Finance 350

Homework 6

Due: Monday, November 24

Question 1 (Did you say *arbitrage* in the APT?)

This homework problem was motivated by a point first made by David Kreps. The example itself is due to Kerry Back. Suppose there are 4 equally likely states and 3 assets and the gross return (one plus the rate of return) matrix is

$$Z = \begin{pmatrix} 1 & 4 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

This market has a factor structure with factor

$$f = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

The beta of each of the risky assets is 1.

1. Show that there is an arbitrage opportunity in this market.
2. Calculate each asset's loading on the factor and show that the residual risk has conditional mean zero
3. Show that the APT pricing relation holds and calculate the factor risk premium.
4. Does the APT still hold when the market is completed by adding the following asset?

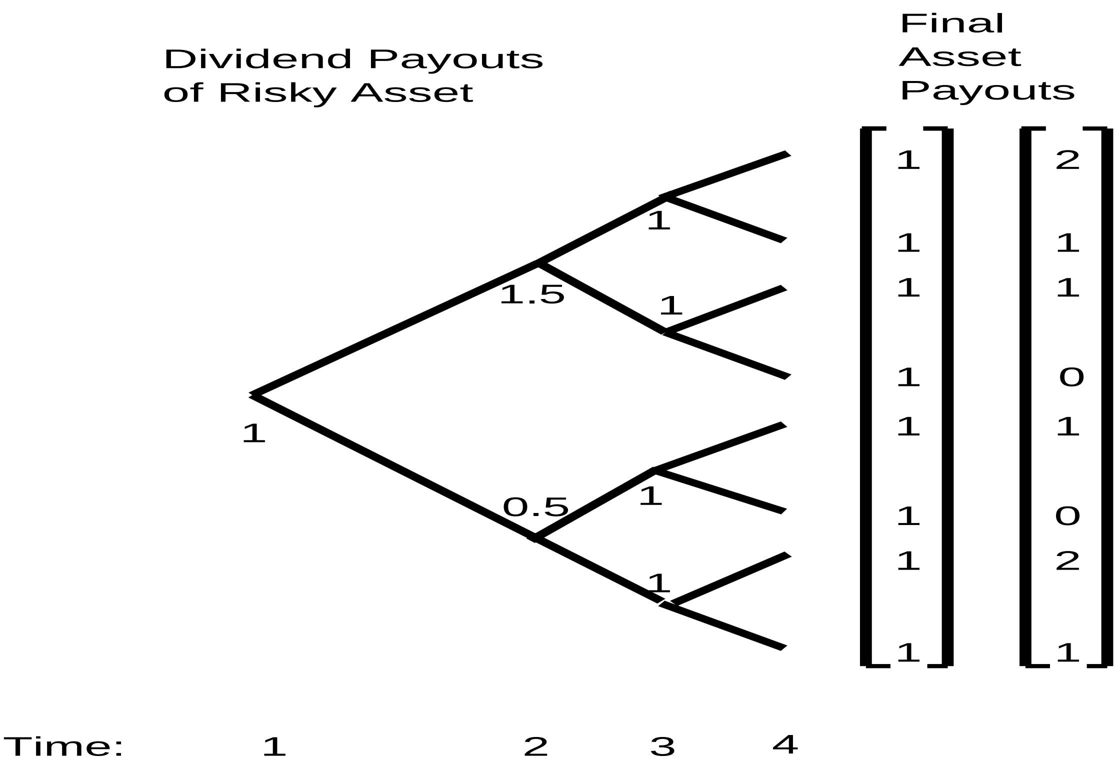
$$\begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}$$

5. What are the state prices in the complete market?

6. What can you conclude about the existence of arbitrage opporunities and the APT?

Question 2

Consider a 2 agent, 3 time period, economy. Uncertainty about the final state of nature is resolved according to the following information filtration:



The conditional probability of any branch is $\frac{1}{2}$ so all states are equally likely. There are two long lived assets, a riskless asset that pays of one unit of consumption at each point in time (including time 1) and a risky asset. The dividend payoff (in units of consumption) of the risky asset is marked at each node in the above filtration. The usual convention that all dividends are paid prior to trade in the period is adopted. The final payoffs of both assets are also marked on the figure.

Both agents have log utility with no time preference, i.e.,

$$U(x) = \log(x(1)) + \sum_{t=2}^{4} E[\log(x(t))].$$

The first agent is endowed with $1\frac{1}{2}$ units of the risky asset. The second agent is endowed with 1 unit of the riskless asset and is short $\frac{1}{2}$ unit of the risky asset (i.e., is endowed with $-\frac{1}{2}$ unit of the risky asset.)

1. Assume that a the full set of pure state claims exist and solve for the resulting complete market equilibrium.
2. Check to see whether at these equilibrium prices, the market is completable. If so, what are the dynamic trading strategies that implement the complete market equilibrium?
3. Since the market is dynamically complete, the introduction of any security can not change the asset span. That is, any security can be dynamically created out of the existing securities. Assume a european call option that expires at time 4 and has a strike price equal to 1 is introduced into the market. Calculate the option price at every node.
4. Describe the dynamic trading strategy that reproduces the option.[1]
5. Now construct a new economy. Assume that the two agents have the following endowments: Both agents have 1 unit of the riskless asset. In addition, the first agent is long $\frac{1}{2}$ unit of the risky asset and the second agent is short $\frac{1}{2}$ unit of the risky asset. To make life easy, assume that there is no consumption until time 4 and therefore the assets pay no dividends. Thus agents now have log preferences over final period consumption. Do parts (1) and (2) for this new economy.
6. Does an equilibrium exist in this new economy?

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[1]This insight, that the option can be created through a dynamic trading strategy due to Black and Scholes. They realized that as a result the option can be priced by the absence of arbitrage alone, i.e., you did not have to know anything about the underlying equilibrium to answer (3). Their famous option pricing formulae is derived by using this principle in a continuous time framework.