News and Liquidity in Markets 
with Asymmetrically-Informed Traders*

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Abstract

We propose an information-based theory to explain time-varying liquidity and establish its link to a variety of patterns in financial markets. We consider a dynamic economy in which sellers’ private information is revealed stochastically over time to a market of traders. The equilibrium involves periods in which liquidity dries up: shares remain in the hands of liquidity-constrained traders despite efficient gains from trade. This leads to an endogenous (information-driven) liquidation cost. Buyers correctly anticipate such costs, driving prices below, and volatility in excess of, their fundamental values. The model predicts that (efficient) fire sales can occur after sufficiently bad news; a trade at a low price reveals information, which then facilitates the trade of additional shares. Additional implications for asset prices and market efficiency are discussed.

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1 Introduction

A broad empirical literature has documented that financial markets are susceptible to periods of illiquidity. The underlying mechanism driving this phenomenon remains an open theoretical question. We propose an information-based theory to explain time-varying liquidity and establish its link to expected returns, prices deviating from fundamentals and excess volatility. The key mechanism is that an information asymmetry, which varies stochastically over time as news is revealed, creates a trading friction that leaves asset shares inefficiently allocated during periods when uncertainty is highest. This leads to a time-varying liquidation cost because selling is sometimes associated with inefficient delay. Traders correctly anticipate such costs, driving prices below fundamentals, and creating an (information-driven) illiquidity discount.

To develop this theory, we enrich the model of Daley and Green (2012) (hereafter DG12), in which there is single, privately-informed seller and news about the asset is revealed gradually over time. First, we introduce idiosyncratic liquidity shocks, which generate both gains from repeated trade and endogenously-determined prices that reflect the extent to which these gains are captured. Second, we add a market size component by including multiple shares, and multiple sellers, of the asset. This generates a realistic feature of many markets; trading behavior of one seller can reveal information to the market that may facilitate or inhibit the trades of other agents. We construct an equilibrium in which there is no trade when uncertainty about the asset’s type is high, the market is perfectly liquid when beliefs are favorable, and a sell-off (or fire sale) can occur when the market is pessimistic.

Implications for Asset Prices and Trade Patterns

Dating back to Shiller (1981), there is an extensive literature that has studied whether prices accurately reflect fundamentals. The general consensus has been that prices frequently diverge from fundamentals and are too volatile relative to what is implied by news about fundamentals. Our model provides a novel channel consistent with both of these phenomena. In the absence of asymmetric information or idiosyncratic liquidity shocks, the asset would trade efficiently at its fundamental value, with volatility driven solely by information about fundamentals. In their (combined) presence, the no-trade region leads to an endogenous liquidation cost driving prices below fundamentals. Further, news about fundamentals affects not only the market beliefs about the fundamental value of the asset, but also the ability to sell the asset in the future. This additional consequence compounds the price reaction and

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1 The literature on these topics is vast and we do not attempt a comprehensive overview here. See Shiller (2003) and Boudoukh et al. (2007) for recent discussions.
generates additional volatility.

Within our model, illiquidity, bid-ask spreads, excess returns, volatility, and trade volume depend on the underlying market belief, and are therefore both time-varying and stochastic. The model predicts that excess returns (attributable to illiquidity) correlate positively with volatility, and move inversely to trade volume and liquidity. This is consistent with studies by Amihud and Mendelson (1986); Brennan and Subrahmanyam (1996); Amihud (2002). Over time, news eliminates the information asymmetry. However, in the short-term, higher quality news reduces liquidity and increases volatility, but may increase or decrease excess returns and liquidity premia. Asymmetric information increases expected returns and bid-ask spreads, while decreasing trade volume. More frequent liquidity shocks reduce prices and increase liquidity premia.

The model generates endogenous “fire sales” in which a trade by one agent causes other agents to sell immediately thereafter. Unlike in much of the theoretical literature, the underlying mechanism is not driven by accounting standards or collateralized lending, rather it is purely informational. In our case, fire sales improve efficiency; the information revealed by the first trade reduces information asymmetries, facilities trade for other shares and leads to higher future liquidity. Moreover, the mere possibility of a fire sale helps stabilize a shaky market providing a lower bound on pessimism in the market and improving the speed at which assets are allocated efficiently. In addition, a small amount of bad news can lead to a drastic reduction in liquidity, which explains another phenomenon often referred to as “liquidity drying up” (Smith 2008; Reuters 2008).

The recent collapse of the private-label mortgage-backed securities (MBS) market is perhaps illustrative. A number of papers have documented the presence of information asymmetries in the market for MBS (Downing et al. 2009). However, prior to the collapse, trade and issuance of mortgage-backed securities occurred in a liquid and well-functioning market, despite the fact that banks issuing these securities had significant information about the underlying collateral that was inaccessible to investors. Beginning in 2006, economic indicators of a decline in the housing market increased uncertainty regarding the value of the collateral, and led to a catastrophic drop in both liquidity and prices. Investors were unwilling to buy these securities or lend against them (even at a substantial discount/haircut) for fear of being stuck with the most “toxic” assets. Perhaps rightly so. As a result, these MBS remained on the balance sheet of numerous large banks despite their need for capital.

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2See a report from the U.S. Treasury on the role of fire sales in the recent financial crisis or Shleifer and Vishny for a survey of the literature on fire sales in financial markets. Wall Street traders and analysts refer to this as “market capitulation” (Zweig 2008; Cox 2008).

3Krishnamurthy (2010) or Brummermeier (2009a) provide a descriptive analysis of how collateralized debt markets malfunctioned in the recent crisis.
Figure 1 illustrates the dramatic decrease in liquidity that private-label MBS experienced following the beginning of a downward trend in housing prices. There was a significant tightening of credit starting in 2008, which certainly played a role in the decline. However, several facts suggest that asset-specific factors (e.g., information frictions) were also responsible. First, private-label MBS issuance as a percentage of the total US bond market experienced a similarly severe decline. Second, issuance of agency-backed MBS, which are less information sensitive than private-label MBS, remained roughly constant over the same time period (despite the dramatic decrease in real estate prices).

\[\text{Private Label Issuance ($B)}\]

\[\text{Fraction of US Bond Market}\]

\[\text{Case Shiller National Home Price Index}\]

**FIGURE 1:** Following a drop in housing prices, private-label MBS issuance and percentage of private label MBS contributing to the US bond market fell drastically (sources SIFMA, S&P).

Both the informational and risk-preference assumptions of the model (we assume traders are risk neutral) align most naturally with the interpretation that the asset market represents an idiosyncratic element of a larger economy. However, the effect on prices is not diversifiable because it is not driven by dislike for risk; rather, by an informational friction that depresses the expected discounted stream of cash flows. The market size, which can also be interpreted as a measure of transparency, plays an important role in determining how quickly private information is revealed to uninformed market participants. In *small* markets, there are periods of time during which no sellers are present and the market learns only by observing news about fundamentals, whereas, in *large* markets the presence of sellers, and therefore the information revealed by their trading behavior, is continuous.

\[\text{4If one were interested in applying the model to a systematic component of the market portfolio, it would be natural to incorporate risk-averse traders to study the pricing impact of aggregate liquidity risk. See Section 7 for comments on this extension.}\]
Implications for Welfare and Market Efficiency

Both the market size and the quality of the news determine how quickly private information is fully revealed (at which point full efficiency is restored). The market size, $N$, can be interpreted literally (i.e., the physical number of tradable shares) or as a measure of market transparency (i.e., the number of shares an agent has information regarding). That is, small $N$ corresponds both to markets for obscure, heterogenous products (e.g., private label mortgage-backed CDOs), in which only a few securities are backed by the underlying collateral, as well as assets with a large number of identical physical shares, but a decentralized marketplace (e.g., corporate bonds prior to the introduction of TRACE). Under this latter interpretation, one can evaluate regulations, such as the introduction (and subsequent extensions) of TRACE, aimed at improving transparency through the availability of transaction data.

Perhaps not surprisingly, our results suggest that large (or more transparent) markets are more efficient than small ones, though this is not driven by the availability of trading partners, but rather by the availability of additional information. In contrast, market efficiency can decrease with the level of news quality. Higher quality news has two opposing effects. First, equilibrium beliefs evolve more rapidly through the no-trade region, and thus the amount of time spent in the inefficient region decreases. Second, better news provides more incentive for sellers of high-type assets to hold out and wait for a better price, thereby increasing the size of the no-trade region. Which effect dominates depends both on the current state and the magnitude of the increase in news quality.

Market efficiency decreases with the arrival rate of shocks because traders liquidate (and incur the cost of doing so) more frequently. This occurs despite the fact that the asset’s fundamental value is unaffected. As the arrival rate of liquidity shocks goes to zero, asset prices converge to fundamental values. Conversely, the efficiency of the market can improve with the severity of the liquidity shocks (i.e., holding costs), even for relatively small increases that preserve strategic considerations. More severe shocks impose larger inefficiencies during periods of no trade, but also decrease the motivation for agents to endure such periods and hence the size of the no-trade region. One implication is that government programs aimed at injecting liquidity into the system and easing the credit constraints of distressed financial institutions have an adverse effect that can actually reduce market efficiency.

5In July 2002, FINRA introduced regulation to improve the transparency of corporate bond markets by requiring all member broker/dealers to report corporate bonds transactions to TRACE, which then makes transaction data publicly available. More recently, TRACE reporting requirements have extended to a broader class of fixed income securities including agency debt and agency backed MBS.
The remainder of the paper is organized as follows. Section 1.1 discusses our work within the context of the theoretical literature. Section 2 presents the general model. Section 3 analyzes small markets \((N = 1)\) and compares it to several benchmark cases. Section 4 extends the analysis to larger markets \((N > 1)\). Section 5 derives implications for asset pricing and trading patterns, and relates our findings to the empirical literature. Section 6 discusses implications for welfare and efficiency. Section 7 discusses the roles and interpretations of the assumptions of the model and concludes.

1.1 Relation to the Theoretical Literature

Our work contributes to two strands of the literature. The first is the literature on asset pricing with asymmetric information pioneered by Grossman and Stiglitz (1980) in a setting where agents are price-takers, and by Kyle (1985); Glosten and Milgrom (1985) in settings with strategic traders, and further developed by Hellwig (1980); Diamond and Verrecchia (1981); Admati (1985); O’Hara (2003); Easley and O’Hara (2004), among others. More recently, in a setting with risk-averse agents, Vayanos and Wang (2012) show that asymmetric information hampers risk sharing, increases ex-ante expected returns, and reduces liquidity. One of our primary contributions to this literature is to introduce gradual information arrival and study its implications for asset prices and efficiency, as well as its interaction with liquidity. The information that agents possess in our model is “long-lived.” Garleanu and Pedersen (2003) study a model with private liquidity shocks and adverse selection. The private information about the asset in their model is short-lived: it pertains only to cash flows arriving next period. They show that allocation costs arise and affect an asset’s required return due to the combination of a trader’s private information about both his liquidity preference and the asset’s cash flows next period. Eisfeldt (2004) also studies a model where information is short-lived and liquidity is determined endogenously.

The second strand is a broad literature studying asset pricing and liquidity in the presence of other frictions. For example Amihud and Mendelson (1986); Constantinides (1986); Vayanos (1998, 2004); Acharya and Pedersen (2005) do so in the presence of exogenous trading costs. Lo et al. (2004) examine trading volume in such a setting. Our friction is non-institutional, in that nothing in the form of the environment prevents efficient trade. Duffie et al. (2005, 2007) study the implications of search and bargaining on asset prices and liquidity. In their model, search frictions (i.e., the lack of a trading partner at any given point in time) generate a liquidation cost, while intermediation leads to bid-ask spreads and novel dynamics. Vayanos and Wang (2007); Vayanos and Weill (2008) develop search-based models that derive liquidity premia due to endogenous concentration of traders in segmented markets. Our model is absent both search and intermediation; a potential seller can contract
directly with potential buyers at any point in time. Rather, it is the informational asymmetry, together with its gradual (but stochastic) dissipation, that generates an endogenous liquidation cost and equilibrium dynamics.

The theoretical groundwork for this paper was developed in [DG12] who study the decision of a privately-informed seller facing a market of buyers where information is revealed gradually over time. In their model, there is a single seller and the asset is traded only once. As mentioned above, we enrich this framework along two dimensions. Extending results from the framework in [DG12] to this setting is a non-trivial exercise. However, we believe the primary contribution of this paper derives from the ability to explain patterns in financial markets through the endogenous price formation, and the implications for asset prices and market efficiency.

The equilibrium we construct is of the signaling-barrier variety; a seller of a low-type asset sells probabilistically at some lower boundary, which prevents beliefs from dropping below the boundary. Thus, not selling at the boundary is a positive signal about the value of the asset. This equilibrium feature is also present in [Bar-Isaac 2003; Gul and Pesendorfer 2011]. Several other papers have considered settings with asymmetric information and news arrival. For example, [Kremer and Skrzypacz 2007] study a dynamic signaling model where a grade about the seller is revealed at some finite time $T$. They show how an endogenous lemons market develops and that trade is always delayed in equilibrium. [Korajczyk et al. 1992; Lucas and McDonald 1990] study the effect of information releases on equity issues in a setting with adverse selection.

2 The Model

In the economy, there is a continuum of agents, indexed by $A \in [0, 1]$, and $N \in \{1, 2, \ldots, \infty\}$ indivisible shares of an asset. The asset has a fixed type $\theta \in \{L, H\}$. At every moment in time $t \in [0, \infty)$, each share $n$ is owned by an agent in the economy. We refer to this agent as the owner of share $n$ at time $t$, formally denoted by $A^t_n$. Each agent can own at most one share, which generates a cash flow to its current owner that depends on $\theta$ and the owner’s liquidity status: either constrained or unconstrained. An unconstrained owner of a type-$\theta$ asset obtains an instantaneous cash flow (or dividend) of $v_\theta$, whereas a constrained owner has positive holding costs and obtains only $k_\theta < v_\theta$. All agents are risk neutral and

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6 We define this ownership process to be left-continuous, meaning $A^t_n$ should be interpreted as the owner at the beginning of “period” $t$.

7 This accommodates both additive and proportional holding costs without imposing either. In addition, because agents are risk neutral, nothing substantive changes if the cash flow is random with mean $v_\theta$ or $k_\theta$, depending on the owner’s liquidity status. Finally, if one wishes to avoid an infinite total cash flow when $N = \infty$, $v_\theta, k_\theta$ can be interpreted as “infinitesimal” quantities (see [Anderson 2008]).
discount future payoffs at rate $r$. Let $V_\theta = \frac{v_\theta}{r}$ and $K_\theta = \frac{k_\theta}{r}$ denote the value of a share being held \textit{ad infinitum} by an unconstrained and constrained agent respectively. We assume that $K_H > V_L$, meaning there is the potential for a “lemons” problem \textit{a la} Akerlof (1970). In other words, holding costs create gains from trade, but are not overly punitive, which preserves strategic concerns at the forefront of our analysis. In contrast, as holding costs grow to infinity, constrained agents effectively become noise traders and trading behavior in the economy is efficient.

Because there are gains from trade only when an owner is shocked, we refer to an owner as a \textit{seller} if she is liquidity constrained, and as a \textit{holder} otherwise. A holder of share $n$ is subject to an observable liquidity shock that arrives according to a Poisson process, $L^n = \{L^n_t : 0 \leq t \leq \infty\}$ with intensity $\lambda$; if $A^n_t$ is a holder at time $t$, then the arrival of the first shock after time $t$ induces a positive holding cost, thereby transforming her into a seller. For simplicity, we assume that subsequent arrivals have no effect on $A^n_t$ (i.e., a seller maintains a positive holding cost indefinitely) and that $\{L^n\}_{n=1}^N$ are mutually independent. Let $I^n_t$ be the indicator that is equal to one if and only if $A^n_t$ is a seller at time $t$.

The market opens at $t = 0$, with each share $n$ owned by an agent, $A^n_0$, whose liquidity status is commonly known. $A^n_0$ knows the asset’s type, potential buyers do not. At every $t \geq 0$, there is an outstanding bid generated from the buyers in the market (unconstrained agents who do not own shares), and any seller can accept the bid in exchange for her share. If a share trades, its new owner learns the asset’s type, and the previous owner exits the economy. If a seller rejects the current bid, she retains her share, receives the flow payoff, and can entertain future bids.

The bid process is a convenient modeling device for aggregated buyer behavior, and relies on the assumptions that shocks are observable and that buyers are numerous and identical. Under these assumptions, the precise mechanism for trading is largely unimportant. In each of the following examples of trading mechanisms, it can be shown that there exists an equilibrium with trading behavior (i.e., the timing of, and price at, each trade) identical to the one on which we focus.

\textbf{Example 2.1. Decentralized or Over-the-Counter Markets:} At every $t \geq 0$, each owner is approached by at least one buyer who can make a purchase offer. Owners observe offers and

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\textsuperscript{8}Relying on rationale that we formalize in \textit{Definition 2.1}, because omnipresent buyers compete with one another, there is no uncertainty about the maximum price they are willing to pay following any given history. It then makes no difference for equilibrium outcomes whether buyers bid this price (as in Example \textit{2.1}), the sellers ask for it (as in Example \textit{2.2} where the bid is not an explicit part of agent strategies), or a market maker facilitates the transaction (as in Example \textit{2.3}). In all cases, a seller simply decides when to \textit{stop}, at which point she is paid the buyers’ value for her share (conditional on her stopping, of course), regardless of whether stopping translates to accepting an offer from a buyer or market maker, or to posting a price equal to the buyers’ (conditional) value for a share, which will then be accepted.
decide if, when and to whom to sell.\footnote{In contrast to Duffie et al. (2005, 2007), this model of over-the-counter markets still features decentralized trading, but there are no search frictions.}

**Example 2.2. Posted Prices:** At every \( t \geq 0 \), each owner posts a price at which she is willing to sell. Buyers observe posted prices and choose if, when and from whom to buy.

**Example 2.3. Facilitation by a Market Maker:** There is a strategic market maker privy to the public history. At every \( t \geq 0 \), the market maker generates bid prices at which he will purchase shares from sellers, and ask prices at which he will sell shares to buyers. He obtains weakly lower flow value from shares of the asset than sellers do and aims to maximize his own expected profit, but competitive pressures drive this profit to zero.

### 2.1 Public Information

A key feature of the model is that news about the asset’s type is continuously and publicly revealed via a Brownian diffusion process, \( X \), where for all \( t \geq 0 \)

\[
X_t = \mu t + \sigma B_t
\]

and \( B \) is a standard Brownian motion. Define the signal-to-noise ratio \( \phi \equiv |\mu_H - \mu_L|/\sigma \), which we assume to be strictly greater than zero. Larger values of \( \phi \) imply higher quality news; \( \phi = 0 \) corresponds to a model without news.

To formalize the information structure, we introduce the probability space \((\Omega, \mathcal{F}, \mathbb{Q})\) in which \( \theta, B, \) and \( \{L^n\}_{n=1}^\infty \) are mutually independent. The state space \( \Omega \) contains all possible \((\theta, B, \{L^n\}_{n=1}^\infty)\) and allows for an independent randomization device. The public history at time \( t \), which also corresponds to the information set of buyers at time \( t \), contains:

- The initial liquidity status of original owners: \( \{I^n_0\}_{n=1}^N \)
- The arrival times of liquidity shocks: \( \{L^n_s : 0 \leq s \leq t\}_{n=1}^N \)
- The history of news: \( \{X_s : 0 \leq s \leq t\} \)
- The history of all past trades

Let \( \mathcal{F}_t \) denote the filtration generated by the public history.\footnote{Notice that while the history of past bids is not explicitly included in the public history, because the bid will be progressively measurable with respect to \( \mathcal{F}_t \), it would be redundant to include it.} Finally, it will be convenient to keep track of the set of (random) times in which at least one share trades, which we denote by \( T \).\footnote{One might argue that it is unreasonable to think agents can keep track of all the information in the public history. This is assumption is not crucial. The equilibrium we construct and analyze is stationary; all relevant information prior to time \( t \) will be encapsulated in a simple state variable (Section 3).}
2.2 The Bid Process and Owner Strategies

The bid process $W = \{W_t, 0 \leq t \leq \infty\}$ is a real-valued stochastic process, progressively measurable with respect to $\mathcal{F}_t$. To prevent trades based solely on the expectation of ever-increasing prices, we impose the standard transversality condition: for any $t, \mathcal{F}_t$, 
$$\lim_{h \to \infty} E[e^{-r_h} W_{t+h} | \mathcal{F}_t] = 0. \tag{12}$$

The asset owners’ information set contains the public history as well as the asset type. In addition, we allow owners to mix by including a randomization device. Let $\{\mathcal{G}_t\}_{t \geq 0}$ denote the filtration generated by the information sets of owners.

