Chapter 5. The Discount Rate

In the last chapter we explained how to value streams of cash flows at different points in time. A key input into the valuation procedure is the interest rate. Since the focus of that chapter was on the cash flows, we did not distract the discussion by questioning where the interest rate comes from. That is the focus of this chapter.

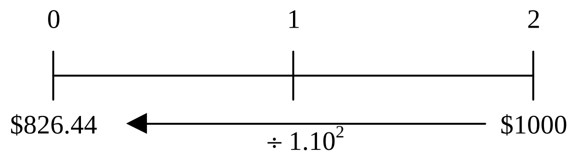
There are, in fact, many different interest rates in the economy. In each application there will always be one correct rate to use. To distinguish this rate from other interest rates in the economy, we will give it a distinct name --- **the discount rate.** To reiterate, the discount rate is the correct rate to use to move a particular cash flow in time.

In many applications the discount rate is easy to calculate. In others it is not. In some sense one of the central open questions in finance is whether it is possible to find a theory that can specify the discount rate in *every* situation. Given that this is still an open research question, there is no way we could completely answer this question in this chapter (or even in this book). What we will do in this chapter is begin to answer the question by demonstrating how to calculate the discount rate in a number of very common applications. All the applications we cover in this chapter occur routinely in investment decisions --- on both a business and a personal level. They will also reoccur throughout the rest of this text.

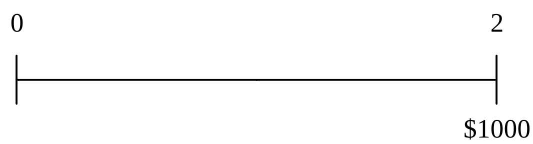
5.1 Computing Discount Rates of Different Period Lengths

When writing down a timeline, one decision we must make is what is the most convenient "period length" to use. In our examples so far, we have generally considered cash flows that occur at yearly intervals, and so have chosen a period length of one year. Selecting a period length equal to the interval between cash flows is often the most convenient choice. Certainly, the period length cannot be longer, but, in principal, it could be shorter. For example, often when the cash flows interval is longer than one year, we often still pick a period length of one year. The reason is that discount rates are often quoted on an annual basis.

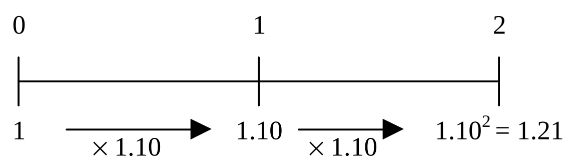
Consider the following simple timeline where we have computed the present value of $1000 received in two years using a discount rate of 10% per year:



In this case we use a two period time line because the discount rate is quoted for the length of one year. If we had used a period length of two years, we could not use 10% as the discount rate. In that case we would need compute the discount rate from the one year interest rate. To see how to compute the discount rate when the period length is two years, we first write down the time line:



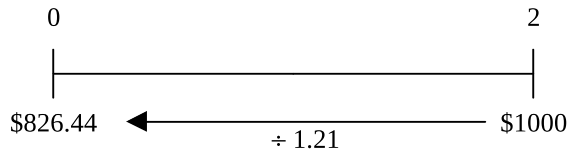
We now use the Law of One Price to compute the discount rate, that is, we compute do-it-yourself rate. The competitive interest rate you can earn at the bank is fixed at 10% per year, so you can create a two year investment by putting you money in the bank for one year and letting it roll over at the end of the year. Here is the timeline:



Since the 10%/annum rate will be fixed over the two years, the value of the investment after 2 years is the future value of $1 in 2-years. So, from the timeline it is clear we will have $1.21 dollars in 2 years for each dollar we invest today. By the Law of One Price, we must get the same return if we instead had chosen to invest our money in a two year account so we must earn 21% interest over the two years. That is, the following interest rates are equivalent

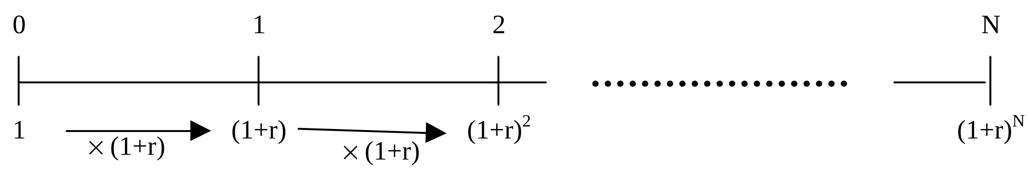
10% per year = 21% per 2-years.

To discount cash flows on a timeline with a 2-year period length, we must 21% use as the discount rate. Putting this in our terminology, the *interest rate* is quoted as 10%/annum, then the *discount rate* using a period length of 2 years is 21%. Computing the present value using a 2 year period length, we get the identical present value as when a one year period length was used:



For convenience, if the discount rate is r, then define the **discount factor** as (1+r). So the discount factor is just one plus the discount rate. Using this terminology it is easy to state the rule we just derived: Take the one-year discount factor and compounded it twice to get the 2-year discount factor: $1.10^2 = 1.21$.

The same logic applies in the general case. If the annual rate of interest is r, we can always manufacture an N-year investment out of N one year investments that are rolled over into each other:



By the law of one price this strategy must yield the same payoff as a N-year investment. So if the N-year interest rate is denoted r_N we must have

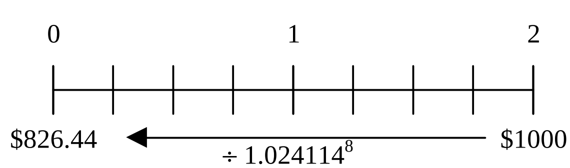
$$1+r_N=(1+r)^N.$$

That is, to get the N-year discount factor, we compound the one-year discount factor N times.

This rule is easy to see when N is an integer. However, it actually works for any N. For example, consider another timeline in which the period length is 3 months, or 1/4 years. In this case, we compute the 3-month discount factor as

$$\text{3-month discount factor} = 1+r_{1/4} = (1.10)^{1/4} = 1.0241136\ldots$$

That is, 10% per year is equivalent to a little more than 2.41% every 3-months. Let's check this on the timeline:



Again, we get the identical answer as before. As long as we use the correct discount rate, the rules of time travel always work! Summing up:

Given a discount rate of *r* per year, the equivalent discount rate r_y per *y*-years is given by compounding $(1+r)$ **for** y **periods:**

$$\mathbf{1 + r_y = (1 + r)^y.}$$

In the discussion thus far, we started with a one-year discount factor. However there is nothing special about 1 year (beyond convenience). All of the above logic would apply if we started with a different period length --- we simply compound the discount rate by the number of original periods that are in the new period length. That is,

Given a discount rate of r_x per *x*-years, the equivalent discount rate r_y per *y*-years is given by compounding $(1+r_x)$ **for** y/x **periods:**

$$\mathbf{1 + r_y = (1 + r_x)^{y/x}.} \qquad \textbf{(1)}$$

For example, take 3 months as our initial period length (x in the above formula). We calculated the 3-month discount rate above to be 2.4114%. Now let us convert this into a two year discount rate (y in the above formula):

$$1+r_2 = (1+r_{1/4})^{\left(\frac{2}{1/4}\right)}$$
$$= 1.0241148^8 = 1.21.$$

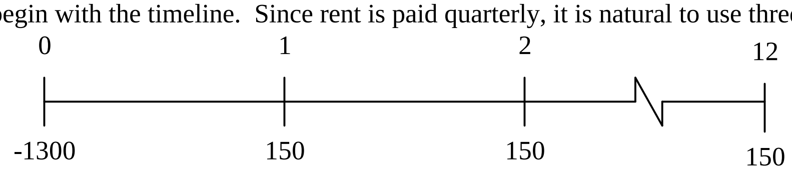
That is, since there are eight 3-month time intervals in 2 years, we must compound the discount factor eight times.

Now that we know how to compute discount rates for different period lengths, the question is, which period length should be used? In the example considered so far, there was no reason to choose a one-year, 2-year, or 3-month time interval – it is purely a matter of convenience. In more complicated problems, however, there is often a "natural" period length corresponding to the frequency of the cash flows. We illustrate this with the following example.

Example 5.1

You own an apartment building in which the tenant pay rent quarterly (every 3 months). You are faced with the following investment decision. If you install new carpeting in each apartment, you believe you will be able to charge $150 per quarter more in rent for the next 3 years. After that, the carpet will again need to be replaced (or the rent lowered). If the cost of new carpet is $1300 per apartment and the market interest rate is 8% per year, what should you do?

As always, we begin with the timeline. Since rent is paid quarterly, it is natural to use three-month periods:



That is, if we spend $1300 today, we will receive an extra $150 per quarter for the next 12 quarters.

Next, we must determine the discount rate to use. Since each period in our timeline corresponds to one-quarter, we need to determine the quarterly discount factor. Since the market rate is 8% per year, the equivalent quarterly rate is given by

$$(1.08)^{1/4} = 1.0194265...$$

or a bit more than 1.0194% per month. Given the discount factor, we can now compute the NPV of this investment. Note that the quarterly cash flows are in the form of a 12 quarter annuity of $150 per quarter. Since the annuity formula does not depend on any specific period length, we can use it to compute the present value just as we did in Chapter 4:

PV($150/period for 12 periods) =

$$150 \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^N}\right) = 150 \times \frac{1}{.0194265}\left(1 - \frac{1}{1.0194265^{12}}\right) = \$1591.90 .$$

Note that we use $N = 12$ for the number of payments, and $r = 1.94265\%$ as the interest rate per payment period. We can use the annuity calculator in the same way,

N	I	PV	PMT	FV
12	1.94265%	-1591.90	150	0

Given this, we compute the NPV of the investment as

NPV = 1591.90 – 1300.00 = $291.90.

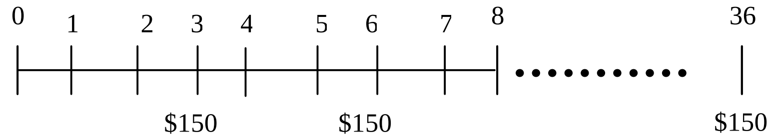
So, we should install the new carpet, earning a profit of $291.90 per apartment.

Caution: There are two common mistakes to watch out for in this type of problem. First, when converting to a monthly interest rate, be careful to record enough digits (and make sure your calculator is set to display them!). Interest rates over short time intervals are generally small, but the effect of leaving of some digits can have noticeable effect. Using $r = 2\%$ gives an NPV of about $5 less – not enough to matter here, but it could be important if the amounts were larger or the time horizon longer.

Another mistake is to convert this to an annual annuity as follows. Since you will receive $150 per month in extra rent, you will receive $150×4 = $600 in extra rent per year. So, the extra rent is "like" a 3-year annuity of $600 per year. Discounting this at 8% per year gives a present value of $1546, for an NPV of $246. This is significantly different from the true answer! The reason for the discrepancy is that when we compute the gain of $600 per year, we are combining cash flows that occur at different points in time (i.e., different months). This is a violates the first rule of time travel --- only cash flows that occur at the same point in time can be combined. In essence, this calculation ignores the fact that the extra rent you receive in the first quarter can be invested and earn interest over the entire year. Ignoring this extra interest results is an underestimate of the true value of the investment.

