Internet Appendix to "An Information-Based Theory of Time-Varying Liquidity

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In this appendix, we provide proofs for Theorems 1 and 3 from Daley and Green (2015). To do this we start with some preliminary results about the analytics of $\mathbb{S}^N$ in Section I. We then prove Theorem 1 in Section II and Theorem 3 in Section III. It will sometimes be useful to invoke the following shorthand: $\bar{F}(z) \equiv \mathbb{E}[F_\theta(z)|z]$ and $\bar{G}(z,\cdot) \equiv \mathbb{E}[G_\theta(z,\cdot)|z]$. We also let $\zeta \equiv (\alpha, \beta, F_L, F_H, G_L, G_H, B)$, denote an arbitrary candidate solution to $\mathbb{S}^N$ as defined in.

I. Preliminary Analysis of $\mathbb{S}^N$

The Seller's Value Function

The solutions to the differential equations in (B3) and (B4) are of the form

$$F_\theta(z) = C_1^\theta e^{q_1^\theta z} + C_2^\theta e^{q_2^\theta z} + K_\theta, \tag{IA.1}$$

where $(C_1^L, C_2^L, C_1^H, C_2^H)$ are unknown constants, $\left(q_1^L, q_2^L\right) = \frac{1}{2}\left(1 \pm \sqrt{1+\frac{8r}{\phi^2}}\right)$, and $\left(q_1^H, q_2^H\right) = \frac{1}{2}\left(-1 \pm \sqrt{1+\frac{8r}{\phi^2}}\right)$.[1] The necessary boundary conditions on the seller's value function ((B7) to (B12) in the main article) can be written as:

$$C_1^L e^{q_1^L \alpha} + C_2^L e^{q_2^L \alpha} + K_L = V_L \tag{IA.2}$$

$$q_1^L C_1^L e^{q_1^L \alpha} + q_2^L C_2^L e^{q_2^L \alpha} = 0 \tag{IA.3}$$

$$C_1^L e^{q_1^L \beta} + C_2^L e^{q_2^L \beta} + K_L = B(\beta) \tag{IA.4}$$

$$C_1^H e^{q_1^H \beta} + C_2^H e^{q_2^H \beta} + K_H = B(\beta) \tag{IA.5}$$

$$q_1^H C_1^H e^{q_1^H \alpha} + q_2^H C_2^H e^{q_2^H \alpha} = 0 \tag{IA.6}$$

$$q_1^H C_1^H e^{q_1^H \beta} + q_2^H C_2^H e^{q_2^H \beta} = B'(\beta). \tag{IA.7}$$

The Buyers' Value Function

LEMMA IA.1: *Suppose that F_L, F_H, G_L, G_H satisfy the conditions imposed by $\mathbb{S}^N$. Then, for all*

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[1] For a reference, see Polyanin and Zaitsev (2003), which can be used to substantiate any of our claims pertaining to closed-form solutions of differential equations.

$z \geq \alpha$, *B*, *as defined by* (B29), *solves the differential equation:*

$$B''(z) + (2p(z) - 1)B'(z) - \frac{2(r+\lambda)}{\phi^2}B(z) = \frac{-2}{\phi^2}\Big(p(z)(\lambda F_H(z) + v_H) + (1-p(z))(\lambda F_L(z) + v_L)\Big). \tag{IA.8}$$

Proof. Let $\eta_2 \equiv \frac{2(r+\lambda)}{\phi^2}$. We abuse notation slightly by omitting the function arguments and using G_θ to refer to $G_\theta(z, 1_{\{N>1\}})$ for N finite and $G_\theta^\infty(z)$ for $N = \infty$. Using this convention, we have from (B29), that $B = pG_H + (1-p)G_L$, which is twice differentiable for $z > \alpha$ (this follows from (B18) and (B19) for $N < \infty$, or (B24) and (B25) for $N = \infty$). Thus,

$$\begin{aligned} B' &= pG_H' + (1-p)G_L' + p'(G_H - G_L), \\ B'' &= pG_H'' + (1-p)G_L'' + 2p'(G_H' - G_L') + p''(G_H - G_L). \end{aligned}$$

Therefore,

$$\begin{aligned} B'' + B' - \eta_2 B &= p\left(G_H'' + G_H' - \eta_2 G_H\right) + (1-p)\left(G_L'' + G_L' - \eta_2 G_L\right) + (p'' + p')(G_H - G_L) + 2p'(G_H' - G_L') \\ &= p\left(G_H'' + G_H' - \eta_2 G_H\right) + (1-p)\left(G_L'' - G_L' - \eta_2 G_L\right) + \\ &\quad (p'' + p')(G_H - G_L) + 2p'(G_H' - G_L') + 2(1-p)G_L'. \end{aligned} \tag{IA.9}$$

Using the functional form of p, the last line of the above can be simplified to

$$\begin{aligned} (p'' + p')(G_H - G_L) + 2p'(G_H' - G_L') &+ 2(1-p)G_L' \\ &= \frac{2}{1+e^z}\left(p'(G_H - G_L) + pG_H + (1-p)G_L\right) \\ &= \frac{2}{1+e^z}B' = 2(1-p)B'. \end{aligned}$$

Substituting the above into (IA.9) and rearranging gives

$$\begin{aligned} B'' + (2p-1)B' - \eta_2 B &= p\left(G_H'' + G_H' - \eta_2 G_H\right) + (1-p)\left(G_L'' + G_L' - \eta_2 G_L\right) \\ &= -\frac{2}{\phi^2}\Big(p(\lambda F_H + v_H) + (1-p)(\lambda F_L + v_L)\Big), \end{aligned}$$

where the second equality follows by substituting in from either (B14) and (B15) (and using the fact that $G_\theta(z,1) = G_\theta(z,0)$ for $z \geq \beta$), or (B24) and (B25), which completes the proof. □