A pure strategy for an owner of share $n$ of a type-$\theta$ asset who becomes a seller at time $t$, hereafter a “($\theta, t$)-seller” of share $n$, is an $\mathcal{F}_h$-adapted stopping time greater than or equal to $t$. A mixed strategy is a stopping time adapted to $\mathcal{G}_h$ (to allow for randomization) and denoted by $\tau_{n, t}^\theta \geq t$. For our analysis, it will be convenient to represent a seller’s (mixed) strategy by the distribution it induces over pure strategies: let $S_{n, t}^\theta = \{\tau_{n, h}^\theta, t \leq h \leq \infty\}$, denote the progressively measurable process with respect to $\mathcal{F}_h$, where

$$S_{n, h}^\theta(\omega) \equiv \Pr(\tau_{n, t}^\theta(\omega) \leq h | \mathcal{F}_h)$$

From the buyer’s perspective, $S_{n, h}^\theta$ keeps track of how much probability mass the ($\theta, t$)-seller of share $n$ has “used up” by time $h$ by assigning positive probability to accepting bids at times $s \in [t, h]$. An upward jump in $S_{n, t}^\theta$ corresponds to the ($\theta, t$)-seller of share $n$ accepting with an atom of mass. For any given sample path, $S_{n, h}^\theta$ is a CDF over a ($\theta, t$)-seller’s acceptance time.

2.3 The Market Belief

Along the equilibrium path, the market belief about the asset type must be consistent with the public history and the equilibrium strategies. We begin by deriving the belief process that updates only based on news and then incorporate the information content from the public history due to strategic effects into a second component. Under a change of variables, the market belief can be represented by the sum of these two processes.

The market begins with a common prior $\pi = \Pr(\theta = H)$. Let $f_\theta$ denote the density function of type $\theta$’s news at time $t$, which is normally distributed with mean $\mu_\theta t$ and variance $\sigma^2 t$. Define $\hat{P}$ to be the belief process for a Bayesian who updates only based on news starting

\[\text{See Brunnermeier (2009b) for a discussion of how the failure of this condition can lead to “bubbles.”} \]
from the prior (i.e., \( \hat{P}_0 = \pi \)).

\[
\hat{P}_t \equiv \frac{\hat{P}_0 f_t^H(X_t)}{\hat{P}_0 f_t^H(X_t) + (1 - \hat{P}_0) f_t^L(X_t)}
\]  

(1)

It is useful to define a new process \( \hat{Z} \equiv \ln(\hat{P}/(1 - \hat{P})) \), which represents the belief in terms of its log-likelihood ratio. Because the mapping from \( \hat{P} \) to \( \hat{Z} \) is injective, there is no loss in making this transformation. By definition,

\[
\hat{Z}_t = \ln \left( \frac{\hat{P}_t}{1 - \hat{P}_t} \right) = \ln \left( \frac{\hat{P}_0}{1 - \hat{P}_0} \right) + \ln \left( \frac{f_t^H(X_t)}{f_t^L(X_t)} \right) + \phi \sigma (X_t - t(\mu_H + \mu_L))
\]  

and thus,

\[
d\hat{Z}_t = -\frac{\phi}{2\sigma} (\mu_H + \mu_L) dt + \frac{\phi}{\sigma} dX_t
\]  

(2)

Now define \( P = \{P_t, 0 \leq t < \infty\} \) to be the equilibrium market belief process. \( P_t \) differs from \( \hat{P}_t \) because it accounts for the possibility and realizations of trades before time \( t \). Define \( Z \equiv \ln(P/(1 - P)) \). As before, there is no loss in making this transformation. Because Bayes rule is linear in log-likelihoods, we can decompose \( Z \) as \( Z = \hat{Z} + Q \), where \( Q \) is the stochastic process that keeps track of the information conveyed by the history of past acceptances and rejections. For example, suppose that in equilibrium, the likelihood of share \( n \) trading at time \( t \) is larger if \( \theta = L \) than if \( \theta = H \). Then, if trade occurs at \( t \), \( Q \) decreases; if trade does not occur at \( t \), then \( Q \) increases.

2.4 Equilibrium Definition

Given \( W \), the problem facing a \((\theta, t)\)-seller of share \( n \) is to select a stopping rule that, for all \( h \geq t \), solves:

\[
\sup_{\tau \geq h} E^\theta \left[ \int_h^\tau e^{-r(s-h)} k_\theta ds + e^{-r(\tau-h)} W_\tau |\mathcal{G}_h} \right] \quad (SP_{\theta,t})
\]

Let \( S_{n,t}^{\theta} = \text{supp}(S_{n,t}^{\theta}) \). We say that \( S_{n,t}^{\theta} \) solves \( (SP_{\theta,t}) \) if each \( \tau \in S_{n,t}^{\theta} \) solves \( (SP_{\theta,t}) \). Now define \( F_{n,\theta,t}(h, \omega) \) to be the expected payoff to the \((\theta, t)\)-seller of share \( n \), who chooses a \( \tau \) that solves \( (SP_{\theta,t}) \), starting from time \( h \geq t \). In addition, because a holder waits to become a seller at some future time \( t \) when the shock arrives, let

\[
G_{\theta}^n(s, \omega) \equiv E^\theta \left[ \int_s^t e^{-r(x-s)} v_\theta dx + e^{-r(t-s)} F_{\theta,t}^n(t, \omega) |\mathcal{G}_s} \right] \quad (3)
\]

denote the expected payoff to the holder of share \( n \) of a type-\( \theta \) asset starting from time \( s \).
Definition 2.1. An equilibrium consists of \( \{S_{\theta,t}^{L,n}, S_{\theta,t}^{H,n}\}_{t \in \mathbb{R}^+, n \in \{1,\ldots,N\}} \), \( W, Z \) such that

1. **Owner Optimality**: Given \( W \), for all \((\theta,t)\) and \( n \), \( S_{\theta,t}^{n} \) solves \((SP_{\theta,t})\).

2. **Belief Consistency**: For any \( t \) and history such that \( \mathcal{F}_t \neq \emptyset \), \( Z_t \) satisfies Bayes rule.

3. **Zero Profit**: If \( \mathcal{F}_t \cap \{ t \in T \} \neq \emptyset \), then, for all shares \( n \) that trade at \( t \),

\[
W_t = E[G^n_\theta(t^+, \omega) | \mathcal{F}_t, t \in T]
\]

4. **No Deals**: If \( I^n_t(\omega) = 1 \), there does not exist a \( q \in \mathbb{R} \) such that

\[
E \left[ G^n_\theta(t^+, \omega) | \mathcal{F}_t, F_{\theta,n}^n(t, \omega) \leq q, A^n_t \neq A^n_{t^+} \right] - q > 0
\]

The first two conditions, **Owner Optimality** and **Belief Consistency**, represent standard criteria: a seller in possession of share \( n \) at time \( t \) must choose a strategy that maximizes her payoff, and beliefs must follow from Bayes rule along the equilibrium path (i.e., \( \mathcal{F}_t \neq \emptyset \)). The interpretation of **Zero Profit** is clear—any executed trade must earn the purchasing buyer zero expected surplus—and is motivated by the interpretation of numerous competitive buyers. If **No Deals** fails, then there exists an offer that will earn a buyer a positive expected payoff; hence, this condition reflects the equilibrium requirement that no buyer can profitably deviate by making an offer that a seller would be willing to accept with positive probability.

In our analysis, we first analyze the case of \( N = 1 \), which serves to highlight the effect of liquidity considerations. We then generalize our analysis to \( N > 1 \) and discuss the implications of market size. As discussed in Section 1, one can interpret \( N = 1 \) as pertaining to decentralized markets or markets for non-standard financial products, whereas larger \( N \) corresponds to more centralized or transparent markets in which many identical shares of the same asset are traded.

### 3 Equilibrium in Small Markets

In this section we provide both an informal description and a formal characterization of the equilibrium for the \( N = 1 \) case and compare the equilibrium to several natural benchmarks. In Section 4 we discuss how behavior must change if \( N > 1 \), and characterize the equilibrium in larger markets. In order to focus on the implications for prices and trade dynamics in Sections 5 and 6, the details of the equilibrium construction are relegated to Appendix A for the interested reader.

Let \( N = 1 \), so there is a single owner at any point in time who owns the entire asset. To simplify notation, we therefore drop the \( n \) index on \( I, S^{\theta,t} \), etc. The equilibrium we construct here is a natural extension of the equilibrium of focus in DG12 (see Benchmark 2
in Section 3.1). It has a Markovian structure; both the market belief, \( Z \), and the owner’s liquidity status, \( I \), follow Markov processes, and strategies are stationary with respect to the current realization of these processes. We use \((z, i)\) when referring to the state variable, which should be interpreted as any \((t, \omega)\) such that \((Z_t(\omega), I_t(\omega)) = (z, i)\). In addition, references to generic \( z \) should be understood as \( z \in \mathbb{R} \), as opposed to the degenerate belief levels \( z = \pm \infty \), unless otherwise stated.\(^{13}\) Let \( w(z, i) \) denote the bid in the state \((z, i)\).

Equilibrium play is characterized by a pair \((\alpha, \beta) \in \mathbb{R}^2, \alpha < \beta\), and an increasing function \( B : \mathbb{R} \rightarrow [V_L, V_H] \). Both \( \alpha \) and \( \beta \) represent important belief thresholds, and \( B(z) \) represents a buyer’s expected value for the asset given the belief \( z \). To describe play, first consider states in which the owner is a seller \((i = 1)\):

- When \( z < \alpha \), the market is said to be *pessimistic*, and the bid is \( w(z, 1) = V_L \). The low-type seller accepts with positive probability, and the high type rejects with probability one. If trade occurs, the market belief jumps immediately to \( z = -\infty \), but if trade does not occur it jumps to \( \alpha \).

- When \( z > \beta \), the market is said to be *optimistic*, and trade is immediate: \( w(z, 1) = B(z) \) and both types of seller accept with probability one.

- When beliefs are intermediate, \( z \in (\alpha, \beta) \), the asset is not traded. The bid is unattractive to either type of seller, and both sides of the market wait for more information to be revealed.

For states in which the owner is a holder \((i = 0)\):

- There is no trade. Beliefs evolve based on the realization of news.

Intuition for the trading dynamics is as follows. When beliefs are favorable (for \( z \) high), a high-type seller has little to gain and a high cost of delay, \( r(B(z) - K_H) \). Therefore, a high-type seller is willing to trade at \( B(z) \) and thus buyers are willing to pay \( B(z) \); trade occurs immediately at buyers’ value. As beliefs become less favorable, the market shuts down and waits for more news. In this region, a high-type seller will not accept \( B(z) \) because the combination of her flow payoff and the option value of trading in the future is more attractive. A low type would be happy to accept \( B(z) \). However, because the high type is not willing to sell at this price, buyers are not willing to trade at \( B(z) \). Buyers would be willing to pay \( V_L \) for the asset, however, the combination of the low type’s flow payoff and the option to trade in the future is more attractive than such a bid. In this region (i.e., \( z \in (\alpha, \beta) \)), any

\(^{13}\)In any equilibrium, after reaching a degenerate belief at time \( t \), \( Z_t \in \{\pm \infty\} \), Lemma \[B.17\] shows that continuation play must be as follows. For all \( h \geq t \), \( Z_h = Z_t \), \( W_h = E[V_\theta | Z_h] \) if \( I_h = 1 \), and sellers accept with probability one if \( W_h \geq K_\theta \) and reject otherwise.
bid that would be accepted would earn the buyer a negative expected payoff and thus trade must not occur in equilibrium.

As the belief decreases, so too does a low-type seller’s option value from waiting. The belief where she is just indifferent between accepting $V_L$ and delaying trade is $\alpha$. For $z < \alpha$ the low-type seller mixes between accepting and rejecting $V_L$ in a way such that, conditional on not observing trade, the market belief jumps instantaneously to $\alpha$, which serves as a lower reflecting barrier for the belief process while $i = 1$. In economic terms, not selling when the owner is constrained, but the market is pessimistic, is an imperfect signal of high asset value.

The following definition formalizes the description provided above.

**Definition 3.1.** For any pair $(\alpha, \beta) \in \mathbb{R}^2, \alpha < \beta$, and measurable $B : \mathbb{R} \to [V_L, V_H]$, define $m_t = \sup \{ s \leq t : I_s = 1 \}$, $Q^\alpha_t = \max \{ \alpha - \inf_{s \leq m_t} \tilde{Z}_s, 0 \}$, $Q^\alpha_{0-} = 0$, and $\Xi(\alpha, \beta, B)$ to be the belief process and strategy profile such that for all $t, h \geq 0$:

$$Z_t = \begin{cases} -\infty & \text{if there exists } s < t \text{ when the asset sold and } Z_s \leq \alpha \\ \hat{Z}_t + Q^\alpha_t & \text{otherwise} \end{cases}$$

$$S^H_{t,h} = \begin{cases} 1 & \text{if there exists } s \in [t, h] \text{ such that } Z_s \geq \beta \\ 0 & \text{otherwise} \end{cases}$$

$$S^L_{t,h} = \begin{cases} 1 & \text{if there exists } s \in [t, h] \text{ such that } Z_s \geq \beta \\ 1 - e^{-(Q^\alpha_{h-} - Q^\alpha_t)} & \text{otherwise} \end{cases}$$

$$W_t = \begin{cases} V_L & \text{if } Z_t \leq \alpha \\ K_L + E^L_t[e^{-rT(\beta,t)}](B(\beta) - K_L) & \text{if } Z_t \in (\alpha, \beta) \\ B(Z_t) & \text{if } Z_t \geq \beta \end{cases}$$

where $E^L_t$ is the expectation with respect to the probability law of the process $Z$ conditional on $\mathcal{F}_t$ and $\theta = L$, and $T(\beta, t) \equiv \inf \{ s \geq t : Z_s \geq \beta \}$.

Our informal description of the equilibrium did not specify the bid in states where trade does not occur because it is not uniquely pinned down. In such states, we have specified $W_t$ to be the highest bid consistent with the candidate being an equilibrium. 

**Theorem 3.2.** There exists an $(\alpha^*, \beta^*, B^*)$, such that $\Xi(\alpha^*, \beta^*, B^*)$ is an equilibrium.

The theorem is established by construction; Appendix A derives necessary conditions that any candidate $\Xi$ must satisfy. Appendix B demonstrates that these conditions are also sufficient and proves existence. Equilibrium value functions will play a key role in

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14 To see this, notice that, in $\Xi(\alpha, \beta, B)$, low types mix between accepting and rejecting at $z = \alpha$. Therefore, *Owner Optimality* will require low types to be indifferent between these strategies, including the one that always rejects at $z = \alpha$, i.e., playing according to $T(\beta, t)$. 

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the subsequent analysis. For each $\theta$, let $F_\theta$ and $G_\theta$ denote the value function of a type-$\theta$ seller and holder respectively (see Figure 7(a) for an illustration of these functions in a particular example). Because the equilibrium is stationary, each function depends only on $z$. In equilibrium, the value functions of buyers, holders, and sellers are intertwined. A buyer who purchases the asset immediately becomes a holder; hence, a buyer’s value for the asset, $B$, depends on a holder’s value. A holder eventually becomes a seller; hence, the holder’s value depends on the seller’s value. Of course, a seller’s value is endogenously determined by the price at which the asset can be sold (i.e., the buyer’s value). Thus, we now turn to characterizing buyers’ value function (and therefore prices) relative to several benchmarks.

**Remark 3.3.** An alternative interpretation of the model is that the asset is a firm with a book (or liquidation) value of assets equal to $V_L$. In addition, the firm has a growth opportunity that pays cash flow $v_H - v_L$, if it is a “high” growth opportunity, and zero otherwise. In this case, a “trade” at a prices equal to $V_L$ (i.e., below $\alpha$) corresponds to liquidating the firm rather than transferring the asset to a new buyer. The only place in which this interpretation leads to a different implication is with regard to trade volume (see Section 5.7).

### 3.1 Benchmark Cases

In this section we establish properties of two natural benchmark cases that will be useful for comparison to the equilibrium described above. An important concept for the benchmarks is that of the fundamental value of the asset.

**Definition 3.4.** For any belief $Z_t$, let $p(Z_t) \equiv \frac{e^{Z_t}}{1 + e^{Z_t}}$ denote the probability assigned to $\theta = H$ given $Z_t$, and $\bar{V}(Z_t) \equiv E[V_\theta|Z_t] = V_L + p(Z_t)(V_H - V_L)$ denote the (expected) fundamental value of a share of the asset. Similarly, let $\bar{K}(Z_t) \equiv E[K_\theta|Z_t]$.

**Benchmark 1: The Symmetric Information Model**

Consider the economy without asymmetric information. For owners to have no private information, it must be that the cash flows are public. This is most easily accomplished by assuming that cash flows and the news process are synonymous (the analysis is unchanged by stochastic cash flows, see footnote 7), and that a seller incurs an additive holding cost that is constant across type: $v_H - k_H = v_L - k_L$ (so the shocked owner does not receive any additional meaningful information). In this symmetric information model, equilibrium behavior is straightforward: (i) $Z_t = \hat{Z}_t$ because nothing can be learned from trading behavior, (ii) buyers are always willing to pay the expected fundamental value of the asset, $W_t = \bar{V}(Z_t)$.

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15 In a recent working paper, Strebulaev et al. (2012) consider an alternative, but related, interpretation of DG12 within a corporate finance context, in which “trade” above $\beta$ corresponds to issuing equity.
and (iii) if $A_t$ is hit with a liquidity shock at time $t$, she sells immediately. Therefore, the asset value to both sellers and holders is equal to $\bar{V}$. Because the asset spends zero time in the possession of constrained agents, the equilibrium is fully efficient and the price reflects fundamentals. This is true regardless of the value of $\lambda$; because there are no institutional trading frictions, liquidity shocks in isolation do not generate inefficiency or cause prices to deviate from fundamentals.

**Benchmark 2: The Model without Resale**

Restoring the information asymmetry, consider the case in which $\lambda = 0$. A holder is never shocked, never becomes a seller, and therefore retains the asset in perpetuity. Hence, a type-$\theta$ holder’s value, $G_\theta$, is simply $V_\theta$. Further, buyers do not face future liquidity concerns. Correspondingly, given any belief $Z_t$, their expected value of possessing the asset, which we have denoted by $B(Z_t)$, is simply the fundamental value, $\bar{V}(Z_t)$. This is the situation considered in [DG12] in which the main result is as follows.

**Theorem 3.5 [DG12].** Given $N = 1$, if $\lambda = 0$, there exists a unique $\Xi$-equilibrium. It has the following properties:

- $G_\theta(z) = V_\theta$ for all $z$.
- $B(z) = \bar{V}(z)$ for all $z$.

Further, this equilibrium is the unique equilibrium among all stationary equilibria satisfying a mild refinement on off-equilibrium path beliefs.

The key take-away is that without liquidity shocks, when a share trades it does so at its fundamental value (conditional on trade). Thus, liquidity concerns are necessary, though not sufficient (recall Benchmark 1), for the divergence of prices from fundamentals.

### 3.1.1 Comparison to Benchmarks

Recall two related features common to both benchmarks: 1) whenever the asset trades, the price equals the fundamental value, and 2) $B(z) = \bar{V}(z)$ for all $z$. Neither of these are true in the presence of both asymmetric information and liquidity shocks.

