic utility will provide the CAPM. A simple example of such an economy would be factors that are independently and identically distributed uniform random variables and agents with cubic utilities.

Unfortunately, none of the distribution-preference pairs that satisfy Proposition 1 are likely to be any more realistic than either one of the two well known conditions. Even if I only define utility functions over positive consumption, no finite order polynomial is both strictly increasing and strictly concave in the positive domain. It is therefore important to determine whether the CAPM can hold under any other restrictions on preferences and distributions alone. The following proposition establishes that if I restrict attention to utility functions that are analytic the answer is no.

First I must define, for any choice set X_κ, the domain of consumption $U_\kappa(\cdot)$ can range over. For any $\tilde{x}_\kappa \in X_\kappa$, denote $\underline{x}_\kappa$ ($\overline{x}_\kappa$) as the essential infimum (supremum) of $\tilde{x}_\kappa$.[2] Then the essential infimum of X_κ is defined as $\underline{X}_\kappa = \inf\{\underline{x}_\kappa | \tilde{x}_\kappa \in X_\kappa\}$. Likewise the essential supremum of X_κ is defined as $\overline{X}_\kappa = \sup\{\overline{x}_\kappa | \tilde{x}_\kappa \in X_\kappa\}$. Intuitively $\overline{X}_\kappa$ ($\underline{X}_\kappa$) denotes the largest (smallest) possible level of consumption available to agent κ. The interval $\mathcal{X}_\kappa = (\underline{X}_\kappa, \overline{X}_\kappa) \subseteq \Re$ is the domain over which $U_\kappa(\cdot)$ ranges.

Proposition 2 *Assume that every agent κ's utility function is analytic inside $\mathcal{X}_\kappa$. Then for every vector of endowments, the CAPM holds if either (i) a $N < \infty$ exists such that every agent's utility function is a polynomial of order N (i.e., $U_\kappa(x) = \sum_{n=0}^{N} a_n^\kappa x^n$) or less and the joint distribution of the factors is pseudo-elliptical of order N; or (ii) the joint distribution of the factors is pseudo-elliptical of order ∞.*

Proof: Again, I only consider the case when $J = 2$ since the proof for arbitrary J follows identical logic. From Corollary 3.2 (and the proof of Proposition 1) I know that for the CAPM to hold (2) must be satisfied for each agent κ when $\tilde{x}$ is admissible, market clearing and for each κ, α^κ is a levered position in the market portfolio.

Consider first the subset of analytic functions in $\mathcal{X}_\kappa$ that are polynomials of order $N < \infty$ or less. Following the logic in the proof of Proposition 1 I have that (2) implies that the following equation must always be satisfied for these utility functions:

$$\sum_{n=0}^{N-1} a_n^\kappa \sum_{i=2}^{n} \theta_\kappa^{n-i} \beta_\kappa^i \begin{pmatrix} n \\ i \end{pmatrix} \Bigg\{ E\left[\tilde{\delta}_1 \tilde{\delta}_2^i\right] b_{m2}^{i+1} +$$

[2] The essential infimum (supremum) of a random variable $\tilde{x}$ is $\sup\{y \in \Re | P(y > x) = 0\}$ ($\inf\{y \in \Re | P(y < x) = 0\}$).

$$\sum_{j=1}^{i} \begin{pmatrix} i \\ j \end{pmatrix} \left[E\left[\tilde{\delta}_1^{j+1}\tilde{\delta}_2^{i-j}\right] - \left(\frac{j}{i-j+1}\right) E\left[\tilde{\delta}_1^{j-1}\tilde{\delta}_2^{i-j+2}\right]\right] b_{m1}^{j} b_{m2}^{i-j+1} - E\left[\tilde{\delta}_1^{i}\tilde{\delta}_2\right] b_{m1}^{i+1} \Bigg\} = 0. \tag{6}$$

Now since the market portfolio always has price 1, its return is the same as its cashflow (i.e., $\tilde{R}_m = \tilde{V}_m$). The market cashflows are, however, completely determined by the total endowment. As a result, both b_{m1} and b_{m2} are completely determined by the total endowment. Since (6) must hold for any endowment allocation, it must hold for any b_{m1} and b_{m2}. However, since (6) can be viewed as a polynomial in b_{m1} and b_{m2}, the only way it can be satisfied for every b_{m1} and b_{m2} is if every coefficient of $b_{m1}^{i} b_{m2}^{i-j}$ is exactly zero. Imposing this condition provides precisely the moment restrictions that define the pseudo-elliptical distributions of order N.[3] That is, (6) is satisfied for any b_{m1} and b_{m2} only if the joint distribution of the factors is pseudo-elliptical of order N.

Next consider the rest of the analytic functions in $\mathcal{X}_\kappa$. Since $U(\cdot)$ is an analytic function that not a finite order polynomial, I have that

$$U_\kappa(x) = \sum_{n=0}^{\infty} a_n^\kappa (x - x_0)^n \tag{7}$$

where, $x \in \mathcal{X}_\kappa$, x_0 is the midpoint of $\mathcal{X}_\kappa$ and for all $n > 0$ there exists an $N > n$ such that $|a_N^\kappa| > 0$. Substituting (7) into (2) implies that (6) must be satisfied for $N = \infty$. This can only occur for any b_{m1} and b_{m2} if the joint distribution of the factors is pseudo-elliptical of order ∞. Thus the CAPM can only hold for an analytic utility function that is not a finite order polynomial if the joint distribution of the factors is pseudo-elliptical of order ∞. ■

The above proposition can be interpreted as restricting preferences by placing conditions on the power series coefficients of agents' utility functions. Similarly, it restricts the distribution of asset returns by placing conditions on the central moments of the joint distribution of the factors.

An immediate corollary of this proposition is that if the class of utility functions is limited to analytic functions that are everywhere strictly concave and strictly increasing, the CAPM only holds (for any endowment allocation) if returns are given by Ross'[4] two-fund separating distributions. Most economists believe that in any realistic model of the economy, at the very least, all agents should prefer more to less and be risk averse. Therefore, Ross' conditions provide the only realistic way to get the CAPM.

[3] To see this note, that for each value of i, the fact that the coefficient on the middle term inside the curly brackets must be zero relates all moments of the same order to each other. However, the coefficients of the first and last terms in the curly brackets are also required to be zero so together with the relation implied by the middle term this implies that all odd order moments are zero.

References

[1] Berk, J. B. (1997), "Necessary Conditions for the CAPM," *Journal of Economic Theory*, **73**: 245-257.

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