LEMMA IA.2: *If* $\hat{B}$ *solves* (IA.8), *then it has the form*

$$\hat{B} = \begin{cases} \bar{F}(z) + \frac{r(\bar{V}(z) - \bar{K}(z))}{r+\lambda} + C_{21}^B \frac{e^{q_1^B z}}{1+e^z} + C_{22}^B \frac{e^{q_2^B z}}{1+e^z} & \text{for } z \in (\alpha, \beta) \\ \bar{V}(z) + C_{31}^B \frac{e^{q_3^B z}}{1+e^z} + C_{32}^B \frac{e^{q_4^B z}}{1+e^z} & \text{for } z > \beta, \end{cases} \tag{IA.10}$$

where $\bar{K}(z) \equiv \mathbb{E}[K_\theta | z]$, $\left(q_3^B, q_4^B\right) = \frac{1}{2}\left(1 \pm \sqrt{1 + \frac{8r}{\phi^2}}\right)$, $\left(q_1^B, q_2^B\right) = \frac{1}{2}\left(1 \pm \sqrt{1 + 8\frac{r+\lambda}{\phi^2}}\right)$, *and* C_{ij}^B *are arbitrary constants.*

Proof. We use $B_3 : [\beta, \infty) \to [V_L, V_H]$ to denote any solution restricted to the domain $z \geq \beta$

(this notation is used in the proof of Theorem 1). For $z \geq \beta$, since $\bar{F}(z) = B(z)$, (IA.8) becomes

$$B'' + (2p(z) - 1)B' - 2\frac{r}{\phi^2}B(z) = -2\frac{r}{\phi^2}\bar{V}(z),$$

which has homogeneous solution of the form $B_{h,3}(z) = C_{31}^B \frac{1}{1+e^z} e^{q_3^B z} + C_{32}^B \frac{1}{1+e^z} e^{q_4^B z}$ and a particular solution $B_{3,p}(z) = \bar{V}(z)$, where $\left(C_{31}^B, C_{32}^B\right)$ are arbitrary constants. Similarly, let $B_2 : [\alpha, \beta] \to [V_L, V_H]$ denote any solution restricted to the domain $z \in [\alpha, \beta]$. For $z \in (\alpha, \beta)$, the homogeneous solution to (IA.8) is of the form $B_{h,2}(z) = C_{21}^B \frac{1}{1+e^z} e^{q_1^B z} + C_{22}^B \frac{1}{1+e^z} e^{q_2^B z}$, where C_{21}^B, C_{22}^B are arbitrary constants. Since F_θ takes the form in (IA.1), the particular solution is $B_{p,2}(z) = \bar{F}(z) + \frac{r(\bar{V}(z) - \bar{K}(z))}{r+\lambda}$.
□

COROLLARY IA.1: *Since B is twice differentiable above α (Lemma IA.1), any B_2, B_3 that are part of a solution to $\mathbb{S}^N$ must satisfy*

$$B_2(\beta) = B_3(\beta) \tag{IA.11}$$
$$B_2'(\beta) = B_3'(\beta) \tag{IA.12}$$

II. Proof of Theorem 1

To prove Theorem 1 we first establish the existence of a solution to $\mathbb{S}^1$ in Lemma IA.3. Together with Theorem 3, this is sufficient for the result. Recall that when $N = 1$, $G_\theta(\cdot, 1)$ does not appear in $\mathbb{S}^1$, so we simplify notation by simply writing $G_\theta(\cdot)$ for $G_\theta(\cdot, 0)$.

LEMMA IA.3: *There exists candidate ζ that solves $\mathbb{S}^1$.*

The proof of Lemma IA.3 involves several steps, which we detail below. By way of overview, in Section II.A we first complete the analytic characterization of $\mathbb{S}^1$, which we began for general N in Section I. Then, in Section II.B we reduce the problem of finding solutions to $\mathbb{S}^1$ to solving a system of two analytic, nonlinear equations. We then demonstrate that a solution to the reduced system, and therefore $\mathbb{S}^1$, exists in Section II.C.

A. *The Analytics of* $\mathbb{S}^1$

In addition to the expressions from Section I, when $N = 1$, Lemma IA.1 extends (via the same proof) to cover all $z \in \mathbb{R}$, not just $z \geq \alpha$. For $z < \alpha$, the homogeneous solution to (IA.8) is of the form $B_{h,1}(z) = C_{11}^B \frac{1}{1+e^z} e^{q_1^B z} + C_{12}^B \frac{1}{1+e^z} e^{q_2^B z}$, where $\left(q_1^B, q_2^B\right) = \frac{1}{2}\left(1 \pm \sqrt{1 + 8\frac{r+\lambda}{\phi^2}}\right)$ and C_{11}^B, C_{12}^B are arbitrary constants. For all $z \leq \alpha$, $F_\theta(z) = F_\theta(\alpha)$ and thus $\bar{F}(z) = p(z)F_H(\alpha) + (1 - p(z))F_L(\alpha)$. This leads to the particular solution $B_{p,1}(z) = \frac{r\bar{V}(z) + \lambda\bar{F}(z)}{r+\lambda}$. Summarizing, we now have that for any $z \notin \{\alpha, \beta\}$, $B(z) = B_1(z)I_{z<\alpha} + B_2(z)I_{z\in(\alpha,\beta)} + B_3(z)I_{z>\beta}$, where