**Proposition 3.6.** If $\lambda > 0$, then in any $\Xi$-equilibrium, $B(z) < \bar{V}(z)$ for all $z$.

The intuition for the result is clear. Because buyers are competitive, $B(z)$ coincides with the total expected discounted stream of cash flows that the asset endows to the economy starting from state $(z, 0)$. In expectation, the asset will spend a positive amount of time
inefficiently allocated as a result of the no-trade region. Buyer’s anticipate these costs causing their value, $B$, to drop below $\bar{V}$. The following proposition provides further insight as to how the economy is affected by an increase in shock frequency.

**Proposition 3.7.** Fix all parameters except $\lambda$, let $\Xi(\alpha_0, \beta_0, \bar{V})$ denote the unique $\Xi$-equilibrium with $\lambda = 0$, and $\Xi(\alpha_1, \beta_1, B)$ be any $\Xi$-equilibrium with $\lambda > 0$. Then, $\beta_1 > \beta_0$, $B(\beta_1) \geq \bar{V}(\beta_0)$, and $\beta_1 - \alpha_1 \geq \beta_0 - \alpha_0$.

From Proposition 3.6 we know that $B < \bar{V}$ when $\lambda > 0$. This means that the asset is worth less to the buyers, which results in a stronger incentive for the high-type seller to hold out, increasing the upper bound of the no-trade region $\beta$. Surprisingly, it turns out that the price at the upper boundary is also higher when $\lambda > 0$: $B(\beta_1) \geq \bar{V}(\beta_0)$. Because the low-type seller always receives the same payoff, $V_L$, at the lower boundary, and the price is higher at the upper boundary, her indifference at $\alpha$ requires the size of the no-trade region, $\beta - \alpha$, to increase.

**4 Equilibrium Behavior in Larger Markets**

Before studying the model with a large number of shares, consider first the case of $N = 2$. Any equilibrium will have two important considerations not present in the $N = 1$ case. Both of which can be considered informational externalities: the trading behavior of one owner affects the value derived by, and hence the trading behavior of, the other owner.

1. First, consider an equilibrium and an on-path history in which both of the owners are sellers ($I^1_t = I^2_t = 1$) and that the equilibrium calls for at least one of them to sell with positive probability if and only if $\theta = L$ (e.g., akin to $z < \alpha$ in an appropriately generalized version of $\Xi$). By Zero Profit, $W_t$ must be $V_L$. Now, if one seller does accept the bid at time $t$, the asset’s type is perfectly revealed to be $L$. The other seller has no further incentive to delay trade and immediately follows suit. This can be interpreted as a fire sale: the trade of one share at a low price induces the trade of other shares at low prices over a short period of time.

2. Next, consider an equilibrium and on-path history in which the owner of share 1 is a holder ($I^1_t = 0$). If $N = 1$, there are no other traders from which the market can learn, and hence beliefs evolve only based on news. With multiple shares that is no longer true. Now, the evolution of the market belief about the asset type, and therefore the holder’s continuation value, depends on the liquidity status of the owner of share 2.

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16Because the model is posed in continuous time “immediately” becomes “instantaneously,” and both shares sell at time $t$ (see Proposition 4.1).
If $I^2_t = 0$, then the belief evolves only based on news until one of them is shocked. However, if $I^2_t = 1$, there can be information content in whether share 2 is traded. For example, if the equilibrium calls for only a low-type seller of share 2 to trade with positive probability at time $t$, then a trade of share 2 reveals $\theta = L$, but no trade increases the belief beyond what is revealed by news alone (all while $I^1_t = 0$). Because this affects the holder’s value function, it affects the buyer value for the asset, and therefore also the seller’s value function in equilibrium.

It is straightforward to see how these considerations extend to arbitrary $N > 1$. The following result formalizes the “fire-sale” aspect discussed above.

**Proposition 4.1.** Fix arbitrary $N > 1$. In any equilibrium, if, on the equilibrium path, any share $n$ trades at time $t$ where $W_t < K_H$, then (i) $W_t = V_L$, (ii) any share $m$ such that $I^m_t = 1$ trades at $t$ as well, and (iii) all times $t’ > t$ such that there exists $m$ with $I^m_{t’} = 1$ also satisfy (i) and (ii).

Having navigated the above considerations, we now describe the generalization of $\Xi$ to the case of multiple shares. For arbitrary $N$, let $\Xi^N(\alpha, \beta, B)$ be the strategy profile and belief process in which each seller follows the strategy described in $\Xi(\alpha, \beta, B)$, with the acceptance/rejection behavior of all contemporaneous low-type sellers being perfectly correlated, $Z$ is the belief process that is Bayesian consistent with this strategy profile, and $W$ is as in $\Xi(\alpha, \beta, B)$. (Note that $\Xi^1$ and $\Xi$ are synonymous.) An $\Xi^N$-equilibrium is described by a system of equations, $S^N$, that is derived in Appendix A.

**Theorem 4.2.** For arbitrary $N \in \{1, 2, \ldots, \infty\}$, an equilibrium of the form $\Xi^N(\alpha, \beta, B)$ is characterized by the system of equations $S^N$ (see (42) in Appendix A). That is, a solution to the equations is both necessary and sufficient for an equilibrium of this form.

The analytic characterization of the equilibrium becomes somewhat more involved for arbitrary finite $N > 1$ as doing so requires an additional state variable because, as discussed above, the value to a holder depends on whether other traders are shocked (see Appendix A.2). This contingency is irrelevant when the market is “well-functioning” (i.e., $z \geq \beta$), and matters most when $z \leq \alpha$, as this is when the behavior of sellers affects the evolution of the belief. Nevertheless, it is straightforward to see that the essential properties of each of the benchmark cases investigated in Section 3.1 extend to larger markets, and that

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17 When $N$ is finite, for $z \in (\alpha, \beta)$, the presence or absence of sellers affects the probability that sellers will be present the next time that $z$ reaches $\alpha$, so also affects the holder value function. However, when $N = \infty$, the probability that there will be sellers present the next time that $\alpha$ is reached is one, regardless of the number of sellers today, so the contingency is again irrelevant.
Propositions 3.6 and 3.7 hold for any \( N \). Another feature of note is that, on average, holders benefit from the presence of sellers in the market. The intuition is that the additional information generated by sellers’ behavior improves the efficiency of future allocations, which translates into higher expected holder values (since buyers make zero profit).

**Proposition 4.3.** In any \( \Xi^N \)-equilibrium, \( N > 1 \), \( E[G_\theta(z, \tilde{t})|z] \) is weakly greater if at least one seller is present than if not.

To understand the effect of market size (or transparency) relative to \( N = 1 \), it seems natural to consider the case with countably infinite shares, \( N = \infty \), which we refer to as \( N \) being large\(^{18} \). This also facilitates a more tractable analysis because the second feature discussed above is now always present: when \( N \) is large, the presence of sellers in the market is continuous. Hence, even though the evolution of the belief is different than in the single-share model, the holder value function again depends only on \( z \).

While the equilibrium in the \( N = 1 \) and \( N = \infty \) case appear qualitatively similar, the key difference is in the information available in the market and hence the evolution of market beliefs. When \( N = 1 \), there are periods of time in which information is revealed only by news, whereas when \( N \) is large, the economy is always learning from the trading behavior of sellers (in addition to the news). In this way, beliefs evolve based on more information in larger markets. It is natural to ask what implications this has for prices, trade dynamics and efficiency, to which we now turn.

5 Implications for Asset Prices, Liquidity and Trade Patterns

The equilibrium in our model generates a number of empirical implications for asset prices and trade patterns in financial markets, which we elaborate upon in this section. We derive a number of familiar measures: bid-ask spread, excess returns, return volatility, illiquidity discount, and volume. Each is state dependent, hence both time-varying and stochastic. We illustrate how each of these objects varies with the underlying market belief (\( z \)) and then discuss the implications for their co-movements. We also illustrate the effect of market size (\( N \)), news quality (\( \phi \)), and frequency of shocks (\( \lambda \)). Along the way, our findings are related to the empirical literature.

To embark on this exercise, we first establish equilibrium bids, asks, and prices. In states where trade occurs almost surely this exercise is trivial. Yet, a defining feature of the equilibrium is periods of no trade in which establishing prices is less obvious.

\(^{18}\)Unlike the \( N = 1 \) case, we do not analytically prove existence for \( N \) large. Though somewhat more involved, we believe that a proof method analogous to the \( N = 1 \) case should apply here as well. Further, we have verified numerically that a solution exists for a wide range of parameters. A similar numerical procedure could be used to locate equilibria for arbitrary \( N \).
5.1 Bid, Ask, and Price

The bid price is clearly specified by Definition 2.1. It is the maximal price that a buyer (or market maker) can propose to a seller that is consistent with the equilibrium. That is, any price higher than the bid price would result in a profitable deviation for a seller. Letting $B(z)$ denote the bid price to a seller in state $z$ of a $\Xi^N$-equilibrium, we immediately have

$$B(z) = F_L(z).$$

Although an ask price is not specified explicitly by the equilibrium definition, it is useful to consider the trading environment in which sellers post ask prices (Example 2.2). The first step is to determine what strategies for the high and low type seller are consistent with a $\Xi^N$-equilibrium under this trading mechanism. Clearly, the strategy is uniquely pinned down for $z \geq \beta$ as both type of sellers must ask for (and trade at) a price of $B(z)$.

For $z \leq \beta$, there are many ask prices that can be consistent with the equilibrium. However, in all of them, two properties must hold: (1) for $z \in (\alpha, \beta)$, the high and low type sellers must pool (i.e., post the same price), and (2) for $z \leq \alpha$, low-type sellers must mix between pooling with high types and posting a price of $V_L$ with the probability necessary to keep beliefs consistent with $\Xi^N$. Because $F_H(z)$ is the minimum bid acceptable to an $H$ seller in state $z$, it seems natural to select this as the high type’s strategy at which point the low type’s strategy is then completely pinned down by the two properties just mentioned.

Given the above strategies, and letting $A(z)$ denote the (expected) ask price observed by the market, we have that

$$A(z) = \begin{cases} 
B(z) & z \geq \beta \\
F_H(z) & z \in (\alpha, \beta) \\
q(z)V_L + (1 - q(z))F_H(z) & z \leq \alpha 
\end{cases}$$

where $q(z) = \frac{p(\alpha) - p(z)}{p(\alpha)}$.

As mentioned earlier, a notion of the price in states where trade may not occur is less obvious. Clearly, any consistent notion of the equilibrium price should fall within the bid price and the ask price, and one could argue that any price process satisfying this condition is consistent with our model. However, imposing a condition from arbitrage pricing theory requires that price equals the expected (net) cash flows from the asset discounted at the risk-adjusted rate, $r^*$. Since all agents in the economy are risk neutral, $r^* = r$.

**Definition 5.1.** For any history, the equilibrium price is the market expectation of future
Lemma 5.2. For any $\Xi^N$-equilibrium, the price, denoted by $P(z)$, is equal to the expectation of the sellers’ value function, $P(z) = E[F_\theta(z)\mid z]$.

Because buyers make zero profit, the proof of the lemma follows immediately from the fact that the owner value functions satisfy the appropriate Bellman equations. Having established bid, ask and price we can turn to the implications; before doing so we discuss briefly our choice of parameters.

5.2 Parametrization

Naturally, the selection of cash flow parameters will depend on the type of asset markets under consideration. For example, one might expect the severity of the adverse selection problem (i.e., $v_H - v_L$) to be larger for more obscure securities (i.e., smaller $N$). Thus, we will consider two different sets of parameters to represent two different types of markets.

For the entirety of this section, we fix the interest rate, $r = 5\%$.

First, we set $N = 1$ (referred to as small markets) and choose cash flow parameters consistent with OTC markets for mortgage-backed securities by setting $v_H = 10\%$ and $v_L = 5\%$ with a holding cost of 2 percentage points ($k_\theta = v_\theta - 0.02$). To motivate this, suppose that “type” corresponds to whether the security will default or prepay. Low type securities eventually default whereas high type securities only prepay. This parametrization is consistent with a 50% recovery rate on defaulted securities as estimated by Moody’s for Ba-A rated tranches of mortgage-backed CDOs (Gluck and Remeza, 2000). It is also in line with loss given default estimates for high LTV residential mortgages (Qi and Yang, 2009).

Next, we let $N = \infty$ (referred to as large markets) and select cash flow parameters consistent with shares in a firm who makes a new investment of uncertain type. The current operations of the firm generate dividends of 3%. A “good” investment ($\theta = H$) will increase dividends (in perpetuity) by 0.5% ($v_H = 3.5\%$) whereas if $\theta = L$, dividends remain constant ($v_L = 3\%$). The holding cost is set 0.5% ($k_\theta = v_\theta - 0.005$).

One should also expect that the appropriate choice of $\lambda$ and $\phi$ to depend on the type of asset and characteristics of its participants (e.g., stocks should be associated with higher $\phi$ than an obscure derivative of a pool of mortgages). However, our motivation here is to clearly demonstrate the effects of these two forces and their potential economic significance.

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19While this definition seems most natural, it may be worthwhile to point out that the results are qualitatively similar when using a price equal to the midpoint between the bid and ask.

20To complete the mapping, note that our model is isomorphic to one in which the asset type is revealed randomly (i.e., default or prepayment occurs) according to a poisson arrival with the same arrival rate for either event; this effectively just increases the discount rate. Further, it could easily be extended to one in which the arrival rates differed.
Therefore, in what follows, we illustrate the effect of changing both shock frequency ($\lambda$) and news quality ($\phi$); hence, these two parameters vary consistently across the both large and small markets. We leave a full calibration of an appropriately enriched version of the model for future work.

### 5.3 Illiquidity

Illiquidity is measured using the (percentage) bid-ask spread:

$$\text{BAS}(z) = \frac{A(z) - B(z)}{P(z)}$$

A few properties are immediate. First, the bid-ask spread is strictly positive in states where trade does not always occur. It is zero in the extremes as the information friction disappears. Finally, $\text{BAS}(z) = 0$ for all $z \geq \beta$, since trade is immediate and price impact is zero.

Figure 2 plots the bid-ask spread for the two sets of parameters discussed above and indicates that illiquidity is higher in small markets and maximized at the lower boundary of the no-trade region. Moving from 2(a) to 2(b) illustrates the comparative static effect of increasing $\lambda$, from 2(a) to 2(c) illustrates the effect of higher $\phi$. The bid-ask spread increases with $\phi$ as higher quality news increases the relative value of high-type shares, driving a larger wedge between what a high-type seller is willing to accept and what a buyer is willing to bid.

### 5.4 The Illiquidity Discount

The illiquidity discount of the asset is measured relative to the symmetric information benchmark in which, upon arrival of a liquidity shock, the asset trades immediately at a price equal to the expected fundamental value, $\bar{V}$ (see Section 3.1). Therefore, we measure the illiquidity discount, denoted by $D$, as the amount (in percentage terms) that prices deviate from
fundamentals:

\[ D(z) = \frac{\bar{V}(z) - P(z)}{V(z)} \]

Figure 3 illustrates that the discount is highly correlated with the bid-ask spread: both are lower in large markets, greatest during periods of no-trade and converge to zero as the uncertainty is resolved. However, in (c), unlike the bid-ask spread, the illiquidity discount decreases with \( \phi \). Within our model, \( D \) is also a measure of market inefficiency, see Section 6, which shows that higher \( \phi \) can also increase the illiquidity discount in some states.

5.5 Excess Returns

Again, consider the symmetric information benchmark in which the prices are equal to fundamentals, \( \bar{V} \). The cash flow to the buyer is \( v_\theta \) and hence the instantaneous return is

\[ dR_t = \frac{d\bar{V}(Z_t) + v_\theta dt}{\bar{V}(Z_t)} \]

Since \( \bar{V}(Z_t) \) is an \( \mathcal{F}_t \)-martingale,

\[ E[dR_t|\mathcal{F}_t] = \frac{0 + r\bar{V}(Z_t)dt}{\bar{V}(Z_t)} = rdt \]

Thus, a buyer’s expected return of purchasing the asset in the symmetric information model is simply the market discount rate, and the excess expected return, \( E[dR_t|\mathcal{F}_t] - rdt \), is zero.

Returning to the model with asymmetric information, and letting \( P_t = P(Z_t) \), instantaneous returns are given by

\[ dR_t = \frac{dP_t + v_\theta dt}{P_t} \] (5)

Taking the \( \mathcal{F}_t \)-expectation gives the following result:
Proposition 5.3. The expected excess return of an arbitrary seller’s share is given by

\[
\frac{1}{dt} E[dR_t | F_t] - r = \begin{cases} 
    r \frac{p(z)}{p(\alpha)} \left[ \bar{V}(z) - \bar{K}(z) \right] / E[F_0(z)|z] & z \leq \alpha \\
    r \left[ \bar{V}(z) - \bar{K}(z) \right] / E[F_0(z)|z] & z \in (\alpha, \beta) \\
    0 & z \geq \beta 
\end{cases}
\]

Excess returns are strictly positive (in some states) despite the fact that all agents are risk neutral. This does not imply that buyers earn excess returns in equilibrium. In fact, conditional on a trade occurring, the expected excess return is zero. It is the information friction that prevents traders from buying despite the appearance of positive excess returns during periods of no trade. As seen in Figure 4, excess returns are highest in the no-trade region, but remain positive for \( z < \alpha \) due to the strictly positive probability that trade will be delayed.

5.6 Volatility

The volatility of prices has both a diffusive component and a jump component (for \( z \leq \alpha \)). The diffusive component is \( \phi P_z \), which follows directly from Ito’s lemma and the volatility of the Brownian component of \( Z \). A comparison to Benchmark 1 is again useful. Recall that in this case, the price is equal to the fundamental value \( \bar{V} \) and therefore the fundamental volatility of prices is given by \( \phi \bar{V} \).

Proposition 5.4. In normally functioning markets (i.e., \( z \geq \beta \)), the equilibrium volatility of prices and returns is strictly greater than the fundamental volatility of prices and returns.

Consistent with the empirical literature, this result says that equilibrium prices are more
volatile than can be explained by information revealed about fundamentals. The mechanism is intuitive. Bad news about fundamentals means both lower expected cash flows as well as higher expected liquidation costs. Thus, bad news gets compounded leading to an additional source of risk and higher volatility in equilibrium than can be justified based solely on fundamentals.

According to the equilibrium dynamics, it can also be shown that volatility and returns can jump at the boundaries of the no-trade region (see Figure 6). This is consistent with Eraker et al. (2003), who find strong evidence for jumps in both volatility and returns. Our model further suggests that a jump in returns should coincide with a jump in volatility (e.g., at \( z = \beta \)), though not the converse (e.g., at \( z = \alpha \)). Last, the model predicts that diffusive volatility is highest in the no-trade region when excess returns and illiquidity are high, while jump volatility peaks when the market is awaiting a sell-off.

### 5.7 Trade Volume

To measure volume independent of market size, we calculate the expected number of times that a share of the asset is traded over an arbitrary length of time given an initial state. Let \( \nu_n^n \) denote the counting process, which keeps track of the number of trades that occur in \([0, t]\) for an arbitrary share \( n \), i.e.,

\[
d\nu_n^n = 1_{\{A_t^n \neq A_{t-}^n\}}, \quad \text{where } \nu_0^- = 0
\]

Clearly, the turnover of a share depends on the liquidity status of its owner, denoted by \( i \). We let \( f \) and \( g \) denote the functions mapping \((t, z)\) to expected trade volume per unit conditional on the share being owned by a seller \((i = 1)\) and holder \((i = 0)\) respectively. That is, \( f(t, z) \equiv E[\nu_t|(Z_0, I_n^n) = (z, 1)] \) and \( g(t, z) \equiv E[\nu_t|(Z_0, I_n^n) = (z, 0)] \). The following proposition characterizes trade volume.

