Waiting for News in the Market for Lemons*

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Abstract

We study a dynamic setting in which stochastic information (*news*) about the value of a privately-informed seller's asset is gradually revealed to a market of buyers. We construct an equilibrium that involves periods of no trade or *market failure*. The no-trade period ends in one of two ways: either enough good news arrives restoring confidence and markets re-open, or bad news arrives making buyers more pessimistic and forcing *capitulation* i.e., a partial sell-off of low-value assets. Conditions under which the equilibrium is unique are provided. We analyze welfare and efficiency as they depend on the quality of the news. Higher quality news can lead to more inefficient outcomes. Our model encompasses settings with or without a standard static adverse selection problem—in a dynamic setting with sufficiently informative news, reservation values arise endogenously from the option to sell in the future and the two environments have the same equilibrium structure.

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1 Introduction

Consider an entrepreneur who is interested in selling her company due to liquidity constraints. Naturally, she is better informed than the market about her company's fundamentals. She would like to sell, yet there is no reason she is forced to do so on any given day, nor can she commit to delaying trade. With every passing day that she retains ownership, she gains (or loses) the day's profit and maintains the option to sell the next day. If trade is in fact delayed, then the market may learn about the value of the firm by observing cash flows, investments, customer base, etc. Herein lies the key innovation of this paper; we introduce an exogenous public information process, *news*, into a model of such settings. We then study the implications for trading behavior, efficiency and welfare.

We model the environment as a game played in continuous time where a risk-neutral seller faces a competitive market. There is common knowledge of gains from trade, but the seller is privately informed about the asset's value (i.e., her type), which may be either high or low. At each point in time prior to trade, the seller receives offers from the market. If an offer is accepted, then the trade is consummated and the game ends. If all offers are rejected, then the seller consumes the type-dependent flow payoff endowed by the asset. Contemporaneously, information about the seller's type is publicly revealed via a Brownian diffusion process with type-dependent drift (μ_θ) and common volatility (σ). The *quality* of the news process is captured by the square of the signal-to-noise ratio, $(\frac{\mu_H - \mu_L}{\sigma})^2$, relative to the common discount rate $r > 0$.

We construct an equilibrium that is stationary in the market belief about the asset type. The equilibrium involves three distinct regions.

(i) When beliefs about the seller are favorable, the market is fully efficient: trade is immediate at a price equal to the expected market value of the asset.

(ii) When beliefs are unfavorable, there is a partial sell-off: a low offer is made, the low type accepts with positive probability, while the high type rejects with probability one. Conditional on a rejection, the market belief about the seller increases, and thus rejecting offers when beliefs are unfavorable serves as a positive signal about the seller's type.

(iii) When beliefs are intermediate, the market dries up: no trade occurs as market participants wait for more information to be revealed before consummating a trade.

If the market belief is in either region (ii) or (iii), then it evolves based on both the realization of news as well as the seller's equilibrium strategy, until either the favorable-belief region

(i) is reached or until it reaches the unfavorable-belief region (ii) *and* the low type's mixing results in acceptance.

We investigate how welfare and efficiency of the equilibrium depend on the quality of the news. As the news quality goes to zero, the (type-dependent) payoffs to each player converge to the payoffs in the static game where all buyers make simultaneous take-it-or-leave-it offers. Only when dynamics are coupled with news arrival is welfare affected. Introducing news mitigates the well-known inefficiency associated with trade breakdown in the market for lemons (Akerlof, 1970), as both types of seller trade *eventually* in equilibrium. However, a new inefficiency develops for intermediate beliefs due to trade being delayed. That is, the introduction of an informative news process gives the high type incentive to wait and the low type incentive to mimic for intermediate beliefs, creating delay (and inefficiency) in markets where fully efficient trade would have transpired in the absence of news. Nevertheless, as the quality of news becomes arbitrarily high, the expected costs from delay go to zero, and the inefficiency disappears. Hence, improving news quality can increase or decrease efficiency depending on the magnitude of the improvement and the initial market belief.

An efficiency-seeking planner who is considering news quality as a policy instrument (e.g., mandatory disclosure or audits, protecting independent journalism, etc.) might be tempted to reason that higher news quality is always (weakly) better as it alleviates the potential inefficiency caused by private information. Our results demonstrate that this is not the case. However, we also show that introducing news improves efficiency precisely when there would have been inefficiency without news.

A relevant consideration in our analysis is the potential presence of a static adverse selection problem—that is, whether the value of the high-type seller's outside option (e.g., the entrepreneur retaining the company forever) is higher than the market value for a low-type asset. If it is, we say that the Static Lemons Condition (SLC) holds. In a single-period model where all buyers make take-it-or-leave it offers, this condition is necessary to generate inefficient trading outcomes.

If the SLC holds, the equilibrium described above is the unique equilibrium among the class of stationary equilibria subject to a mild refinement on off-path beliefs. If the SLC fails, the same equilibrium exists provided the news quality is sufficiently high. Despite the absence of a traditional adverse selection problem, an *endogenous* market for lemons develops because the option to sell in the future drives the seller's continuation value above the market's willingness to pay. Trade is delayed despite the fact that buyers are always willing to offer more than the high type's outside option.[1] While other forms of equilibria can exist when

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[1] A seller's *outside option* is her payoff from never trading, whereas her *continuation value* is her expected payoff from rejecting the current offer (given the history) in equilibrium.

the SLC fails, each involves a region of no-trade provided news quality is sufficiently high.

When news quality is low, the difference between the two cases is more striking. The SLC implies that the high type is never tempted to sell when the market belief about her asset is unfavorable—foregoing offers carries no opportunity cost until the market belief improves. When the SLC fails, the high type's opportunity cost is positive regardless of the market belief. With low enough news quality, the potential benefit from waiting is outweighed by this cost, and the unique equilibrium involves immediate trade for all beliefs (similar to Swinkels (1999)).

We investigate the robustness of our findings by considering alternative specifications. In Section 6, we replace the Brownian news process with a compound Poisson process with type-dependent arrival rates. As in other models (e.g., Abreu et al. (1991)), the equilibrium varies depending on whether absence of an arrival is a positive or negative signal. However, in both cases the equilibrium bears resemblance to the equilibrium with Brownian news. In fact, each case illustrates separate key features present in the Brownian-news model. In the Supplemental Appendix, we pose the game in discrete time (with news modeled by a random walk) under the SLC. We demonstrate that when the time between offers is short, the unique equilibrium exhibits the same three regions: immediate trade, no trade and partial sell-off. As the time between offers goes to zero, the equilibrium converges to the equilibrium of the continuous-time game.

Our paper contributes to both the literature on dynamic adverse selection (Janssen and Roy (2002), Hörner and Vieille (2009)) and on dynamic signaling (Nöldeke and van Damme (1990), Swinkels (1999), Kremer and Skrzypacz (2007)). In the spirit of Spence (1973), the models of dynamic signaling often present educational signaling as their lead example: the asset is the seller's labor, and the time prior to trade is the time she spends in school (incurring a flow cost while doing so). In this context, the news process in our model can be interpreted as the student's grades. We provide a unified framework for analyzing both environments in a continuous-time setting and explore the impact of information about the seller's type being gradually revealed to the market. A more detailed discussion of how our findings relate to these two strands of literature is postponed until Section 7.

The equilibrium in our model has similar features to three recent works. Most notably, Gul and Pesendorfer (2011) (with asymmetric information) analyze the incentives for a political party to campaign on an issue when doing so reveals unbiased information about the issue to voters. Bar-Isaac (2003) investigates learning and reputation in a model where a privately-informed monopolist decides whether to sell in each period. Lee and Liu (2010) consider a reputation game where, in each period, a short-lived plaintiff makes a take-it-or-leave-it settlement demand to a long-run defendant who is privately informed of her liability.

The common thread is that the high type never succumbs when beliefs are below a lower threshold that is endogenously determined by the low type's indifference condition. This perseverance serves as an imperfect signal that boosts reputation because the low type mimics only with some probability. The low type's mixed strategy exactly offsets negative information at the lower threshold, which acts as a reflecting barrier on the equilibrium belief process. A distinguishing feature of our model is that there is also an endogenously-determined *upper* barrier at which point the high type decides to consummate a trade. The location of the lower barrier influences the high type's decision and similarly the location of the upper barrier influences the indifference condition of the low type (as the offer acceptable to the high type will only be tendered when the buyers (correctly) anticipate that she will accept it). Thus the two barriers must be determined simultaneously.

Within a static context, Levin (2001) shows that decreasing the information asymmetry between buyers and sellers can increase or decrease efficiency. This result is qualitatively similar to our result regarding increasing the quality of news, however, the mechanisms by which these results obtain are quite different.[2] Daley and Green (2011b) analyze a static signaling model in which the sender chooses a costly signal that also generates a stochastic grade.[3] The present dynamic model relaxes the (implicit) assumption that the seller can commit to delay trade (just as Nöldeke and van Damme (1990) and Swinkels (1999) do to the standard Spence signaling model).

Though our model is stark compared to the intricacies of financial markets, the forces described above may help to explain some of the peculiar trading patterns observed in the recent financial crisis. For example, bad news can cause an otherwise well-functioning market to shut down entirely. The recent collapse of the mortgage-backed securities market is perhaps most relevant. Prior to 2007, trade and issuance of private label mortgage-backed securities and collateralized debt obligations occurred in a liquid and well-functioning market. Indicators of a decline in the housing market increased uncertainty in the value of the underlying collateral and led to a catastrophic drop in both liquidity and prices.[4] Investors were unwilling to buy these securities or lend against them (even at a substantial

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[2] It should be noted that Levin (2001) focuses on the effect of reducing the *seller's* private information, whereas the seller's private information is fixed in our model. In his model, the release of public information exogenously shifts the demand curve (depending on the realization). Whether this increases or decreases the likelihood of trade (efficiency) depends on both the information structure and the likelihood of trade in the absence of public information. In our model, trade always occurs eventually. Increasing the quality of news affects the dynamic incentives of the seller, which endogenously leads to an increase/decrease in the amount of costly delay and hence less/more efficient outcomes.

[3] Feltovich et al. (2002) study a similar setting and identify conditions under which "countersignaling" equilibria exist.

[4] See Krishnamurthy (2010) or Brunnermeier (2009) for a descriptive analysis of how debt markets malfunctioned in the recent crisis.

discount/haircut) for fear of being stuck with the most "toxic" assets. As the crisis deepened, some banks were eventually forced to capitulate, selling these securities for a fraction of their notional value.[5] In related work (Daley and Green, 2011a), we have begun a formal investigation of these phenomena by enriching our framework to capture key features of financial markets. Preliminary results indicate that three analogous trading regions develop, lending support to the interpretation given above.

In the next section, we introduce the model and our notion of equilibrium. In Section 3, we construct and verify the equilibrium described above. Section 4 contains comparative static, welfare and efficiency results. In Section 5, we refine our equilibrium notion to include stationarity and a restriction on off-path beliefs. We show that the equilibrium described above is unique when the SLC holds, but may not be when it fails. Section 6 considers an alternative specification where news arrives according to a Poisson process with type-dependent arrival rate. We conclude with a discussion of our results in the context of the literature in Section 7. Several technical preliminaries are provided in Appendix A. Proofs for the main results of the paper are located in Appendix B. The Supplemental Appendix analyzes a discrete-time analog.