Consider the implications of not using a period length of one quarter in Example 5.1. Clearly, you can never use a period length *longer* than the period of the cash flows --- for instance you cannot show quarterly cash flows on a timeline that has yearly time intervals (see the second caution in the example if this is not immediately clear). However as we have already noted, you could, in principal, use a time period *shorter* than one quarter.

The following timeline shows the cash flows in Example 5.1 using a period length of one month:



The timeline illustrates an immediate problem --- the cash flows are not an annuity anymore. That is, we defined an annuity in the last chapter as a fixed cash flow *every period* for a certain number of periods. In this case the cash flows occur every *fourth* period, not every period. So the annuity formula cannot be used to value the cash flows in this case. Given this reality it makes much more sense to use a period length that equals the period length of the cash flows so the formulae derived in the last chapter can all be used.

One final note of caution. The preceding discussion made no reference to the discount rate. That is, the point applies even if the annual interest rate is converted to a 1 month discount rate. Even if the discount rate is computed correctly, the fact still remains that the annuity formula cannot be used. In light of this, we recommend always picking the time period on the timeline to match the period length of the cash flows.

5.2 Understanding Interest Rate Quotes: APR's and EAR's

Up to now we have stated all interest rates as the amount of interest earned over the period length specified. For example, 8% per annum means that you will have 8% more money at the end of one year if you invest. Although we will continue to use this simple convention, not all interest rates are quoted this way. Let us take a specific case.

On June 12, 2001, Discover Bank advertised savings accounts with the following interest rate: "5.85% Annual Percentage Rate with Daily Compounding."

Note that this quote is stated in terms of an **annual percentage rate**, or **APR**. This type of interest rate quote is actually quite common and has a very specific meaning. The APR quote specifies both an interest rate (5.85%) and a compounding interval (daily). The important thing to remember about an APR is that it is not an actual interest rate. Instead, it is a way of stating the interest paid *without* considering compounding.

For example, if a bank quotes a rate of "6% APR with monthly compounding," the account does *not* pay 6% interest each year. Instead, it pays this interest on a monthly basis. That is, you will earn

$$6\%/12 = 0.5\% \text{ every month.}$$

So in this case the discount rate for one month period length is 0.5%. Thus, an APR with monthly compounding is actually a way of quoting a monthly interest rate, rather than an annual one. If you invested in this bank for one year you would have

$$1.005^{12} = 1.06167\ldots$$

by the end of the year .

How do APR's work in general? Again, the first step is to remember that the APR itself is not a true annual interest rate. Interest is not paid yearly, but is instead paid at the end of each compounding period. To determine the interest paid each period or the discount rate, we divide the APR by the number of compounding periods per year, which we denote by *k*. That is,

$$\text{interest per period} = \frac{\text{APR}}{k \text{ periods/year}} \ .$$

Remember that this is the way APR rate quotes work. All investors understand this and so also understand that the actual interest earned over the year will include compounding and so will not equal the APR. Naturally, they are often want to know what the annual interest that will be paid will be equal to. Banks quote this rate as the **effective annual rate**, or **EAR**.[1]

The effective annual rate is the true interest earned over a one year time interval, that is, *it is the discount rate for a period length of one year*. Thus, to determine the equivalent annual rate from the APR we must convert a discount rate from one time interval into another. Lets revisit the example in which the rate is quoted as a "6% APR with monthly compounding." In that case we saw that this quote corresponded to a monthly interest rate of 0.5%. Since there are 12 months in a year, the discount factor must be compounded 12 times to convert it to a one year discount factor (using formula **(1)**).

$$1.005^{12} = 1.06167\ldots$$

or about 6.17%. This is the effective annual rate (EAR), the discount rate for a period length of one year (or, alternatively, the true interest you will earn in one year) that corresponds to the APR quote. Remember, the EAR is just an interest rate quote, it does not necessarily imply that the correct period length to use is one year. Of course, when the correct period length to use is one year, then the EAR is the appropriate discount rate.

In general, if we wish to convert an interest rate with a period length of $\frac{1}{k}$ of a year, we must compound it for each of the *k* compounding periods. This means that the general formula for calculating the EAR from an APR quote is:

$$(1 + \text{EAR}) = \left(1 + \frac{\text{APR}}{k}\right)^k .$$

Figure 5.1~~Figure 5.1~~ applies this formula to compare the effect of different compounding frequencies for a 6% APR.

[1] The effective annual rate is also often referred to as the **equivalent annual rate (EAR), annual percentage yield (APY) or effective annual yield (EAY).**

Compounding	*k*	Discount Rate	Effective Annual Rate
Annual	1	6% / year	6.000%
Semiannual	2	$\frac{6\%}{2}$ = 3% / 6 months	6.090% ($1.03^2 = 1.0609$)
Monthly	12	$\frac{6\%}{12}$ = .5% / month	6.168% ($1.005^{12} = 1.061677\ldots$)
Daily	365	$\frac{6\%}{365}$ = .016438...% / day	6.183% ($1.00016438^{365} = 1.06183\ldots$)

Figure 5.1: Different compounding intervals for a 6% APR.

Note that as the compounding frequency *k* increases, so does the effective annual yield. The reason for this is that the compounding frequency determines how often the interest is paid, and the sooner it is paid the sooner you can earn interest on interest.

Example 5.2

What is the EAR for the account offered by Discover Bank?

The APR quoted above is 5.85% with daily compounding. Therefore,

$$(1 + \text{EAR}) = \left(1 + \frac{.0585}{365}\right)^{365} = 1.0602399\ldots$$

Thus, the EAR is about 6.02%.

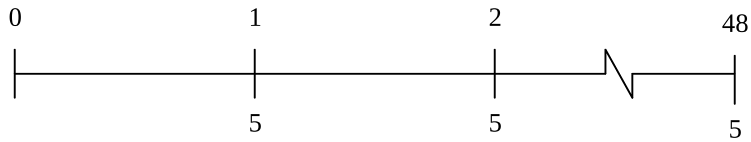
If the APR is quoted without a specific compounding period, then the compounding period is always assumed to be the same as the period of the cash flows. The important thing to remember about APR's is that they are not true interest rates, and so *should not be used directly to discount cash flows*. Rather, we must first convert them to a true interest rate before discounting.

Example 5.3

You are considering the purchase of a new digital answering machine. The new machine costs $150, and will last 4 years. Alternatively, your local phone company offers voice mail for $5 per month, paid at the end of each month. If the interest on your savings is 5% APR with semiannual compounding, how do the costs of these two alternatives compare?

The $150 cost of the new machine is already in present value terms. To make the cost of the voice mail service comparable, we must convert it to a present value as well.

As always, we begin with the time line. The cost of the voice mail service is $5 per month for the next 48 months:



These cash flows are a 48-month of $5 per month. Thus, we can compute the present value using the annuity formula. To do so, we need the discount rate in monthly terms as well.

Given an interest rate of 5% APR with semiannual compounding, how do we determine a monthly rate? First, note that this APR really means that we earn interest of 5%/2 = 2.5% every 6 months. This is a true, 6-month interest rate. How do we convert a 6-month rate to an equivalent one-month rate? Using the formula of Section 5.1, we compound the 6-month rate by 1/6:

$$(1.025)^{1/6} = 1.0041239$$

or about 0.4124% per month.

Using the annuity formula, $PV = 5 \times \frac{1}{.004124}\left[1 - \frac{1}{1.004124^{48}}\right] = \$217.33$.

We can also use this rate in the annuity calculator:

48	0.4124%	217.33	-5	0
N	**I**	**PV**	**PMT**	**FV**

So, paying $5 per month for 48 months is equivalent to paying $217.33 today. Thus, the present value cost of the voice mail service is significantly more than the answering machine.

Compounding and Regulation Q

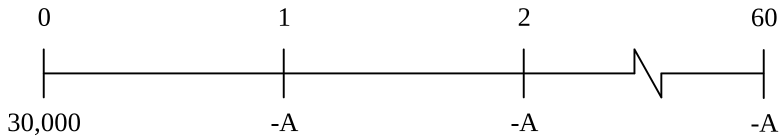
BoxStoryText

5.3 Application: Mortgages and Loans

APR's are also used when banks quote interest rates for loans. For example, most consumer loans, such as mortgages and car loans, have monthly payments and are quoted based on monthly compounding. For example, typical terms for a $30,000 new car loan might be "6.75% APR for 60 months." This means that the loan will be repaid with 60 equal monthly payments, computed using a 6.75% APR with monthly compounding.

5.3.1 Computing the Loan Payment

The example in the preceeding paragraph is typical of most loan quotes --- banks quote the interest rate but usually not the loan payment. To calculate the loan payment you equate the outstanding balance on the loan with the present value of the loan payments using the *loan interest rate* as the discount rate. Consider the timeline for the above loan:



From the timeline, we can see that the bank is exchanging $30,000 today for a 60-month annuity of $*A* per month. In order for the bank to be willing to do so, the two must have equal value when evaluated using the loan interest rate as the discount rate. Because the loan payments occur at monthly intervals, we use a one-month time interval. We therefore need to calculate the one month discount rate. Since the 6.75% APR is based on monthly compounding, the corresponding one month discount rate is 6.75%/12 = 0.5625%. Using the formula for an annuity payment derived in Chapter 4, Section XX, the payment A is given by

$$A = \frac{30,000}{\frac{1}{0.005625}\left(1 - \frac{1}{(1+0.005625)^{60}}\right)} = 590.50\,.$$

Alternatively, we can solve for the payment *A* using the annuity calculator,

60	6.75%/12	30,000	-590.50	0
N	**I**	**PV**	**PMT**	**FV**

or $590.50 per month. This is the monthly payment that will be required for the loan.

This works not only initially, but during the life of the loan as well. The outstanding balance on the loan is the amount of cash the bank is willing to take today to payoff the loan. This must be equivalent to the remaining payments that the bank expects to receive. Thus, the general relationship is

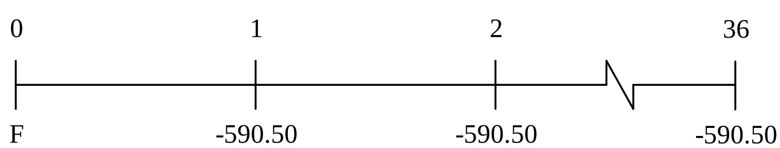
Loan Balance = PV(Remaining Payments)

using the loan interest rate as the discount rate.

Example 5.4

Suppose you purchased the car using the loan above. Now, two years later, you want to sell the car to get the new 2-seater convertible you have been dreaming about. How much do you owe on the loan?

Here is the new timeline:



We need to compute the remaining balance on the loan. This is just the present value of the remaining 60 – 24 = 36 payments, evaluated using the loan interest rate as the discount rate:

$$F = \frac{590.50}{0.005625}\left(1 - \frac{1}{(1+0.005625)^{36}}\right) = \$19{,}195.19$$

Alternatively, we could have used the annuity calculator:

36	6.75%/12	19,195.19	-590.50	0
N	**I**	**PV**	**PMT**	**FV**

So you owe $19,195.19 on the loan.