$$\begin{aligned} B_1(z) &\equiv \frac{r\bar{V}(z) + \lambda\bar{F}(z)}{r+\lambda} + C_{11}^B \frac{e^{q_1^B z}}{1+e^z} + C_{12}^B \frac{e^{q_2^B z}}{1+e^z} \\ B_2(z) &\equiv \bar{F}(z) + \frac{r(\bar{V}(z) - \bar{K}(z))}{r+\lambda} + C_{21}^B \frac{e^{q_1^B z}}{1+e^z} + C_{22}^B \frac{e^{q_2^B z}}{1+e^z} \\ B_3(z) &\equiv \bar{V}(z) + C_{31}^B \frac{e^{q_3^B z}}{1+e^z} + C_{32}^B \frac{e^{q_4^B z}}{1+e^z}, \end{aligned} \tag{IA.13}$$

and $\left(C_{11}^B, C_{12}^B, C_{21}^B, C_{22}^B, C_{31}^B, C_{32}^B\right)$ are arbitrary constants to be pinned down by boundary conditions.

When $N = 1$, (B14) and (B15) imply that G_H and G_L must be differentiable everywhere (not just above β). Therefore, B must also inherit this property, which leads to the following additional boundary conditions:

$$B_1(\alpha) = B_2(\alpha) \tag{IA.14}$$

$$B_1'(\alpha) = B_2'(\alpha). \tag{IA.15}$$

B. Reducing the System

For any fixed $(\alpha, \beta) \in \mathbb{R}^2, \alpha < \beta$, (IA.2) to (IA.7) and (IA.11) to (IA.15) imply a system of ten linear equations in the ten variables $(C_1^L, C_2^L, C_1^H, C_2^H, C_{11}^B, C_{12}^B, C_{21}^B, C_{22}^B, C_{31}^B, C_{32}^B)$. The system of equations is linearly independent and therefore has a unique solution parameterized by (α, β).

To pin down (α, β), two remaining boundary conditions must be satisfied. Letting $C_{12}^B(\alpha, \beta)$ and $C_{31}^B(\alpha, \beta)$ be the parameterized solution values obtained from the linear subsystem, (B30) and (B31) are satisfied if and only if

$$C_{12}^B(\alpha, \beta) = 0, \tag{IA.16}$$

$$C_{31}^B(\alpha, \beta) = 0. \tag{IA.17}$$

Finally, to incorporate (B14) to (B17), starting from arbitrary time t such that $I_t = 0$, let $\tau = \inf\{s \geq t : I_s = 1\}$. Notice then that if B, F_L, F_H satisfy (B30) and (IA.1)-(IA.7), the functions $\hat{G}_L, \hat{G}_H$, defined as

$$\hat{G}_\theta(z) \equiv \mathbb{E}_z^\theta \left[\int_t^\tau e^{-r(s-t)} v_\theta ds + e^{-r(\tau - t)} F_\theta(\hat{Z}_\tau) \right],$$

solve (B14) to (B17) by construction. Therefore, to prove existence of a solution to the entire system, $\mathbb{S}^1$, it is sufficient to show that there exists a pair $\alpha < \beta$ such that (IA.16) and (IA.17) hold.

C. Existence of Solution to the Reduced System

Having reduced $\mathbb{S}^1$ to (IA.16) and (IA.17), we now prove the following lemma, giving the desired result, Lemma IA.3, as an immediate corollary.

LEMMA IA.4: *There exists $(\alpha, \beta) \in \mathbb{R}^2$, $\alpha < \beta$, such that* IA.16 *and* IA.17 *simultaneously hold.*

COROLLARY IA.2: *There exists a solution to $\mathbb{S}^1$.*

The proof of Lemma IA.4 relies on several additional lemmas. Throughout, we make use of the following change of variables: for any $\alpha < \beta$, let $A \equiv e^\alpha \in \mathbb{R}_{++}$, $D \equiv e^{\beta - \alpha} \in (1, \infty)$, $x \equiv \sqrt{1 + \frac{8r}{\phi^2}} > 1$, and $y \equiv \sqrt{1 + \frac{8(r+\lambda)}{\phi^2}} > x$. Finally, let $\hat{C}_{12}^B(A, D)$ and $\hat{C}_{31}^B(A, D)$ refer to the constants as functions of (A, D) (e.g., $\hat{C}_{12}^B(e^\alpha, e^{\beta-\alpha}) = C_{12}(\alpha, \beta)$).

DEFINITION IA.1: *Define the correspondence $D_{12} : R_{++} \rightrightarrows (1, \infty)$ as, for all $A > 0$, $D_{12}(A) = \left\{D : \hat{C}_{12}^B(A, D) = 0\right\}$.*

DEFINITION IA.2: *Define the correspondence $D_{31} : R_{++} \rightrightarrows (1, \infty)$ as, for all $A > 0$, $D_{31}(A) = \left\{D : \hat{C}_{31}^B(A, D) = 0\right\}$.*

LEMMA IA.5: *For all $A > 0$,* (i) *$D_{12}(A) \neq \emptyset$,* (ii) *$\inf D_{12}(A) > 1$, and* (iii) *$\sup D_{12}(A) < \infty$.*