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[5] For example, on July 29, 2008 Merrill Lynch sold $30.6 billion notional of collateralized debt obligations to Lone Star Funds at 22 cents on the dollar (Keogh and Shenn, 2008).

2 The Model

There is one seller holding an asset of type $\theta \in \{L, H\}$ and a competitive market of [potential] buyers.[6] The seller knows her type while buyers do not. Let $\pi_0 \in (0, 1)$ denote the prior probability that buyers assign to $\theta = H$. The game is played in continuous time, starting at $t = 0$ with an infinite horizon. While the seller is in possession of the asset she receives a private flow payoff of K_θ. The seller discounts future payoffs at finite interest rate $r > 0$.

At every time t, the seller receives private offers from the buyer side of the market. If the seller accepts an offer of m at time t, the trade is executed and the game ends. The type-dependent (average) payoff to the seller is then

$$r\left(\int_0^t e^{-rs} K_\theta ds + e^{-rt} m\right)$$

The factor r outside the parentheses normalizes payoffs to the same scale as flow payoffs. Therefore, a type-θ seller's payoff from holding the asset *ad infinitum* is $\int_0^\infty e^{-rs} r K_\theta ds = K_\theta$.

Buyers have a common value for a type-θ asset, normalized to flow scale, and denoted by V_θ, with $V_H > V_L$. There is common knowledge of gains from trade: $V_\theta > K_\theta$ for each θ. The payoff to a buyer whose offer of m is accepted is $V_\theta - r\,m$. The interpretation is that this is the average payoff to a buyer who, after purchase, consumes a flow of V_θ forever and discounts at rate r (see Remark 2.2 for a complete micro-foundation). Henceforth, we refer to "offers" as *being* their normalized value (i.e., not m, but $r\,m$). All players are risk-neutral and maximize their expected payoff.

If the seller rejects all current offers, she retains the asset, receives the flow payoff and can entertain future offers. In addition, news about the seller's type is gradually revealed—as we discuss below. A heuristic description of the timing is depicted in Figure 1.

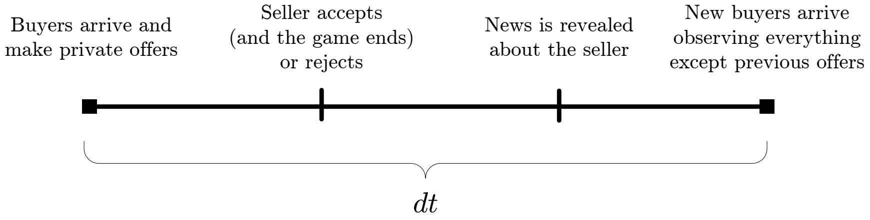

FIGURE 1 – Heuristic Timeline of a Single "Period"

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[6] We use "the seller holds an asset of type θ" interchangeably with "the seller is of type θ." Similarly, any references to "buyers" or "the market" are equivalent.

2.1 News Arrival

News about the seller's asset is revealed via a Brownian diffusion process. Regardless of type, the seller starts with an initial score X_0, normalized to 0. The news process then evolves according to

$$dX_t = \mu_\theta dt + \sigma dB_t \tag{1}$$

where $B = \{B_t, \mathcal{F}_t, 0 \leq t \leq \infty\}$ is standard Brownian motion on the canonical probability space $\{\Omega, \mathcal{F}, \mathcal{Q}\}$. At each time t, the entire history of news, $\{X_s, 0 \leq s \leq t\}$, is publicly observable.[7] Without loss of generality, $\mu_H \geq \mu_L$. The parameters μ_H, μ_L and σ are common knowledge. Define the signal-to-noise ratio $\phi \equiv (\mu_H - \mu_L)/\sigma$. When $\phi = 0$, the news is completely uninformative. Larger values of ϕ imply more informative news. In what follows, we assume that $\phi > 0$, unless otherwise stated. Central to the analysis will be the informativeness of news *relative* to the rate of discounting, r. Therefore, define $\gamma \equiv \phi^2/r$ to be the *quality* of the news.

Remark 2.1. The results of our model remain unchanged if the flow payoff is stochastic with type-dependent mean K_θ. Further, matching our motivating example of the entrepreneur, the flow payoff to the seller could **be** the incremental news process, in which case $K_\theta = \mu_\theta$. (Under this specification, results and comparative statics that rely on values of, or changes to, news quality γ, should be interpreted as relying on values of, or changes to, σ or r. This interpretation maintains a clean separation between news quality and flow-payoff terms.)

2.2 Strategies

Rather than define strategies for each buyer, we model the buyer side of the market as a real-valued stochastic process $W = \{W_t, 0 \leq t \leq \infty\}$ adapted to the filtration $(\mathcal{H}_t)_{t\geq 0}$, where $\mathcal{H}_t$ is the σ-algebra generated by $\{X_s, 0 \leq s \leq t\}$. W_t represents the highest offer made at time t. This approach will simplify our exposition and can be micro-founded by Remark 2.2.[8] Both X and W are stochastic processes defined over the probability space $\{\Omega', \mathcal{H}, \mathcal{P}\}$, where $\Omega' = \Omega \times \Theta$, $\mathcal{H} = \mathcal{F} \times 2^\Theta$ and $\mathcal{P} = \mathcal{Q} \times \nu$, where ν is the measure over $\Theta \equiv \{L, H\}$ defined implicitly by π_0.

A pure strategy for the type-θ seller is an $\mathcal{H}_t$-adapted stopping time $\tau_\theta(\omega) : \Omega' \to \mathbb{R}_+ \cup \{\infty\}$. A mixed strategy for the seller is a distribution over such times, which can be represented as a stochastic process $S^\theta = \{S^\theta_t, 0 \leq t \leq \infty\}$ also adapted to $(\mathcal{H}_t)_{t\geq 0}$. The

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[7] The news process and flow payoffs are not verifiable and cannot be contracted upon. The most obvious reasons for this assumption are the standard issues with verification and contractibility. There are others. In many applications, the news ends after trade occurs. For example, when a student leaves school to enter the workforce she no longer receives grades.

[8] We thank the editor and an anonymous referee for this suggestion.

process must be right-continuous and satisfy $0 \leq S^\theta_t \leq S^\theta_{t'} \leq 1$ for all $t \leq t'$. $S^\theta(\omega)$ is a CDF over the type-θ seller's acceptance time on $\mathbb{R}_+ \cup \{\infty\}$ along the sample path $X(\omega, \theta)$. A discontinuous increase in S^θ corresponds to acceptance with an atom of probability mass.[9]

Remark 2.2. The offer process W (when coupled with conditions 3 and 4 of Definition 2.1 below) is a convenient modeling device for the following strategic situation. There is an infinite horizon, and trade does not end the game. At each time t, two or more short-lived buyers arrive and simultaneously make private offers to the owner of the asset. A trade transfers the rights to the asset's future flow payoffs to the buyer. Each buyer earns a flow payoff of V_θ from owning the asset and discounts future payoffs at rate r. After the initial sale, the asset will not be re-traded because the purchasing buyer learns the asset type upon its transfer of ownership (Milgrom and Stokey, 1982).[10] The (average) payoff to a buyer who purchases the asset with offer $m = \frac{w}{r}$ is therefore $V_\theta - w$.

2.3 The Market Belief

At every time t, if trade has not yet occurred, buyers assign a probability to the asset being of high value. Along the equilibrium path, the market belief is conditioned on both the entire path of past news and on the fact that trade has not yet occurred. It will be convenient to separate these two sources of information. Let f^θ_t denote the density of X_t conditional on θ, which for $t > 0$ is normally distributed with mean $\mu_\theta t$ and variance $\sigma^2 t$.[11] Let $S^\theta_{t^-} \equiv \lim_{s \uparrow t} S^\theta_s$ (which is well defined for $t > 0$ given that S^θ is bounded and non-decreasing), and specify that $S^\theta_{0^-} = 0$. Here, "at time t" should be interpreted to mean *before* observing the seller's decision at time t, which is why left limits are appropriate. If $S^L_{t^-} \cdot S^H_{t^-} < 1$, then the game is on the equilibrium path, and the probability the market assigns to $\theta = H$ is defined by Bayes Rule as

$$\frac{\pi_0 f^H_t(X_t)(1 - S^H_{t^-})}{\pi_0 f^H_t(X_t)(1 - S^H_{t^-}) + (1 - \pi_0) f^L_t(X_t)(1 - S^L_{t^-})} \tag{2}$$

Taking the log-likelihood ratio of (2) results in

$$Z_t = \underbrace{\ln\left(\frac{\pi_0}{1 - \pi_0}\right) + \ln\left(\frac{f^H_t(X_t)}{f^L_t(X_t)}\right)}_{\hat{Z}_t} + \underbrace{\ln\left(\frac{1 - S^H_{t^-}}{1 - S^L_{t^-}}\right)}_{Q_t} \tag{3}$$

Hence, Z is the stochastic process that tracks the market belief in terms of its log-likelihood

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[9] This definition of a strategy will suffice for our notion of equilibrium (Definition 2.1) used in Section 3, which is analogous to a Bayesian Nash Equilibrium. In Section 5, our equilibrium notion is strengthened, and we generalize the definition of a strategy to consider play after off-equilibrium path events.

[10] To be more precise, any equilibrium in which trade occurs after the initial sale is payoff equivalent to the equilibrium in which no trade occurs after the initial sale.

[11] Let $f^H_0 = f^L_0$ be the Dirac delta function.

ratio. The transformation from belief as a probability to a log-likelihood ratio is injective, and therefore without loss, and will simplify the analysis.

By working in log-likelihood space we are able represent Bayesian updating as a linear process, and the market belief as the sum of two components, $Z = \hat{Z} + Q$, as seen in (3). Notice that the two component processes separate the two sources of information to the market. $\hat{Z}$ is the belief process for a Bayesian who updates *only based on news* starting from $\hat{Z}_0 = Z_0 = \ln\left(\frac{\pi_0}{1-\pi_0}\right)$. Q is the stochastic process that keeps track of the information conveyed by the fact that the seller has rejected all past offers.[12]

For any interval of time over which there is zero probability of trade in equilibrium, the law of motion of Z will be identical to that of $\hat{Z}$. Because such *no-trade* intervals will be a crucial part of our equilibrium analysis, it is useful to understand how beliefs evolve during them.

$$\hat{Z}_t = \ln\left(\frac{\pi_0}{1-\pi_0}\right) + \ln\left(\frac{f_t^H(X_t)}{f_t^L(X_t)}\right) = \hat{Z}_0 + \frac{\phi}{\sigma}X_t - \frac{\phi}{2\sigma}(\mu_H + \mu_L)t$$

and thus

$$d\hat{Z}_t = -\frac{\phi}{2\sigma}(\mu_H + \mu_L)dt + \frac{\phi}{\sigma}dX_t \tag{4}$$

Inserting the law of motion from equation (1) into equation (4) gives a probabilistic representation of how beliefs based solely on news evolve from the perspective of the privately-informed seller, which we denote by $\hat{Z}^\theta$. The high type expects to receive good news making $\hat{Z}^H$ a submartingale (5). The low type is expectant of bad news making $\hat{Z}^L$ a supermartingale (6).

$$d\hat{Z}_t^H = -\frac{\phi}{2\sigma}(\mu_H + \mu_L)dt + \frac{\phi}{\sigma}(\mu_H dt + \sigma dB_t) = \frac{\phi^2}{2}dt + \phi dB_t \tag{5}$$

$$d\hat{Z}_t^L = -\frac{\phi}{2\sigma}(\mu_H + \mu_L)dt + \frac{\phi}{\sigma}(\mu_L dt + \sigma dB_t) = -\frac{\phi^2}{2}dt + \phi dB_t \tag{6}$$

Because $\hat{Z}^\theta$ is a linear transformation of the news process, it is also a Brownian diffusion process, retaining desirable properties, such as stationary independent increments. This makes the analysis more tractable than working with the corresponding non-linear processes in probabilities.