5.3.2 Paying Off Loans: An Investment Opportunity

Debt, in the form of student loans, car loans, credit card debt, and home mortgages, is a part of most individuals' lives. One important thing to remember as an investor is that often, the best investment opportunity is right at home: paying off debt.

For example, suppose you have a $100 balance on an outstanding loan. The loan is due next year and has an interest rate of 10% (EAR). Consider what happens to your cash flows if you pay off the loan:

	0	1
Keep loan	0	-110
Pay off loan	-100	0
Change in Cash Flows	-100	+110

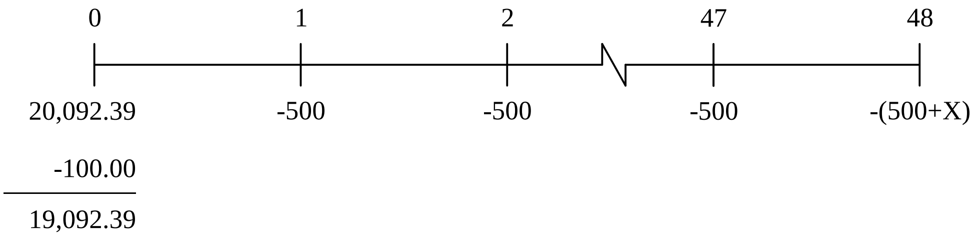
As the timeline shows, paying off the loan means you will have $100 less today, but $110 more in one year. Therefore, paying off the loan is like investing $100 and getting it back with 10% interest one year later.

In general, paying off a loan is equivalent to investing the money and earning an interest rate equal to the loan rate for the term of the loan. This is often a much higher rate than you would earn investing your money in a savings account or other similar investment opportunities.

Example 5.5

You have an outstanding student loan with required payments of \$500 per month for the next 4 years. The interest rate on the loan is 9% APR (monthly). You are considering making an extra prepayment of \$100 today. What effect will this have?

We begin with the timeline:



Although the timeline shows the current outstanding balance, this is something we need to calculate by calculating the PV of the remain cash flows of the current loan. Given payments of \$500/month for the next 4×12 = 48 months and a loan rate of 9%/12 = 0.75% per month, we can use the annuity calculator to get[2]

N	I	PV	PMT	FV
48	0.75%	20,092.39	-500	0

Thus, your remaining balance is \$20,092.39.

If you prepay an extra \$100 today, your will lower your remaining balance to \$20,092.39 – 100 = \$19,992.39. Though your balance is reduced, with most loans your required monthly payment does not change. Instead, you will pay off the loan faster; that is, it will reduce the payments you need to make at the very end of the loan.

What is the amount of difference in the final payment, labeled *X* on the timeline? Since the PV of the remaining cash flows must always equal the outstanding balance we have that:

$$19{,}092.39 = \frac{500}{0.0075}\left(1 - \frac{1}{(1+0.0075)^{48}}\right) + \frac{X}{1.0075^{48}}$$

Note that we have used the timeline to value the remaining payments as the PV of an annuity plus the PV of the difference in the final payment. Solving this equation for X gives

$$19{,}092.39 = 20{,}092.39 + \frac{X}{1.0075^{32}}$$

$$\Rightarrow X = -143.14$$

So the final payment will be lower by \$143.14 dollars.

You can also use the annuity calculator to derive this. If you prepay \$100 today, then your future balance in period 47 will be

N	I	PV	PMT	FV
47	0.75%	19,992.39	-500	-354.20

To pay off the loan in period 48 you will need to pay this balance plus one period of interest

$354.20 × (1.0075) = \$356.86 final payment.

[2] Alternatively, we can use the annuity formula: $PV = \frac{500}{0.0075}\left(1 - \frac{1}{1.0075^{48}}\right) = 20{,}092.39$

Therefore, the benefit of prepaying \$100 today is a final payment in 48 months which is reduced by \$500 – 356.86 = \$143.14.

Prepaying the loan therefore let's us exchange \$100 today for \$143.14 in 4 years, which is just like investing the \$100 at the loan rate:

$$\$100 \times (1.0075)^{48} = \$143.14 \text{ !}$$

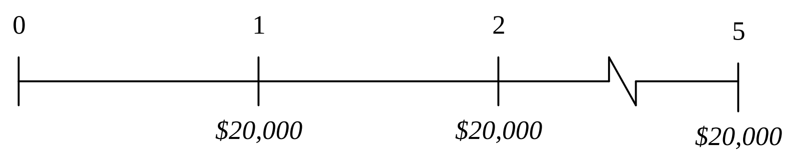
5.3.3 When Calculating the Discount Rate the Question Matters

One of the important lessons of in the last two subsections is that the discount rate not only depends on the characteristics of the cash flows, but *also on the question that is being asked.* It is quite possible that in two situations the same cash flows might be discounted at a different rate because the question that is posed differs. Let's see this in the context of an example.

Example

You have a loan outstanding at a local bank that is 5 years old. Under the original terms of the loan, it required that you repay the loan in by making 10 annual payments of \$20,000. The interest rate on the loan is 8% (APR, annual compounding). The loan contract allows you to repay the loan in full at any time by returning the outstanding principal. If you wanted to repay the loan today, how much would you have to pay.

The timeline of the remaining payments on the loan is:



The outstanding balance on this loan is the present value of these cash flows using the loan interest rate as the discount rate, that is, 8%:

$$PV = \frac{20{,}000}{0.08}\left(1 - \frac{1}{1.08^5}\right) = 79{,}854.20$$

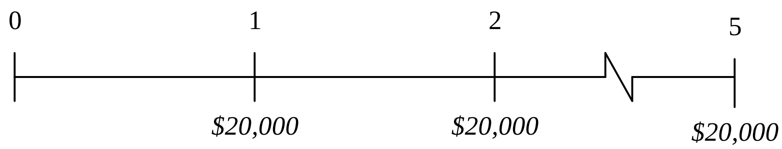
So in this case the correct discount rate to use is 8%.

Contrast this example with the following example.

Example

Now that you know it will cost you \$79,854.20 to pay off the loan, you are trying to evaluate whether paying it off is a good idea. Currently banks are quoting an interest rate of 7% (APR, annual compounding) for a 5 year level pay loan. Does it make sense to pay off your existing loan and replace it with a new loan.

In one way the decision is rather obvious. If you replace the existing loan with new 5 year loan you effectively lower your interest rate by 1%. Let's confirm this explicitly by calculating the NPV. We begin with the timeline for the new loan (it is identical to the timeline in the previous example:



To find out how much the new bank will allow you to borrow (the loan principal) we calculate the present value of these payments using the interest rate on the new loan as the discount rate:

$$PV = \frac{20,000}{0.07}\left(1 - \frac{1}{1.07^5}\right) = 82,003.95$$

Since you only owe $79,854.20 on the old loan the NPV of this transaction is 82,003.95-79,854.20=$2,149.75 (so by refinancing you will be able to pocket this amount of money). Note that in this case the discount rate is 7%, the rate you can get on your alternative financing.

Even though the cash flows are the same in the two examples, the discount rate is different. In the first case the discount rate is specified in the loan contract. You have no control over it. In the second case, the discount rate is the rate on the very best alternative financing you can find. You have some control over it. The harder you choose to look the more likely you will be able to find a bank willing to make the loan to you at a lower rate. How hard you choose to look is a personal issue, and so the correct discount rate in this case might very well vary from person to person depending on what alternative uses for the funds they might have available to them.

As these two examples demonstrate, the correct discount rate to use not only depends on the characteristics of the cash flows, but also the question that is being posed. Furthermore, there are other cases, besides these, in which the question is important. For example, since investors are subject to different tax rates (if they are subject to tax at all), as we will explain in Section 5.6, they will often use a different discount rate to present value the same cash flows.

5.4 The Term Structure of Interest Rates

Although we have not been explicit about this, thus far, in our formulas and calculations, we have assumed that interest rates remain constant (they do not vary). We do this for simplicity because this assumption implies that the same interest rate applies if you want to exchange money today for money in one year, or if you want to exchange money today for money in 10 years. While convenient, this is not consistent with common experience. Interest rates change over time, and banks offer different interest rates depending upon how long you wish to save (or borrow) money.

Term	APR (daily)	EAR
6 months	3.70%	3.77%
1 year	4.05%	4.13%
1 ½ years	4.25%	4.34%
2 years	4.50%	4.60%
2 ½ years	4.75%	4.86%
3 years	4.95%	5.07%
4 years	5.35%	5.50%
5 years	5.85%	6.02%
7 years	5.65%	5.81%
10 years	5.65%	5.81%

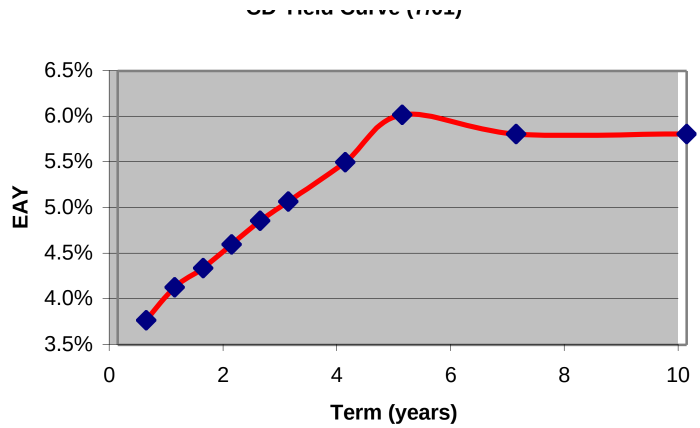
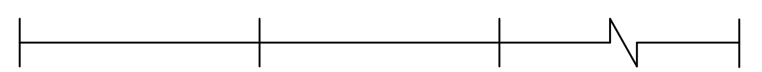


Figure 5.2~~3~~: CD interest rates (July, 2001)

Figure 5.3~~Figure 5.4~~ shows the interest rates quoted on savings accounts for Discover® Bank in July, 2001. These accounts, called **certificates of deposit** or “CDs,” require that the money is invested for a fixed length of time. Note that the interest rate paid varies with the term of the CD. In particular, the interest rises with the term, up to a length of 5 years. This is a very common pattern. This relationship between the investment term and the interest rate is generally referred to as the **term structure of interest rates**. It is sometimes plotted as a graph as shown on the right. This graph is referred to as a **yield curve**.

In our examples thus far, we have assumed that the interest rate is equal for different terms, or equivalently, that the yield curve is a flat, horizontal line. But this is rarely the case in practice. How does this effect our time value of money calculations?

Suppose you wish to determine the present value of a project that generates safe cash flows of $1000 in one year, and $1000 in 5 years:



Let’s consider first the cash flow in one year. What is the present value of this cash flow? Recall that the present value is equivalent to the “do it yourself” cost of replicating that cash flow. Since we can earn 4.13% on a 1-year CD, if we invest

$1000/1.0413 = $960.34 today,

we will have $960.34 × 1.0413 = $1000 in one year. Thus, the present value of the first cash flow is $960.34. That is, we discount the one-year cash flow by the one-year interest rate to determine its present value.

Why don’t we use the higher, 5-year interest rate to discount the one-year cash flow? The reason is, if we invest in the 5-year CD to earn the 5-year rate, we cannot withdraw the money after one year. Thus, the 5-year CD cannot be used to replicate a one-year cash flow, and so the Law of One Price cannot be used to justify using the 5-year CD rate to discount a 1 year cash flow.