Proof. Fix (α, β) and solve (IA.2) to (IA.7) and (IA.11) to (IA.15) for C_{12}^B. Making the change of variables gives

$$\hat{C}_{12}^B(A, D) = A^{\frac{1}{2}(y-1)}\left(A \times T_1(D) + Q_1(D)\right),$$

where, for all $D > 1$, $T_1(D) = \frac{\sum_{i=1}^5 \kappa_i D^{g_i}}{1+l_1 D^x} < 0$, $Q_1(D) = \frac{\sum_{i=1}^7 \upsilon_i D^{h_i}+m}{1+l_1 D^x} > 0$, $l_1, g_i, h_i, m > 0$, $\max_i\{g_i\} = x + \frac{1}{2}(y+1)$, $\bar{\kappa} = \kappa_{\arg\max_i\{g_i\}} < 0$, $\frac{\sum_i \kappa_i}{1+l_1} = -\left(\frac{x^2-1}{2y(1+y)}\right)$, $\sum \upsilon_i + m = 0$, $\max_i\{h_i\} = \frac{3}{2}x + \frac{1}{2}y > \max_i\{g_i\}$, and $\bar{\upsilon} = \upsilon_{\arg\max_i\{h_i\}} > 0$. From this, we have that

$$\lim_{D\to 1} T_1(D) = \frac{\sum_i \kappa_i}{1+l_1} = -\left(\frac{x^2-1}{2y(1+y)}\right) < 0, \qquad \lim_{D\to\infty} T_1(D) = -\infty,$$
$$\lim_{D\to 1} Q_1(D) = \frac{\sum_i \upsilon_i + m}{1+l_1} = 0, \qquad \lim_{D\to\infty} Q_1(D) = \infty.$$

Fixing any $A > 0$, $\lim_{D\to\infty} \hat{C}_{12}^B(A, D) = \infty$ (since $\max_i\{h_i\} > \max_i\{g_i\}$ and $\bar{\upsilon} > 0$) and $\lim_{D\to 1} \hat{C}_{12}^B(A, D) = -\left(\frac{x^2-1}{2y(1+y)}\right) A^{\frac{1}{2}(1+y)} < 0$ (since $\lim_{D\to 1} Q_1(D) = 0$). Property (i) follows from the continuity of $\hat{C}_{12}^B$ and the intermediate value theorem; (ii) and (iii) follow from the fact that both limits are bounded from zero. □

LEMMA IA.6: *For all* $A > 0$, (i) $D_{31}(A) \neq \emptyset$, (ii) $\inf D_{31}(A) > 1$, *and* (iii) $\sup D_{31}(A) < \infty$.

Proof. Fix (α, β) and solve (IA.2) to (IA.7) and (IA.11) to (IA.15) for C_{31}^B. Making the the change of variables gives

$$\hat{C}_{31}^B(A, D) = A^{-\frac{1}{2}(x+1)}\left(A \times T_2(D) + Q_2(D)\right)$$

where $T_2(D) = \frac{\sum_{i=1}^4 \kappa_i D^{g_i}}{D^x(1+l_2 D^x)}$, $Q_2(D) = \frac{\sum_{i=1}^5 \upsilon_i D^{h_i}}{D^x(1+l_2 D^x)}$, $l_2, g_i, h_i, m > 0$, $\max_i\{g_i\} = 2x + 1$, $\bar{\kappa} = \kappa_{\arg\max_i\{g_i\}} > 0$, $\max\{h_i\} = 2x$, $\bar{\upsilon} = \upsilon_{\arg\max_i\{h_i\}} > 0$, $\sum \upsilon_i = 0$, and $\sum \kappa_i = -\left(\frac{x+1}{x-1}\right) < 0$. From this, we have that

$$\lim_{D\to 1} T_2(D) = -\left(\frac{1+x}{2x}\right) < 0, \qquad \lim_{D\to\infty} T_2(D) = \infty,$$
$$\lim_{D\to 1} Q_2(D) = 0, \qquad \lim_{D\to\infty} Q_2(D) = \frac{(V_L - K_L)(x-1)^2}{2x^2} > 0.$$

Fixing any $A > 0$: $\lim_{D\to\infty} \hat{C}_{31}^B(A, D) = \infty$ and $\lim_{D\to 1} \hat{C}_{31}^B(A, D) = -\left(\frac{1+x}{2x}\right) A^{\frac{1}{2}(1-x)} < 0$. Property (i) follows from the continuity of $\hat{C}_{31}^B$ and the intermediate value theorem; (ii) and (iii) follow from the fact that both limits are bounded away from zero. □

LEMMA IA.7: *Let* $d_{12} : \mathbb{R}_{++} \to (1, \infty)$ *be an arbitrary function such that, for all* $A > 0$, $d_{12}(A) \in D_{12}(A)$. *Then,*

1. $\lim_{A\to 0} d_{12}(A) = 1$, *and*
2. $\lim_{A\to\infty} d_{12}(A) = \infty$.

Proof. Recall that $\hat{C}_{12}^B(A, D) = A^{\frac{1}{2}(y-1)}\left(A \times T_1(D) + Q_1(D)\right)$, and therefore $A \times T_1(d_{12}(A)) + Q_1(d_{12}(A)) = 0$ for all $A > 0$.

(1) For any D, $\lim_{A\to 0} A \times T_1(D) + Q_1(D) = Q_1(D)$. Since $Q_1(D) > 0$ for all $D > 1$ and $\lim_{D\to 1} Q_1(D) = 0$, it must be that $\lim_{A\to 0} d_{12}(A) = 1$.

(2) Suppose $\lim_{A\to\infty} d_{12}(A) = 0$. Then $A \times T_1(d_{12}(A)) + Q_1(d_{12}(A)) \to -\infty$, a contradiction. Therefore, $\lim_{A\to\infty} d_{12}(A) > 0$. Since $T_1(D) < 0$ for all D, $\lim_{A\to\infty} A \times T_1(D) = -\infty$ and thus in order to have $A \times T_1(d_{12}(A)) + Q_1(d_{12}(A)) = 0$, it must be that $Q_1(d_{12}(A)) \to \infty$, which requires that $\lim_{A\to\infty} d_{12}(A) = \infty$. □

LEMMA IA.8: *Let* $d_{31} : \mathbb{R}_{++} \to (1,\infty)$ *be an arbitrary function such that, for all* $A > 0$, $d_{31}(A) \in D_{31}(A)$. *Then,*

1. $\liminf_{A\to 0} d_{31}(A) > 1$, *and*
2. $\limsup_{A\to\infty} d_{31}(A) < \infty$.