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[12] Similar to our remark in footnote 9, the specification of Q_t when $S_{t^-}^L = S_{t^-}^H = 1$ is immaterial given the equilibrium concept given by Definition 2.1. Again, we address this issue in Section 5, when using the equilibrium concept given by Definition 5.2.

2.4 Equilibrium

Given any W, the seller faces an optimal stopping problem.[13]

$$\sup_{\tau \geq 0} E^{\theta}\left[\int_0^{\tau} e^{-rs} r K_{\theta} ds + e^{-r\tau} W_{\tau}\right] \tag{SP_θ}$$

Recall that S^{θ} is a distribution over stopping times. Let $\mathcal{S}^{\theta} = supp(S^{\theta})$. We say that S^{θ} solves (SP_{θ}) if all $\tau \in \mathcal{S}^{\theta}$ solve (SP_{θ}).[14] For any (t, ω) such that $S^{\theta}_{t-}(\omega) < 1$, there exists $\tau \in \mathcal{S}^{\theta}$ such that $\tau(\omega) \geq t$. For any such τ, let

$$F_{\theta}(t, \omega) = E^{\theta}\left[\int_t^{\tau} e^{-rs} r K_{\theta} ds + e^{-r(\tau - t)} W_{\tau} | \mathcal{H}_t\right]$$

be the $\mathcal{H}_t$-measurable function denoting the type-θ seller's expected payoff starting from time t conditional on his information set. Let τ^* denote the (random) time at which trade occurs.[15] Our equilibrium notion is as follows.

Definition 2.1. An *equilibrium* of the game is a quadruple (S^L, S^H, W, Z), such that

1. Seller Optimality: Given W, S^{θ} solves the type-θ seller's problem *(SP_{θ})*
2. Belief Consistency: For all t such that $S^L_{t-} \cdot S^H_{t-} < 1$, Z_t is given by (3)
3. Zero Profit: If there exists a $\tau \in \mathcal{S}^L \cup \mathcal{S}^H$ such that $\tau(\omega) = t$ for some ω, then $W_t = E[V_{\theta} | \mathcal{H}_t, \tau^* = t]$
4. No (Unrealized) Deals: For all θ, t, ω such that $S^{\theta}_{t-}(\omega) < 1$: $F_{\theta}(t, \omega) \geq E\left[V_{\theta'} | \mathcal{H}_t, V_{\theta'} \leq V_{\theta}\right]$

The first two conditions, Seller Optimality and Belief Consistency, are standard. Because the buyer side of the market is modeled as a (non-strategic) offer process, we replace the usual individual payoff-maximization criterion with conditions 3 and 4. The interpretation of the Zero Profit condition is clear (any executed trade must earn the purchasing buyer zero expected surplus), and is motivated by the interpretation of Bertrand competition among buyers. Notice that, given S^H, S^L, the condition may not completely pin down W along the equilibrium path. Following histories where both types reject, any offer will suffice (provided both types find it optimal to reject it). To interpret the No Deals condition, note that if $F_{\theta}(t, \omega) < E\left[V_{\theta'} | \mathcal{H}_t, V_{\theta'} \leq V_{\theta}\right]$, then there exists an offer that will earn a buyer a positive

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[13] Although Z does not appear in (SP_{θ}) directly, its law will affect the seller's problem in equilibrium through the restrictions placed on W (i.e., the Zero Profit and No Deals conditions of Definition 2.1).

[14] That is, for any $\tau_{\theta} \in \mathcal{S}^{\theta}$, $E^{\theta}\left[\int_0^{\tau_{\theta}} e^{-rs} r K_{\theta} ds + e^{-r\tau_{\theta}} W_{\tau_{\theta}}\right] = \sup_{\tau \geq 0} E^{\theta}\left[\int_0^{\tau} e^{-rs} r K_{\theta} ds + e^{-r\tau} W_{\tau}\right]$.

[15] The (conditional) distribution of τ^* is easily derivable from S^L, S^H, and π_0, however a formal definition requires enriching the probability space to incorporate the seller's mixing and is omitted for parsimony.

expected payoff—namely, any offer between $F_\theta(t,\omega)$ and $E[V_{\theta'}|\mathcal{H}_t, V_{\theta'} \leq V_\theta]$. Hence, the No Deals condition reflects the equilibrium requirement that no buyer can profitably deviate by making an offer that the seller would accept with positive probability. [16]

3 Equilibrium Construction

In this section we construct the equilibrium of interest. We begin by defining a two-parameter class of candidate equilibria. Let $\Psi(z) \equiv E[V_\theta|Z_t = z]$.

Definition 3.1. For any pair $(\alpha,\beta) \in \mathbb{R}^2, \alpha < \beta$, let $Q_t^\alpha = \max\{\alpha - \inf_{s\leq t}\hat{Z}_s, 0\}$. Define $\Xi(\alpha,\beta)$ to be the belief process and strategy profile:

$$Z_t = \hat{Z}_t + Q_t^\alpha \tag{7}$$

$$S_t^H = \begin{cases} 1 & \textit{if there exists } s \leq t \textit{ such that } Z_s \geq \beta \textit{ and } W_s \geq \Psi(Z_s) \\ 0 & \textit{otherwise} \end{cases} \tag{8}$$

$$S_t^L = \begin{cases} 1 & \textit{if there exists } s \leq t \textit{ such that } Z_s \geq \beta \textit{ and } W_s \geq \Psi(Z_s) \\ 1 - e^{-Q_t^\alpha} & \textit{otherwise} \end{cases} \tag{9}$$

$$W_t = \begin{cases} \Psi(Z_t) & \textit{if } Z_t \geq \beta \\ V_L & \textit{if } Z_t < \beta \end{cases} \tag{10}$$

If $\Xi(\alpha,\beta)$ is an equilibrium, then Belief Consistency requires that $Z_t = \hat{Z}_t + Q_t^\alpha$ along the equilibrium path (see (3)). Definition 3.1 specifies that Z follows the same process off the equilibrium path as well.

The belief process $Z_t = \hat{Z}_t + Q_t^\alpha$ is a time-homogenous $\mathcal{H}_t$-Markov process, meaning the candidate equilibrium has a stationary structure, with the market belief serving as the state variable. We will use z when referring to the state variable as opposed to the stochastic process Z (i.e., if $Z_t = z$, then the game is "in state z, at time t"). In addition, when referring to generic states, z, we mean $z \in \mathbb{R}$, as opposed to the degenerate belief states $z = \pm\infty$, unless otherwise stated.[17] For convenience, let $w(Z_t) = W_t$. Under $\Xi(\alpha,\beta)$, if $z \geq \beta$, then $w(z) = \Psi(z)$ and the offer is accepted by both types. When the belief is between α and β, $w(z) = V_L$, which is rejected by both types, making this a *no-trade* region.[18] When $z \leq \alpha$, the offer is V_L, the high type rejects, and the low type mixes precisely such that, conditional

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[16] Within the context of Remark 2.2, this rationale relies on the fact that offers are *private*. It is assumed that the offer affects $F_\theta(t,\omega)$ only if it will be accepted. Private offers eliminate the possibility of signaling through rejection of high offers (Nöldeke and van Damme (1990), Hörner and Vieille (2009)).

[17] Continuation play after reaching a degenerate belief at time t, $Z_t \in \{\pm\infty\}$, is trivial: for all $t' \geq t$, $Z_{t'} = Z_t$, $W_{t'} = \Psi(Z_t)$, and $S_{t'}^\theta = S_{t-}^\theta + (1 - S_{t-}^\theta)1_{\{\exists\, s \in [t,t'] : W_s \geq K_\theta\}}$.

[18] As noted previously, because both types reject, w is not uniquely pinned down for $z \in (\alpha,\beta)$. The specification used is chosen purely for simplicity.

on rejection, $(Z_t)_{t>0}$ never falls below α. That is, α is a lower reflecting barrier for Z.

The main result of this section (Theorem 3.1) is that there exists a unique (α^*, β^*) such that $\Xi(\alpha^*, \beta^*)$ is an equilibrium if either the Static Lemons Condition holds (i.e., $K_H > V_L$, see Definition 3.2) or γ is high enough. To demonstrate this, we first derive necessary properties of and construct the value functions for each type of seller for a fixed (α, β). We then show there is a unique pair, (α^*, β^*), consistent with our equilibrium notion.

3.1 Value Functions

Given the stationary structure, the value functions for each type of seller depend only on the current state. Let us abuse notation slightly and write $F_\theta(z)$ for the type-θ seller's value in state z.

Outside the no-trade region, $z \notin (\alpha, \beta)$, value functions are straightforward to compute. For any $z \geq \beta$, both types trade at $\Psi(z)$ so $F_L(z) = F_H(z) = \Psi(z)$. For any $z < \alpha$, V_L is offered, and the low type mixes, implying she must be indifferent, hence $F_L(z) = V_L$. Further, at any $z < \alpha$, conditional on rejection, the belief will jump to α, so $F_L(\alpha) = V_L$, and $F_H(z) = F_H(\alpha)$, where $F_H(\alpha)$ is determined in the subsequent analysis.