**Proposition 5.5.** When \( N = 1 \), for any \( t > 0 \), the expected trade volume satisfies

\[
f(t, z) : \begin{cases} 
  f = \frac{p(\alpha) - p(z)}{p(\alpha)}(1 + \lambda t) + \frac{p(z)}{p(\alpha)}f(t, \alpha) & \text{for } z \leq \alpha \\
  f_t = \frac{\phi^2}{2} \left[ (2p(z) - 1)f_z + f_{zz} \right] & \text{for } z \in (\alpha, \beta) \\
  f = 1 + g & \text{for } z \geq \beta 
\end{cases}
\]  

(6)

\[
g(t, z) : \quad g_t = \lambda(f - g) + \frac{\phi^2}{2} \left[ (2p(z) - 1)g_z + g_{zz} \right] & \text{for all } z
\]

(7)

\[22\text{Numerical results suggest that the proposition also holds for } z < \beta.\]
with the boundary conditions

\[
\lim_{z \to \pm \infty} f(t, z) = 1 + \lambda t \\
\lim_{z \to \pm \infty} g(t, z) = \lambda t
\]

and the initial conditions

\[
f(0, z) = \begin{cases} 
\frac{p(\alpha) - p(z)}{p(\alpha)} & z \leq \alpha \\
0 & z \in (\alpha, \beta) \\
1 & z \geq \beta 
\end{cases}
\]

\[g(0, z) = 0\]

When \( N = \infty \), the above holds except that (7) becomes

\[
g(t, z) : \begin{cases} 
 g(t, z) = \frac{p(\alpha) - p(z)}{p(\alpha)} \lambda t + \frac{p(z)}{p(\alpha)} g(t, \alpha) & \text{for } z < \alpha \\
 g_t = \lambda (f - g) + \frac{\phi^2}{2} [(2p(z) - 1)g_z + g_{zz}] & \text{for } z \geq \alpha
\end{cases}
\]

Using this result, one can easily solve the system of PDEs numerically to calculate expected trade volume, which we illustrate in Figure 5. In the symmetric information benchmark, independent of market size, volume is constant in \( z \) and equal to the \( \pm \infty \) boundaries of the asymmetric information model (i.e., \( i + \lambda t \)). As expected, the model predicts that information asymmetries lead to lower trade volume. This is in line with Easley et al. (1996), who document the large number of infrequently traded stocks (averaging less than one trade per day) and find them more likely to be associated with information-based trades and higher bid-ask spreads. Volume is relatively high when the market is sufficiently optimistic or sufficiently pessimistic. It is for intermediate beliefs that volume drops, which seems consistent with anecdotal evidence of traders waiting for uncertainty to be resolved before entering the market.

Figure 5 also suggests that volume (on a per share measure) is higher in large markets; the observability of many traders’ behavior “speeds up” the belief process and decreases the amount of time it takes for each trader to sell.

Remark 5.6. A “trade” at price \( V_L \) corresponds to liquidation under the alternative interpretation of the model given in Remark 3.3. In this case, \( g, f \to 0 \) as \( z \to -\infty \), trade volume is strictly monotonic in \( z \), and thus positively correlated with the price. This is consistent with both the time-series and cross-sectional results documented by Cochrane (2002).
Figure 5: Trade volume for $t = 1$

Figure 6 illustrates the relationship between bid-ask spread, excess returns, volatility and volume; this implies a rich set of testable implications. As mentioned earlier, all of these measures are time-varying and stochastic. Explanations for stochastic volatility and time-varying returns have appeared in the literature in a variety of other settings. Unlike this literature, we also derive predictions for volume and liquidity as well as their co-movements with returns and volatility. For example, our model also predicts that excess returns (due to illiquidity) move inversely to trade volume and liquidity. This is consistent with studies by Amihud and Mendelson (1986); Brennan and Subrahmanyam (1996); Amihud (2002). While the literature has debated the relationship between returns and volatility, consistent with (e.g., French et al. (1987)), our model predicts a positive relationship.

To obtain cross-sectional predictions, one can interpret higher prices as corresponding to higher market-to-book ratios (i.e., growth stocks), whereas lower prices correspond to value stocks. That is, the market is optimistic (high $z$) about growth stocks (high price-to-book) relative to value stocks (low price-to-book). Notice that prices are increasing in $z$ and excess returns are lowest for $z \geq \beta$ and strictly positive for $z \leq \beta$. Thus, our model provides a micro-founded mechanism to help explain the value premium. This mechanism, i.e., that value stocks are less liquid and therefore command a premium, is consistent with Lee and Swaminathan (2000), who identify an inverse link between turnover and future returns.

Our model provides a number of additional predictions: for example, volume should be high when the market is sufficiently optimistic and price levels are relatively high, or when the market is sufficiently pessimistic and prices are relatively low; volume drops, liquidity dries up, and expected returns are highest during periods of greatest uncertainty. Developing an additional set of conditions and assumptions, we predict that:

\[ \lambda = 0.25, \phi = 0.5 \]
\[ \lambda = 2, \phi = 0.5 \]
\[ \lambda = 0.25, \phi = 1 \]

Examples include: regime switching models with a representative agent such as David (1997), Veronesi (1999), David and Veronesi (2008, 2009); Banerjee and Green (2012), where uninformed investors update their beliefs about the whether other investors in the market are informed; Collin-Dufresne and Fos (2012), where noise trader volatility is stochastic and persistent.

Chordia et al. (2001) find evidence for both a positive relationship and time variation in liquidity and trading activity.
accurate proxy for market optimism (or pessimism) in the cross-section remains a potential challenge to testing some of these predictions, though prices or market-to-book ratios should serve well in the time-series.

6 Welfare and Efficiency

In this section we investigate the equilibrium welfare and efficiency properties of the economy and how they vary with key parameters. Except when otherwise noted, figures use the first parametrization discussed in Section 5.2 with \( \phi = 0.5 \) and \( \lambda = 0.25 \).

Welfare

We will explore welfare via the value functions \( F_L, F_H, G_L, G_H \). By construction, the (expected) payoff to every agent in the economy, except for initial owners, is zero; any agent not initially endowed with a share either never obtains it or pays a price for it that exactly equals the expected value it generates for her. Hence, the welfare properties we describe can be thought of either strictly as the welfare of \( A_0^\alpha \), depending on the prior and her initial liquidity status, or, more loosely, as the welfare of any owner once in possession of the asset (her gross welfare).

Figure 7(a) illustrates value functions relative to \( \bar{V} \). Notice that \( B < \bar{V} \) (Proposition 3.6), and that both high-type owner value functions lie (weakly) above \( B \), while both low-type value functions lie (weakly) below \( B \). Not surprisingly, the value to the owner of a high-value asset is strictly higher before being hit by a liquidity shock. However, the same is not true for the low-type asset. When beliefs are favorable, a low-type holder would prefer to become a seller because the low-type seller is able to pool with the high-type seller, trading at a price...
well above $V_L$.

**Efficiency**

To explore the efficiency of the economy we will compare the equilibrium value derived from a share of the asset to the amount derived if the share is always efficiently allocated (i.e., $\bar{V}$). Because all buyers earn zero profit, the discounted expected value of a share is $E[F_\theta(z)|z]$ or $E[G_\theta(z)|z]$, depending on whether the owner is a seller or a holder. We look at the percentage efficiency loss per share by defining the following:

$$L^F \equiv \frac{\bar{V}(z) - E[F_\theta(z)|z]}{\bar{V}(z)} \quad L^G \equiv \frac{\bar{V}(z) - E[G_\theta(z)|z]}{\bar{V}(z)}$$

Notice that $L^F$ is identical to the illiquidity discount (Section 5), i.e., the divergence of prices relative to the symmetric information benchmark. The equivalence follows from: i) within each model, the price is the discounted expected cash flow, and ii) in the symmetric information case, there are no trading frictions, so the asset is always efficiently allocated (Section 3.1). Figure 7(b) shows both measures of efficiency loss are positive for all $z$ (this follows from Proposition 3.6), single-peaked with maximal inefficiency occurring at some $z \in (\alpha, \beta)$, and tend to zero as $z \to \pm \infty$.

### 6.1 Comparative Statics

**Market Size:** As discussed in Section 4 one key difference between large ($N = \infty$) and small ($N = 1$) markets pertains the information available to uninformed market participants. In large markets, there are always sellers present providing a channel, in addition to the news process, through which learning can take place. In contrast, in small markets, there are periods during which no sellers are present, and the market relies solely on news.
Intuitively, this “extra” information in the market should reduce the information asymmetry and render larger markets more efficient. Indeed, both Proposition 4.3 and our numerical results (see Figure 8(b)) are consistent with this hypothesis. However, this intuition ignores the endogenous response of agents (i.e., the decision of when to sell). Indeed, extra information in the form of better news quality, can decrease market efficiency (see directly below) due to the distortion in sellers’ incentive to wait for more news (and a higher price).

The different effects of the two types of information on efficiency can be explained by the different circumstances in which the information is revealed; in the case of higher \( \phi \), additional information is revealed in all states and therefore has first-order consequences on the expected benefit to the seller of waiting for a higher price; in the case of higher \( N \), additional information is revealed for \( z \leq \alpha \), which has a non-trivial level effect, but an insignificant effect on a seller’s marginal consideration at \( z = \beta \). As indicated in Figure 8(a), the majority of the extra surplus generated in large markets is captured by holders, who no longer endure periods in which beliefs drift below \( \alpha \).

\textbf{Remark 6.1.} The remaining comparative static effects that we present are qualitatively similar regardless of the market size. Therefore, in what follows, we fix \( N = 1 \).

\textbf{News Quality:} One might think that, because the market is fully efficient under symmetric information, increasing news quality will bring the market closer to the benchmark and improve efficiency. As alluded to directly above and illustrated by Figure 9(c), this is not necessarily the case.\(^{25}\) As \( \phi \) increases, the inefficiency decreases for lower states but increases in higher states. Increasing \( \phi \) “speeds things up,” which generates two offsetting effects on efficiency. First, beliefs move more quickly through the no-trade region reducing the amount of time that shares are inefficiently allocated. Second, high-type sellers expect

\(^{25}\) See DG12 for a formal proof of this result when \( N = 1 \) and \( \lambda = 0 \).

\(^{26}\) For ease of exposition, Figures 9(c), 10(c), and 11(c) only depict \( \mathcal{L}^G \). The results for \( \mathcal{L}^F \) are similar.
good news to be revealed more quickly and therefore have more incentive to wait; both $\beta$ and the size of the no-trade region increase. Whether increasing new quality leads to more or less efficient markets depends on both the initial state and the initial quality.

Not surprisingly, improved news quality benefits the high-type owner (both holder and seller) and hurts the low-type owner. As $\phi \to \infty$, $\beta \to \infty$, as the high-type seller waits to be almost perfectly identified before trading, but the expected time to trade after being hit by a shock goes to zero. Therefore, $G_{\theta}, F_{\theta}$ converge to $V_{\theta}$, and $B$ converges to $\bar{V}$.\(^{27}\)

**Shock Frequency:** Figure 10(a,b) illustrates how the value functions depend on the arrival rate of the shocks, $\lambda$. When $\lambda \approx 0$, as in panel (a), the economy approximates the $\lambda = 0$ benchmark of Section 3.1. A holder expects to retain the asset for a long duration, so $G_{\theta} \approx V_{\theta}$. In turn, $B(z) = E[G_{\theta}(z) | z] \approx \bar{V}(z)$, meaning the bid approximates fundamental value when $z \geq \beta$.

Endogenous liquidation costs arise due to the no-trade region as exiting a position is often associated with holding the asset while being constrained. When $\lambda$ increases, as in panel (b), traders face the prospect of costly liquidation more frequently (in expectation). Buyers correctly anticipate these higher future liquidation costs, lowering $B$. Both forces negatively impact the high-type holder, lowering $G_{H}$. However, as we saw above, a low-type holder is anxious for the shock to arrive when it facilitates pooling with the high type; for high values of $z$, $G_{L}$ increases with $\lambda$.

As depicted in Figure 10(c), inefficiency increases with shock frequency, providing a clean manifestation of the underlying economics. Recall that neither the efficient value, $\bar{V}$, nor the ability of a symmetrically-informed economy to achieve this value, is affected by $\lambda$. Hence, the result follows only because the equilibrium behavior of the agents in the economy intro-

\(^{27}\)The convergence is uniform for $G_{H}, F_{H}, B$, but only pointwise for $G_{L}, F_{L}$.
Figure 10: Value functions and market efficiency as they depend on the frequency of liquidity shock arrivals.

Borrowing Costs: In each of the figures thus far $k_\theta = v_\theta - \delta$, where $\delta = 0.02$. That is, the shock induces an additive holding cost of two percent. Let us maintain the additivity assumption, and consider the effect of varying $\delta$.

The following trade-off arises. An increased holding cost means sellers receive lower net cash flows whilst in possession of the asset, hurting their welfare. However, this implies that the high-type seller is more willing to sell, decreasing $\beta$, which in turn makes the equilibrium more efficient for high levels of $z$ (Figure 11(c)). In Figure 11(a,b), $G_H$ and $F_H$ are (substantially) lower under the higher holding cost ($\delta = 3\%$) for low $z$, but (slightly) higher when $\delta = 3\%$ for high $z$. For the low type, however, the efficiency force completely trumps the change in cash flows as $F_L$ and $G_L$ are (at least weakly) higher under the higher holding cost for all $z$. Said succinctly, increased holding costs promote efficient trade and may result in a Pareto improvement, but if not, the welfare cost is borne solely by the high-type asset owners.

These results imply that government policies aimed at “easing” liquidity constraints of distressed financial institutions can have detrimental side-effects on the economy. Indeed, part of the motivation for Federal Reserve credit easing during the financial crisis was aimed at “preventing a liquidation of assets at distressed prices to avoid destabilizing affects,” (Carlson et al., 2009). However, mitigating destabilization may come at the cost of slower (efficient) reallocation.

Time: Over time the news process and trading behavior reveal evermore information about

\[ \text{No qualitative differences arise when using a proportional holding cost structure, i.e., } k_\theta = \delta v_\theta. \]
\( \theta \) to the economy. Consequently, the asset type is eventually learned (up to any arbitrary precision) and the long-run, steady-state distribution of the market belief is degenerate; either \( \pm \infty \). It follows that inefficiency disappears as \( t \to \infty \) with probability one. One might be tempted to conjecture a stronger claim: that expected inefficiency is decreasing in time. This is not true. One way to see this result is to transform the belief back into probabilities, \( p \), rather than log-likelihoods, \( z \), and plot the transformed \( \mathcal{L}^G, \mathcal{L}^F \) as a function of \( p \). This is shown in Figure 12. Notice that \( \mathcal{L}^G \) is convex for \( p > p(\beta) \). Further, starting from \( p > p(\beta) \), \( P \) is a continuous martingale and \( \mathcal{L}^G = \mathcal{L}^F \). By Jensen’s inequality, starting from a belief of \( p \) at time \( t \), the expected inefficiency at time \( t' > t \) is higher if \( t' - t \) is small enough.

An intuition for this is as follows: starting from such a \( p > p(\beta) \), we know that there is probability \( 1 - p \) that the limit distribution will consist of the degenerate belief that \( \theta = L \). That is, there is probably \( (1 - p) \) that \( \theta = L \), and if so, it will be found out eventually. However, doing so will be costly, in terms of efficiency. If indeed \( \theta = L \), then the market belief will, in expectation, smoothly decrease, going first through the region where inefficiency is much higher before it gets lower again (and eventually disappears).
7 Discussion of Assumptions and Final Remarks

The model entails a number of simplifying assumptions, which are made primarily to facilitate a tractable analysis and to keep the intuition for the main forces accessible. Below we interpret these assumptions as well as discuss the robustness of the model and equilibrium to various generalizations and/or extensions.

Liquidity Shocks: Observability of shocks is an important feature of both the model and its application. While this assumption is restrictive, we believe it has a natural interpretation. Observable liquidity shocks corresponds to a marketplace dominated by large institutional traders or banks whose liquidity needs are transparent. Investors must be able to discern traders with a credible reason for trading from speculators. For example, in the recent financial crisis, it was not difficult to identify firms with liquidity needs (e.g., Bear Stearns, Lehman Brothers, AIG).

A model with unobservable liquidity shocks corresponds to a marketplace either dominated by private firms (e.g., hedge funds) or one in which the identity of trading partners remains anonymous (e.g., dark pools), where the motivation for trading is often unclear. If shocks are unobservable, then, in favorable market conditions, the holder of a share of a low-value asset would prefer to sell before being hit by a shock, breaking the equilibrium. The moral is that an observable shock provides the owner with a credible reason to liquidate. Without this, buyers face an even more severe exposure to the lemons problem.

Finally, we only concern ourselves with shocks that increase the liquidity preference of the owner. Including “reverse” shocks—ones that turn sellers back into holders—would change little qualitatively. It would simply decrease the gains from trade, as the seller may return to being a holder if she rejects bids, and this would, of course, happen sometimes on the equilibrium path. In addition, our modeling of the buyer side of the market makes no mention of the potential that some buyers may be liquidity constrained. This is not important. Provided there are sufficiently many unconstrained buyers, the presence of constrained buyers will have no effect.

Learning: In our model, the asset type is fixed, and a new owner learns the type perfectly upon purchasing a share. These assumptions are made primarily for tractability. Allowing the asset type to switch over time would reduce high types’ incentive to delay, increase low types’ incentive to pool, and change the evolution of market beliefs, but the key forces would persist. In addition, an analysis similar to ours would also apply to a model in which the purchasing buyer obtains some noisy, binary signal of the asset quality that is subsequently, gradually revealed to the market through news. The model would become more complicated, though, if asset owners obtain a noisy signal and subsequently learn additional information.
from news because one would need to keep track of multiple sets of beliefs. In such a model, enough bad news could induce even a “high-initial-signal” owner to sell at low prices. Formal analysis of such a model is left for future work.

**Risk Neutrality:** One can interpret the assumption that agents are risk neutral either literally, or as an “as if” stand-in for risk-averse traders that have hedged their idiosyncratic risk in this asset with a portfolio of other holdings. If, instead, agents were averse to the risk imbued by this asset (e.g., because the asset represents a market portfolio), there would be two off-setting effects. On one hand, risk aversion will incent sellers to trade more quickly, shrinking the no-trade region and reducing the effect of information asymmetry. On the other hand, sufficient good news becomes more valuable as it not only increases the mean of traders’ expectations, but also reduces the variance, providing more incentive for sellers to delay trade. Which of these forces dominates, and the implications for the interaction of risk premia and liquidity, seems a promising direction for subsequent research.

**Binary Types:** Our results are derived from a setting with and binary asset types; yet, the key forces would persist in a more general environment. Regardless of the number of types, trade remains inefficient provided holding costs are not overly punitive—as higher type sellers have incentive to wait for news when beliefs are not favorable—and thus prices remain below fundamentals and selling at a low price reveals negative information about $\theta$, which can facilitate future trades.

**Final Remarks**

We have presented a model that features news arrival and liquidity shocks in a market with asymmetrically-informed traders. The interaction between news arrival and liquidity shocks generates (informational) liquidation costs within an otherwise frictionless environment. In larger or more transparent markets, the presence of other informed traders generates informational externalities, which can lead to fire sales. The framework provides a unified setting from which to draw implications for asset prices, volatility, illiquidity, trade patterns, welfare, and efficiency.
References


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A The Equilibrium Characterization System

In this section we present the system that characterizes an equilibrium of the form $\Xi^N$, for arbitrary $N \in \{1, 2, \ldots, \infty\}$, on which Theorem 4.2 relies. The equations govern the necessary optimality and interdependency properties of the equilibrium value functions of sellers, holders, and buyers. Taking the buyer value, $B$, as given, the analysis in Section A.1 follows closely the treatment in DG12. Subsequent analysis shows how to derive and characterize the holder value functions and, ultimately, $B$ using additional equilibrium arguments.