Proof. Recall that $\hat{C}^B_{31}(A,D) = A^{-\frac{1}{2}(x+1)}\left(A \times T_2(D) + Q_2(D)\right)$, and, therefore, $A \times T_2(d_{31}(A)) + Q_2(d_{31}(A)) = 0$ for all $A > 0$.

(1) For any $\epsilon > 0$, let $N_\epsilon = \{(A,D) \in \mathbb{R}_{++} \times (1,\infty) : \|(A,D) - (0,1)\| < \epsilon\}$. Let (A_ϵ, D_ϵ) denote an arbitrary point such that $(A_\epsilon, D_\epsilon) \in N_\epsilon$. To prove the first result, it suffices to show that there exists $\epsilon > 0$ such that $\hat{C}^B_{31}(A,D) < 0$ for any (A_ϵ, D_ϵ). Recall that $\lim_{D\to 1} \hat{C}^B_{31}(A,D) = -\left(\frac{1+x}{2x}\right) A^{\frac{1}{2}(1-x)} < 0$. Further $Q_2(D_\epsilon)$ is arbitrarily close to $-\left(\frac{1+x}{2x}\right)$, implying that $\lim_{A\to 0} A \times T_2(D_\epsilon) + Q_2(D_\epsilon) = Q_2(D_\epsilon)$, and therefore $\lim_{A\to 0} \hat{C}^B_{31}(A, D_\epsilon) = \lim \frac{Q_2(D_\epsilon)}{A^{\frac{1}{2}(1+x)}}$. Note that Q_2 is continuously differentiable in D. Taking the derivative of Q_2 evaluated at $D = 1$ gives $Q_2'(1) = \frac{(V_L - K_H)(x^2-1)}{4x} < 0$ (since $K_H > V_L$). Hence, $Q_2(D_\epsilon) < 0$. Therefore, $\bigtriangledown \hat{C}^B_{31}|_{(A=0,D=1)} = \left(\frac{\partial \hat{C}^B_{31}}{\partial A}, \frac{\partial \hat{C}^B_{31}}{\partial D}\right)|_{(A=0,D=1)} < 0$. Using a Taylor expansion, $\hat{C}^B_{31}(A,D) \approx \hat{C}^B_{31}(0,1) + \bigtriangledown \hat{C}^B_{31}(0,1) \cdot (A, D-1) < 0$ for any $(A,D) \in N_\varepsilon$ and ε sufficiently small.

(2) Since $\max_i\{g_i\} = 2x+1 > \max\{h_i\} = 2x$, for (A,D) arbitrarily large $|A \times T_2(D)| >> |Q_2(D)|$. If $T_2(D) \neq 0$, then $|A \times T_2(D)|$ becomes arbitrarily large with A. Hence, $\lim_{A\to\infty} T_2(d_{31}(A)) = 0$. Therefore, $\limsup_{A\to\infty} d_{31}(A) < \infty$. □

LEMMA IA.9: *For any two values* $a < a'$ *both in* $\mathbb{R}_{++}$, *there exist two continuous paths,* p_{12}, p_{31}, *such that for each* $i \in \{12, 31\}$,

1. $p_i : [0,1] \to [a,a'] \times (1,\infty)$,
2. *for all* $t \in [0,1]$, $\hat{C}^B_i(p_i(t)) = 0$, *and*
3. $p^1_i(0) = a$, *and* $p^1_i(1) = a'$, *where* $p^j_i(t)$ *denotes the* j^{th} *component of* $p_i(t)$.

Proof. Fix an $i \in \{12, 31\}$ and any two values $a < a'$, both in $\mathbb{R}_{++}$. Define the set $S \equiv \{(A,D) : A \in [a,a'], D = D_i(A)\}$. Because $\hat{C}^B_i$ is a uniformly continuous co-lipschitz mapping, S can be partitioned into a finite collection of disjoint, closed, and connected components, denoted $\hat{S}$ (Maleva (2005))—implying that there exists an $\epsilon > 0$ such that

$$\min_{\substack{s,s' \in \hat{S} \\ s \neq s'}} \left(\min_{\substack{(A,D)\in s \\ (A',D')\in s'}} ||(A,D) - (A',D')|| \right) > \epsilon.$$

Now let $\underline{D}$ be any value in $(1, \inf\{D_i(a)\})$ and $\overline{D}$ be any value in $(\sup\{D_i(a')\}, \infty)$. Hence, $\hat{C}^B_i(a, \underline{D}) < 0$ and $\hat{C}^B_i(a', \overline{D}) > 0$ (from the proofs of Lemmas IA.5 and IA.6). Then, if $\hat{S}$ does not contain a component connecting (a,d) to (a',d') for some values of $d \in D_i(a), d' \in D_i(a')$, there exists a continuous path $q : [0,1] \to [a,a'] \times (1,\infty)$ such that $q(0) = (a, \underline{D})$, $q(1) = (a', \overline{D})$, and $\{t : q(t) \in S\} = \emptyset$. However, this violates the intermediate value theorem (since q is continuous with $q(0) < 0$, $q(1) > 0$) implying that for any two values $a < a'$, there exists a continuous path p_i satisfying (1) to (3) of the lemma. □

Proof of Lemma IA.4. This follows nearly immediately from Lemmas IA.7 through IA.9. Using the notation from Lemma IA.9, as $a \to 0$ and $a' \to \infty$, from Lemmas IA.7 and IA.8, $p_{12}^2(0) < p_{31}^2(0)$ and $p_{12}^2(1) > p_{31}^2(1)$. Because the paths are continuous, they must intersect. By construction, any intersection is a solution to (IA.16) and (IA.17) simultaneously. □

III. Proof of Theorem 3

To prove Theorem 3 we begin with some preliminary results in Section III.A, and then provide the main verification argument in Section III.B.