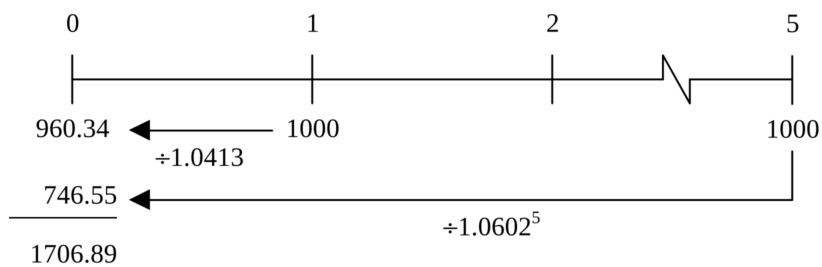
Now consider the 5-year cash flow. If we want to have \$1000 for sure in 5-years, we can invest money today in a 5-year CD. How much will we need to invest? In this case,

$$\$1000/1.0602^5 = \$746.55 \text{ today.}$$

That is, \$746.55 is the do-it-yourself cost of having \$1000 in 5-years, and so is the present value of the 5-year cash flow.

What about the "do-it-yourself" strategy of investing the money in a one-year CD, and then "rolling over" or reinvesting in the CD every year for 5 years? The problem with this strategy is that although we know we will earn the one-interest rate of 4.13% the first year, we don't know what interest rate we will receive in the following years. Thus, there is no way to know for sure how much we need to invest today in order to have \$1000 in year 5. Stated another way, whatever amount we invest today will produce a risky amount in year 5, depending upon future interest rates. Thus, this "rollover" strategy cannot be used to replicate a certain cash flow of \$1000 in year 5.

Putting these results together, we have shown that the right calculation of the present value for this example is



Thus, the present value is \$1706.89, which corresponds to the do-it-yourself strategy of investing \$960.34 in a one-year CD, and \$746.55 in a 5-year CD.

The previous example demonstrates that when computing the present value of a cash flow in year n, we must use the interest rate corresponding to a n-year investment. It is that interest rate that tells us the exchange rate for money in year 0 with money in year n.

In general, then, given a sequence of cash flows C_0, C_1, ..., C_n, and the term structure of interest rates r_1, r_2, ..., r_n, the NPV of the cash flows is calculated as

$$\text{NPV} = C_0 + \frac{C_1}{1+r_1} + \frac{C_2}{(1+r_2)^2} + ... + \frac{C_n}{(1+r_n)^n}.$$

Warning! All of our shortcuts for computing present values (annuity and perpetuity formulas, the annuity calculator) are based on having a constant interest rate (a flat yield curve) --- each cash flow at each date is *discounted at the same rate*. When interest rates are not constant, these formulas are not necessarily appropriate. Whether they can be used depends on the question being asked. For example, if we are trying to calculate the IRR, the formulas work in all cases, even when interest rates are not constant and the yield curve is not flat. The reason, of course, is that the IRR is the *single* rate that discounts all the cash flows to get an NPV of zero. Similarly, if you are calculating the monthly payment on a mortgage, since the interest rate on the mortgage is constant over the life of the mortgage, the formula will give the correct answers. However, if your object is to value a cash flow stream, then since each cash flow at each date must be discounted at the rate appropriate for that maturity, the cash flows must be treated separately, so the formulae will give inaccurate answers.

Example 5.6

Compute the present value of a 5-year annuity of $1000 per year, given the term stucture in Figure 5.3~~Figure 5.4~~. What is the future value of the annuity?

To compute the present value, we use the general formula and discount each cash flow by the corresponding interest rate:

$$PV = \frac{1000}{1.0413} + \frac{1000}{1.0460^2} + \frac{1000}{1.0507^3} + \frac{1000}{1.0550^4} + \frac{1000}{1.0602^5} = \$4290.20 \text{ today.}$$

Note that we cannot use the annuity formula here – after all, if we did, which interest rate would we choose? Having computed the present value, however, we can now solve for the IRR using the annuity formula:

$$\begin{aligned}\$4290.20 &= \frac{1000}{(1+r)} + \frac{1000}{(1+r)^2} + \frac{1000}{(1+r)^3} + \frac{1000}{(1+r)^4} + \frac{1000}{(1+r)^5} \\ &= \frac{1000}{r}\left(1 - \frac{1}{(1+r)^5}\right)\end{aligned}$$

The annuity calculator can be used to solve explicitly for r:

5	5.33%	-4290.20	1000	0
N	**I**	**PV**	**PMT**	**FV**

Thus, the corresponding IRR is 5.33%. Notice that this is not equal to any of the interest rates in Figure 5.3~~Figure 5.4~~, but rather is a type of average of them.

Computing the future value of the annuity is easy once we have the present value. We simply compute the future value in year 5 of $4290.20 today. Since we are moving the cash flow forward from year 0 to year 5, we must compounding using the 5-year interest rate. Therefore,

$$FV = \$4290.20 \times 1.0602^5 = \$5746.67 \text{ in year 5.}$$

To sum up, to decide whether the formula (or annuity calculator) is appropriate to answer a question, the key criterion is whether it is appropriate to discount every cash flow at the same rate. If this is appropriate, then the formula (or calculator) can be used. Otherwise, each cash flow must be discounted separately as the appropriate rate.

5.5 Discounting Risky Cash Flows

An important determinant of the discount rate for a particular set of cash flows is the riskiness of the cash flows. At this early stage, simplicity is important, and so in most cases we abstract from this issue and just consider riskless cash flows. However, it is important to bear in mind that in reality most cash flows contain risk. Consequently, in this section we briefly consider how the discount rate is affected by the riskiness of the cash flows. A complete treatment of this subject will however be delayed to Part X of this book.

Consider a project in which you invest $900 today in exchange for a cash flow of $1050 in one year. If the one-year bank rate is 5% (EAR), we would compute the NPV as

NPV = -$900 + $1050 / 1.05 = $100,

which looks like a good deal. But suppose that this investment is in a dot-com startup with a questionable future, and the $1050 cash flow that you are supposed to receive is a promised, but not a guaranteed payment. Is it reasonable to discount the promised future cash flow using the bank interest rate to compute the NPV? Would you be willing to take this project?

The issue here is that the 5% bank rate represents a way to exchange money today for money in the future with no risk. But the promise of $1050 is not a no risk cash flow. The risks do not match. Recall that the reason that 5% is the appropriate rate to use to discount riskless cashflows is that at this rate you can "do-it-yourself" --- we can create our own riskless investment. So by the Law of One Price all riskless investments must earn 5%. However, since we cannot create a risky investment from riskless cashflows, 5% is not the appropriate rate to use. Our risky project needs to be evaluated using the discount rate we could earn on another investment of "similar risk."

We don't yet have the tools to measure or compare risks; we will develop those tools later, in Chapters ??. For now, we simply point out that, in the same way that interest rates vary with the term of the cash flows (as depicted by the yield curve), interest rates also vary with the risk of the cash flows. These differences in rates for different risks are often referred to as **credit spreads**. As an example, we illustrate below the interest rate you would receive in July 2001 for a variety of one-year investments.

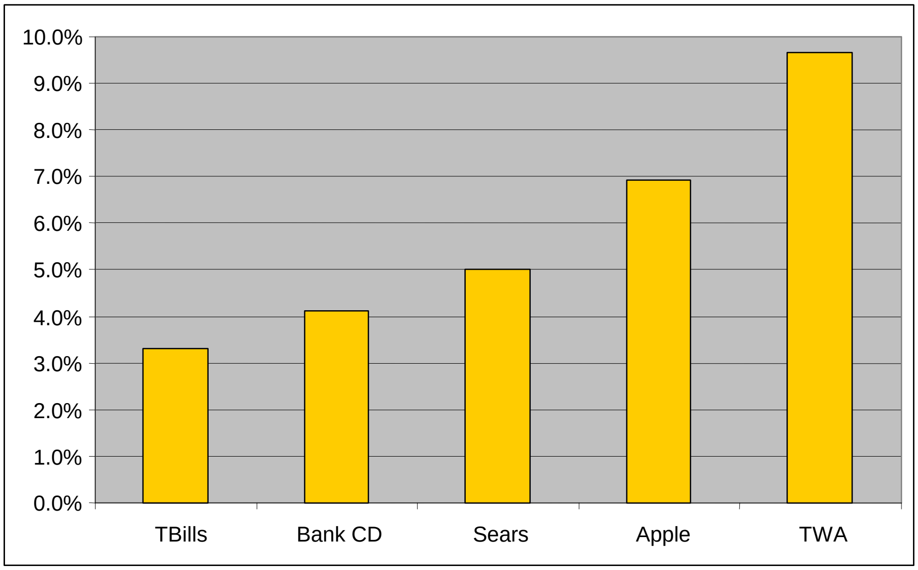


Figure 5.34: Interest Rates on Investments of Differing Risk (July, 2001)

Note that Treasury Bills (like a deposit with the US Government) have the lowest interest rate and have virtually no risk. Bank CD's are often a bit higher, but expose you to the risk that the bank could default.[3] Loaning money to corporations is even riskier, since they could go bankrupt, and so these loans typically have even higher interest rates associated with them. (At the time of this chart, TWA was in financial distress.)

So, when discounting cash flows, it is important to match both the term and the risk of the cash flows when determining the interest rate. We have seen how to match the term using comparable investments with the same term as each cashflow. As for risk, we will develop some sophisticated tools later in this text. For now, we will have to rely on intuitive estimates.

Example 5.7

Your firm is a supplier of Apple computer. You are owed $200,000, due immediately. The account manager asks to reschedule the payment, so that you will receive $212,000 in one year. The manager claims this is a good deal for you given current Treasury Bill interest rates are below 4%. Is it really a good deal?

Using an interest rate of 4%, we would compute a present value of

PV = $212,000 / 1.04 = $203,846 today,

which does indeed look like a good deal. The problem, of course, is that the $212,000 in one year is risky – there is some chance we will not be paid. So it is inappropriate to use a low risk Treasury or bank rate to discount this cash flow.

Absent other tools, in this case we could use the fact that Apple is paying almost 7% interest on its other outstanding debt of maturity of 1 year. Thus, if we wanted to create a cash flow in one year that has similar risk, we could invest the $200,000 today in Apple debt. This would give us a promise in one year of

[3] In the U.S., depositors are protected against bank default (up to $100,000) by the federal government, but if the bank does default there is still risk as to when the money will ultimately be paid.

FV = \$200,000 × 1.07 = \$214,000 in one year (with risk of Apple default).

Comparing future values, receiving the \$200,000 today and investing it in Apple ourselves is clearly better than the promise of only \$212,000 offered by the account manager. Alternatively, since the 7% interest rate reflects a similar risk investment, we can use it to compute a present value:

PV = \$212,000 / 1.07 = \$198,131 today.

So, this is not a good deal for us – we are giving up close to \$2000 by agreeing to reschedule.

5.6 The Effect of Taxes

Until now, we have neglected the fact that most investment opportunities come with a hidden partner: the government. That is, the government takes a share of the gains from investment through taxes on interest, income and capital gains.