A.1 The Seller Value Function

For any $N$, fix a candidate $\Xi^N(\alpha, \beta, B)$ such that $B$ is differential for $z \geq \beta$. Due to the stationary structure of the candidate equilibrium, the state $z$ is sufficient to compute the seller’s payoff. Therefore, without loss, fix $t = 0$ and let $T(\beta) \equiv \inf\{s \geq 0 : Z_s \geq \beta\}$. Note that $T(\beta)$ is the (pure) strategy prescribed by $\Xi^N$ for the high type and in the set of strategies prescribed by $\Xi^N$ for the low type. Therefore, for each $\theta$, the equilibrium value function must be consistent with this strategy:

$$F_\theta(z) = E^\theta_\Xi \left[ \int_0^{T(\beta)} e^{-rt} k_\theta dt + e^{-rT(\beta)} B(\beta) \right] \quad (11)$$

where $E^\theta_\Xi$ is the expectation over the process $Z$ under the probability law starting at $z$ and conditional on $\theta$ ($Q^\theta_\Xi$). For $z \in (\alpha, \beta)$, the seller waits and $Z$ evolves according to news. Therefore,

$$F_\theta(z) = k_\theta dt + e^{-rdt} E^\theta_\Xi \left[ F_\theta(z + d\hat{Z}) \right] \quad (12)$$

Applying Ito’s lemma to $F_\theta$, using the law of motion of $\hat{Z}$, and taking the expectation conditional on $\theta$, (12) implies a differential equation that $F_\theta$ must satisfy for all $z \in (\alpha, \beta)$. Namely, for a high-type seller

$$rF_H(z) = k_H + \frac{\phi^2}{2} \left( F_H''(z) + F_H'(z) \right) \quad (13)$$

and for a low-type seller

$$rF_L(z) = k_L + \frac{\phi^2}{2} \left( F_L''(z) - F_L'(z) \right) \quad (14)$$

The equilibrium specifies that for all $z \geq \beta$, both types of seller trade immediately at $w(z, \vec{i}) = B(z)$. Therefore,

$$F_H(z) = F_L(z) = B(z) \quad \forall z \geq \beta \quad (15)$$

For all states $(z, \vec{i})$ such that $z \leq \alpha$, low-type sellers mix, and the equilibrium belief jumps instantaneously to $\alpha$ conditional on no trade. Therefore,

$$F_H(z) = F_H(\alpha), \quad F_L(z) = F_L(\alpha), \quad \forall z \leq \alpha \quad (16)$$

$^29$ $B$ will also turn out to be increasing, continuous, and differentiable almost everywhere. These are not required for the present analysis, but may help provide intuition for the arguments.
There are six boundary conditions that help pin down the seller’s equilibrium value function in the interior of the no-trade region. Four of these are physical conditions that must be satisfied for the equilibrium value functions to be consistent with (11). The value matching conditions are straightforward:

\[ F_L(\beta^-) = B(\beta) \tag{17} \]
\[ F_H(\beta^-) = B(\beta) \tag{18} \]

where \( g(x^+) \) (\( g(x^-) \)) is used to denote the right (left) limit of the function \( g \) at \( x \). For the high type, the belief process reflects at \( z = \alpha \), therefore, the value function must satisfy

\[ F'_H(\alpha^+) = 0 \tag{19} \]

(see Harrison (1985, §5)). According to \( \Xi^N \), the low type mixes at the lower boundary such that \( Z \) is killed at the lower boundary at a rate of \( \kappa = 1 \), implying that \( F_L \) must satisfy the Robin boundary condition

\[ F'_L(\alpha^+) = F_L(\alpha) - V_L \tag{20} \]

(see Harrison (2013, §9)). The remaining conditions are equilibrium conditions required to ensure that both Owner Optimality and No Deals hold.

\[ F'_L(\alpha^+) = 0 \tag{21} \]
\[ F'_H(\beta^-) = B'(\beta) \tag{22} \]

The equilibrium argument for (21) is as follows. According to \( \Xi \), the low type mixes between accepting \( V_L \) at \( \alpha \) and rejecting. Therefore, she must be indifferent between these two strategies. The first strategy implies a payoff at \( \alpha \) of \( F_L(\alpha) = V_L \). Using the stopping rule \( T(\beta) \) implies that \( F'_L(\alpha) = 0 \) (since \( Z \) reflects conditional on rejection). In order to be consistent with indifference, both must hold. Note that in conjunction with (20), any two conditions of these conditions imply the third.

To see why (22) must hold, suppose that \( F'_H(\beta^-) < B'(\beta) \) and consider the following deviation: reject at \( z = \beta \) and continue to reject until \( z = \beta + \epsilon \) for some arbitrarily small \( \epsilon > 0 \). Instead of accepting \( B(\beta) \), the high type attains a convex combination of \( B(\beta + \epsilon) \) and \( F_H(\beta - \epsilon) \), which lies strictly above \( B(\beta) \), implying the deviation is profitable. On the other hand, if \( F'_H(\beta^-) > B'(\beta) \), then the high type would prefer to accept sooner (\( F_H(\beta - \epsilon) < B(\beta - \epsilon) \)) and thus buyers will have a profitable deviation, violating No Deals.\(^{31}\)

It remains to determine the buyer value function, \( B \), which, in turn, requires deriving a holder’s value for a share of the asset.

\(^{30}\)That the killing rate is \( \kappa = 1 \) follows from the definition of \( Q^\alpha \) in \( \Xi^N \).

\(^{31}\)The necessity of high-type-seller indifference at \( \beta \), and therefore (22), hinges on the specification of off-equilibrium-path beliefs imposed by \( \Xi^N \). Regardless of this specification, the weaker condition \( F'_H(\beta^-) \leq B'(\beta) \) is necessary. Equilibria in which \( F'_H(\beta^-) < B'(\beta) \) can be sustained only by imposing “threat beliefs” for off-equilibrium-path rejections (i.e., the probability assigned to a high type decreases following an unexpected rejection). A mild refinement on off-equilibrium-path beliefs, namely that beliefs cannot decrease following an unexpected rejection, makes (22) necessary.\(^{DG12}\)
A.2 The Holder Value Function for Finite $N$

A holder waits to be hit by a shock, and then becomes a seller. As discussed in Section 4, the evolution of the belief during this time depends on whether any other owners are sellers or not. Therefore, define $G_\theta(z, \max(\tilde{t}))$ to be the equilibrium payoff of a type-$\theta$ holder given belief $z$ when no other owners are shocked, $\max(\tilde{t}) = 0$, and when at least one other owner is shocked, $\max(\tilde{t}) = 1$.

We proceed by constructing $G_\theta$ based on the structure of $\Xi^N$. Consider first $G_\theta(z, 0)$. $Z$ evolves based solely on the realization of news, and the holder is waiting until either she is shocked and becomes a seller, or another owner is shocked and her value function becomes $G_\theta(\cdot, 1)$.

$$G_\theta(z, 0) = v_\theta dt + \lambda dt F_\theta(z) + (N - 1)\lambda dt G_\theta(z, 1) + (1 - N\lambda dt)e^{-\lambda dt}E^\theta \left[ G_\theta(z + d\hat{Z}_t, 0) \right]$$

(23)

As $[12]$ did for the seller, $[23]$ implies a differential equation that $G_\theta(\cdot, 0)$ must satisfy for all $z$. Namely, for a high-type and low-type holder respectively,

$$rG_H(z, 0) = v_H + \lambda(F_H(z) + N[G_H(z, 1) - G_H(z, 0)] - G_H(z, 1)) + \frac{\theta^2}{2}(G_H'(z, 0) + G_H''(z, 0))$$

(24)

$$rG_L(z, 0) = v_L + \lambda(F_L(z) + N[G_L(z, 1) - G_L(z, 0)] - G_L(z, 1)) - \frac{\theta^2}{2}(G_L'(z, 0) - G_L''(z, 0))$$

(25)

As $z \to \pm \infty$, the belief becomes degenerate, and the effect of news on equilibrium beliefs goes to zero. A holder waits for the next shock to come. Therefore,

$$\lim_{z \to \infty} G_\theta(z, 0) = \frac{r V_\theta + \lambda \lim_{z \to \infty} F_\theta(z) + (N - 1)\lambda \lim_{z \to \infty} G_\theta(z, 1)}{r + N\lambda} \quad \theta \in \{L, H\}$$

(26)

$$\lim_{z \to -\infty} G_\theta(z, 0) = \frac{r V_\theta + \lambda \lim_{z \to -\infty} F_\theta(z) + (N - 1)\lambda \lim_{z \to -\infty} G_\theta(z, 1)}{r + N\lambda} \quad \theta \in \{L, H\}$$

(27)

Remark A.1. When $N = 1$, $[24]$, $[27]$ simplify, and all $G_\theta(\cdot, 1)$ terms drop out. In this case, the entire system has no dependence on $G_\theta(\cdot, 1)$. With only one share, there is never a history in which both a holder and a seller exist simultaneously. Thus, the analysis of $G_\theta(\cdot, 1)$ is not relevant for the $N = 1$ case, and notation was simplified to $G_\theta(\cdot) = G_\theta(\cdot, 0)$.

Now consider $G_\theta(z, 1)$. $Z$ evolves based on the realization of both news and the trading behavior of the sellers. We first state the characterization and then explain.

$$G_L(z, 1) = \begin{cases} 
q_L(z|\alpha)V_L + (1 - q_L(z|\alpha))G_L(\alpha, 1) & \text{for } z < \alpha \\
\frac{1}{r} \left( v_L + \lambda(F_L(z) - G_L(z, 1)) + \frac{\theta^2}{2}(G_L'(z, 1) - G_L''(z, 1)) \right) & \text{for } z \in [\alpha, \beta] \\
G_L(z, 0) & \text{for } z \geq \beta 
\end{cases}$$

(28)

$$G_H(z, 1) = \begin{cases} 
G_H(\alpha, 1) & \text{for } z < \alpha \\
\frac{1}{r} \left( v_H + \lambda(F_H(z) - G_L(z, 1)) + \frac{\theta^2}{2}(G_H'(z, 1) + G_H''(z, 1)) \right) & \text{for } z \in [\alpha, \beta] \\
G_H(z, 0) & \text{for } z \geq \beta 
\end{cases}$$

(29)

When $z < \alpha$, if $\theta = L$, trade occurs with positive probability, which reveals that $\theta = L$ and leads to a common value of $V_L$ for all owners. Conditional on $\theta = L$, this occurs with probability $q_L(z|\alpha) \equiv \frac{p(\alpha) - p(z)}{p(\alpha)(1 - p(z))}$. If no sellers sell, then $z$ jumps to $\alpha$ yielding $G_\theta(\alpha, 1)$.

For $z > \alpha$, beliefs evolve based only on news, so the equations are analogous to those derived previously.
Note that, for $z \geq \beta$, all sellers sell immediately, meaning sellers are “present” for an arbitrarily short amount of time. Further, there is no information content gleaned from a sale. Thus, $G_\theta(z, 1) = G_\theta(z, 0)$, which implies the following two value matching conditions.

$$G_H(\beta^-, 1) = G_H(\beta, 0)$$  \hspace{1cm} (30)
$$G_L(\beta^-, 1) = G_L(\beta, 0)$$  \hspace{1cm} (31)

The behavior of $Z$ at the $\alpha$ requires that

$$G'_H(\alpha^+, 1) = 0$$  \hspace{1cm} (32)
$$G'_L(\alpha^+, 1) = G_L(\alpha, 1) - V_L$$  \hspace{1cm} (33)

Similar to (19), (32) is due to the reflecting boundary of $Z$ (for high-type owners) when there is at least one seller present. Similar to (20), (33) is the Robin condition, which must be satisfied because at $\alpha$, for a low-type holder when sellers are present, the process $Z$ is either reflected (if the sellers reject, yielding the holder $G_L(\alpha, 1)$) or killed (if the sellers accept, yielding the holder $G_L(-\infty, 0) = V_L$).

A.3 The Holder Value Function for Countably Infinite $N$

As $N \to \infty$, any smooth solution to (24)-(25) requires that $|G^\theta(z, 1) - G^\theta(z, 0)| \to 0$ for all $z$. In the limit ($N = \infty$), beliefs always account for the presence of sellers (see Section 4). Thus, we no longer need to distinguish the two cases and simply let $G^\infty_\theta$ be the type-$\theta$ holder’s value function, which is given by:

$$G^\infty_L(z) = \begin{cases} q_L(z|\alpha)V_L + (1 - q_L(z|\alpha))G^\infty_L(\alpha) & \text{for } z < \alpha \\ \frac{1}{r} \left( v_L + \lambda (F_L(z) - G^\infty_L(z)) - \frac{\sigma^2}{2} (G^\infty_L(z) - G^\infty''_L(z)) \right) & \text{for } z \geq \alpha \end{cases}$$  \hspace{1cm} (34)

$$G^\infty_H(z) = \begin{cases} G^\infty_H(\alpha) & \text{for } z < \alpha \\ \frac{1}{r} \left( v_H + \lambda (F_H(z) - G^\infty_H(z)) + \frac{\sigma^2}{2} (G^\infty_H(z) + G^\infty''_H(z)) \right) & \text{for } z \geq \alpha \end{cases}$$  \hspace{1cm} (35)

This form for $G^\infty$ comes from the fact that above $\alpha$, the belief evolves solely based on news and a holder is simply waiting to get shocked, but once $\alpha$ is reached the behavior of the sellers in the market affects the belief just as described immediately above. Finally, the boundary conditions, which all come from similar argument to those previous given, are:

$$\lim_{z \to \infty} G^\infty_\theta(z) = \frac{rV_\theta + \lambda \lim_{z \to \infty} F_\theta(z)}{r + \lambda}, \quad \text{for } \theta \in \{L, H\}$$  \hspace{1cm} (36)
$$G^\infty_L(\alpha^+) = G_L(\alpha, 1) - V_L$$  \hspace{1cm} (37)
$$G^\infty_H(\alpha^+) = 0$$  \hspace{1cm} (38)

A.4 The Buyer Value Function

Finally, we derive the buyer value function. After purchasing a share of the asset, a buyer becomes a holder and therefore a buyer’s (unconditional) value for a share is the expected holder value. For finite $N$, this will depend on whether there are sellers present after the share is purchased. Because trade occurs at $B$ only when $z \geq \beta$, this dependence has no implications for on-path equilibrium play (as $G_\theta(z, 1) = G_\theta(z, 0)$ for all such $z$). Nevertheless, this dependence is important for checking whether profitable off-path deviations
exist. Therefore, let

\[ B(z) \equiv \begin{cases} \max_{j \in \{0, 1\}} \{E[G_\theta(z, j)|z]\} & \text{for } N < \infty \\ E[G_\theta^\infty(z)|z] & \text{for } N = \infty \end{cases} \]  

(39)

which will ensure that, when checking whether a buyer has a profitable deviation, it is unnecessary to
distinguish whether sellers are present following such a deviation (i.e., demonstrating No Deals using \( B \)
as defined by (39) is sufficient). From (39) we see that \( B \) is differentiable above \( \beta \) (since all \( G \)s are), as we
assumed at the outset. In addition, for any finite \( N \), we have that

\[ \lim_{z \to \infty} B(z) = \frac{rV_H + \lambda \lim_{z \to \infty} F_H(z) + (N - 1)\lambda \lim_{z \to \infty} G_H(z, 1)}{r + N\lambda} = V_H \]  

(40)

where the first equality is implied by (26) and (29), and the second by (15) and \( B \) bounded. Similarly

\[ \lim_{z \to -\infty} B(z) = \frac{rV_L + \lambda \lim_{z \to -\infty} F_L(z) + (N - 1)\lambda \lim_{z \to -\infty} G_L(z, 1)}{r + N\lambda} = V_L \]  

(41)

Analogous arguments establish these same boundary conditions when \( N = \infty \).

A.5 Summary and Restatement of Characterization Results

Collecting the relevant equations for each case, define the system of equations \( S^N \) as follows.

\[ S^N \equiv \begin{cases} (13)-(22), \quad (24)-(27), \quad (39) & \text{for } N = 1 \\ (13)-(22), \quad (24)-(33), \quad (39) & \text{for } 1 < N < \infty \\ (13)-(22), \quad (34)-(38), \quad (39) & \text{for } N = \infty \end{cases} \]  

(42)

For convenience, we restate the theorems from Sections 3 and 4.

Restatement of Theorem 4.2. For arbitrary \( N \in \{1, 2, \ldots, \infty\} \), an equilibrium of the form \( \Xi^N(\alpha, \beta, B) \) is
characterized by the system of equations \( S^N \) (see (42)). That is, a solution to the equations is both necessary
and sufficient for an equilibrium of this form.

Restatement of Theorem 3.2. There exists an \( (\alpha^*, \beta^*, B^*) \), such that \( \Xi^1(\alpha^*, \beta^*, B^*) \) is an equilibrium.
(Given Theorem 4.2, this is equivalent to: a solution to \( S^1 \) exits.)

B Proofs for Sections 3 and 4

Our first objective is to prove the two theorems in Sections 3 and 4: Theorem 3.2 in §B.2, Theorem 4.2
(which requires the proof of Proposition 4.3) in §B.3. To do this we start with some preliminary results
about the analytics of \( S^N \) in §B.1. The proofs of all other results from Sections 3 and 4 are found in §B.4. It
will sometimes be useful to invoke the following shorthand: \( \bar{F}(z) \equiv E[F_\theta(z)|z] \) and \( \bar{G}(z, \cdot) \equiv E[G_\theta(z, \cdot)|z] \).

We also let \( \zeta \equiv (\alpha, \beta, F_L, F_H, G_L, G_H, B) \), denote an arbitrary candidate solution to \( S^N \).

B.1 Preliminary Analysis of \( S^N \)

For any \( N \), the following hold.
The Seller’s Value Function

The solutions to the differential equations in (13) and (14) are of the form\(^{32}\)

\[ F_\theta(z) = C_1^\theta e^{\alpha z} + C_2^\theta e^{\beta z} + K_\theta \]  

(43)

where \((C_1^L, C_2^L, C_1^H, C_2^H)\) are unknown constants, \((q_1^L, q_2^L) = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{8r}{\sigma^2}} \right)\), and \((q_1^H, q_2^H) = \frac{1}{2} \left( -1 \pm \sqrt{1 + \frac{8r}{\sigma^2}} \right)\).

The necessary boundary conditions on the seller’s value function (17)-(22) can be written as:

\[ C_1^L e^{\alpha z} + C_2^L e^{\beta z} + K_L = V_L \]  

(44)

\[ q_1^L C_1^L e^{\alpha z} + q_2^L C_2^L e^{\beta z} = 0 \]  

(45)

\[ C_1^H e^{\alpha z} + C_2^H e^{\beta z} + K_L = B(\beta) \]  

(46)

\[ C_1^H e^{\alpha z} + C_2^H e^{\beta z} + K_L = 0 \]  

(48)

\[ q_1^H C_1^H e^{\alpha z} + q_2^H C_2^H e^{\beta z} = B'(\beta) \]  

(49)

The Buyers’ Value Function

**Lemma B.1.** Suppose that \(F_L, F_H, G_L, G_H\) satisfy the conditions imposed by \(S^N\). Then, for all \(z \geq \alpha, B\), as defined by (46), solves the differential equation:

\[ B''(z) + (2p(z) - 1)B'(z) - \frac{2(r + \lambda)}{\phi^2} B(z) = \frac{-2}{\phi^2} \left( p(z) (\lambda F_H(z) + v_H) + (1 - p(z)) (\lambda F_L(z) + v_L) \right) \]  

(50)

**Proof.** Let \(\eta_2 = \frac{2(r + \lambda)}{\phi^2}\). We abuse notation slightly by omitting the function arguments and using \(G_\theta\) to refer to \(G_\theta(z, 1_{\{N > 1\}})\) for \(N\) finite and \(G_\theta^\infty(z)\) for \(N = \infty\). Using this convention, we have from (39), that \(B = pG_H + (1 - p)G_L\), which is twice differentiable for \(z > \alpha\) (follows from (28), (29) for \(N < \infty\), or (34), (35) for \(N = \infty\)).

\[ B' = pG_H' + (1 - p)G_L' + p'(G_H - G_L) \]

\[ B'' = pG_H'' + (1 - p)G_L'' + 2p'(G_H' - G_L') + p''(G_H - G_L) \]

And therefore

\[ B'' + B' - \eta_2 B = p (G_H'' + G_L'' - \eta_2 G_H) + (1 - p) (G_L'' + G_H' - \eta_2 G_L) \]

\[ + (p'' + p')(G_H - G_L) + 2p'(G_H' - G_L') \]

\[ = p (G_H'' + G_L'' - \eta_2 G_H) + (1 - p) (G_L'' + G_H' - \eta_2 G_L) \]

\[ + (p'' + p')(G_H - G_L) + 2p'(G_H' - G_L') + 2(1 - p)G_L' \]  

(51)

\(^{32}\)For a reference, see Polyavin and Zaitsev (2003), which can be used to substantiate any of our claims pertaining to closed-form solutions of differential equations.
Using the functional form of \( p \), the last line of the above can be simplified:

\[
(p'' + p')(G_H - G_L) + 2p'(G'_H - G'_L) + 2(1 - p)G'_L
= \frac{2}{1 + e^{\frac{r}{\phi}}} (p'(G_H - G_L) + pG_H + (1 - p)G_L)
= \frac{2}{1 + e^{\frac{r}{\phi}}} B' = 2(1 - p)B'
\]

Substituting the above into (51) and rearranging gives

\[
B'' + (2p - 1)B' - \eta_2 B = p (G''_H + G'_H - \eta_2 G_H) + (1 - p) (G''_L + G'_L - \eta_2 G_L)
- \frac{2}{\phi^2} \left( p(\lambda F_H + v_H) + (1 - p)(\lambda F_L + v_L) \right)
\]

where the second equality follows by substituting in from either [24] and [25] (and using the fact that \( G_\theta(z, 1) = G_\theta(z, 0) \) for \( z \geq \beta \), or [31]-[35], which completes the proof.