A. *Preliminaries*

FACT IA.1: *If ζ is a solution to $\mathbb{S}^N$, then*

$$F_\theta(z) = \mathbb{E}_z^\theta\left[\int_0^{T(\beta)} e^{-rt}k_\theta dt + e^{-rT(\beta)}B(\beta)\right] = K_\theta + \mathbb{E}_z^\theta\left[e^{-rT(\beta)}\right](B(\beta) - K_\theta), \tag{IA.18}$$

where $T(\beta) = \inf\{t : Z_t \geq \beta\}$ and $\mathbb{E}_z^\theta$ is the expectation over the process Z under the law $\mathcal{Q}_z^\theta$.

Proof. By construction. □

DEFINITION IA.3: *For any C^2 function $f : \mathbb{R} \to \mathbb{R}$, $MB_H(f(z)) \equiv \frac{\phi^2}{2}(f'(z) + f''(z)) - r(f(z) - K_H)$.*

LEMMA IA.10: *If ζ is a solution to $\mathbb{S}^N$, then $MB_H(B_3(z)) < 0$ for all $z \geq \beta$.*

Proof. We first establish three inequalities for all $z \geq \beta$: (i) $B_3'(z) > \bar{V}'(z)$, (ii) $B_3(z) < \bar{V}(z)$, and (iii) $q_1^H(B_3(\beta) - K_H) > B_3'(\beta)$. To establish (i) and (ii), recall from (IA.13) that

$$B_3(z) = \bar{V}(z) + C_{32}^B \frac{e^{q_4^B z}}{1 + e^z},$$

where $C_{32}^B, q_4 < 0$. Therefore,

$$B_3'(z) = \bar{V}'(z) + C_{32}^B \frac{e^{zq_4^B}}{(e^z + 1)^2}(q_4^B(1 + e^z) - e^z) > \bar{V}'(z).$$

Thus, $B_3(z) < \bar{V}(z)$ and $B_3'(z) > \bar{V}(z)$.

To establish (iii), for a given β, we can solve boundary conditions (IA.5) and (IA.7), obtaining

$$C_1^H(\beta) = \frac{B_3'(\beta) + q_2^H(K_H - B_3(\beta))}{q_1^H - q_2^H}e^{-q_1^H\beta}, \tag{IA.19}$$

$$C_2^H(\beta) = \frac{-\left(B_3'(\beta) + q_1^H(K_H - B_3(\beta))\right)}{q_1^H - q_2^H}e^{-q_2^H\beta}. \tag{IA.20}$$

Next, using boundary condition (IA.6), we arrive at the correspondence

$$\mathcal{B}_H(\alpha) = \{\beta \in \mathbb{R} : \beta \geq \alpha, \alpha = A_H(\beta)\},$$

where $A_H(\beta) = \frac{1}{q_1^H - q_2^H} \ln\left(\frac{-q_2^H}{q_1^H}\frac{C_2^H(\beta)}{C_1^H(\beta)}\right)$. Because $\frac{-q_2^H}{q_1^H} > 0$, any real solution requires $\text{sgn}(C_1^H(\beta)) = \text{sgn}(C_2^H(\beta))$. Since $F_H'(\beta) = B_3'(\beta) > 0$ and $\text{sgn}(F_H') = \text{sgn}(C_1^H)$, it must be that $C_1^H(\beta), C_2^H(\beta) > 0$. Finally, $C_2^H(\beta) > 0$ and (IA.20) imply that $q_1^H(B_3(\beta) - K_H) > B_3'(\beta)$.

Having established (i) to (iii), for any $C_{32}^B > 0$, because $B_3 < \bar{V}$ and $B_3' > \bar{V}'$, if $q_1^H(B_3(z) - K_H) > B_3'(z)$ then, $q_1^H(\bar{V}(\beta) - K_H) > \bar{V}'(\beta)$. Therefore, since β satisfies $q_1^H(B_3(z) - K_H) > B_3'(z)$, it must be that $\beta > \beta_H \equiv \inf\left\{x : q_1^H(\bar{V}(z) - K_H) > \bar{V}'(z), \forall z > x\right\}$. Lemma B.3 of Daley and Green (2012) shows that $MB_H(\bar{V}(z)) < 0$ for all $z \geq \underline{\beta}_H$. Lemma IA.10 then follows from the fact that $MB_H(B_3(z)) \leq MB_H(\bar{V}(z))$ for all z, which can be seen by differentiating $MB_H(B_3(z))$ with respect to C_{32}^B to get $-\frac{e^{z(1-x)/2}(x-1+e^z(1+x))}{(1+e^z)^3} < 0$, where $x = \sqrt{1 + 8r/\phi^2} > 1$. □

B. *Verification*

Proof of Theorem 3. The necessity of the equations in $\mathbb{S}^N$ is demonstrated in Appendix B of the main article. For sufficiency, we show that if ζ solves $\mathbb{S}^N$, then $\Xi^N(\alpha, \beta, B)$ satisfies the four requirements from Definition 1. Below, a separate proof is provided for each condition. □