To illustrate the effect of taxes, consider an investment that pays 8% interest (EAR). If you invest \$100 at the start of the year, you will earn \$8 in interest at year end. In the U.S., this interest is taxable as income. If you are in the 40% tax bracket, this means that you will owe

(40% income tax rate) × (\$8 interest) = \$3.20 tax liability.

Thus, your net after-tax return is \$8 – 3.20 = \$4.80. This is equivalent to earning 4.80% interest and not paying any taxes. We call 4.80% the **equivalent after-tax return** of the investment.

In general, consider an investment paying interest rate r and suppose the tax rate is τ. Then for each dollar invested,

0		1	
1	→	$1 + r$	pre-tax
		$-\tau \times r$	tax due
		$1 + r(1 - \tau)$	net after-tax

Thus, the equivalent after-tax return is $r(1-\tau)$.

This same adjustment for taxes can be applied to loans. The interest on loans is in some cases tax-deductible. This means that you do not have to pay taxes on money you made but used to make an interest payment. As far as the government is concerned, it is as if you did not make this money. Such interest payments are often referred to as a *tax shield* because you are shielding this money from taxes.

For individuals, the most common tax shield is interest paid on home mortgages and some types of student loans. For corporations, interest payments on almost all forms of debt issued by the firm are shield from taxes (see Chapter ???).

When interest is tax-deductible, paying interest on a loan reduces your overall tax burden. For example, suppose you borrow $100 for one year using a tax-deductible student loan with an interest rate of 8% EAR. Then at the end of the year you will owe $100 plus $8 in interest. The interest payment of $8 can be deducted from your income for tax purposes. If you are in the 40% tax bracket, this reduces your total taxes by 40% × $8 = $3.20. Combining these, we get a net after-tax expense of

$100 (loan principal) + $8 (loan interest) – $3.20 (tax savings) = $104.80.

Thus, the effective after-tax interest rate on the loan is only 4.80%. This is the same after-tax rate we found above: $r\,(1-\tau) = 8\%\,(1-40\%) = 4.80\%$.

Another way to think about it is to realize that if you chose not to borrow, only $4.80 of the $8 you would have spent on interest would have been yours to keep --- the rest would have gone to the government. On the other hand when you borrow, you get to use the full $8 to make the interest payment, which means you are not giving the government $3.20, so you are effectively $3.20 richer.

To summarize: if the interest on an investment is taxable, or if the interest on a loan is tax deductible, then the

$$\text{Equivalent After-Tax Return} = r\,(1-\tau),$$

where r is the pre-tax return and the tax rate is τ.

How do taxes affect our NPV calculations? There are two issues. First, when writing down the cash flows from an investment opportunity in a timeline, we must take care to include the tax liabilities it will generate. After all, we want to determine whether the investment is good for us, the investor, and so we need to include all of the cash flows that we will pay or receive as a result of the investment. Second, these after-tax cash flows should be discounted at the after-tax interest rate – this is the rate at which we can exchange money today for money in the future.

The way to understand the reasons for these adjustments is to use the Law of One Price. To replicate ("do-it-yourself") a taxable investment we need to first replicate the cash flows we actually receive. That means we have to replicate the after tax cash flows. Second we need to generate these cash flows ourselves. The only way to do that is to use the rate we actually receive, the *after tax* interest rate or return.

The easy way to remember what adjustments to make is to always stick with the numbers you actually receive --- a bank might post a before tax interest rate, but since you actually receive the after tax rate, that is what you use. Similarly, since you only receive the after tax cash flows, that is what you need to use.

Example 5.8

Your employer is going to pay you a $10,000 bonus at the end of the year. You can either receive the bonus on December 31 or on January 1. If the interest rate on your savings is 8% (APR, monthly) and your tax rate is 40%, how much is it worth to you to receive it on January 1 versus December 31?

Absent taxes, there is almost no difference between the two – there is one day's worth of interest, but this is negligible. (Indeed, in either case it is likely that you would deposit the check the same day!)

The big difference between these is due to taxes. If the payment is received in December, it will be taxed this year, but if it is received in January, it will be taxed in the following year. That is, by delaying the check for 24 hours you can delay the tax payment by one year!

How can we quantify this benefit? The tax liability in either case is (40%)($10,000) = $4000. The gain from delaying the tax is the difference between paying $4000 today and the PV of paying it in one year. To compute the PV, we need to determine the correct discount rate. Given an interest rate of 8% (APR, monthly), we first compute the EAR of

$$(1 + 8\%/12)^{12} = 1.083, \text{ or } 8.30\% \text{ EAR.}$$

The equivalent after-tax return is then (8.30%)(1 – 40%) = 4.98%. This is the discount rate to use, because this is what you actually earn when you invest money. Using this rate we get

$$\text{PV(\$4000 paid in one year)} = 4000/1.0498 = \$3810 \text{ paid today.}$$

In other words, paying $4000 in one year is equivalent to paying $3810 today. This is a savings of $190, or 1.9%, from deferring the payment.

To check this result, note that if you deposit $3810 in the bank today, you will have

$$\$3810 \times 1.083 = \$4126 \text{ in one year.}$$

You will then owe tax on the interest your earned of (40%) (4126 – 3810) = $126. After paying taxes on the interest earned, you will have exactly $4000 left in the account.

Note: The above calculation is based on two important assumptions. First, we have assumed that your income tax rate is 40% in both years. If your tax bracket is likely to change, then the tax liability would be different each year. In our example above, as long as your tax rate will not increase by more than 1.9%, there is an advantage to deferring the taxes until the next year.

The second important assumption is that taxes are paid on an annual basis. In some circumstances, taxes must be paid more frequently, for example quarterly. This would have two effects. First, delaying the payment would only delay the tax liability by 3 months. Second, we would compute the after-tax return using the corresponding 3-month interest rate, rather than the EAR.

5.6.1 Gains from Deferral: Capital Gains Taxes

So far, we have considered the tax treatment of interest payments, which (in the U.S.) are taxed as ordinary income on an annual basis. Many investment opportunities do not involve interest. Instead, the gains you receive come from selling an asset (like stock, or real estate) for a higher price than you paid for it. Such gains are called **capital gains**.

Capital gains are taxed differently from interest income in two important respects. First, they are taxed at the capital gains tax rate, which is often lower than the tax rate on interest income. Second, the taxes are usually paid in the year that you sell the asset and **realize** the gain, rather than each year that you own the asset. For these two reasons, earning income through capital gains is generally preferred to earning it as ordinary interest.

The first benefit, a lower tax rate, is obvious. The second benefit of only paying tax once the gains are realized, is more subtle. Let's consider a simple example. Suppose you are a real estate investor buying and selling undeveloped land. Suppose the land that you buy increases in value by 10% each year, and the capital gains tax rate is 20%. For a one-year investment of $100,000, you would earn

0		1	
100,000	→ $\times 1.10$	110,000	sale of land
		-(20%)(10,000)	Capital Gain Tax
		108,000	after-tax proceeds

This is equivalent to earning the after-tax rate of 8% on your investment. This is just like the case of ordinary interest in the previous section, since

$$r\,(1-\tau) = 10\%\;(1-.20) = 8\% \text{ equivalent after-tax return.}$$

If we then invested the proceeds in a new piece of land the next year, the calculation would be identical and after two years we would have,

$$\$108{,}000 \times 1.08 = \$116{,}640.$$

But now consider a two-year investment. In this case, our cash flows are

0		2	
100,000	→ $\times 1.10^2$	121,000	sale of land
		-(20%)(21,000)	Capital Gain Tax
		116,800	after-tax proceeds

This is $160 higher than we would have with a sequence of one-year investments. Where does this extra money come from? With the sequence of one-year investments, we pay $2000 in tax in the first year. But with the two-year investment, this tax is not paid until the end of the second year when the gain is realized. Deferring this tax payment is valuable, since we earn 8% after-tax on our money in the second year. That is,

$$(\$2000 \text{ tax in the first year}) \times (8\% \text{ after-tax return}) = \$160 \text{ benefit.}$$

In general, the longer we hold the investment, the longer we delay payment of the capital gains tax. Given the time value of money, delaying a tax payment is beneficial.

Another way to express the benefit is to determine the equivalent annual after-tax return for the two-year investment. In this case, it is about 8.074%, since[4]

$$\$100{,}000 \times 1.08074^2 = \$116{,}800.$$

Thus, with capital gains taxes, if we hold the investment longer than one year, our effective after-tax return exceeds $r\,(1-\tau)$ due to the benefit of deferral. Figure 5.4~~Figure 5.6~~ shows how the equivalent after tax return changes for different holding periods. Note that as the holding period grows, so does the after-tax return.

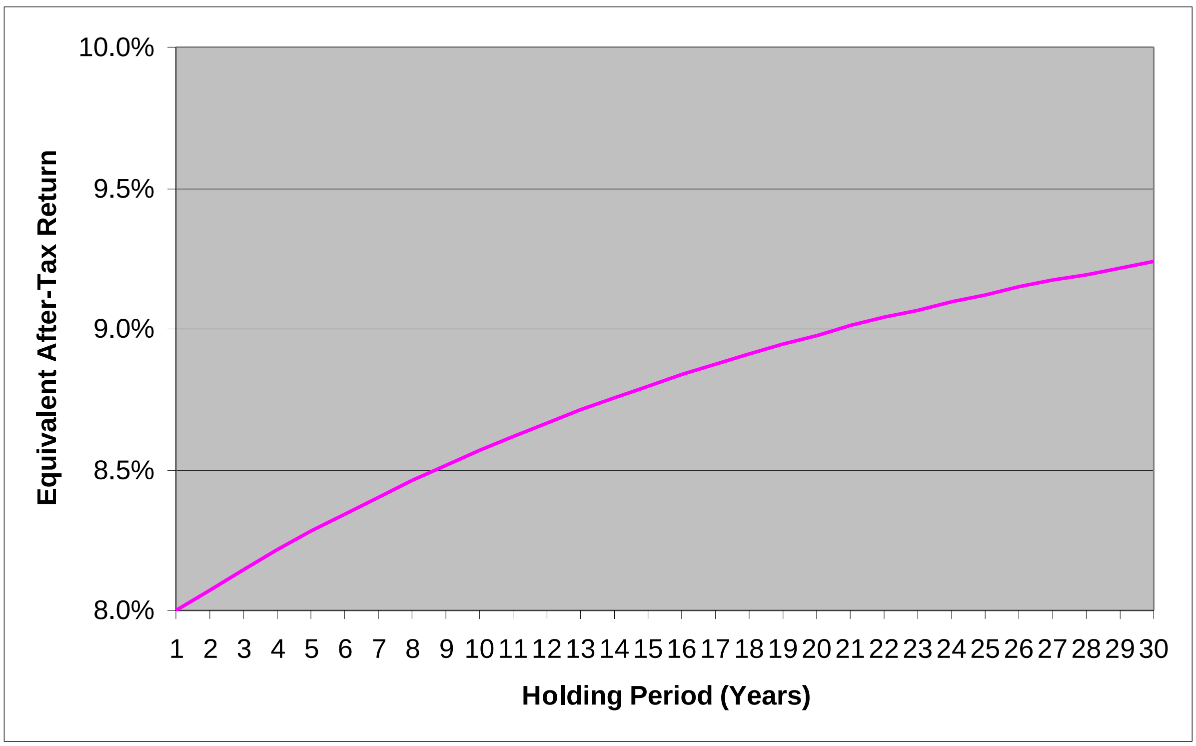


Figure 5.4~~6~~: Equivalent After-Tax Return on an Investment with a 10% Pre-Tax Return and a 20% Capital Gains Tax Rate.