**Lemma B.2.** If \( \hat{B} \) solves (50), then it has the form

\[
\hat{B} = \begin{cases} 
\hat{F}(z) + \frac{r(V(z) - K(z))}{r + \lambda} + C_{B1} e^{q_1^B z} + C_{B2} e^{q_2^B z} & \text{for } z \in (\alpha, \beta) \\
\hat{V}(z) + C_{B1} e^{q_1^B z} + C_{B2} e^{q_2^B z} & \text{for } z > \beta
\end{cases}
\]

where \((q^B_1, q^B_2) = \frac{1}{2} \left( 1 \pm \sqrt{1 + \frac{8\pi}{\epsilon^2}} \right)\) and \(C_{ij}^B\) are arbitrary constants.

**Proof.** We use \( B_3 : [\beta, \infty) \rightarrow [V_L, V_H] \) to denote any solution restricted to the domain \( z \geq \beta \) (a notation whose usefulness will become apparent in the proof of Theorem 3.2). For \( z \geq \beta \), since \( \hat{F}(z) = B(z) \), (50) becomes

\[
B'' + (2p(z) - 1)B' - \frac{r}{\phi^2} B(z) = -\frac{2}{\phi^2} \hat{V}(z)
\]

which has homogeneous solution of the form \( B_{h,3}(z) = C_{B1} e^{q_1^B z} + C_{B2} e^{q_2^B z} \) and a particular solution \( B_{3, p}(z) = \hat{V}(z) \) where \((C_{B1}^B, C_{B2}^B)\) are arbitrary constants. Similarly, let \( B_2 : [\alpha, \beta] \rightarrow [V_L, V_H] \) to denote any solution restricted to the domain \( z \in [\alpha, \beta] \). For \( z \in (\alpha, \beta) \), the homogenous solution to (50) is of the form

\( B_{h,2}(z) = C_{B1} e^{q_1^B z} + C_{B2} e^{q_2^B z} \), where \( C_{B1}^B, C_{B2}^B \) are arbitrary constants. \( F_\theta \) takes the form in (43), which gives a particular solution of \( B_{p,2}(z) = \hat{F}(z) + \frac{r(V(z) - K(z))}{r + \lambda} \).

**Corollary B.3.** Since \( B \) is twice differentiable above \( \alpha \) (Lemma B.1), any \( B_2, B_3 \) that are part of a solution to \( S^N \) must satisfy:

\[
B_2(\beta) = B_3(\beta)
\]

\[
B_2'(\beta) = B_3'(\beta)
\]

**B.2 Proof of Theorem 3.2**

To prove Theorem 3.2, we first establish the existence of a solution to \( S^1 \) in Lemma B.4. Together with Theorem 4.2 this is sufficient for the result. Recall that when \( N = 1 \), \( G_\theta(\cdot, 1) \) does not appear in \( S^1 \), so we simplify notation by simply writing \( G_\theta(\cdot) \) for \( G_\theta(\cdot, 0) \).
Lemma B.4. There exists candidate ζ that solves $S^1$.

The proof of Lemma B.4 involves several steps, which we detail below. By way of overview, in §B.2.1 we first complete the analytic characterization of $S^1$, which we began for general $N$ in §B.1. Then, in §B.2.2 we reduce the problem of finding solutions to $S^1$ to solving a system of two analytic, non-linear equations. We then demonstrate that a solution to the reduced system, and therefore $S^1$, exists in §B.2.3.

B.2.1 The Analytics of $S^1$

In addition to the expressions from §B.1, when $N = 1$, Lemma B.1 extends (via the same proof) to cover all $z \in \mathbb{R}$, not just $z \geq \alpha$. For $z < \alpha$, the homogenous solution to (50) is of the form $B_{h,1}(z) = C_{11}^B \frac{1}{1+p} e^{p_1^B z} + C_{12}^B \frac{1}{1+e^z} e^{q_2^B z}$, where $(q_1^B, q_2^B) = \frac{1}{2} \left( 1 \pm \sqrt{1+8r^2/\alpha^2} \right)$ and $C_{11}^B, C_{12}^B$ are arbitrary constants. For all $z \leq \alpha$, $F_0(z) = F_0(\alpha)$ and thus $F(z) = p(z) F_H(\alpha) + (1-p(z)) F_L(\alpha)$. This leads to the particular solution $B_{p,1}(z) = \frac{rV(z) + \lambda F(z)}{r + \lambda}$. To summarize, for any $z \notin \{\alpha, \beta\}$, $B(z) = B_1(z) I_{z < \alpha} + B_2(z) I_{z \in (\alpha, \beta)} + B_3(z) I_{z > \beta}$, where

$$B_1(z) \equiv \frac{rV(z) + \lambda F(z)}{r + \lambda} + C_{11}^B \frac{e^{p_1^B z}}{1+e^z} + C_{12}^B \frac{e^{q_2^B z}}{1+e^z}$$

$$B_2(z) \equiv \frac{rV(z) - \overline{K}(z)}{r + \lambda} + C_{21}^B \frac{e^{p_1^B z}}{1+e^z} + C_{22}^B \frac{e^{q_2^B z}}{1+e^z}$$

$$B_3(z) \equiv \frac{\overline{V}(z) + C_{31}^B e^{q_2^B z}}{1+e^z} + C_{32}^B \frac{e^{q_2^B z}}{1+e^z}$$

and $(C_{11}^B, C_{12}^B, C_{21}^B, C_{22}^B, C_{31}^B, C_{32}^B)$ are arbitrary constants to be pinned down by boundary conditions.

When $N = 1$, (24) and (25) imply that $G_H$ and $G_L$ must be differentiable everywhere (not just above $\beta$). Therefore, $B$ must also inherit this property, which leads to the following additional boundary conditions:

$$B_1(\alpha) = B_2(\alpha) \quad (56)$$

$$B_1'(\alpha) = B_2'(\alpha) \quad (57)$$

B.2.2 Reducing the System

For any fixed $(\alpha, \beta) \in \mathbb{R}^2, \alpha < \beta$, (44)-(49) with (53)-(57) imply a system of ten linear equations in the ten variables $(C_{11}^B, C_{12}^B, C_{11}^H, C_{12}^H, C_{11}^B, C_{12}^B, C_{21}^B, C_{22}^B, C_{31}^B, C_{32}^B)$. The system of equations is linearly independent and therefore has a unique solution parameterized by $(\alpha, \beta)$.

To pin down $(\alpha, \beta)$, two remaining boundary conditions must be satisfied. Letting $C_{12}^B(\alpha, \beta)$ and $C_{31}^B(\alpha, \beta)$ be the parameterized solution values obtained from the linear subsystem, (40) and (41) are satisfied if and only if

$$C_{12}^B(\alpha, \beta) = 0 \quad (58)$$

$$C_{31}^B(\alpha, \beta) = 0 \quad (59)$$

Finally, to incorporate (24)-(27), starting from arbitrary time $t$ such that $I_t = 0$, let $\tau = \inf\{ s \geq t : I_s = 1 \}$. 46
Notice then, that if \( B, F_L, F_H \) satisfy \((40)\) and \((43)-(49)\), the functions \( \hat{G}_L, \hat{G}_H \) defined as

\[ \hat{G}_\theta(z) \equiv E^\theta_z \left[ \int_0^T e^{-r(s-t)}v_0ds + e^{-r(T-t)}F_\theta(Z_t) \right] \]

solve \((24)-(27)\) by construction. Therefore, in order to prove existence of a solution to the entire system, \( S^1 \), it is sufficient to show that there exists a pair \( \alpha < \beta \) such that \((58)\) and \((59)\) hold.

### B.2.3 Existence of Solution to the Reduced System

Having reduced \( S^1 \) to \((58)\) and \((59)\), we now prove the following lemma, giving the desired result, Lemma B.4 as an immediate corollary.

**Lemma B.5.** There exists \((\alpha, \beta) \in \mathbb{R}^2, \alpha < \beta\), such that \((58)\) and \((59)\) simultaneously hold.

**Corollary B.6.** There exists a solution to \( S^1 \).

The proof of Lemma B.5 relies on several additional lemmas. Throughout, we will make use of the following change of variables. For any \( \alpha < \beta \), let \( A \equiv e^\alpha \in \mathbb{R}_{++}, D \equiv e^{\beta-\alpha} \in (1, \infty) \) and \( x \equiv \sqrt{1 + \frac{8\alpha}{\phi^2}} > 1, y \equiv \sqrt{\frac{1 + \frac{8\alpha}{\phi^2}}{\phi^2}} > x \). Finally, let \( \hat{C}_{12}^B(A, D) \) and \( \hat{C}_{31}^B(A, D) \) refer to the constants as functions of \( (A, D) \) (e.g., \( \hat{C}_{12}^B(e^\alpha, e^{\beta-\alpha}) = C_{12}(\alpha, \beta) \)).

**Definition B.7.** Define the correspondence \( D_{12} : R_{++} \ni (1, \infty) \) as, for all \( A > 0 \), \( D_{12}(A) = \{ D : \hat{C}_{12}^B(A, D) = 0 \} \).

**Definition B.8.** Define the correspondence \( D_{31} : R_{++} \ni (1, \infty) \) as, for all \( A > 0 \), \( D_{31}(A) = \{ D : \hat{C}_{31}^B(A, D) = 0 \} \).

**Lemma B.9.** For all \( A > 0 \), i) \( D_{12}(A) \neq \emptyset \), ii) inf \( D_{12}(A) > 1 \), and iii) sup \( D_{12}(A) < \infty \).

**Proof.** Fix \((\alpha, \beta)\) and solve \((44)-(49), (53)-(57)\) for \( C_{12}^B \). Making the the change of variables gives

\[ \hat{C}_{12}^B(A, D) = A^{\frac{1}{2}(y-1)} (A \times T_1(D) + Q_1(D)) \]

where, for all \( D > 1 \), \( T_1(D) = \sum_i \frac{\kappa_iD^{\nu_i}}{1+\nu_iD^{\nu_i}} < 0 \), \( Q_1(D) = \sum_i v_iD^{\nu_i+m} > 0 \), and \( l_i, g_i, h_i, m > 0, \max_i\{g_i\} = x + \frac{1}{2}(y+1), \kappa = \kappa_{\arg\max_i\{g_i\}} < 0, \sum \frac{\kappa_i}{1+\nu_i} = -\left( \frac{x^2-1}{2y(1+y)} \right), \sum v_i + m = 0, \max_i\{h_i\} = \frac{2}{3}x + \frac{1}{2}y > \max_i\{g_i\}, \bar{v} = v_{\arg\max_i\{h_i\}} > 0 \). From this, we have that

\[ \lim_{D \to 1} T_1(D) = \frac{\sum_i \kappa_i}{1+\nu_i} = -\left( \frac{x^2-1}{2y(1+y)} \right) < 0, \quad \lim_{D \to \infty} T_1(D) = -\infty \]

\[ \lim_{D \to 1} Q_1(D) = \frac{\sum v_i + m}{1+\nu_i} = 0, \quad \lim_{D \to \infty} Q_1(D) = \infty \]

Fixing any \( A > 0 \), \( \lim_{D \to \infty} \hat{C}_{12}^B(A, D) = -\left( \frac{x^2-1}{2y(1+y)} \right) A^{\frac{1}{2}(y+1)} < 0 \) (since \( \max_i\{h_i\} > \max_i\{g_i\} \) and \( \bar{v} > 0 \)) and \( \lim_{D \to 1} \hat{C}_{12}^B(A, D) = -\left( \frac{x^2-1}{2y(1+y)} \right) A^{\frac{1}{2}(1+y)} < 0 \) (since \( \lim_{D \to 1} Q_1(D) = 0 \)). Properties (i)-(iii) follow: (i) from the continuity of \( \hat{C}_{12}^B \) and the intermediate value theorem; (ii) and (iii) from the fact that both limits are bounded away from zero.

**Lemma B.10.** For all \( A > 0 \), i) \( D_{31}(A) \neq \emptyset \), ii) inf \( D_{31}(A) > 1 \), and iii) sup \( D_{31}(A) < \infty \).
Proof. Fix \((\alpha, \beta)\) and solve (44)-(49), (53)-(57) for \(C^B_{31}\). Making the the change of variables gives

\[
\hat{C}^B_{31}(A, D) = A^{-\frac{1}{2}(x+1)} (A \times T_2(D) + Q_2(D))
\]

where \(T_2(D) = \sum_{i=1}^{31} \kappa_i \frac{D_{hi}}{D_{hi}(1+D_{hi}^m)}\), \(Q_2(D) = \sum_{i=1}^{31} v_i D_{hi}\), and \(l_2, g_i, h_i, m > 0\) with \(\max_i \{g_i\} = 2x + 1\), \(\bar{\kappa} = \kappa_{\arg \max_i \{g_i\}} > 0\). \(\max_i \{h_i\} = 2x\), \(\bar{v} = v_{\arg \max_i \{h_i\}} > 0\), \(\sum v_i = 0\), \(\sum \kappa_i = -\left(\frac{x+1}{x-1}\right) < 0\). From this, we immediately have that

\[
\lim_{D \to 1} T_2(D) = -\frac{1 + x}{2x} < 0, \quad \lim_{D \to \infty} T_2(D) = \infty
\]

\[
\lim_{D \to 1} Q_2(D) = 0, \quad \lim_{D \to \infty} Q_2(D) = \frac{(V_L - K_L)(x - 1)^2}{2x^2} > 0
\]

Fixing any \(A > 0\): \(\lim_{D \to \infty} \hat{C}^B_{31}(A, D) = \infty\) and \(\lim_{D \to 1} \hat{C}^B_{31}(A, D) = -\left(\frac{1 + x}{2x}\right) A^{\frac{1}{2}(1-x)} < 0\). Properties (i)-(iii) follow: (i) from the continuity of \(\hat{C}^B_{31}\) and the intermediate value theorem; (ii) and (iii) from the fact that both limits are bounded away from zero.

\[\square\]

Lemma B.11. Let \(d_{12} : \mathbb{R}_+ \to (1, \infty)\), be an arbitrary function such that, for all \(A > 0\), \(d_{12}(A) \in D_{12}(A)\). Then,

1. \(\lim_{A \to 0} d_{12}(A) = 1\), and
2. \(\lim_{A \to \infty} d_{12}(A) = \infty\)

Proof. Recall that \(\hat{C}^B_{12}(A, D) = A^{\frac{1}{2}(x-1)} (A \times T_1(D) + Q_1(D))\), and, therefore, \(A \times T_1(d_{12}(A)) + Q_1(d_{12}(A)) = 0\) for all \(A > 0\).

(1) For any \(D\), as \(\lim_{A \to 0} A \times T_1(D) + Q_1(D) = Q_1(D)\). Since \(Q_1(D) > 0\) for all \(D > 1\) and \(\lim_{D \to 1} Q_1(D) = 0\), it must be that \(\lim_{A \to 0} d_{12}(A) = 1\).

(2) Suppose \(\lim_{A \to \infty} d_{12}(A) = 0\). Then \(A \times T_1(d_{12}(A)) + Q_1(d_{12}(A)) \to -\infty\), a contradiction. Therefore, \(\lim_{A \to \infty} d_{12}(A) > 0\). Since \(T_1(D) < 0\) for all \(D\), \(\lim_{A \to \infty} A \times T_1(D) = -\infty\) and thus in order to have \(A \times T_1(d_{12}(A)) + Q_1(d_{12}(A)) = 0\), it must be that \(Q_1(d_{12}(A)) \to \infty\) which requires that \(\lim_{A \to \infty} d_{12}(A) = \infty\).

\[\square\]

Lemma B.12. Let \(d_{31} : \mathbb{R}_+ \to (1, \infty)\), be an arbitrary function such that, for all \(A > 0\), \(d_{31}(A) \in D_{31}(A)\). Then,

1. \(\liminf_{A \to 0} d_{31}(A) > 1\),
2. \(\limsup_{A \to \infty} d_{31}(A) < \infty\)

Proof. Recall that \(\hat{C}^B_{31}(A, D) = A^{-\frac{1}{2}(x+1)} (A \times T_2(D) + Q_2(D))\), and, therefore, \(A \times T_2(d_{31}(A)) + Q_2(d_{31}(A)) = 0\) for all \(A > 0\).

(1) For any \(\epsilon > 0\), let \(N_\epsilon = \{(A, D) \in \mathbb{R}_+ \times (1, \infty) : \| (A, D) - (0, 1) \| < \epsilon\}\). Let \((A_\epsilon, D_\epsilon)\) denote an arbitrary point such that \((A_\epsilon, D_\epsilon) \in N_\epsilon\). To prove the first result, it suffices to show that there exists \(\epsilon > 0\) such that \(\hat{C}^B_{31}(A, D) < 0\) for any \((A_\epsilon, D_\epsilon)\). Recall that \(\lim_{D \to 1} \hat{C}^B_{31}(A, D) = -\left(\frac{1 + x}{2x}\right) A^{\frac{1}{2}(1-x)} < 0\). Further \(Q_2(D_\epsilon)\) is arbitrarily close to \(-\left(\frac{1 + x}{2x}\right)\) implying that \(\lim_{A \to 0} A \times T_2(D_\epsilon) + Q_2(D_\epsilon) = Q_2(D_\epsilon), \)
and, therefore, that \( \lim_{A \to 0} \hat{C}_3^B(A, D_*) = \lim_{A \to \varepsilon} \frac{Q_2(D_\varepsilon)}{A^2(1 + \varepsilon)}. \) Note that \( Q_2 \) is continuously differentiable in \( D \). Taking the derivative of \( Q_2 \) evaluated at \( D = 1 \) gives \( Q_2'(1) = \frac{(V_L - 4K_H)(\alpha^2 - 1)}{4\varepsilon} < 0 \) (since \( K_H > V_L \)). Hence \( Q_2(D_\varepsilon) < 0 \). Therefore, \( \nabla \hat{C}_3^B|(A=0, D=1) = \left( \begin{array}{c} \frac{\partial \hat{C}_3^B}{\partial A} \\ \frac{\partial \hat{C}_3^B}{\partial D} \end{array} \right) \big|_{(A=0, D=1)} < 0 \). Using a Taylor expansion, \( \hat{C}_3^B(A, D) \approx \hat{C}_3^B(0, 1) + \nabla \hat{C}_3^B(0, 1) \cdot (A, D - 1) < 0 \) for any \((A, D) \in N_\varepsilon \) and \( \varepsilon \) sufficiently small.

(2) Since \( \max_i \{g_i\} = 2x + 1 > \max_i \{h_i\} = 2x \), for \((A, D)\) arbitrarily large \( |A \times T_2(D)| >> |Q_2(D)| \).

If \( T_2(D) \neq 0 \), then \( |A \times T_2(D)| \) becomes arbitrarily large with \( A \). Hence \( \lim_{A \to \infty} T_2(d_{31}(A)) = 0 \). Therefore, \( \limsup_{A \to \infty} d_{31}(A) < \infty \).