For all $z \in (\alpha, \beta)$ the seller rejects $w(z)$ and takes her continuation payoff. Over an arbitrarily short interval of time, the probability of exiting the no-trade region is negligible.

$$F_\theta(z) \approx rK_\theta dt + e^{-rdt} E_z^\theta \left[F_\theta(z + dZ) \right] \tag{11}$$

Applying Ito's formula to the right hand side of (11) gives

$$F_\theta(z) \approx rK_\theta dt + e^{-rdt} E_z^\theta \left[F_\theta(z) + F'_\theta(z) dZ + \frac{1}{2} F''_\theta(z) (dZ)^2 \right]$$

In the no-trade region, beliefs evolve based only on the news. Substituting in the law of motion ($dZ_t = d\hat{Z}_t$), taking the expectation and the limit as $dt \to 0$, yields a second-order differential equation for each type seller's value function in the no-trade region.

$$F_L(z) = K_L - \frac{\gamma}{2} \left(F'_L(z) - F''_L(z) \right) \tag{12}$$

$$F_H(z) = K_H + \frac{\gamma}{2} \left(F'_H(z) + F''_H(z) \right) \tag{13}$$

The differential equations have closed-form solutions

$$F_L(z) = C_1^L e^{q_1^L z} + C_2^L e^{q_2^L z} + K_L \tag{14}$$

$$F_H(z) = C_1^H e^{q_1^H z} + C_2^H e^{q_2^H z} + K_H \tag{15}$$

where $(q_1^L, q_2^L) = \frac{1}{2}\left(1 \pm \sqrt{1+8/\gamma}\right)$, $(q_1^H, q_2^H) = \frac{1}{2}\left(-1 \pm \sqrt{1+8/\gamma}\right)$ and the four constants, C_i^θ, are yet to be determined.[19]

The constants are pinned down by four boundary conditions, three of which are *value-matching* conditions.

$$F_L(\alpha^+) = V_L \tag{16}$$

$$F_L(\beta^-) = \Psi(\beta) \tag{17}$$

$$F_H(\beta^-) = \Psi(\beta) \tag{18}$$

where $f(x^-)$ and $f(x^+)$ denote the left and right limit of f at x, respectively. Finally, the belief process is reflecting at $z = \alpha$ for the high type.[20] A necessary condition is then:

$$F_H'(\alpha^+) = 0 \tag{19}$$

For any given (α, β), (16)–(19) pin down the four unknown constants, determining F_L and F_H in the no-trade region, which completes the construction of the seller's value function generated by $\Xi(\alpha, \beta)$.

3.2 Identifying Equilibrium α and β

We are left to determine which (if any) values of (α, β) are consistent with equilibrium. The key will be two *smooth pasting* conditions, which are required to ensure that the seller's strategy solves (SP_θ) and that W satisfies the No Deals condition.

First, consider the high type at $z = \beta$. If $F_H'(\beta^-) < \Psi'(\beta)$, then a convex combination of $F_H(\beta - \epsilon)$ and $\Psi(\beta + \epsilon)$ is strictly greater then $F_H(\beta) = \Psi(\beta)$. This implies that the high type can profitably deviate by rejecting at all $z \in [\beta, \beta + \delta)$ for sufficiently small δ. On the other hand, if $F_H'(\beta^-) > \Psi'(\beta)$, then there exists an ϵ such that $F_H(\beta - \epsilon) < \Psi(\beta - \epsilon)$, violating No Deals. Next, consider the low type at $z = \alpha$. Because $F_L(\alpha) = V_L$ and Z reflects at α, if $F_L'(\alpha^+) > 0$ then rejecting at α leads to a strictly higher payoff violating her indifference. On the other hand, if $F_L'(\alpha^+) < 0$, then again No Deals is violated (there exists $\epsilon > 0$ such that $F_L(\alpha + \epsilon) < V_L$).

The arguments above imply the following two additional boundary conditions are neces-

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[19] Polyanin and Zaitsev (2003, p. 215)

[20] See Harrison (1985, Chap. 5) for a discussion of necessary boundary conditions for a function of a reflected process.

sary for $\Xi(\alpha, \beta)$ to be an equilibrium.[21]

$$
\begin{aligned}
F_L'(\alpha^+) &= 0 && (20)\\
F_H'(\beta^-) &= \Psi'(\beta) && (21)
\end{aligned}
$$

The question then is whether these conditions can be satisfied simultaneously. To answer this, in Appendix B, we construct two mappings: B_L and B_H. For each type, $B_\theta(\alpha)$ is the value of β such that the strategy S^θ as given in $\Xi(\alpha, \beta)$ solves the type-θ sellers problem starting from any z. Thus, the three boundary conditions pertaining to type θ are satisfied at each $(\alpha, B_\theta(\alpha))$. Further, any intersection of B_L and B_H is a solution to the system (16)–(21). An intuitive interpretation is as follows. The low type is all too eager to accept $\Psi(\beta)$ at β. The relevant part of her stopping problem is when to give up and accept V_L—this is $B_L^{-1}(\beta)$.[22] For the high type, the problem is the reverse. She never accepts V_L, and must decide where to accept $\Psi(z)$.[23] Where she will optimally accept is influenced by the location of the reflecting barrier α—this is $B_H(\alpha)$. Hence, the optimal stopping *policy* of each type influences the stopping *problem* faced by the other.

Whether an intersection exists depends on parameters. As discussed in the Introduction, of particular relevance is whether a standard static adverse selection problem can arise depending on the market belief. We define the condition as follows:

Definition 3.2. The *Static Lemons Condition (SLC)* holds if and only if $K_H > V_L$.

Given a prior π_0, trade breaks down in the static model when $K_H > E[V_\theta|\pi_0]$. Hence, the SLC implies that there exists at least *some* non-degenerate π_0 such that there is market breakdown in a static setting. We define it in this way because we will characterize the equilibrium for all possible beliefs.

Lemma 3.1.

- *If the SLC holds, then there exists a unique pair* (α^*, β^*) *such that* $\beta^* = B_L(\alpha^*) = B_H(\alpha^*)$.
- *If the SLC does not hold, then there exists a* $\underline{\gamma} > 0$ *such that for all* $\gamma > \underline{\gamma}$*, there exists*

---

[21] See Shiryaev (1978, Sect. 3.8) for a more formal treatment of the necessity of smooth pasting conditions or Dixit (1993, Sect. 4.1) for a more intuitive exposition.

[22] B_L is strictly increasing (Lemma B.1), making this well defined.

[23] While we have specified that $w(z) < \Psi(z)$ for all $z < \beta$, this can only be part of equilibrium if $F_H(z) \geq \Psi(z)$. In other words, it must be that $w(z) = \Psi(z)$ if the high type will accept it, so we can think of the high type as deciding when to stop and accept $\Psi(z)$.

a unique pair (α^*, β^*) *such that* $\beta^* = B_L(\alpha^*) = B_H(\alpha^*)$.[24]

Which brings us to the main result of this section:

Theorem 3.1.

- *If the SLC holds, then* $\Xi(\alpha^*, \beta^*)$ *is an equilibrium.*
- *If the SLC does not hold and* $\gamma > \underline{\gamma}$, *then* $\Xi(\alpha^*, \beta^*)$ *is an equilibrium.*

Notice that Theorem 3.1 does not specify an equilibrium for all parameter configurations (namely, if the SLC fails and $\gamma \leq \underline{\gamma}$). Using a constructive proof, we demonstrate equilibrium existence for all parameters in Proposition 5.1. As $\Xi(\alpha^*, \beta^*)$ is our main equilibrium of interest, we postpone this analysis until Section 5.

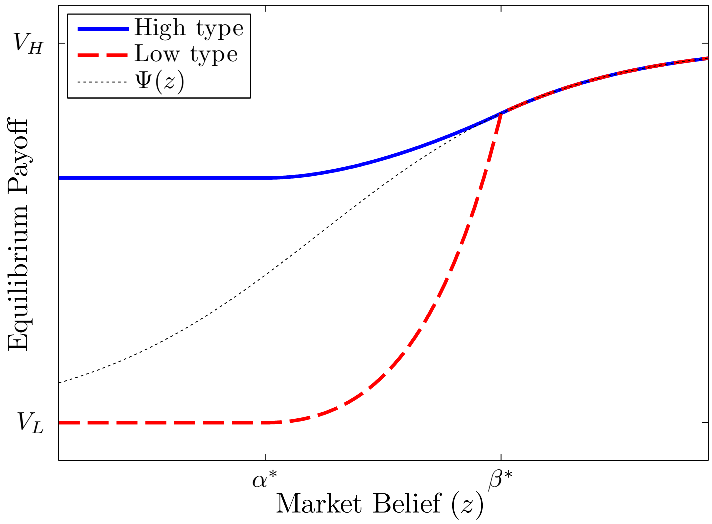

FIGURE 2 – Equilibrium Value Functions under $\Xi(\alpha^*, \beta^*)$

Intuition for the trading dynamics of $\Xi(\alpha^*, \beta^*)$ is as follows. When beliefs are very favorable the seller has little to gain and a high cost from waiting. Trade occurs immediately at expected market value. As beliefs become less favorable, the market shuts down and waits for more news before making serious offers. In this region, the high type will not accept $\Psi(z)$ because the combination of her flow payoff and the option value of trading in the future is more attractive. The low type would be happy to accept $\Psi(z)$; however the combination of *her* flow payoff and the option to trade in the future is more attractive than an offer of V_L.

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[24] The closed-form expression for $\underline{\gamma}$, derived in Lemma B.2 (Appendix B.3), is $\underline{\gamma} = \frac{2}{\underline{q}_1(1+\underline{q}_1)}$ where $\underline{q}_1 \equiv \frac{V_H+V_L-2K_H-2\sqrt{(V_L-K_H)(V_H-K_H)}}{V_H-V_L}$.

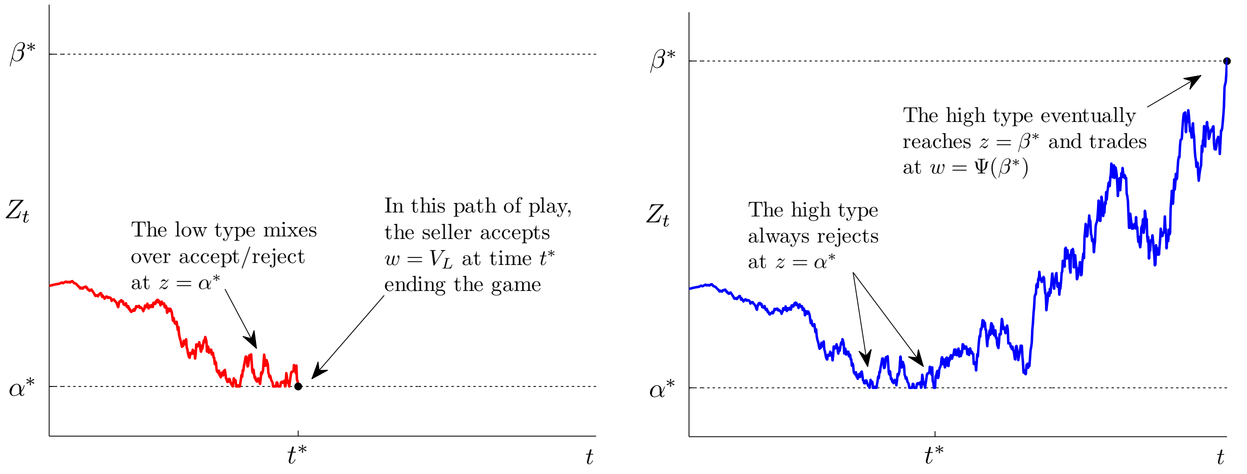
Any offer that would be accepted would also earn the buyer a negative expected payoff. As the belief decreases, so too does the low type's option value. The belief where she is just indifferent between accepting V_L and delaying trade is α^*. For $z < \alpha^*$ the low-type seller mixes between accepting and rejecting V_L in a way such that, conditional on not observing trade, the market belief jumps instantaneously to α^*. In economic terms, not selling when the market is pessimistic is an imperfect signal of high value.