Proof of Condition 2 (Belief Consistency). To prove *Belief Consistency* we need the following generalizations of objects from Definition 2 for arbitrary N. Let $m_t^n = \sup\{s \leq t : I_s^n = 1\}$, $\overline{m}_t = \sup_n(m_t^n)$, $Q_t^{\alpha,n} = \max\{\alpha - \inf_{s \leq m_t^n} \hat{Z}_s, 0\}$, $Q_t^\alpha = \max\{\alpha - \inf_{s \leq \overline{m}_t} \hat{Z}_s, 0\}$, and $Q_{0-}^\alpha = Q_{0-}^{\alpha,n} = 0$. Then, if there does not exist $s \in [t, h]$ such that $Z_s \geq \beta$, $S_{n,h}^{L,t} = 1 - e^{-\left(Q_h^{\alpha,n} - Q_{t-}^{\alpha,n}\right)}$ (and is perfectly correlated with all other sellers present at time h), and we still have $Z_t = \hat{Z}_t + Q_{t-}^\alpha$.

Now, fix an arbitrary on-path history up to time t. We will show that $Z_t = \hat{Z}_t + Q_{t-}^\alpha$ is Bayesian consistent with the strategy profile in $\Xi^N(\alpha, \beta, B)$. There are two cases: there exists $s < t$ such that $W_s = V_L$ and a share of the asset sold, or no such $s < t$ exists. For the first case, notice that only the low type accepts a bid of V_L on path, and hence such an action perfectly reveals $\theta = L$ and the belief correctly becomes degenerate for all future times. For the second case, we argue by induction. Let $M_t = \sup\{s \leq \overline{m}_t : I_s^n = 0 \ \forall\ n\}$ if such a time exists and zero otherwise. For $t = 0$, $Q_{0-}^\alpha = 0$, so $Z_0 = \ln\left(\frac{P_0}{1-P_0}\right)$, as it should. Next, let $t > 0$ and assume that $Z_s = \hat{Z}_s + Q_{s-}^\alpha$ for all $s \leq M_t$. Then, for any $s \in [M_t, \underline{m}_t]$, Bayes rule (in log-likelihood form) requires that

$$\begin{aligned}
Z_s &= Z_{M_t} + \ln\left(\frac{f_{s-M_t}^H(X_s - X_{M_t})}{f_{s-M_t}^H(X_s - X_{M_t})}\right) + \ln\left(\frac{1-0}{1-(1-e^{-(Q_{s-}^\alpha - Q_{M_t-}^\alpha)})}\right) \\
&= Z_{M_t} + (\hat{Z}_s - \hat{Z}_{M_t}) + (Q_{s-}^\alpha - Q_{M_t-}^\alpha) \\
&= \hat{Z}_{M_t} + Q_{M_t-}^\alpha + (\hat{Z}_s - \hat{Z}_{M_t}) + (Q_{s-}^\alpha - Q_{M_t-}^\alpha) = \hat{Z}_s + Q_{s-}^\alpha.
\end{aligned}$$

If $t = \overline{m}_t$, the argument is finished. If $t > \overline{m}_t$, then $I_s^n = 0$ for all n and $s \in (\overline{m}_t, t]$, meaning that all shares are owned by holders over this period and Bayes rule requires that $Z_t = Z_{\overline{m}_t} + (\hat{Z}_t - \hat{Z}_{\overline{m}_t})$. Equivalently, Q_t must equal $Q_{\overline{m}_t}$, which is precisely what Q^α prescribes. □

Proof of Condition 3 (Zero Profit). In Ξ^N, there are two cases in which trade occurs. First, when $Z_t \geq \beta$, all sellers trade with probability one, so for each transacted share n,

$$\mathbb{E}[G^n_\theta(t^+, \omega)|\mathcal{F}_t, t \in \mathcal{T}] = B(Z_t) = W_t,$$

satisfying the condition. Second, when $z \leq \alpha$, only low-type sellers trade with positive probability. Such a trade reveals $\theta = L$, so

$$\mathbb{E}[G^n_\theta(t^+, \omega)|\mathcal{F}_t, t \in \mathcal{T}] = V_L = W_t,$$

satisfying the condition. □

Proof of Condition 4 (No Deals). To demonstrate *No Deals*, we first demonstrate three inequalities. If ζ solves $\mathbb{S}^N$, then (i) $F_L \geq V_L$, (ii) $F_H \geq B$, and (iii) $F_H \geq F_L$.

(i) Clearly the statement is true for $z \geq \beta$. Note that $F'_L(\alpha) = 0$ and $F''_L(z) > 0$ for all $z \in [\alpha, \beta)$. Hence, $F'_L(z) > 0$ for all $z \in (\alpha, \beta)$. The result then follows from $F_L(\alpha) = V_L$.

(ii) Clearly the statement is true for $z \geq \beta$. To see that the statement holds for $z < \beta$, fix any β and $C^B_{32} < 0$ and solve (IA.14) and (IA.15) for C^B_{21} and C^B_{22} and (IA.19) and (IA.20) for C^H_1 and C^H_2. Then, by direct calculation, $C^H_1 e^{q^H_1 z} + C^H_2 e^{q^H_2 z} + K_H > B_2(z)$ for all $z < \beta$. This verifies $F_H > B$ for $z \in [a, b)$. All that remains is to show that (ii) holds for $z < \alpha$. We break the argument into two cases.