Lemma B.13. For any two values \( a < a' \) both in \( \mathbb{R}^+ \), there exist two continuous paths, \( p_{12}, p_{31} \), such that for each \( i \in \{12, 31\} \),

1. \( p_i : [0, 1] \to [a, a'] \times (1, \infty) \)
2. for all \( t \in [0, 1] \), \( \hat{C}_i^B(p_i(t)) = 0 \)
3. \( p_i(0) = a \), and \( p_i(1) = a' \), where \( p_i(t) \) denotes the \( i \)-th component of \( p_i(t) \)

Proof. Fix an \( i \in \{12, 31\} \) and any two values \( a < a' \), both in \( \mathbb{R}^+ \). Define the set \( S \equiv \{(A, D) : A \in [a, a'], D = D_i(a)\} \). Because \( \hat{C}_i^B \) is a uniformly continuous co-Lipschitz mapping, \( S \) can be partitioned into a finite collection of disjoint, closed connected components, denoted \( \hat{S} \) (Maleva, 2005)—implying that there exists an \( \varepsilon > 0 \) such that

\[
\min_{s, s' \in \hat{S}} \left( \min_{(A, D) \in S} \frac{\| (A, D) - (A', D') \|}{(A, D') \in S} \right) > 2\varepsilon
\]

Now let \( D \) be any value in \( (\inf \{D_i(a)\}) \) and \( \overline{D} \) be any value in \( (\sup \{D_i(a')\}, \infty) \). Hence, \( \hat{C}_i^B(a, D) < 0 \) and \( \hat{C}_i^B(a', D) > 0 \) (from the proofs of Lemmas B.9 and B.10). Then, if \( \hat{S} \) does not contain a component connecting \( (a, D) \) to \( (a', D') \) for some values of \( D \in D_i(a), D' \in D_i(a') \), there exists a continuous path \( q : [0, 1] \to [a, a'] \times (1, \infty) \) such that \( q(0) = (a, D), q(1) = (a', D) \), and \( \{t : q(t) \in S\} = \emptyset \). However, this violates the intermediate value theorem (since \( q \) is continuous with \( q(0) < 0, q(1) > 0 \) implying that for any two for any two values \( a < a' \), there exists a continuous path \( p_i \) satisfying (1)-(3) of the Lemma.

Proof of Lemma B.13. This follows nearly immediately from Lemmas B.11 through B.13. Using the notation from Lemma B.13 as \( a \to 0 \) and \( a' \to \infty \), from Lemmas B.11 and B.12 \( p_{12}^2(0) < p_{31}^2(0) \) and \( p_{12}^2(1) > p_{31}^2(1) \). Because the paths are continuous, they must intersect. By construction, any intersection is a solution to \( (68) \) and \( (69) \) simultaneously.

B.3 Proof of Theorem 4.2

To prove Theorem 4.2, we begin with some preliminary results in B.3.1 including a proof of Proposition 4.3 and then provide the main verification argument in B.3.2.

B.3.1 Preliminaries

Fact B.14. If \( \zeta \) is a solution to \( S^N \), then

\[
F_\theta(z) = E^{\theta}_z \left[ \int_0^{T(\beta)} e^{-rT(\beta)} k_\theta dt + e^{-rT(\beta)} B(\beta) \right] = K_\theta + E^{\theta}_z \left[ e^{-rT(\beta)} \right] (B(\beta) - K_\theta)
\]
where $T(\beta) = \inf \{ t : Z_t \geq \beta \}$, $E^\theta_z$ is the expectation over the process $Z$ under the law $Q^\theta_z$.

Proof. By construction.

Definition B.15. For $C^2$ function $f : \mathbb{R} \to \mathbb{R}$, $MB_H(f(z)) = \frac{q^2}{2} (f'(z) + f''(z)) - r(f(z) - K_H)$.  

Lemma B.16. If $\zeta$ is a solution to $SN$, then $MB_H(B_3(z)) < 0$ for all $z \geq \beta$.

Proof. We first establish three inequalities for all $z \geq \beta$: i) $B'_3(z) > \bar{V}'(z)$, ii) $B_3(z) < \bar{V}(z)$, and iii) $q^H_1(B_3(\beta) - K_H) > B'_3(\beta)$. To establish (i) and (ii), recall from (55) that

$$B_3(z) = \bar{V}(z) + C^B_{32} e^{q^0_4 z}$$

where $C^B_{32}, q_4 < 0$. Therefore,

$$B'_3(z) = \bar{V}'(z) + C^B_{32} \frac{e^{q^0_4 z}}{(e^z + 1)^2} (q^B_1(1 + e^z) - e^z) > \bar{V}'(z)$$

Thus, $B_3(z) < \bar{V}(z)$ and $B'_3(z) > \bar{V}(z)$.

To establish (iii), for a given $\beta$, we can solve boundary conditions (47) and (49), obtaining

$$C'^H_1(\beta) = \frac{B'_3(\beta) + q^H_1(K_H - B_3(\beta))}{q^H_1 - q^2_1} e^{-q^H_2 \beta}$$

$$C'^H_2(\beta) = -\frac{(B'_3(\beta) + q^H_1(K_H - B_3(\beta)))}{q^H_1 - q^2_1} e^{-q^H_2 \beta}$$

(61)

(62)

Next, using boundary condition (48), we arrive at the correspondence

$$B_3(\alpha) = \{ \beta \in \mathbb{R} : \beta \geq \alpha, \alpha = A_H(\beta) \}$$

where $A_H(\beta) = \frac{1}{q^H_1 - q^2_1} \ln \left( \frac{-q^H_1 C'^H_1(\beta)}{q^2_1 C'^H_1(\beta)} \right)$. Because $-\frac{q^H_1}{q^2_1} > 0$, any real solution requires $\text{sgn}(C'^H_1(\beta)) = \text{sgn}(C'^H_1(\beta))$. Since $F'_H(\beta) = B'_3(\beta) > 0$ and $\text{sgn}(F'_H) = \text{sgn}(C'^H_1)$, it must be that $C'^H_1(\beta), C'^H_2(\beta) > 0$. Finally, $C'^H_2(\beta) > 0$ and (52) imply that $q^H_1(B_3(\beta) - K_H) > B'_3(\beta)$.

Having established (i)-(iii), for any $C^B_{32} > 0$, because $B_3 < \bar{V}$ and $B'_3 > \bar{V}'$, if $q^H_1(B_3(z) - K_H) > B'_3(z)$ then, $q^H_1(\bar{V}(\beta) - K_H) > \bar{V}'(\beta)$. Therefore, since $\beta$ satisfies $q^H_1(B_3(z) - K_H) > B'_3(z)$, it must be that $\beta > \beta_H \equiv \inf \{ x : q^H_1(\bar{V}(z) - K_H) > \bar{V}'(z), \forall z > x \}$. Lemma B.3 of [DG12] shows that $MB_H(\bar{V}(z)) < 0$ for all $z \geq \beta_H$. Lemma B.16 then follows from the fact that $MB_H(B_3(z)) \leq MB_H(\bar{V}(z))$ for all $z$, which can be seen by differentiating $MB_H(B_3(z))$ with respect to $C^B_{32}$ to get $-e^{x(1-e)/2} / (1+e^x)^2 < 0$, where $x = \sqrt{1 + 8r/\sigma^2} > 1$.

Proof of Proposition 4.3. We will show that, in any $\Xi^N$-equilibrium, $E[G_\theta(z, 1)|z] \geq E[G_\theta(z, 0)|z]$ $\forall z, N \notin \{1, \infty\}$. Without loss we start at $t = 0$, with arbitrary initial state $(z, \bar{i})$ such that some share $n$ satisfies $I^n_0 = 0$. Let $\tau = \inf \{ t : I^n_t = 1 \}$. From the structure of $\Xi^N$, for each $\theta$,

$$G_\theta(z, \max(\bar{i})) = E^\theta \left[ \int_0^T e^{-r_t} v_\theta dt + e^{-r_T} F_\theta(Z_T) | (Z_0, \bar{i}_0) = (z, \bar{i}) \right]$$
So,
\[
E[G_\theta(z, \max(\bar{\bar{i}}))|z] = E \left[ E^\theta \left[ \int_0^\tau e^{-rt} v_0 dt + e^{-rt} F_\theta(Z_\tau)|\{(Z_0, \bar{\bar{I}}_0) = (z, \bar{\bar{i}})\} \right] \right] \\
= E \left[ \int_0^\tau e^{-rt} v_0 dt |z \right] + E \left[ e^{-rt} E^\theta \left[ F_\theta(Z_\tau)|\{(Z_0, \bar{\bar{I}}_0) = (z, \bar{\bar{i}})\} \right] \right] 
\]
where the second equality follows from the independence of the shock and news processes.

Let \( M \) be the set of shares, other than \( n \), initially owned by holders and \( \tau' = \inf\{t : I_t^m = 1, m \in M\} \). Finally, define the process \( \bar{Z} \) on \([0, \tau]\) as \( \bar{Z}_0 = z \) and
\[
d\bar{Z}_t = \begin{cases} 
\frac{d\bar{\bar{Z}}_t}{dt} & \text{for } t < \tau' \\
\frac{d\bar{Z}_t}{dt} & \text{for } t \geq \tau'
\end{cases}
\]
Now consider the two cases at \( t = 0 \): \( \max(\bar{\bar{i}}) = 1 \) (i.e., there exists a seller) and \( \max(\bar{\bar{i}}) = 0 \). In the latter, by construction, \( Z_\tau = \bar{Z}_\tau \). In the former, \( Z_\tau = \bar{Z}_\tau + Q_{\min(\tau, \tau')} \). That is, given any \( \tau, \tau' \) and path of \( X \) on \([0, \tau]\), the difference between \( Z_\tau \) in the two cases is due to updating based on the information content of the low-type sellers’ trading behavior prior to \( \min\{\tau, \tau'\} \) in the \( \max(\bar{\bar{i}}) = 1 \) case versus no such information in the \( \max(\bar{\bar{i}}) = 0 \) case. Notice that this additional information is binary in nature (either the sellers traded or they did not), with one realization (trade) perfectly revealing that \( \theta = L \) and the other (no trade) increasing the belief that \( \theta = H \). Hence, from (63) it is sufficient to show that any such signal increases the value of \( E[F_\theta(Z_\tau)|\tau, \bar{Z}_\tau] \).

To show this result, it will be convenient to transition back to beliefs as probabilities \( p = \frac{e^s}{1+e^s} \). Let \( f_\theta \) be the type-\( \theta \) seller’s value function and \( \bar{f}(p) = E[f_\theta(p)|p] \). For any \( p \in (0, 1) \), let \( R_p \) be the ray from the point \((0, V_L)\) through the point \((p, \bar{f}(p))\), and for any \( p' \leq p \), \( R_p(p') \) be the value such \( R_p \) also passes through \((p', R_p(p'))\). Using standard value-of-information arguments, it suffices to show that, for any pair \( p' \leq p \), \( \bar{f}(p') \leq R_p(p') \).

This last requirement is established via the following properties of \( \bar{f} \) (demonstrated below): i) \( \bar{f}(0) = V_L \), \( \bar{f}(1) = V_H \), \( \bar{f}(p) \leq \bar{V}(p) \) for all \( p \), and \( \bar{f} \) is continuous and increasing, ii) \( \bar{f} \) is convex below \( p(\beta) \) and concave above \( p(\beta) \), and iii) \( \bar{f}'(p) > \bar{V}'(p) \) for all \( p > p(\beta) \). Given just (i) and (ii), the result is immediate for \( p \leq p(\beta) \). For \( p > p(\beta) \), property (iii) implies that \( \bar{f} \) is steeper than \( R_p \) at all points \( p' \in [p(\beta), p) \), so \( \bar{f} \leq R_p \) in this region. Finally, they cannot intersect below \( p(\beta) \), since \( \bar{f} \) does intersect \( R_{p(\beta)} \) in this region and \( R_{p(\beta)} \) lies everywhere below \( R_p \) due to \( \bar{f} \) increasing.

i) By constructions in Section A

ii) That \( \bar{f} \) is convex below \( p(\beta) \) is immediate for \( p < p(\alpha) \) since \( \bar{f} \) is linear in this region.

For \( p \in (p(\alpha), p(\beta)) \), making the change of variables from (43) and taking the expectation over \( \theta \), we get
\[
\bar{f} = c_1(1-p) \left( \frac{p}{1-p} \right)^{q_1^L} + c_2(1-p) \left( \frac{p}{1-p} \right)^{q_2^L} + pk_H + (1-p)k_L
\]
where \( c_i = C_i^H + C_i^L > 0 \). Taking the second derivative gives
\[
\bar{f}'' = c_1 \frac{1}{p^2} q_1^L \left( q_1^L - 1 \right) \left( \frac{p}{1-p} \right)^{q_1^L} + \frac{1}{p^2} c_2 q_2^L \left( q_2^L - 1 \right) \left( \frac{p}{1-p} \right)^{q_2^L} > 0
\]
where the inequality follows from \( q_1^Z > 1, q_2^Z < 0. \)

For \( p \geq p(\beta) \), recall that \( F_L(z) = F_H(z) = B_3(z) \). Making the change of variables from \( B_3 \) using \( C_{32}^B = 0 \) (which it must be in \( S^N \)), we get that

\[
\tilde{f}(p) = pV_H + (1 - p)V_L + C_{32}^B(1 - p) \left( \frac{p}{1 - p} \right)^{q_2^Z}
\]

recalling that \( C_{32}^B < 0 \) and taking the second derivative in \( p \) gives the result.

iii) Follows by taking the first derivative of (64) and noting that \( C_{32}^B, q_2^Z < 0. \)

### B.3.2 Verification

**Proof of Theorem 4.2** The necessity of the equations in \( S^N \) is demonstrated in Appendix A. For sufficiency, we show that if \( \zeta \) solves \( S^N \), then \( \Xi^N(\alpha, \beta, B) \) satisfies the four requirements from Definition 2.1. Below, a separate proof is provided for each condition.

**Proof of Condition 2 (Belief Consistency).** To prove Belief Consistency we will need the following generalizations of objects from Definition 3.1 for arbitrary \( N \). Let \( m^n_l = \sup\{s \leq t : I^n_s = 1\} \), \( \overline{m}_t = \sup_n(m^n_l) \), \( Q^{n,\alpha}_l = \max\{\alpha - \inf_{s \leq m^n_l} \hat{Z}_s, 0\} \), \( Q^n_\alpha = \max\{\alpha - \inf_{s \leq \overline{m}_t} \hat{Z}_s, 0\} \), and finally \( Q^n_{\alpha,n} = Q^{n,\alpha}_l = 0 \) still. Then, if there does not exist \( s \in [t, h] \) such that \( Z_s \geq \beta \), \( S_{L,t}^n = 1 - e^{-\left(Q^n_{\alpha,n} - Q^n_{\alpha,n}\right)} \) (and is perfectly correlated with all other sellers present at time \( h \)), and we still have \( Z_t = \hat{Z}_t + Q^n_{\alpha,-} \).

Now, fix an arbitrary on-path history up to time \( t \), and we will show that \( Z_t = \hat{Z}_t + Q^n_{\alpha,-} \) is Bayesian consistent with the strategy profile in \( \Xi^N(\alpha, \beta, B) \). There are two cases: i) there exists \( s < t \) such that \( W_s = V_L \) and a share of the asset sold, or ii) no such \( s < t \) exists. For the first case, notice that only the low type ever accepts a bid of \( V_L \) on path, hence such an action perfectly reveals \( \theta = L \), and the belief correctly becomes degenerate for all future times. For the second case, we argue by induction. Let \( M_t = \sup\{s \leq \overline{m}_t : I^n_s = 0 \ \forall \ n\} \) if such a time exists and zero otherwise. For \( t = 0 \), \( Q^n_{\alpha,-} = 0 \), so \( Z_0 = \ln \left( \frac{1}{1 - \beta} \right) \) as it should. Next, let \( t > 0 \) and assume that \( Z_s = \hat{Z}_s + Q^n_{\alpha,-} \) for all \( s \leq M_t \). Then, for any \( s \in [M_t, \overline{m}_t] \), Bayes rule (in log-likelihood form) mandates that

\[
Z_s = Z_{M_t} + \ln \left( \frac{f^H_{M_t}(X_s - X_{M_t})}{f^L_{M_t}(X_s - X_{M_t})} \right) + \ln \left( \frac{1 - 0}{1 - (1 - e^{-\left(Q^n_{\alpha,-} - Q^n_{\alpha,-}\right)})} \right)
\]

\[
= Z_{M_t} + (\hat{Z}_s - \hat{Z}_{M_t}) + (Q^n_{\alpha,-} - Q^n_{\alpha,-})
\]

\[
= \hat{Z}_s + Q^n_{\alpha,-} + (\hat{Z}_s - \hat{Z}_{M_t}) + (Q^n_{\alpha,-} - Q^n_{\alpha,-}) = \hat{Z}_s + Q^n_{\alpha,-}
\]

If \( t = \overline{m}_t \), the argument is concluded. If \( t > \overline{m}_t \), then \( I^n_s = 0 \) for all \( n \) and \( s \in (\overline{m}_t, t] \), meaning all shares are owned by holders over this period and Bayes rule mandates that \( Z_t = Z_{\overline{m}_t} + (\hat{Z}_t - \hat{Z}_{\overline{m}_t}) \). Equivalently, \( Q_t \) must equal \( Q_{\overline{m}_t} \), which is precisely what \( Q^n \) prescribes.

**Proof of Condition 3 (Zero Profit).** In \( \Xi^N \), there are two cases in which trade occurs. First, when \( Z_t \geq \beta \), all sellers trade with probability one, so for each transacted share \( n \),

\[
E[G^n_\theta(t^+, \omega)|F_t, t \in T] = B(Z_t) = W_t
\]

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satisfying the condition. Second, when \( z \leq \alpha \), only low-type sellers trade with positive probability. Such a trade reveals \( \theta = L \), so

\[
E[G_0^\theta(t^+,\omega)|\mathcal{F}_t, t \in \mathcal{T}] = V_L = W_t
\]

satisfying the condition.

\( \square \)

**Proof of Condition 4 (No Deals).** To demonstrate No Deals, we first demonstrate three inequalities. If \( \zeta \) solves \( S^N \), then i) \( F_L \geq V_L \), ii) \( F_H \geq B \), and iii) \( F_H \geq F_L \).

(i) Note that \( F_L'(\alpha) = 0 \) and \( F_L''(z) > 0 \) for all \( z \in [\alpha, \beta) \). Hence, \( F_L'(z) > 0 \) for all \( z \in (\alpha, \beta) \). The result then follows from \( F_L(\alpha) = V_L \).

(ii) Clearly the statement is true for \( z \geq \beta \). To see that the statement holds for \( z < \beta \), fix any \( \beta \) and \( C_{12}^B < 0 \) and solve (56) for \( C_{12}^H \) and (61)-(62) for \( C_1^H, C_2^H \). Then, by direct calculation, the implied value function \( C_1^H e^{\eta t}z + C_2^H e^{\eta t}z + K_H > B_2(z) \) for all \( z < \beta \). This verifies \( F_H > B \) for \( z \in [a, b] \). All that remains is to show (ii) holds for \( z < \alpha \). We break the argument into two cases.

- For \( N = 1 \), in order to solve (56), (57), \( C_{11}^B \) must be strictly positive implying that \( B_1 \) is increasing on \( (-\infty, \alpha) \), therefore \( B_1(z) \leq B_1(\alpha) = B_2(\alpha) < C_1^H e^{\eta \alpha} \alpha + C_2^H e^{\eta \alpha} + K_H = F_H(z) \).

- For \( N > 1 \), note that \( B(z) \) is a convex combination of \( V_L \) and \( B_2(\alpha) > V_L \). Therefore, \( B(z) < B_2(\alpha) \) for \( z < \alpha \). Recalling that \( B_2(\alpha) = F_H(\alpha) = F_H(z) \) for \( z < \alpha \) implies the desired result.

(iii) This is implied by the following: 1) Fact B.14, 2) \( K_H \geq K_L \), 3) \( B(\beta) = F_H(\beta) > K_H \) (which follows from \( C_1^H, C_2^H > 0 \), see the proof of Lemma B.16) and 4) \( E_2^\varepsilon[e^{-rT(\beta)}] \geq E_1^\varepsilon[e^{-rT(\beta)}] \), because, for any \( t \) and \( z \), the distribution of \( Z_t \) under the law \( Q_z^\varepsilon \) weakly first-order stochastically dominates the analogous distribution under \( Q_z^L \).