The characterization of the equilibrium implies that the low-type seller trades with probability zero at $z = \alpha^*$. This is true; the low type cannot trade with an atom at α^* because then, conditional on rejection, beliefs would instantaneously jump upward, in which case the low type would have strictly preferred to reject at α^*. On the other hand, if the low-type seller never traded at the lower boundary then beliefs would sometimes drift below α^*, which takes strictly positive time. This would impose a cost on the low type and cause F_L to drop below V_L violating No Deals. Clearly neither of these can be part of the equilibrium. Hence, the low type's strategy at the lower boundary acts as a regulator for the equilibrium belief process. She mixes in a way such that prior to trade, the equilibrium belief process never drops below α^*.

Proposition 3.1. *Starting at $Z_t = \alpha^*$, the probability that the low type trades at a price of V_L before time $t + \Delta$ is approximately $\phi\sqrt{2\Delta/\pi}$ for Δ small.*

To understand the behavior of the equilibrium belief process at α^*, it is useful to consider a discrete-time analog of the game where news arrives according to a binary random walk (as studied in the Supplemental Appendix). Suppose that at time t, $Z_t = \alpha^*$. In the next short period of time (Δ), either good news or bad news will be revealed about the seller. If good news arrives, Bayesian updating leads to $Z_{t+\Delta} = \alpha^*_+ > \alpha^*$ and no trade occurs. If bad news arrives, updating leads to $Z_{t+\Delta} = \alpha^*_- < \alpha^*$. At $Z_{t+\Delta} = \alpha^*_-$ the low type accepts V_L with probability $\frac{p(\alpha^*)-p(\alpha^*_-)}{p(\alpha^*)(1-p(\alpha^*_-))}$, where $p(z) = e^z/(1+e^z)$. Conditional on rejection, beliefs jump back to α^*. As $\Delta \to 0$, α^*_- and $\alpha^*_+ \to \alpha^*$, which becomes a reflecting barrier for the continuous-time process Z until trade occurs.

Figure 3 illustrates play according to $\Xi(\alpha^*, \beta^*)$ for a single sample path of the news process. In the left panel, $\theta = L$ and the seller rejects V_L at $z = \alpha^*$ until time t^* at which point she accepts. In the right panel, $\theta = H$ and the seller rejects offers until reaching β^*. Because the belief process reflects off the lower barrier, the high type imperfectly signals her value to the market by rejecting offers when beliefs are unfavorable.



(A) The low type may eventually trade at $z = \alpha^*$ (B) The high type never accepts an offer for $z \leq \beta^*$

FIGURE 3 – Equilibrium dynamics for a fixed sample path

3.3 Verification

We now verify that $\Xi(\alpha^*, \beta^*)$ constitutes an equilibrium. By construction, the belief process is consistent with the specified strategies on the equilibrium path. To see this, note that if $S^L_{t^-} \cdot S^H_{t^-} < 1$ then $S^H_{t^-} = 0$ and $S^L_{t^-} = 1 - e^{-Q^{\alpha^*}_t}$. Therefore, $Z_t = \hat{Z}_t + Q^{\alpha^*}_t$ satisfies Belief Consistency. The Zero Profit condition is immediate since only the low type trades with positive probability for $z \leq \alpha^*$ where the offer is V_L, and both types trade with probability one for $z \geq \beta^*$ where the offer is $\Psi(z)$.

Next, we demonstrate that No Deals is satisfied. For $z \geq \beta^*$, this is immediate as $F_L(z) = F_H(z) = \Psi(z)$. The following lemma establishes that No Deals is satisfied for all z in the no-trade region.

Lemma 3.2. *The value functions implied by* $\Xi(\alpha^*, \beta^*)$ *satisfy:*

- $F_L(z) > V_L$ *for all* $z \in (\alpha^*, \beta^*)$
- $F_H(z) > \Psi(z)$ *for all* $z \in (\alpha^*, \beta^*)$

For all $z \leq \alpha^*$, $F_L(z) = V_L$, and $F_H(z) = F_H(\alpha^*) > \Psi(\alpha^*) \geq \Psi(z)$, verifying that No Deals is satisfied in all states.

Finally, we argue that, for each type, the seller's strategy is optimal. To do so, fix W and Z as in $\Xi(\alpha^*, \beta^*)$. The seller's problem (SP_θ) can be written as

$$F^*_\theta(z) = \sup_{\tau \geq 0} E^\theta_z \left[\int_0^\tau e^{-rt} r K_\theta dt + e^{-r\tau} w(Z_\tau) \right] \tag{22}$$

Recall that the seller's payoff from following S^θ starting from some initial state z is $F_\theta(z)$. Since S^θ is a feasible strategy, $F_\theta^*(z) \geq F_\theta(z)$. We wish to demonstrate that $F_\theta^*(z) = F_\theta(z)$. To do so, consider adding to the game a third-party intermediary who offers to "buy" the type-θ seller's problem for $F_\theta(Z_t)$ at any time $t \geq 0$. That is, in the amended game with the intermediary, at every time t, the type-θ seller receives the offer W_t from the buyers *and* the offer $F_\theta(Z_t)$ from the intermediary. Her value function in the stopping problem with the intermediary is

$$G_\theta^*(z) = \sup_{\tau \geq 0} E_z^\theta \left[\int_0^\tau e^{-rt} r K_\theta dt + e^{-r\tau} \max\{w(Z_\tau), F_\theta(Z_\tau)\} \right] \tag{23}$$

Clearly, the intermediary cannot make the seller worse off, so $G_\theta^*(z) \geq F_\theta^*(z)$. The next result implies that the intermediary also does not make the seller any better off.

Lemma 3.3. *In the game with the intermediary, the type-θ seller can do no better than to accept F_θ immediately: $G_\theta^* = F_\theta$ for $\theta = L, H$.*

It follows that *in the true game* a profitable deviation from S^θ does not exist; if a higher expected payoff was attainable in the true game, then it also would have been attainable in the game with the intermediary.

4 News Quality, Welfare and Efficiency

This section examines how properties of the equilibrium $\Xi(\alpha^*, \beta^*)$ depend on the quality of news γ. We first introduce measures of surplus and efficiency. The seller can guarantee herself a payoff of K_θ, even in the absence of buyers. Therefore let $\Pi^S(z)$ denote the surplus obtained by the seller side of the market in state z

$$\Pi^S(z) \equiv p(z)(F_H(z) - K_H) + (1 - p(z))(F_L(z) - K_L)$$

where $p(z) = e^z/(1 + e^z)$. Because buyers make zero expected profit, $\Pi^S(z)$ is also the *total* surplus attained in state z. Due to common knowledge of gains from trade, the efficient outcome is to trade immediately, resulting in a potential surplus of

$$\Pi^*(z) \equiv \Psi(z) - p(z)K_H - (1 - p(z))K_L$$

Hence, $\Pi^*(z) - \Pi^S(z) \geq 0$, and any strictly positive difference is the efficiency loss from the expected delay of trade.

Using $\Pi^*(z) - \Pi^S(z)$ to measure efficiency presents the following problem. Because Π^* can vary with z, we may wish to interpret $\Pi^*(z) - \Pi^S(z) = 1$ differently if $\Pi^*(z) = 2$ than if $\Pi^*(z) = 200$. Therefore, define the percentage loss in efficiency as a function of z by

$$\mathcal{L}(z) \equiv \frac{\Pi^*(z) - \Pi^S(z)}{\Pi^*(z)}$$

We first characterize the limit properties of the equilibrium as news quality becomes arbitrarily high or low. We then illustrate the differences in equilibrium behavior, welfare and efficiency between a world with no news ($\gamma = 0$) and a world with news ($\gamma > 0$). The section concludes with a discussion of how the properties of $\Xi(\alpha^*, \beta^*)$ vary with γ more generally, relying in part on numerical findings.

4.1 In the Limit

The following proposition characterizes the limit properties of the $\Xi(\alpha^*, \beta^*)$ as news quality becomes arbitrarily high. Let $\overset{pw}{\to}$ and $\overset{u}{\to}$ denote pointwise and uniform convergence respectively.

Proposition 4.1. *In* $\Xi(\alpha^*, \beta^*)$, *as* $\gamma \to \infty$,

1. $\beta^* \to \infty$
2. $\alpha^* \to \ln\left(\frac{V_L - K_L}{V_H - K_H}\right)$
3. $F_H \overset{u}{\to} V_H$
4. $F_L \overset{pw}{\to} V_L$
5. $\mathcal{L} \overset{u}{\to} 0$

Properties 1 and 3–5 are intuitive: as news quality becomes arbitrarily large, the high type waits until the market is virtually sure of her type, each type expects a payoff arbitrarily close to her true market value, and inefficiency is eliminated. The disparity between the strength of convergence for F_L and F_H is due to the fact that, even for large γ, $F_L(z) = \Psi(z)$ for all $z \geq \beta^*$, meaning the convergence of F_L to V_L is only pointwise.

Property 2 answers a question that was less obvious *a priori*. Namely, what becomes of the partial sell-off feature of $\Xi(\alpha^*, \beta^*)$ as γ limits to infinity? Intuition might suggest that as news quality becomes arbitrarily high, the market will rely solely on the news to separate the types (corresponding to $\alpha^* \to -\infty$). In fact, as $\gamma \to \infty$, there are two countervailing effects: $\beta^* \to \infty$ (Proposition 4.1(1)) *and* $(\beta^* - \alpha^*) \to \infty$. The reason that $(\beta^* - \alpha^*) \to \infty$ as $\gamma \to \infty$ is that the expected time spent in any finite-width no-trade region goes to zero. Hence, for the necessary value-matching conditions (16) and (17) to hold, $(\beta^* - \alpha^*)$ must tend to infinity. That $\lim_{\gamma\to\infty} \alpha^*$ is finite demonstrates that neither effect completely dominates the other in the limit. Even as $\gamma \to \infty$, the market relies both on partial sell-offs and on the news to separate the types.