5.7 Inflation: Real versus Nominal Interest Rates

We have seen thus far how money saved today will accumulate over time due to the power of compounding. And, as your savings grow over time, it is natural to think you are getting wealthier. But the amount of money in your bank account is not the only

[4] This is just the internal rate of return (IRR) of the after-tax cash flows on the two-year investment.

measure of financial wealth. After all, you cannot consume money – it is only useful to buy other goods. The amount you can buy depends both on the amount of money you have, as well as the prices that you must pay for goods.

In most modern economies there is a tendency for prices to rise over time. This **inflation** in prices means that goods cost more in the future than they do now. As Figure 5.5~~Figure 5.8~~ shows, goods that would have cost about $25 in 1970 cost about $100 in the year 2000, and are likely to cost close to $130 in the year 2010.

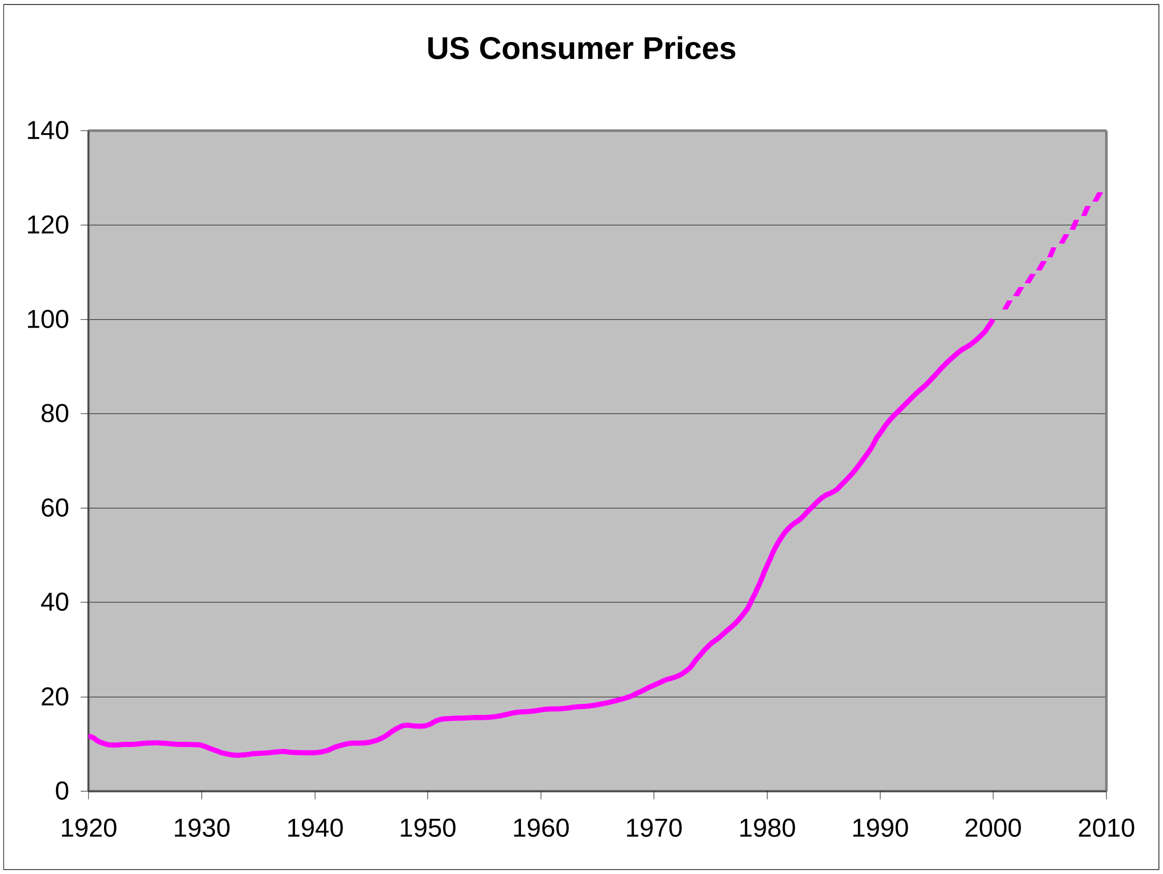


Figure 5.5~~8~~: Caption

How does inflation affect our savings? Even though our money is growing over time, goods are becoming more expensive. So you can only use the money to buy more goods in the future if your money grows faster than prices.

Consider the following example. Eating at your favorite lunch spot costs $10, and you expect this price to rise by 5% next year. If you save $1000 in the bank today, in an account paying 10% interest (EAR), how many lunches can you afford next year?

Date:	0		1	Growth
Money	$1000	→	$1100	10%

Lunch Price	\$10.00	→	\$10.50	5%
Number of Dinners	1000/10 = 100	→	1100/10.50 = 104.76	4.76%

As the table shows, though your money grows by 10%, due to the 5% rate of inflation, your "purchasing power" only grows by 4.76%.

Thus, when inflation is present the interest rate paid on your savings only tells you the rate of growth of your money, but not your purchasing power. Economists refer to the bank interest rate as the **nominal interest rate.** The rate of growth of your purchasing power is called the **real interest rate**. This rate is not a rate that is posted at the bank, but is one we can compute by adjusting for the effect of inflation. Intuitively, we need to discount the rate of growth of our money by the rate of growth of prices to see how our purchasing power has changed:

$$\frac{\text{growth in money}}{\text{growth in prices}} = \text{growth in purchasing power}.$$

Using π to represent the rate of inflation, and r_r to be the real interest rate, we can write this as the formula:

$$\frac{1+r}{1+\pi} = 1 + r_r.$$

Let's check this with our example. There, the (nominal) interest rate is 10%, the rate of inflation is 5%, and so the real interest rate is computed as 1.10/1.05 = 1.0476, or 4.76%.

Note that the above formula can be rewritten,

$$r_r = \frac{1+r}{1+\pi} - 1 = \frac{r-\pi}{1+\pi}.$$

When inflation rates are low, this formula is often approximated as $r_r \cong r - \pi$, the real rate equals the nominal rate minus the rate of inflation. In our example, 10% – 5% = 5%, which is a slight overestimate of the true real rate of 4.76%.

5.7.1 Discounting with Inflation

Determining the real interest rate can be useful for two reasons. First, as we have argued above, it measures the true benefit of saving in terms of real purchasing power. Second, the real interest rate is an "exchange rate" for purchasing power from today to tomorrow. Therefore, just like the nominal interest rate is used to discounting *money* in the future, the real interest rate can be used to discount *purchasing power* in the future.

Specifically, when writing down cash flows in a timeline, it is important to be clear whether the cash flows are actual amounts, or whether they are based on today's prices. If the cash flows are actual amounts, we say they are in **nominal terms** and they should by discounted or compounded using the **nominal** interest rate. This is the case in all of our examples thus far. If cash flows are based on today's prices, and the true amounts will differ due to inflation, we say that they are in **real terms** (they are in terms of real purchasing power). Cash flows in real terms must be discounted or compounded at the **real** interest rate.

We illustrate this with the following example:

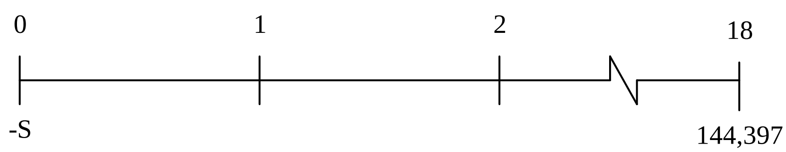
Example 5.9

Your first child is just born, and you decide to open a savings account to help pay for her college tuition. Based on current tuition rates, you would like to have $60,000 saved. However, you expect tuition costs to increase by 5% per year for the next 18 years. If the account pays a fixed 10% interest rate (EAR), how much do you need to deposit today to reach your goal?

There are two ways to solve this problem. The first is to compute the goal in terms of the actual amount of money we will need. In this case, $60,000 is based on today's prices. But with inflation, the true amount you will need is

$$\$60{,}000 \times (1.05)^{18} = \$144{,}397 \text{ in 18 years.}$$

Now we have the usual timeline:

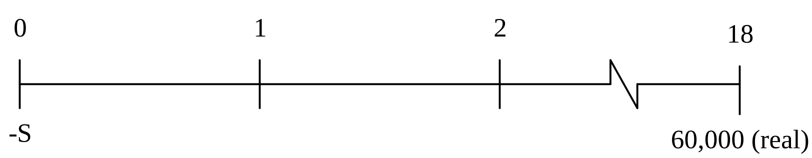


Our problem is to determine how much to deposit now in order to be able to take out $144,397 from the bank in 18 years. This is just the present value at the (nominal) interest rate of 10%,

$$S = \frac{144,397}{1.10^{18}} = \$25,971 .$$

Thus, we need to deposit $25,971 today to reach our goal.

The second way to solve this problem is to write the timeline in real terms,



That is, we need to save today so that we will have $60,000 in real purchasing power in 18 years. To solve this, we again compute the present value, but now we must use the real interest rate. Given the 10% nominal rate and the 5% rate of inflation, the real rate is 1.10 / 1.05 = 1.04762 or 4.762%. Thus, we compute the present value,

$$S = \frac{60,000}{1.04762^{18}} = \$25,971 ,$$

which is identical to the first approach.

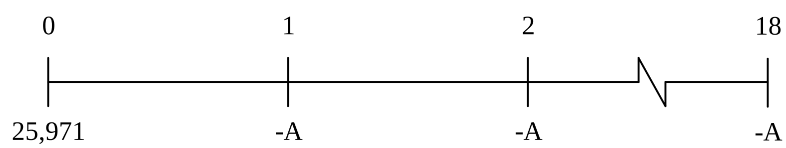
The example illustrates that there are two ways to solve a problem when some of the cash flows are stated in terms of today's prices. We can either adjust them for inflation and convert them to nominal terms, in which case then can be discounted at the nominal interest rate; or we can leave them in real terms and compute the real interest rate to use for discounting.

Often, it is simply a matter of convenience whether we compute the problem in real or nominal terms. Sometimes, though, it is desirable to work in real terms. An example of this is computing savings plans. In that case, it is natural to assume that individuals are willing to sacrifice and save a constant amount of purchasing power each year. That is, we specify the savings plan as a real (rather than a nominal) annuity. We illustrate this by continuing the previous example:

Example 5.10

Based on the previous calculation, you need to deposit $25,971 today to fund your child's education. Unfortunately, you do not have that amount today, and so you decide to save an annual amount for her each year for the next 18 years. What is an appropriate savings plan to reach your goal?

Our savings plan must be equivalent to $25,971 today. That is, it must have the same present value. The timeline is given below:



That is, we need to find the annuity payment that is equivalent to 25,971 today. Using the annuity calculator and the nominal interest rate, [5]

18	10%	25,971	-3167	0
N	**I**	**PV**	**PMT**	**FV**

we will need to save $3167 per year in nominal terms.