Therefore, No Deals is satisfied because

- If \( q \geq F_H(z) \), then \( E[G_\theta(z,\cdot)|z, F_\theta(z) \leq q] = B(z) \leq q \).

- If \( q \in [F_L(z), F_H(z)] \), then \( E[G_\theta(z,\cdot)|z, F_\theta(z) \leq q] = V_L \leq q \).

- If \( q < F_L(z) \), \( \{\theta : F_\theta(z) \leq q\} = \emptyset \).

\( \square \)

**Proof of Condition 1 (Owner Optimality).** To see that that the strategies given in \( \Xi^N \) are optimal for a seller (of arbitrary share \( n \)), we need to show that starting from any \( (t, Z_t) \) such that \( P_n = 1 \), \( T(\beta) = \inf\{s \geq t : Z_s \geq \beta\} \) solves \( S_{P_{n,t}} \) and that both \( T(\beta) \) and \( T(\alpha, \beta) = \inf\{s \geq t : Z_s \notin (\alpha, \beta)\} \) solve \( S_{P_{L,t}} \). Due to stationarity, we can normalize \( t \) to zero. Start with \( \theta = H \), and define

\[
F_H^{**}(z) = \sup_{\tau \geq 0} E_2^\varepsilon \left[ \int_0^\tau e^{-rt}k_{\theta}dt + e^{-r\tau}F_H(Z_{\tau}) \right]
\]

By (ii) in the proof of Condition 4, \( F_H(z) \geq B(z) \). Therefore

\[
F_H^{**}(z) \geq F_H(z) = \sup_{\tau \geq 0} E_z^\varepsilon \left[ \int_0^\tau e^{-rt}k_{\theta}dt + e^{-r\tau}B(Z_{\tau}) \right]
\]

Define \( f_\theta(t, z) = (1 - e^{-rt}) + e^{-rt}F_\theta(z) \), which is \( C^2 \) on \( U \equiv \mathbb{R} \setminus \{\alpha, \beta\} \). By Ito’s formula

\[
f_H(t, Z_t) = f_H(0, Z_0) + \int_0^t A^H f_H(s, Z_s) I(Z_s \in U) ds + \int_0^t \phi e^{-rs} F_H'(Z_s) dB_s + \int_0^t e^{-rs} F_H'(\alpha) dQ_s^\alpha
\]

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Using that $A^H f_H(t, z) = 0$ for all $z \in (\alpha, \beta)$ (by construction), and $A^H f_H(t, z) = e^{-rt} MB_H(B_3(z)) < 0$ for all $z \geq \beta$ (by Lemma B.16), we can conclude that

$$f_H(t, Z_t) \leq f_H(0, Z_0) + M_t = F_H(z) + M_t$$

where $M$ is a martingale given by $\int_0^t \phi e^{-rs} F_H(Z_s) dB_s$. Taking the $Q^H_t$-expectation, using that $F_H$ is bounded (by construction), the optional stopping theorem gives $F_H^*(z) \leq F_H(z)$. Since the high type can attain $F_H(z)$ by following the strategy $T(\beta)$, we can conclude that $F_H^*(z) = F_H(z)$ and hence $S^{H,t}$ solves $(SP_{H,t})$ for all $t$.

For the low type, we first demonstrate that both: 1) $T(\beta)$, and 2) $T(\alpha, \beta)$ achieve an expected payoff equal to $F_L(z)$ starting from any initial $Z_0 = z$. Let $F_{L,j}(z)$ denote the expected payoff from playing according to the pure strategy $(j)$ for $j = 1, 2$ starting from $Z_0 = z$. The case for $j = 1$ is covered by Fact B.14. For $z \in (\alpha, \beta)$, $F_{L,2}$ must solve (14) and therefore is of the form (43). Clearly, $F_{L,2}$ must satisfy value-matching at both $\alpha$ and $\beta$, i.e., (44) and (46), implying the constants are uniquely pinned down and $F_{L,2}(z) = F_L(z)$ for all $z \in (\alpha, \beta)$. Verifying that $F_{L,2}(z) = F_L(z)$ for $z \notin (\alpha, \beta)$ is immediate.

That $F^*_L(z) = F_L(z)$ follows the same steps as the case for $\theta = H$ after noting that (14) implies that $A^L f_L = 0$ for all $z \in (\alpha, \beta)$ and $A^L f_L < A^H f_H < 0$ for all $z > \beta$, which completes the proof.

### B.4 Remainder of Proofs for Sections 3 and 4

**Proof of Theorem 3.3.** Let $t^0 = \inf\{t \geq 0 : I_t = 0\}$. Because $\lambda = 0$, a holder never transitions to a seller, meaning that the asset is held in perpetuity by a holder after $t^0$. Hence, i) $G_\theta(z) = \int_0^\infty e^{-rt} v_\theta dt = \frac{z}{r} = V_\theta$, ii) $B(z) = E[G_\theta(z)|z] = \hat{V}(z)$, for all $z$ (from [39]). If $A_0$ is a holder, then $t^0 = 0$ and the model is trivial. Finally, if $A_0$ is a seller, then $B = \hat{V}$ implies that model is identical to that of DG12 and the result follows from Lemma 3.1 and Theorems 3.1 and 5.1 found therein.

**Proof of Proposition 3.6.** Define $\Pi(z, i)$ to be the $F_i$-expected discounted sum of all agents’ utilities starting from state $(Z_t, I_t) = (z, i)$. Because agents’ utilities are quasi-linear in money, transfers have no affect on $\Pi$, and

$$\Pi(z, i) = E\left[\int_0^\infty e^{-rt} (I_t k_\theta + (1 - I_t) v_\theta) dt | Z_0 = z, I_0 = i\right]$$

$$= E\left[\int_0^\infty e^{-rt} v_\theta dt | Z_0 = z\right] + E\left[\int_0^\infty e^{-rt} (I_t (k_\theta - v_\theta)) dt | Z_0 = z, I_0 = i\right]$$

$$= \hat{V}(z) - (v_\theta - k_\theta) E\left[\int_0^\infty e^{-rt} I_t dt | Z_0 = z, I_0 = i\right]$$

In addition, because all buyers earn zero expected profit, all of this value goes to the current owner.

$$\Pi(z, i) = i E[G_\theta(z)|z] + (1 - i) E[G_\theta(z)|z]$$

From [39],

$$B(z) = E[G_\theta(z)|z] = \Pi(z, 0)$$

Because $v_\theta > k_\theta$, to prove that $B(z) < \hat{V}(z)$, it is sufficient to argue that

$$E\left[\int_0^\infty e^{-rt} I_t dt | Z_0 = z, I_0 = 0\right] > 0$$

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But this is nearly immediate from the structure of the equilibrium. Let \( t^1 \) be the arrival of the first shock, and \( t^2 \geq t^1 \) be the time of the first sale thereafter. Hence, if \( Z_{t^1} \in (\alpha, \beta) \), then \( \text{Prob}(t^2 > t^1) = 1 \). Finally, because the shock arrives in finite time with probability 1, and, because \( Z \) follows a diffusion while \( I = 0 \), there is positive probability that \( Z_{t^1} \in (\alpha, \beta) \), giving

\[
0 < E \left[ \int_{t^1}^{t^2} e^{-rt} dt \right] = E \left[ \int_{0}^{\infty} e^{-rt} I_t dt \right] = E \left[ \int_{0}^{\infty} e^{-rt} I_0 dt \right] = 0
\]

Proof of Proposition 3.7. Algebraic manipulation of Fact B.14 yields that, for any equilibrium \( \Xi(\alpha, \beta, B) \), \( F_{t^1}(\alpha) = K_L + E_{t^1}^L[e^{-rT(\beta)}](B(\beta) - K_L) \). Boundary conditions on the low type seller’s value function then implies \( E_{t^1}^L[e^{-rT(\beta)}] = \frac{q^T_{\beta \alpha} - q^T_{\beta \alpha}}{q^T_{\beta \alpha} - q^T_{\beta \alpha}} \) (see DG12). Therefore, \( B(\beta_1) \geq \bar{V}(\beta_0) \iff (\beta_1 - \alpha_1) \geq (\beta_0 - \alpha_0) \). In addition, Proposition 3.6 shows that \( B < \bar{V} \), meaning \( B(\beta_1) \geq \bar{V}(\beta_0) \implies \beta_1 > \beta_0 \).

For the purpose of contradiction, suppose that \( B(\beta_1) < \bar{V}(\beta_0) \), and therefore \( (\beta_1 - \alpha_1) < (\beta_0 - \alpha_0) \). Recalling the functional form of \( B_3 \) from (55), \( B(\beta_1) < \bar{V}(\beta_0) \) implies that \( \bar{V}(\beta_1) + C_{B,32}^B \bar{V}(\beta_1) < \bar{V}(\beta_0) \), where \( C_{B,32}^B < 0 \). This yields

\[
\frac{B'(\beta_1)}{B(\beta_1) - K_H} = \frac{\bar{V}'(\beta_1)}{\bar{V}(\beta_1) + C_{B,32}^B \bar{V}(\beta_1) - K_H} > \frac{\bar{V}'(\beta_0)}{\bar{V}(\beta_0) - K_H}
\]

(65)

However, using Fact B.14 as above, condition (22) rearranges to

\[
\frac{d}{dz} E_{z=\beta}^H[e^{-rT(\beta)}] = \frac{B'(\beta)}{B(\beta) - K_H}
\]

(66)

If \( (\beta_1 - \alpha_1) < (\beta_0 - \alpha_0) \), then by direct calculation \( \frac{d}{dz} E_{z=\beta_1}^H[e^{-rT(\beta_1)}] < \frac{d}{dz} E_{z=\beta_0}^H[e^{-rT(\beta_0)}] = \frac{\bar{V}'(\beta_0)}{\bar{V}(\beta_0) - K_H} < \frac{B'(\beta_1)}{B(\beta_1) - K_H} \), where the inequality comes from (65), but violates (66), completing the proof.

The following lemma will be used in the proof of Proposition 4.1.

Lemma B.17. If \( Z_t \) is degenerate, then for all \( t' \geq t \) and any \( n \), if \( I_{t'} = 1 \), then share \( n \) trades at \( W_{t'} = \bar{V}_0 \).

Proof. Fix any equilibrium and history such that the belief is degenerate on type \( \theta \) at time \( t_0 \). We first show that there cannot exist on-path continuation play in which a share of the asset trades at a price strictly greater than \( \bar{V}_0 \) at any \( t \geq t_0 \). To do so, fix a share \( n \) and let \( t_1, t_2, t_3, \ldots \) be the times that the share trades for the the first, second, third, etc., times after \( t_0 \) (with \( t_{j+1} = \infty \) if the asset does not trade more than \( j \) times). By Zero Profit, for any \( j \),

\[
W_{t_j} = E \left[ \int_{t_j}^{t_{j+1}} e^{-r(t-t_j)} \left( v_{t_0} + I_0^t(k_{t_0} - v_{t_0}) \right) dt + e^{-r(t_{j+1}-t_j)} W_{t_{j+1}} \bigg| \mathcal{F}_{t_j} \right]
\]

Substituting in the analogous expression for \( W_{j+1} \), then \( W_{j+2} \), etc., and applying the law of iterated expec-

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tations, we get that for arbitrary integer \( \kappa \)

\[
W_{t_j} = E \left[ \int_{t_j}^{t_j+\kappa} e^{-r(t-t_j)} (v_\theta + I^n(k_\theta - v_\theta)) dt \mid \mathcal{F}_{t_j} \right] + E \left[ e^{-r(t_j+\kappa-t_j)} W_{t_j+\kappa} \mid \mathcal{F}_{t_j} \right] \tag{67}
\]

Because any pair of trades must be separated by a shock arrival, as \( \kappa \to \infty \), \( (t_j+\kappa - t_j) \xrightarrow{a.s.} \infty \). Therefore, by the transversality condition on \( W \), the last term in (67) goes to zero in the limit, and we have

\[
W_{t_j} = E \left[ \int_{t_j}^{\infty} e^{-r(t-t_j)} (v_\theta + I^n(k_\theta - v_\theta)) dt \mid \mathcal{F}_{t_j} \right] \leq V_\theta
\]

The argument above further implies that, at the degenerate belief, \( G_\theta, F_\theta \leq V_\theta \). Hence, by Owner Optimality, if a seller is present at time \( t \), \( W_t \leq V_\theta \). We now show that these bounds are tight. Let \( F_\theta, G_\theta \) be the infimums of \( F_\theta \) and \( G_\theta \) over all possible on-path histories after reaching the degenerate belief. By definition,

\[
G_\theta \geq E[(1 - e^{-r(\tau - t)})|G_t| V_\theta + E[E|G_t| F_\theta]
\]

where, starting from arbitrary time \( t \), \( \tau \geq t \) is the time of the next shock. Now suppose there exists an on-path history such that \( F_\theta < G_\theta \). This clearly violates No Deals. Therefore, \( F_\theta \geq G_\theta \). This is consistent with (68) if and only if \( F_\theta = G_\theta = V_\theta \). Hence, \( F_\theta = V_\theta \). Since the share never trades for a price greater than \( V_\theta \), this can only happen if the share always trades at price \( V_\theta \) as soon as its owner is shocked.

**Proof of Proposition 4.4.** By Owner Optimality and \( K_H > V_L \), it is never on-path for a high-type seller to accept a bid less than \( K_H \). Thus, only low types can be trading at \( t \), and Belief Consistency requires that \( Z_{t'} = -\infty \) for all \( t' > t \). Lemma B.17 and Zero Profit then require that \( W_t = V_L \), which establishes (i). Next, suppose (ii) fails. Then there exists \( \hat{t} > t \) such that there is positive probability that the seller of share \( m \) retains the asset up to \( \hat{t} \). Since \( Z_{t'} = -\infty \) for all \( t' > t \), this violates Lemma B.17 so (ii) must hold. Finally, since \( Z_{t'} = -\infty \) for all \( t' > t \), (iii) is a direct implication of Lemma B.17.

C Proofs for Section 5

**Proof of Lemma 5.3.** The proof of Proposition 3.6 shows that the \( \mathcal{F}_t \)-expected discounted cash flow derived from a seller-owned share of the asset is given by \( E[F_\theta(z) \mid z] \). For the remainder of this appendix we drop the superscript \( n \) and let \( i \) (\( I_t \)) denote the liquidity status (process) of the owner of an arbitrary share. The next two propositions will require the use of the following result.

**Lemma C.1.** Let \( f : \mathbb{R} \times \{0,1\} \to \mathbb{R} \) denote an arbitrary function that is twice differentiable in its first argument almost everywhere. Let \( A \) denote (infinitesimal) generator of \( (Z_t, I_t) \) under \( Q \) (i.e., the public measure). For all states such that \( z > \alpha \), we have that

\[
Af(z,i) = \frac{\partial^2}{2} \left[ (2p(z) - 1) f_z(z,i) + f_{zz}(z,i) \right] + (1 - i) \lambda(f(z,1) - f(z,0)) \tag{69}
\]

In the case that \( N = 1 \), the above also holds for all \( z \) when \( i = 0 \).
Proof. For all such states: (i) $dZ_t = d\hat{Z}_t$ and equation (2) gives that $E[dZ_t|\mathcal{F}_t] = \frac{\phi^2}{2}(2p(Z_t) - 1)dt$, and (ii) $I_t$ follows a jump process for all such states with arrival $\lambda$ and fixed jump size $(1 - \lambda)$. The result then follows from Applebaum (2004, Theorem 3.3.3).

Proof of Proposition 5.3. We break the proof into two cases:

1. For $z > \alpha$, using (69), we have that

$$E[dP_t|\mathcal{F}_t] = \frac{\phi^2}{2} \left((2p - 1)P_z + P_{zz}\right)dt$$

$$= \frac{\phi^2}{2} \left(p(F_H' + F_H''_z) + (1 - p)(F_H''_L - F_L')\right)dt$$

For $z \in (\alpha, \beta)$, $F_H$ and $F_L$ satisfy (13) and (14). Substituting this in gives:

$$E[dP_t|\mathcal{F}_t] = r(\bar{F}(z) - \bar{K}(z))dt$$

taking the $\mathcal{F}_t$-expectation we get that

$$\frac{1}{dt} (E[dR_t|\mathcal{F}_t] - r) = r \left(\frac{\bar{F}(z) - \bar{K}(z) + \bar{V}(z)}{F(z)} - 1\right) = \frac{r}{F(z)} \left(\bar{V}(z) - \bar{K}(z)\right)$$

2. For $z \leq \alpha$, with probability $\frac{p(\alpha) - p(z)}{p(\alpha)}$ the price jumps down to $V_L$. With probability $\frac{p(z)}{p(\alpha)}$ the price jumps up to $\bar{F}(\alpha + dZ_t)$. Thus,

$$\frac{1}{dt} E[dP_t|\mathcal{F}_t] = \frac{p(\alpha) - p(z)}{p(\alpha)} \left((V_L - P_t) + \bar{P}(\alpha) + E[dP_t|(Z_t, I_t) = (\alpha, 1)] - P(z)\right)$$

$$= \frac{p(z)}{p(\alpha)} r(\bar{F}(z) - \bar{K}(z))$$

where the second equality uses the fact that $F_L(z) = F_L(\alpha) = V_L$. The expression given in the proposition for this case follows immediately.

Proof of Proposition 5.4. Note that for any $N$, $C_3^R = 0$ from (40) and $C_{32}^R < 0$ by Proposition 3.6 (which shows that $B < \bar{V}$ and can easily be extended for $N > 1$ by the same proof). Thus, it is clear from inspection of (52) that $B_3^R(z) > \bar{V}(z)$ for all $z > \beta$, implying the result for price volatility. Combining the two inequalities on $B$ above implies the statement for return volatility.

Proof of Proposition 5.5. The boundary conditions are uniquely pinned down by equilibrium play; when beliefs are degenerate, trade occurs immediately upon arrival of a shock (Lemma B.17). For the remainder of the proof, we break the state space into four different regions enumerated below. With the exception of the region in which $i = 0$, the arguments below are independent of $N$.

1. For $z \leq \alpha, i = 1$: with probability $\frac{p(\alpha) - p(z)}{p(\alpha)}$ trade occurs ($du_t = 1$) and the state transitions to $(-\infty, 0)$. With probability $\frac{p(z)}{p(\alpha)}$ trade does not occur and the state transitions to $(\alpha, 1)$. Therefore,

$$f(t, z) = \frac{p(\alpha) - p(z)}{p(\alpha)} \left(1 + \lim_{z \to -\infty} g(t, z)\right) + \frac{p(z)}{p(\alpha)} f(t, \alpha) = \frac{p(\alpha) - p(z)}{p(\alpha)} (1 + \lambda t) + \frac{p(z)}{p(\alpha)} f(t, \alpha)$$

where the second inequality follows from the boundary condition on $g$. 57
2. For $z \in (\alpha, \beta), i = 1$: $d\nu_t = 0 \text{ w.p.1.}$ $(f(0, z) = 0)$ Applying the Kolmogorov backward equation (e.g., Applebaum (2004, p. 164)) using the generator from (69) gives $f_t$ in (6).

3. For $z \geq \beta, i = 1$: $d\nu_t = 1 \text{ w.p.1.}$ (thus $f(0, z) = 1$) and the new owner is a holder. Thus $f(t, z) = 1 + g(t, z)$.

4. For $i = 0$:

   (i) When $N = 1$, for all $z$: $d\nu_t = 0 \text{ w.p.1.}$ (thus $g(0, z) = 0$). Again, applying the Kolmogorov backward equation using the generator from (69) gives $g_t$ in (6).

   (ii) When $N = \infty$, (i) holds for all $z > \alpha$ yielding the second equation in (10). For $z < \alpha$, with probability $\frac{p(\alpha) - p(z)}{p(\alpha)}$ another trader sells, the asset type is revealed to be low, and all future trade occurs immediately when the shock arrives (i.e., at rate $\lambda$). With complimentary probability, no other traders sell, in which case $z$ transitions to $\alpha$. Taking the expectation yields the first equation in (10).