We now turn to the other extreme: $\gamma \to 0$. Such a discussion is valid only if the SLC holds as no equilibrium of the Ξ-form exists if the SLC fails and γ is arbitrarily small (see Proposition 5.2). Define $\underline{z}$ to be the unique z such that $\Psi(z) = K_H$.[25]

Proposition 4.2. *If the SLC holds, then in $\Xi(\alpha^*, \beta^*)$, as $\gamma \to 0$,*

1. $\beta^* \to \underline{z}$
2. $\alpha^* \to \underline{z}$
3. $F_H \overset{u}{\to} \max\{K_H, \Psi(z)\}$
4. $F_L \overset{pw}{\to} \begin{cases} V_L & \text{if } z < \underline{z} \\ \Psi(z) & \text{if } z > \underline{z} \end{cases}$
5. $\mathcal{L} \overset{pw}{\to} \begin{cases} \frac{p(z)(V_H - K_H)}{\Pi^*(z)} & \text{if } z < \underline{z} \\ 0 & \text{if } z > \underline{z} \end{cases}$

To understand these results, consider the seller's *expected discount factor* to reach β^*. Let $T(\beta^*) = \inf\{t : Z_t \geq \beta^*\}$. Starting from any state z, the expected discount factor, $E_z^\theta[e^{-rT(\beta^*)}]$, captures how worthwhile it is to wait until Ψ is offered. In Appendix A.1, we show that $F_\theta(z) = K_\theta + E_z^\theta[e^{-rT(\beta^*)}](\Psi(\beta^*) - K_\theta)$ for all $z \leq \beta^*$. Now, suppose that $(\beta^* - \alpha^*)$ does not tend to zero as $\gamma \to 0$. Then, for any z below the limit β^*, $E_z^L[e^{-rT(\beta^*)}] \to 0$, implying that $F_L(z) \to K_L$, violating No Deals. Hence, the no-trade region must collapse.[26]

Maintaining low-type indifference also implies that $E_{\alpha^*}^L[e^{-rT(\beta^*)}]$ does *not* converge to one (otherwise $\lim_{\gamma\to 0} F_L(\alpha^*) = \Psi(\beta^*) > V_L$). Finally, as $\gamma \to 0$ the expected discount factors of the two types converge to one another—when news quality is low, there is little difference in the news each type expects to be revealed. Hence, in the limit the high type still has non-trivial waiting costs starting at α^*. This is why the no-trade region must collapse to $\underline{z}$: for the high type to be willing to forego $\Psi(z)$ in the no-trade region, even though $\Psi(\beta^*)$ is only arbitrarily better, it must be that neither are much better than her holding value, K_H.

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[25] Therefore, $\underline{z}$ exists if and only if the SLC holds, and is equal to $\ln\left(\frac{K_H - V_L}{V_H - K_H}\right)$.

[26] Given that the left and right limits of the limit of F_L disagree at $\underline{z}$, it should be expected that there are some delicacies here, which we omit for parsimony. However, it is immediate that $F_L(\underline{z})$ tends to a limit in $[V_L, K_H]$, and that for any subsequence of γ converging to 0 such that $\alpha^* \geq \underline{z}$ for all γ, $F_L(\underline{z}) \to V_L$. In addition, in the equilibrium constructed for $\gamma = 0$, the low type's value at $\underline{z}$ is V_L.

We can construct an equilibrium that delivers precisely these payoffs, and is a natural extension of $\Xi(\alpha^*, \beta^*)$, when the SLC holds and $\gamma = \phi = 0$. The equilibrium is nearly given by $\Xi(\underline{z}, \underline{z})$, requiring only a modification of W to[27]

$$W_t = \begin{cases} \Psi(Z_t) & \text{if } Z_t > \underline{z} \text{ or both } Z_t = \underline{z} \text{ and } t \geq \mathsf{t} \\ V_L & \text{otherwise} \end{cases}$$

where t is a Poisson random variable with arrival rate $\frac{r(V_L - K_L)}{K_H - V_L}$. The random arrival of the "high offer" when $Z_t = \underline{z}$ is akin to having buyers play mixed strategies. The specific arrival rate is necessary to ensure that the low type is indifferent between accepting V_L at $Z_0 \leq \underline{z}$ or waiting for the high offer.[28]

One can notice that the limit value functions, and, equivalently, the value functions endowed by the $\gamma = 0$ equilibrium, are identical to the payoffs that the seller would obtain in Akerlof's *static* model. When the prior is $p(z)$: for $z < \underline{z}$ the high type's payoff is K_H because she does not sell and the low type gets her market value of V_L, while for $z > \underline{z}$ both types trade and receive payoff $\Psi(z)$.[29] In a heuristic sense, the static model corresponds to $\gamma = 0$ because the agents are arbitrarily impatient, meaning the limit value functions match value functions from the limit game, regardless of how $\gamma \to 0$.

4.2 Introducing News

Assume the SLC holds. Having just seen equilibrium properties when $\gamma = 0$, we wish to understand the welfare and efficiency implications of introducing news (i.e., $\gamma > 0$) and thereby transitioning to $\Xi(\alpha^*, \beta^*)$, with $\beta^* > \max\{\underline{z}, \alpha^*\}$.[30]

Intuition may suggest that a market with better information should be more efficient. After all, asymmetry of information between buyers and the seller is the sole cause of delay, and indeed the market becomes fully efficient in the limit (Proposition 4.1). This intuition proves correct if (and only if) the market would have suffered from inefficiency, due to a static adverse selection problem, in the absence of news: that is, if $z < \underline{z}$. In fact, the introduction of news weakly improves the welfare of both types if $z < \underline{z}$.

News is detrimental to efficiency for all $z \in (\underline{z}, \beta^*)$; introducing news creates a new incentive for the high type to wait, expecting a higher offer tomorrow and leading to $\beta^* > \underline{z}$. States in $(\underline{z}, \beta^*)$ transition from fully efficient trade (when $\gamma = 0$) to *no trade* and strictly positive efficiency loss from delay (when $\gamma > 0$). Thus, news *decreases* efficiency for these

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[27] Notice that the profile $\Xi(\underline{z}, \underline{z})$ emits well-defined beliefs and seller strategies from (7), (8) and (9).

[28] The verification that this constitutes an equilibrium is straightforward.

[29] Given our assumption that buyers are on the "long side" of the market.

[30] For any $\gamma > 0$, Lemma 3.1 shows that $\beta^* > \alpha^*$, and Lemma B.2 shows that $\beta^* > \underline{z}$. Section 4.3 shows that α^* can be greater or less than $\underline{z}$.

intermediate beliefs by providing incentive for the high type to delay trade to attain a higher offer. While efficiency may increase or decrease depending on the market belief, the high type's value function is everywhere (at least) weakly higher with news. The low type can be made better or worse off because news negatively influences the market belief regarding her type, but can also decrease her loss from inefficient delay.

The following corollary summarizes the implications of introducing news and follows routinely from the structure of the $\Xi(\alpha^*, \beta^*)$ and the $\gamma = 0$ equilibrium.

Corollary 4.1. *Fix all parameters such that the SLC holds. Let $F_H^0, F_L^0, \mathcal{L}^0$ correspond to the value functions and percentage loss for the $\gamma = 0$ equilibrium. Let $F_H, F_L, \mathcal{L}$ correspond to the equilibrium value functions and percentage loss for $\Xi(\alpha^*, \beta^*)$ given an arbitrary $\gamma > 0$. Then,*

1. *Efficiency:*
    - (a) $\mathcal{L}(z) < \mathcal{L}^0(z)$ *for* $z \leq \underline{z}$
    - (b) $\mathcal{L}(z) \geq \mathcal{L}^0(z)$ *for all* $z > \underline{z}$, *where the inequality is strict if and only if* $z \in (\underline{z}, \beta^*)$
2. *Welfare:*
    - (a) $F_H(z) \geq F_H^0(z)$, *where the inequality is strict if and only if* $z < \beta^*$
    - (b) *If* $\alpha^* \geq \underline{z}$, *then* $F_L(z) \leq F_L^0(z)$ *for all* z, *where the inequality is strict if and only if* $z \in (\underline{z}, \beta^*)$
    - (c) *If* $\alpha^* < \underline{z}$, *then* $F_L(z) \geq F_L^0(z)$ *for all* $z \leq \underline{z}$ *and* $F_L(z) \leq F_L^0(z)$ *for all* $z > \underline{z}$, *where the inequalities are strict if and only if* $z \in (\alpha^*, \beta^*)$

4.3 General Changes in News Quality

We now investigate how the properties of $\Xi(\alpha^*, \beta^*)$ vary with γ more generally. Our first result is that both the upper boundary and the width of the no-trade region must increase with γ.

Proposition 4.3. *β^* and $(\beta^* - \alpha^*)$ are strictly increasing in γ.*

As shown in Figure 4, α^* can be increasing, decreasing or even non-monotonic in γ. That α^* can increase or decrease should not be surprising since, as demonstrated by Propositions 4.1 and 4.2, $\lim_{\gamma\to 0} \alpha^* > \lim_{\gamma\to\infty} \alpha^*$ if and only if $K_H - V_L > V_L - K_L$.

Because of their interdependence, intuition for how α^* and β^* vary with news quality is a little more subtle than it may first appear. Start with $\Xi(\alpha_0^*, \beta_0^*)$ for some $\gamma_0 > 0$ and consider an increase in news quality to γ_1. If Q and W remained as in $\Xi(\alpha_0^*, \beta_0^*)$, both types of seller would deviate from their strategies in $\Xi(\alpha_0^*, \beta_0^*)$:

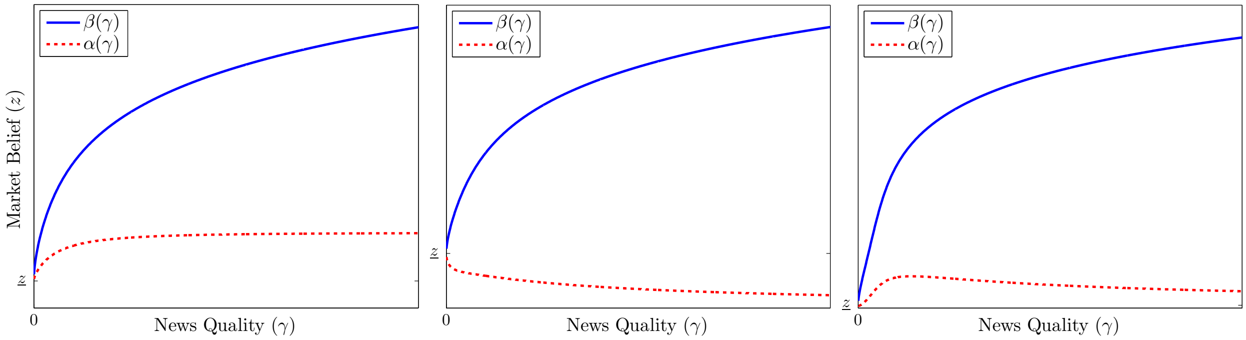
- Because the high type expects the belief to increase more quickly under γ_1, she strictly prefers to reject at $z = \beta_0^*$.
- Because the low type's expected discount factor to reach β_0^* starting from α_0^*, $E^L_{\alpha_0^*}[e^{-rT(\beta_0^*)}]$, increases, she strictly prefers to reject V_L at $z = \alpha_0^*$.

Consider now keeping the lower boundary fixed at α_0^*, but adjusting the upper boundary to where the high type is indifferent between accepting Ψ or not (i.e., $B_H(\alpha_0^*|\gamma_1) > \beta_0^*$). This has two effects on the low type's incentives at $z = \alpha_0^*$. First, it decreases the expected discount factor for the low type starting from $z = \alpha_0^*$, making acceptance of V_L at $z = \alpha_0^*$ more attractive. Second, because Ψ is increasing, it leads to a higher offer upon reaching the upper boundary, making acceptance of V_L at $z = \alpha_0^*$ less attractive.