- For $N = 1$, in order to solve (IA.14) and (IA.15), C^B_{11} must be strictly positive, implying that B_1 is increasing on $(-\infty, \alpha)$, and therefore $B_1(z) \leq B_1(\alpha) = B_2(\alpha) < C^H_1 e^{q^H_1 \alpha} + C^H_2 e^{q^H_2 \alpha} + K_H = F_H(z)$.
- For $N > 1$, note that $B(z)$ is a convex combination of V_L and $B_2(\alpha) > V_L$. Therefore, $B(z) < B_2(\alpha)$ for $z < \alpha$. Recalling that $B_2(\alpha) < F_H(\alpha) = F_H(z)$ for $z < \alpha$ implies the desired result.

(iii) This is implied by the following: 1) Fact IA.1, 2) $K_H \geq K_L$, 3) $B(\beta) = F_H(\beta) > K_H$ (which follows from $C^H_1, C^H_2 > 0$; see the proof of Lemma IA.10), and 4) $\mathbb{E}^H_z[e^{-rT(\beta)}] \geq \mathbb{E}^L_z[e^{-rT(\beta)}]$, because, for any t and z, the distribution of Z_t under the law $\mathcal{Q}^H_z$ weakly first-order stochastically dominates the analogous distribution under $\mathcal{Q}^L_z$.

Therefore, *No Deals* is satisfied because,

- If $y \geq F_H(z)$, then $\mathbb{E}[G_\theta(z, \cdot)|z, F_\theta(z) \leq y] = B(z) \leq y$.
- If $y \in [F_L(z), F_H(z))$, then $\mathbb{E}[G_\theta(z, \cdot)|z, F_\theta(z) \leq y] = V_L \leq y$. □
- If $y < F_L(z)$, $\{\theta : F_\theta(z) \leq y\} = \emptyset$.

Proof of Condition 1 (Owner Optimality). To see that the strategies given in Ξ^N are optimal for a seller (of arbitrary share n), we need to show that starting from any (t, Z_t) such that $I^n_t = 1$, $T(\beta) = \inf\{s \geq t : Z_s \geq \beta\}$ solves $(SP_{H,t})$, and that both $T(\beta)$ and $T(\alpha, \beta) = \inf\{s \geq t : Z_s \notin (\alpha, \beta)\}$ solve $(SP_{L,t})$. Due to stationarity, we can normalize t to zero. Start with $\theta = H$, and define

$$F^{**}_H(z) \equiv \sup_{\tau \geq 0} \mathbb{E}^\theta_z \left[\int_0^\tau e^{-rt} k_\theta dt + e^{-r\tau} F_H(Z_\tau)\right].$$

By (ii) in the proof of Condition 4, $F_H(z) \geq B(z)$. Therefore,

$$F^{**}_H(z) \geq F^*_H(z) \equiv \sup_{\tau \geq 0} \mathbb{E}^\theta_z \left[\int_0^\tau e^{-rt} k_\theta dt + e^{-r\tau} B(Z_\tau)\right].$$

Define $f_\theta(t,z) = \left(1 - e^{-rt}\right) + e^{-rt}F_\theta(z)$, which is C^2 on $U \equiv \mathbb{R}\backslash\{\alpha,\beta\}$. By Ito's formula,

$$f_H(t,Z_t) = f_H(0,Z_0) + \int_0^t \mathcal{A}^H f_H(s,Z_s) I\left(Z_s \in U\right) ds + \int_0^t \phi e^{-rs} F_H'(Z_s) dB_s + \int_0^t e^{-rs} F_H'(\alpha) dQ_s^\alpha.$$

Using the fact thhat $\mathcal{A}^H f_H(t,z) = 0$ for all $z \in (\alpha,\beta)$ (by construction), and $\mathcal{A}^H f_H(t,z) = e^{-rt} MB_H(B_3(z)) < 0$ for all $z \geq \beta$ (by Lemma IA.10), we can conclude that

$$f_H(t,Z_t) \leq f_H(0,Z_0) + M_t = F_H(z) + M_t,$$

where M is a martingale given by $\int_0^t \phi e^{-rs} F_H'(Z_s) dB_s$. Taking the $\mathcal{Q}_z^H$-expectation, using the fact that F_H' is bounded (by construction), the optional stopping theorem gives $F_H^{**}(z) \leq F_H(z)$. Since the high type can attain $F_H(z)$ by following the strategy $T(\beta)$, we can conclude that $F_H^*(z) = F_H(z)$ and hence $S^{H,t}$ solves $(SP_{H,t})$ for all t.

For the low type, we first demonstrate that both $T(\beta)$ and $T(\alpha,\beta)$ achieve an expected payoff equal to $F_L(z)$ starting from any initial $Z_0 = z$. Let $F_{L,j}(z)$ denote the expected payoff from playing according to the pure strategy (j) for $j = 1,2$ starting from $Z_0 = z$. The case for $j = 1$ is covered by Fact IA.1. For $z \in (\alpha,\beta)$, $F_{L,2}$ must solve (B4) and therefore is of the form (IA.1). Clearly, $F_{L,2}$ must satisfy value-matching at both α and β, (i.e., (IA.2) and (IA.4)), implying the constants are uniquely pinned down and $F_{L,2}(z) = F_L(z)$ for all $z \in (\alpha,\beta)$. Verifying that $F_{L,2}(z) = F_L(z)$ for $z \notin (\alpha,\beta)$ is immediate.

That $F_L^*(z) = F_L(z)$ follows the same steps as the case for $\theta = H$ after noting that (B4) implies that $\mathcal{A}^L f_L = 0$ for all $z \in (\alpha,\beta)$ and $\mathcal{A}^L f_L < \mathcal{A}^H f_H < 0$ for all $z > \beta$, which completes the proof. □

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