This is not the only savings plan that will work, however. In fact, though this savings plan is a constant dollar amount, the real burden is changing over time. Due to inflation, the last payment of $3167 in year 18 is in a real sense much easier to make because it represents much less purchasing power than the first payment does. Another way to think about it is that you would be giving up far less by not consuming in year 18 than in 1 because the price of goods has gone up so your payment will buy less.

An alternative savings plan is to give up a constant amount of your purchasing power each year. The time line is the same, though now each payment *A* is in real terms. We solve for *A* again using the annuity calculator, but now using the real interest rate:[6]

[5] Alternatively, we could use the formula for the payment of an annuity:

$$C = \frac{25{,}971 \times 0.1}{\left(1 - \frac{1}{1.1^{18}}\right)} = 3{,}167$$

18	4.762%	25,971	-2181	0
N	**I**	**PV**	**PMT**	**FV**

So, we can also reach our goal by saving $2181 in real terms – that is, in terms of today's prices – each year. This means that we will actually need to save

$\$2181 \times 1.05 \quad = \$2290$ at the end of the first year,

$\$2181 \times 1.05^2 = \$2405$ at the end of the second year, …

$\$2181 \times 1.05^{18} = \$5249$ at the end of the 18th year.

Even though the amounts are increasing, the real burden is the same each year. That is, each year the bundle of goods each payment could be used to buy remains the same.

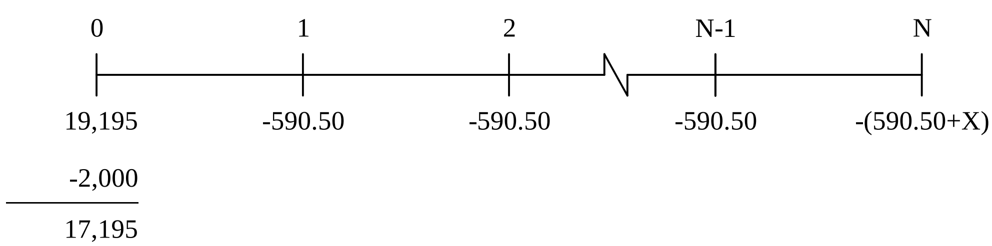
5.8 Putting It All Together

In this section we will consider a series of examples that illustrate how the effects in this chapter interrelate with each other.

Example 5.11

(Continued from Example 5.4) You have come to our senses and decided that the convertible is an impractical fantasy. So, you plan to keep your car and pay the monthly payments of 590.50 for three more years. But your spouse suggests that you use the $2000 you saved for the down payment on the convertible and use it to pay off some of the car loan, rather than keeping it in the bank. How will this effect your remaining payments if you do so? If the rate you will earn at the bank is fixed 3% (APR, monthly), how much will you make by following her advice?

For most car loans and mortgages, if you pay some of the balance early it does not decrease your monthly payment. Instead, your the monthly payments will stay the same (except for possibly the last one), but you will make fewer of them. If we pay an extra $2000 today, how many car payments will we still need to make? What will our last payment be? In terms of the time line, we need to determine the *length* of the annuity *N* and the amount of the final payment that will pay off our debt:



Note that X is the difference between our current payment and the last payment. This is a complicated problem to solve and the way to do it is to break it into pieces.

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[6] Alternatively, using the formula for the payment of an annuity:

$$C = \frac{25{,}971 \times 0.04762}{\left(1 - \frac{1}{1.04762^{18}}\right)} = 2{,}181$$

We begin by ignoring the final partial payment and ask how many payment of $590.50 would be required to pay off the loan with the prepayment. We can determine N by computing how long an annuity of 590.50 per month is equivalent to our reduced balance of $17,195. The simplest way to do this is to just use the annuity calculator:[7]

N	I	PV	PMT	FV
31.89	6.75%/12	17,195	-590.50	0

That is, at will take more than 31 payments, but less than 32 full payments. Thus, to pay off the loan, we will need to make 31 more full payments, and then a final, partial payment. So in the timeline, $N = 32$.

What is the amount of difference in the final payment, X? We can calculate this by equating the outstanding balance to the PV of the cash flows. Note from the timeline that we can calculated the present value of the cash flows by breaking them into a 32 period annuity with payment $590.50 plus a final payment of X in 32 periods:

$$17{,}195 = \frac{590.50}{0.005625}\left(1 - \frac{1}{(1+0.005625)^{32}}\right) + \frac{X}{1.005625^{32}}$$

$$17{,}195 = 17{,}248.73 + \frac{X}{1.005625^{32}}$$

$$\Rightarrow X = -64.29$$

So the final payment will be lower by $64.29 dollars.

Another way to find X is to use the annuity calculator. Suppose instead that we pay the full monthly amount of $590.50 in month 32. Then, unlike a normal loan, the value upon the last payment will not be zero because we will have overpayed. Thus the loan will have a positive value at the end. We can solve for this value on the annuity calculator as follows:

N	I	PV	PMT	FV
32	6.75%/12	17,195	-590.50	64.29

That is, if we pay 32 monthly payments, we will have overpayed by 64.30. Thus, the correct final payment is equal to 590.50 – 64.29 = $526.21.

Now that we know the payment if you choose to prepay, we can evaluate whether prepaying makes sense. Your choice involves two possible payment plans for the loan. The first is to keep your $2000 in the bank and pay 36 more monthly payments. The second is to use the $2000 to pay down the loan, and then pay only 31 more payments, plus 1 partial payment. The two payment schemes are illustrated below:

---

[7] Alternatively, you can use the annuity formula to compute the same value:

$$17{,}195 = \frac{590.50}{0.005625}\left(1 - \frac{1}{(1+0.005625)^{N}}\right)$$

$$\Rightarrow N = -\frac{\ln\left(1 - \frac{17{,}195 \times 0.005625}{590.50}\right)}{\ln(1.005625)} = 31.89$$

	0	1	32	33	36
Don't Prepay		-590.50	-590.50	-590.50	-590.50
Prepay	-2000	-590.50	-526.44	0	0
Δ cash flow	-2000	0	64.06	590.50	590.50

By looking at the difference in the payment schemes, it is clear that prepaying costs money today, but saves money at the end of the loan. To determine the benefit from prepaying, we need to compute the net present value. Since we are determining the advantage relative to keeping the money in the savings account, we use the savings account interest rate of 3%/12 = 0.25% per month as the discount rate:

$$\text{PV}= -2000+\frac{64.29}{1.0025^{32}}+\frac{590.50}{1.0025^{33}}+\frac{590.50}{1.0025^{34}}+\frac{590.50}{1.0025^{35}}+\frac{590.50}{1.0025^{36}}=\$226.41$$

So, prepaying your loan is like having an extra $226.41 in the bank today. The reason for this gain is that you only earn 3% in the bank, whereas paying down the loan is like earning the loan interest rate of 6.75% on your money. Take your spouse to dinner!

Next, let's consider how a tax shield will affect you.

Example 5.12

Assume the government suddenly decided to change the tax code to allow individuals to deduct interest on car loans (this is not currently the case). Your tax rate is 40%. Assuming that you are two years into paying off the original car loan , (i.e., you still owe monthly payments of $590.50 for three more years) how much better off are you by this change in the tax code?

By allowing you to deduct the interest payments, the government is allowing you to shield your interest payments from taxes. So the first step in solving this problem is to work out how much interest you pay every period. We therefore need to calculate the outstanding balance at the beginning of each period. This is equal to the outstanding balance at the end of the previous period. The outstanding balance on the loan at the end of period *n-1* (when there are *(36-n+1)* payments remaining) is:

$$P=\frac{C}{r}\left(1-\frac{1}{(1+r)^{36-n+1}}\right)$$

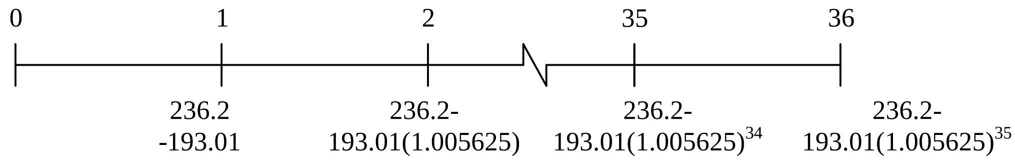
The interest owed in period n is this balance multiplied by the interest rate:

$$P\times r=\frac{C}{r}\left(1-\frac{1}{(1+r)^{36-n+1}}\right)\times r=C\left(1-\frac{1}{(1+r)^{36-n+1}}\right)=590.50\left(1-\frac{1}{1.005625^{36-n+1}}\right)$$

The tax shield in period n is the interest paid multiplied by the tax rate:

$$\begin{aligned}\text{Tax Shield in period } n &= 590.50\left(1-\frac{1}{1.005625^{36-n+1}}\right)\times 0.4 = 236.2\left(1-\frac{1}{1.005625^{36-n+1}}\right)\\ &= 236.2-\frac{236.2(1.005625)^{n-1}}{1.005625^{36}}\\ &= 236.2-193.01(1.005625)^{n-1}\end{aligned}$$

Plotting this on a timeline gives:



So the tax shield is a 36 period annuity of $236.2 minus a 36 period growing annuity with an initial cash flow of $193.01 growing at .5625% per period. Calculating the present value using the after-tax discount rate of 0.5625(1-.4)=.3375% gives:

$$
\begin{aligned}
PV &= \frac{236.2}{0.003375}\left(1-\frac{1}{1.003375^{36}}\right)-\frac{193.01}{0.003375-0.005625}\left(1-\frac{1.005625^{36}}{1.003375^{36}}\right) \\
&= 790.47
\end{aligned}
$$

So the change in the tax code has increased your wealth by $790.47

This preceding example is more realistic than you might think. Many people simultaneously have a large equity position in their houses and a car loan. They could easily capture this NPV by increasing their mortgage loan on their house (which under current tax law enjoys a tax deduction for interest paid) and use the proceeds to pay off the car loan. By converting the car loan into a mortgage they can effectively make the interest on their car loan tax deductible.

5.9 Summary

The discount rate is the correct rate to use to move a particular cash flow in time.

The discount factor is one plus the discount rate.

To convert a discount rate from one period length to another, compound the discount rate factor.

Interest rates are generally quoted as an *annual percentage rate* (APR). The APR is not a true annual interest rate, but must first be converted into a discount rate.

The *effective annual rate* (EAR) is the true interest earned over a one year time interval, that is, it is the discount rate for a period length of one year.

Loan cash flows are computed so that the present value of the future payments, when discounted at the loan interest rate, equals the loan balance.

Paying off a loan is equivalent to earning the loan interest rate for the term of the loan.

If interest rates are not constant over time, when computing the present value of a cash flow it is necessary to use a discount rate that corresponds to the term of the cash flow.

Discount rates vary depending on the risk of the investment. When discounting cash flows, the discount rate used must correspond to the riskiness of the cash flows.

If the interest rate r earned on an investment is taxable at tax rate τ, the equivalent after-tax interest rate is given by $r\,(1-\tau)$. This rate, $r\,(1-\tau)$, is the correct discount rate for taxable investments. The NPV of such investments should be calculated using after-tax cash flows and this discount rate.