Depending on parameters, these forces can have different relative strengths and can lead to α_1^* being greater than, less than, or equal to α_0^*. However, even if these forces result in a decrease in α^*, which (in isolation) makes waiting less attractive for the high type, it is never strong enough to undo the primary effect; increasing γ gives the high type more incentive to wait leading to a higher β^*.

Let us conclude with a discussion of the welfare and efficiency implications of general changes in news quality. Again, consider a change from $\gamma_0 > 0$ to $\gamma_1 > \gamma_0$. We partition into two cases: either $\alpha_1^* \geq \alpha_0^*$ or not. For the first case we have the following:

Proposition 4.4. *Consider two news qualities $\gamma_0 < \gamma_1$, such that $\Xi(\alpha_i^*, \beta_i^*)$ is an equilibrium when $\gamma = \gamma_i$. Let F^i_θ and $\mathcal{L}^i$ denote the respective value functions and percentage loss in equilibrium for $\gamma = \gamma_i$. Then, if $\alpha_1^* \geq \alpha_0^*$,*



(A) $K_L = -\frac{1}{2}$, $K_H = \frac{1}{4}$ (B) $K_L = -\frac{1}{10}$, $K_H = \frac{1}{4}$ (C) $K_L = -\frac{1}{20}$, $K_H = \frac{1}{20}$

FIGURE 4 – Equilibrium boundaries as they depend on γ for three different values of (K_L, K_H), with $V_L = 0, V_H = 1$, and γ ranging from 0 to 20.

1. *For all* z, $F^1_H(z) \geq F^0_H(z)$, *where the inequality is strict if and only if* $z < \beta^*_1$.

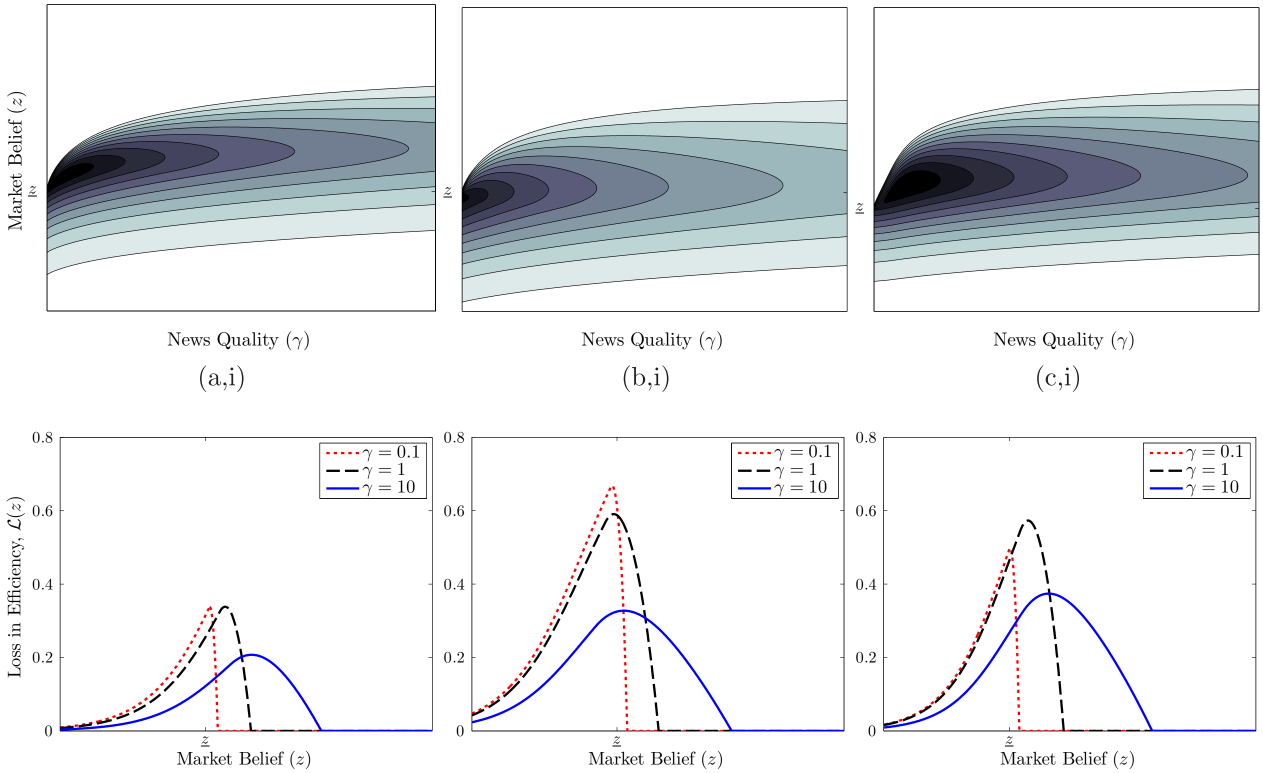
2. *For all* z, $F^1_L(z) \leq F^0_L(z)$, *where the inequality is strict if and only if* $z \in (\alpha^*_0, \beta^*_1)$.

Moreover, there exists a $z' \in (\alpha^*_0, \beta^*_0)$ *such that*

3. $\mathcal{L}^1(z) < \mathcal{L}^0(z)$ *for* $z < z'$.

4. $\mathcal{L}^1(z) > \mathcal{L}^0(z)$ *for* $z \in (z', \beta^*_1)$.

As in Corollary 4.1, an increase in news quality is weakly beneficial (detrimental) to the high (low) type, and inefficiency decreases for low beliefs and increases for intermediate ones. If $\alpha^*_1 < \alpha^*_0$, the analysis is less tractable, and we turn to numerical results. In all such examples we computed, the only change from Proposition 4.4 is in (2). Instead of the low type being weakly worse off for all beliefs, there exists $z'' \in (\alpha^*_0, \beta^*_0)$ such that $F^1_L(z) \geq F^0_L(z)$ for all $z \leq z''$ and $F^1_L(z) \leq F^0_L(z)$ for all $z > z''$ with the inequalities strict if and only if $z \in (\alpha^*_1, z'') \cup (z'', \beta^*_1)$, which can be viewed as the generalization of Corollary 4.1(2c).



(a,i) (b,i) (c,i)

(a,ii) (b,ii) (c,ii)

FIGURE 5 – The top row shows a contour of $\mathcal{L}$ as it depends on z and γ with darker shades indicating greater loss. The bottom row illustrates $\mathcal{L}$ for three different levels of news quality.

Figure 5 illustrates many of the properties we have just discussed for the same three parameter specifications used in Figure 4. The top row shows a contour of the inefficiency as it depends on z and γ with darker shades corresponding to greater loss. Notice that the inefficiency is most severe for small γ and $z \approx \underline{z}$. As γ increases the inefficiency becomes more diffuse over states and less severe in the most inefficient ones. The bottom row of the figure illustrates the loss in efficiency as a function of the market belief for three different levels of news quality. Notice that the loss in surplus shifts towards higher beliefs as γ increases. As most apparent in Figure 5(c,ii), an increase in the quality of the news from $\gamma = 0.1$ to $\gamma = 1$ provides little gain in surplus for $z < \alpha^*_{\gamma=0.1}$ and substantial loss in surplus for $z \in (\beta^*_{\gamma=0.1}, \beta^*_{\gamma=1})$.

5 Stationary, Belief-Monotone Equilibria

$\Xi(\alpha^*, \beta^*)$ is not the unique equilibrium according to Definition 2.1. In this section we define a *Stationary, Belief-Monotone* (SBM) equilibrium and demonstrate three main results:

(i) If the SLC holds, then $\Xi(\alpha^*, \beta^*)$ is the unique SBM-equilibrium.

(ii) Without the SLC, other SBM-equilibria can exist. We investigate how behavior can differ from $\Xi(\alpha^*, \beta^*)$ and identify features common to all SBM-equilibria. In particular, a no-trade region must exist as long as news quality is high enough, and equilibrium payoffs converge to the full-information (and fully efficient) payoffs as news quality becomes arbitrarily high.

(iii) Without the SLC, and with sufficient news quality, $\Xi(\alpha^*, \beta^*)$ is the unique SBM-equilibrium with the property that "good news" (i.e, $d\hat{Z} > 0$) is never harmful to the seller.

Informally, an equilibrium is stationary if, after any history, the current belief is a sufficient summary of all past play. It is belief monotone if rejection by the seller is not inferred to be a signal of low asset value.

Without Belief Monotonicity many unappealing equilibria can be sustained by "threat beliefs" (e.g., a seller who does not accept immediately is considered to be a low type with probability one). For example, suppose the SLC holds and consider the candidate $\Xi'(\alpha, \beta)$ which is identical to $\Xi(\alpha, \beta)$ in all respects *except* that off the equilibrium path $Z_t = -\infty$. For any $\beta \in (\underline{z}, \beta^*)$, $\Xi'(B_L^{-1}(\beta), \beta)$ is an equilibrium. In this equilibrium, the high type is "forced" to trade sooner (i.e., at $z \in (\beta, \beta^*)$) because failure to do so convinces buyers that the seller is the low type for sure (nullifying the effect of any future news).[31] We restrict

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[31] For a non-belief-monotone equilibrium when the SLC does not hold: $W_0 = \Psi(Z_0)$, $W_t = V_L$ for all $t > 0$, and $S_0^H = S_0^L = 1$ is an equilibrium profile for any γ supported by the (consistent) off-path belief $Z_t = -\infty$

attention to stationary equilibria for tractability.

5.1 Definitions and Preliminaries

We first enrich the definition of strategy to specify behavior off the equilibrium path.

Definition 5.1. A *strategy* for a type-θ seller is a family of stochastic processes $\{S^{\theta,t}\}_{t\in\mathbb{R}_+}$ satisfying the following:

1. Measurable Distribution Functions: For all $t_0 \geq 0$, $S^{\theta,t_0} = \{S_t^{\theta,t_0}, t_0 \leq t \leq \infty\}$ is a right-continuous, non-decreasing, $\mathcal{H}_t$-adapted process on $[0,1]$.
2. Self-Consistency: For any $t_1 \geq t_0 \geq 0$, let $S_{t_1^-}^{\theta,t_0} \equiv \lim_{s\uparrow t_1} S_s^{\theta,t_0}$ and specify that $S_{t_0^-}^{\theta,t_0} = 0$. If $S_{t_1^-}^{\theta,t_0} < 1$, it must be that for all $t \geq t_1$,

$$S_t^{\theta,t_1} = \frac{S_t^{\theta,t_0} - S_{t_1^-}^{\theta,t_0}}{1 - S_{t_1^-}^{\theta,t_0}}$$

For any $t \geq 0$, $S^{\theta,t}(\omega)$ is a CDF over the type-θ seller's acceptance time on $[t,\infty]$ along the sample path $X(\omega,\theta)$, *given* that the game has not stopped at any time $t' < t$ (regardless of whether this is on the equilibrium path or not). Let $\mathcal{S}^{\theta,t} = supp(S^{\theta,t})$. In words, Self-Consistency mandates the following natural condition. If, given a fixed sample path of X, at time t_0 the seller chose a CDF that assigned positive probability to rejecting all offers before time $t_1 \geq t_0$, then the CDF she chooses at t_1 must be the CDF from time t_0 conditioned on having reached t_1. Notice that S^θ, which we have used to denote the seller's strategy prior to now, is simply $S^{\theta,0}$. With this extended notion of strategy, we can extend the notion of belief consistency from (3) as follows. Beliefs will be consistent starting from any history if for all $t_1 \geq t_0 \geq 0$ such that $S_{t_1^-}^{L,t_0} \cdot S_{t_1^-}^{H,t_0} < 1$

$$Z_{t_1} = Z_{t_0} + \ln\left(\frac{f_{t_1-t_0}^H(X_{t_1}-X_{t_0})}{f_{t_1-t_0}^L(X_{t_1}-X_{t_0})}\right) + \ln\left(\frac{1-S_{t_1^-}^{H,t_0}}{1-S_{t_1^-}^{L,t_0}}\right) \tag{24}$$

Self-Consistency ensures that (24) produces the same value of Z_{t_1} for any such t_0.