The real interest rate represents the growth in purchasing power over time, and it should be used whenever cash flows are expressed in real terms (i.e., based on today's prices).

Given a nominal interest rate r and a rate of inflation π, the real interest rate r_r is given by $1+r_r=(1+r)/(1+\pi)$.

5.9.1 Appendix: Continuous Compounding (Advanced)

In fact, compounding can be more frequent than daily, attaining even higher effective annual yields. As we move from daily, to hourly ($k = 24\times365$), to compounding every second ($k = 60\times60\times24\times365$), we approach the limit of **continuous compounding**, in which we compound every instant ($k = \infty$). For this case, we cannot apply the formula above directly to compute the discount rate from the APR. In this case the discount rate for a period length of one year, that is, the EAR., is given by the following formula,

$$(1 + \text{EAR}) = e^{\text{APR}},$$

where the mathematical constant[8] $e = 2.71828....$ Once you know the EAR you can use the formulae derived in Section 5.1 to compute the discount rate at any other period length.

Alternatively, if we know the EAR and want to find the corresponding continuously compounded APR, we can invert the above equation by taking the natural logarithm of both sides:

$$ln(1 + \text{EAR}) = ln(e^{\text{APR}}) = \text{APR}.$$

[8] The constant e raised to a power is also written as the function *exp*. That is, $e^{\text{APR}} = exp(\text{APR})$. This function is built-in to most spreadsheets and calculators.

Continuously compounded rates are not just a mathematical oddity. Indeed, banks and other financial institutions do make use of them. They have a number of key advantages, including: (i) they offer the highest possible EAR for a given APR, and (ii) in settings in which cash flows can arrive continuously (rather than in discrete periods), they are much easier to work with.

Example 5.13

What is the EAR corresponding to a 6% APR with continuous compounding? If you invest $10,000 at this rate, how much will you have in 5 years? How does this compare with daily compounding?

To convert the APR to an EAR, we use the formula,

$$(1 + \text{EAR}) = e^{.06} = (2.71828\ldots)^{0.06} = 1.0618365\ldots,$$

so the EAR is about 6.18365%. The future value of our investment in 5 years time is given by

$$\text{FV} = \$10{,}000\times(1 + \text{EAR})^5 = \$10{,}000\times(e^{.06})^5 = \$10{,}000\times e^{.06\times5} = \$10{,}000\times e^{.30} = \$13{,}498.59.$$

Alternatively, we could have calculated this using the EAR above as

$$\text{FV} = \$10{,}000\times1.0618365^5 = \$13{,}498.59.$$

The first approach is more direct if you don't need to compute the EAR for other reasons.

With daily compounding, we earn interest of 6%/365 per day. In this case, the future value in 5 years or 5×365 days is

$$\text{FV} = \$10{,}000\times\left(1+\frac{.06}{365}\right)^{5\times365} = \$13{,}498.25.$$

As we can see, the difference between the two is very small, and can be ignored for most practical purposes unless the amount invested is very large or the time horizon very long.

Example 5.14

Suppose an account pays a 10% APR with semi-annual compounding. What is the equivalent continuously compounded APR?

First, we must convert the 10% APR to a true interest rate. Since it is based on semi-annual compounding, it really means that we earn 5% every 6 months. Compounding this for the year gives

$$(1 + \text{EAR}) = 1.05^2 = 1.1025,$$

or an EAR of 10.25%. To find the corresponding continuously compounded APR, we use the second formula,

$$ln(1.1025) = 9.758\% = \text{APR},$$

That is, the equivalent continuously compounded APR is 9.758%.

Reality Check: Since the continuous compounding is better than semi-annual compounding (we earn interest on interest faster), for the two rates to be equivalent the continuously compounded rate should be smaller, which it is.

Continuous compounding is especially useful when the cash flows themselves occur continuously. For example, consider the cash flows of an internet retailer that sells books online. Suppose the firm forecasts revenues of $10M this year. Those $10M will be received throughout the year, not at year end. That is, the firm expects to earn $10M / 365 = $27,397 per day, or $27,397 / 24 = $1142 per hour from internet sales. These cash flows could approximated as daily annuities, or better as hourly annuities, or even more finely. Rather than do that, it is simpler and more natural to think of the $10M as being paid *continuously* throughout the year.

Our formulas for annuities and perpetuities can be extended to the case of continuous cash flows. Rather than do this, we present here a simple method for adjusting the cash flows when they are paid continuously. In particular, a cash flow of C per period paid continuously during the period is equivalent to a cash flow of $\hat{C}$ paid at the end of the period, where

$$\hat{C} = C \times \frac{r}{r_{cc}} ,$$

r is the per period discount rate, and $r_{cc} = ln(1+r)$ is the equivalent continuously compounded APR. Thus, in our example above, if the interest rate is 8% per year, which is equivalent to $ln(1.08) = 7.696\%$ continuously compounded, then revenues of $10M earned continuously during the year is equivalent to revenues of

$$\hat{C} = 10M \times \frac{r}{r_{cc}} = 10M \times \frac{8\%}{7.696\%} = \$10.395M$$

paid at the end of the year. This amount is larger to reflect the fact that we can start earning interest on the cash flows paid during the year. We will earn an extra 8% on the cash flows paid at the start of the year, declining to 0% for the cash flows paid at the very end of the year. Note that the adjustment is roughly the average of these, or about 4%.

This formula can be extended to the case in which the cash flows are also growing continuously, at rate g per period. In that case, the adjustment becomes

$$\hat{C} = C \times \frac{r-g}{r_{cc} - g_{cc}} \quad \approx \quad C \times \left(1 + \tfrac{1}{2}(r+g)\right),$$

where $g_{cc} = ln(1+g)$ is the continuously compounded growth rate, and C is the cash flow rate at the start of the period. The approximation has the same intuition as described above – we are averaging the effect of interest and growth on the start and end of the year cash flows – and it works well if the interest rate and growth rate are not too far apart.

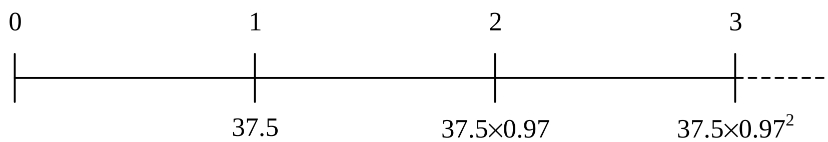
Example 5.15

Your firm is considering buying an oil rig. The rig will initially produce oil at a rate of 30M barrels of oil per year, and you have a long term contract that allows you to sell the oil at a profit of $1.25 per barrel. If rate of oil production from the rig declines by 3% per year, and the discount rate is 10% per year (EAR), how much would you be willing to pay for the rig?

According to the estimates, the rig will generate profits at an initial rate of

(30M barrels per year)×($1.25 / barrel) = $37.5M per year.

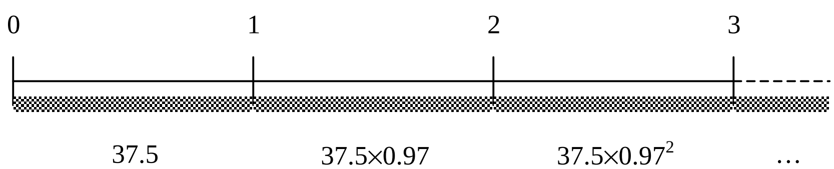
If we imagine the cash flows occurring at the end of the year, the timeline looks like



where we include the fact that the rate of oil flow, and hence the rate of profits declines by 3% each year. The profits of the rig are in the form of a growing perpetuity, with a negative growth rate $g = -3\%$. Using the growing perpetuity formula, the profits have a present value of

PV(profits) = 37.5 / (r – g) = 37.5 / (10% + 3%) = $288M.

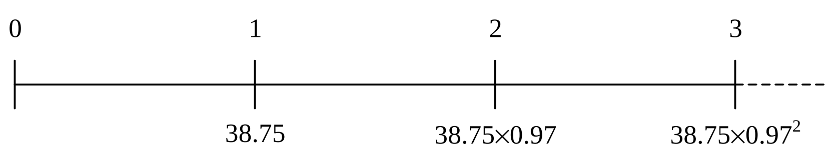
The problem with the above calculation is that it does not take into account the fact that the oil flows continuously throughout the year. Thus, the profits will start being earned immediately. In addition, the rate of production will also start declining immediately, as the oil reserve is depleted. That is, the true timeline is better represented as



where the grey bar represents the fact that the cash flows are paid continuously during the year, at the starting rate indicated for each. To convert this to a standard timeline, we need to use the conversion formula above. That is, the continuous cash flow the first year, starting at a rate of $37.5M per year and growing at continuously at rate of $g = -3\%$ per year, is equivalent to an end of year cash flow of

$$\hat{C} = 37.5 \times \frac{r - g}{r_{cc} - g_{cc}} = 37.5 \times \frac{10\% + 3\%}{9.53\% + 3.05\%} = 38.75,$$

where 9.53% = *ln*(1.10) and –3.05% = *ln*(.97). This yields the equivalent annual timeline:



Now we can once again compute the PV using the growing perpetuity formula,

PV(profits) = 38.75 / (r – g) = 38.75 / (10% + 3%) = $298M.

Thus, we should be willing to pay up to $298M for the rig.

Note: Using the approximation, since $(r + g) = 7\%$, we would adjust the cash flows to $37.5\times(1.035)$ = $38.8M, for a present value of 38.8 / 0.13 = $298.5M, which is quite close to our answer above.

Remark: In many problems we will encounter, cash flows occur throughout the period rather than at the end of the period. In such cases, an adjustment like the one here is most accurate. However, the difference in the resulting present value is not large (in this case, 288M vs. 298M is about 3% difference), and so the adjustment is often ignored when only a ballpark estimate is required.

5.10 Chapter Problems

1. In the U.S. you are only required to withhold what you paid last year in taxes. Each year Hannah receives her annual bonus on January 15. Her employer pays her a relatively low monthly salary that never increases. All pay increases are reflected in her annual bonus. Hannah optimizes her withholding each year by making sure that she only withholds exactly what she paid in taxes the previous year. Then on April 15, the day taxes for the *previous* year are due, she pays the difference. This means that she effectively does not pay the taxes on her annual pay increases until they are actually due, 16 months after the date she actually receives her paycheck. If her tax rate is 40%, the (16 month) before tax interest rate is 5% (EAR), and this year's pay increase was $10,000, how much has she saved by following this strategy as opposed to having her employer withhold the taxes when she is actually paid?

   Solution:
   Tax on bonus = (annual bonus)(tax rate) = ($10,000)(0.4) = $4,000
   Discount rate = $1.05^{16/12} - 1 = 0.0672$
   Equivalent After Tax Rate = $r(1\text{-}T) = 0.0672(1\text{-}0.4) = 4\%$
   PV of tax on bonus if paid in April '02 = $4,000/1.04 = $3,846.15
   PV of tax on bonus if paid in January '01 = $4,000
   NPV = $4,000 - $3,846.15 = $153.85
   Thus Hannah saves $153.85 by delaying the payment of her taxes on the bonus she receives every year.