For any given t, if the game has not yet ended (regardless of whether this is on the equilibrium path or not), the seller faces an optimal stopping problem.

$$\sup_{\tau\geq t} E^\theta\left[\int_t^\tau rK_\theta e^{-rs}ds + e^{-r(\tau-t)}W_\tau|\mathcal{H}_t\right] \tag{$SP_{\theta,t}$}$$

---

for all $t > 0$.

We can now define an SBM-equilibrium.

Definition 5.2. An *SBM-equilibrium* of the game is a quadruple $(\{S^{L,t}\}_{t\in\mathbb{R}_+}, \{S^{H,t}\}_{t\in\mathbb{R}_+}, W, Z)$, where $Z = \hat{Z} + Q$, such that

1. Seller Optimality: Given W, for every t, $S^{\theta,t}$ solves the type-θ seller's problem *($SP_{\theta,t}$)*
2. Belief Consistency: For all $t_1 \geq t_0 \geq 0$ such that $S^{L,t_0}_{t_1^-} \cdot S^{H,t_0}_{t_1^-} < 1$, Z_{t_1} is consistent as given by (24)
3. Zero Profit: For any $t_0 \geq 0$, if there exists a $\tau \in \mathcal{S}^{L,t_0} \cup \mathcal{S}^{H,t_0}$ such that $\tau(\omega) = t$ for some ω, then $W_t = E[V_\theta|\mathcal{H}_t, \tau^* = t]$
4. No Deals: For all θ, t, ω: $F_\theta(t,\omega) \geq E[V_{\theta'}|\mathcal{H}_t, V_{\theta'} \leq V_\theta]$
5. Stationarity: $W_t = w(Z_t)$ for some Borel-measurable function w, and Z is a time-homogenous $\mathcal{H}_t$-Markov process
6. Belief Monotonicity: Q is non-decreasing

Notice that the first four conditions are just the extensions of the conditions in Definition 2.1 to behavior and beliefs off the equilibrium path. Stationarity requires that both the current offer and the evolution of beliefs depend only on the current belief.[32] While it is conventional to define stationarity as a restriction on strategies, which then has implications for beliefs through the equilibrium consistency condition, it is much cleaner for us to do the reverse.[33] We continue to use z to refer to the state variable (distinguishing it from the belief process Z). In order to both motivate and interpret Belief Monotonicity, consider the following lemma which establishes basic properties of trading behavior. Let an *S-equilibrium* be defined by conditions 1-5 in Definition 5.2 (i.e., Belief Monotonicity is relaxed), and notice that because of the stationarity of the seller's problem we can write value functions as functions of the state variable $F_\theta(z)$.

Lemma 5.1. *The following properties are true in any S-equilibrium.*

a. For any state z, $F_L(z) \leq \Psi(z)$ and $F_H(z) \leq V_H$.

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[32] This implies that Z is a time-homogenous Markov process with respect to the seller's information as well. For any t, s, because the distribution of Z_{t+s} given $\mathcal{H}_t$ depends only on Z_t, the distribution of Z_{t+s} given $\mathcal{H}_t$ and θ depends only on Z_t and θ, since $X(\cdot,\theta)$ has stationary, independent increments.

[33] That is, an alternative condition for Stationarity would replace the restriction on Z with a more notationally cumbersome restriction on $\{S^{L,t}\}_{t\in\mathbb{R}_+}, \{S^{H,t}\}_{t\in\mathbb{R}_+}$. However, because (24) is undefined following an unexpected rejection, we would still need an additional condition that, roughly speaking, requires the evolution of beliefs following an unexpected rejection to depend only on the belief at the time of the rejection. We therefore find condition 5 more useful.

b. *In any state z, there are only two prices at which trade can occur: V_L and $\Psi(z)$.*

c. *For any state z at time t, if $w(z) = \Psi(z)$, then $S_{t_1}^{L,t_0} = S_{t_1}^{H,t_0} = 1$ for all $t_0 \leq t \leq t_1$.*

d. *For any times $t_0 \leq t_1$, if $w(z) = V_L$ at t_1 and $S_{t_1^-}^{L,t_0} < 1$, then $S_{t_1}^{L,t_0} < 1$.*

If the first property failed, then buyers would be subsidizing the seller violating Zero Profit. The next three properties give structure to how equilibrium trading behavior must transpire. For example, in any state, the high type either rejects or accepts w with probability one—she does not mix. Further, if the high type accepts so does the low type (with probability one). Hence, if an equilibrium prescribes rejection with positive probability in a certain state, Belief Consistency mandates that the act of rejection cannot decrease the market's belief. Thus, for any realization of X, Q is non-decreasing on the equilibrium path; on path, rejection is always a weakly positive signal of high value. We refer to this property as Belief Monotonicity *on path.* Condition 6 in Definition 5.2 extends this notion off the equilibrium path as well.

One last preliminary property will be useful for establishing further results.

Lemma 5.2. *In any SBM-equilibrium, F_L is continuous on $\mathbb{R}$.*

5.2 Results

Our first result is the existence of an SBM-equilibrium. The proof is constructive, relying on much of the subsequent analysis. We state the result here to avoid any ambiguity on the matter.

Proposition 5.1. *An SBM-equilibrium exists.*

Under the SLC

Definition 3.1 does not specify seller strategies off the equilibrium path. Given that $Z_t = \hat{Z}_t + Q_t^{\alpha^*}$ both on and off the equilibrium path, for $\Xi(\alpha^*, \beta^*)$ to be an SBM-equilibrium, strategies off the equilibrium path must be consistent with Z in the sense of (24). Therefore, extend the definition of $\Xi(\alpha^*, \beta^*)$ such that for all $0 < t_0 \leq t$,

$$S_t^{H,t_0} = \begin{cases} 1 & \text{if there exists } s \in [t_0, t] \text{ such that } Z_s \geq \beta^* \text{ and } W_s \geq \Psi(Z_s) \\ 0 & \text{otherwise} \end{cases}$$

$$S_t^{L,t_0} = \begin{cases} 1 & \text{if there exists } s \in [t_0, t] \text{ such that } Z_s \geq \beta^* \text{ and } W_s \geq \Psi(Z_s) \\ 1 - e^{-\left(Q_t^{\alpha^*} - Q_{t_0}^{\alpha^*}\right)} & \text{otherwise} \end{cases}$$

Theorem 5.1. *If the SLC holds, then $\Xi(\alpha^*, \beta^*)$ is the essentially unique SBM-equilibrium.*

The uniqueness is qualified with "essentially" only because the off-path evolution of Z when $z > \beta^*$, and $w(z)$ for $z \in (\alpha^*, \beta^*)$ are not uniquely pinned down. This indeterminacy has no effect on equilibrium outcomes or payoffs.

A brief intuition for the theorem is as follows. Lemma 5.1(b) establishes that, in any SBM-equilibrium, the highest offer that can be made in state z is $\Psi(z)$. Therefore, if $z < \underline{z}$, then the high type rejects any equilibrium offer. The same cannot be true for the low type, because then Belief Consistency would mandate that beliefs evolve based only on the realization of news when $z < \underline{z}$. But then $\lim_{z \to -\infty} E_z^L \left[e^{-rT(\underline{z})}\right] = 0$, implying that $\lim_{z \to -\infty} F_L(z) = K_L < V_L$, violating No Deals. Simply put, given that high type is holding out at least until $\underline{z}$, there are beliefs low enough such that the low type is willing to accept V_L.

On the opposite extreme but using a similar logic, there must exist a belief $z > \underline{z}$ such that the high type is willing to accept $\Psi(z)$. Of course, the low type is willing to accept as well, and therefore trade occurs with probability one in such states. Hence, there exist two states $z_1 < z_2$ such that $F_L(z_1) = V_L$ and $F_L(z_2) = \Psi(z_2)$. The continuity of F_L (Lemma 5.2) implies that there exists an interval $(\alpha, \beta) \subset (z_1, z_2)$ such that $V_L < F_L(z) < \Psi(z)$ for all $z \in (\alpha, \beta)$. The only behavior consistent with these conditions and Lemma 5.1 is for this interval to be a no-trade region. Belief Monotonicity together with Seller Optimality implies that the high type must be indifferent between rejecting or accepting $\Psi(\beta)$ at β (in contrast to the non-belief-monotone equilibria presented at the beginning of this section). The next step in the proof shows that *outside* of the no-trade region, only the behavior prescribed by $\Xi(\alpha, \beta)$ is consistent with SBM-equilibrium. Lastly, Lemma 3.1 establishes that (α, β) must be (α^*, β^*).

Without the SLC

Without the SLC, $\Xi(\alpha^*, \beta^*)$ may not be unique among SBM-equilibria (depending on parameters). To begin with, (α^*, β^*) does not exist unless γ is large enough. Notice that if $\gamma = 0$ (and the SLC fails), then our model is the (two-type) continuous-time analog of Swinkels (1999). Correspondingly, the unique SBM-equilibrium outcome is immediate trade, at expected market value, regardless of the market belief. This result holds even for positive, but small, values of γ.

Proposition 5.2. *If the SLC does not hold, then for all γ small enough, the essentially unique SBM-equilibrium is $Z_t = \hat{Z}_t$, $W_t = \Psi(Z_t)$ and $S_t^{L,t_0} = S_t^{H,t_0} = 1$ for all $t \geq t_0 \geq 0$.*

Without the SLC, if news quality is poor the market is fully efficient: $F_H = F_L = \Psi$ and $\mathcal{L} = 0$. Notice that these correspond to the analogous limit expressions when the SLC does hold *and* $z > \underline{z}$. Hence, as $\underline{z} \to -\infty$ (i.e., as the static adverse selection problem disappears for each prior) the limit expressions for value functions and efficiency under the SLC converge to those obtained when the SLC fails.

For γ large enough, this equilibrium cannot be sustained. To see why consider the following stopping problem, which we refer to as the *naive-market game*. Let $W_t = \Psi(\hat{Z}_t)$ for all t (i.e, beliefs depend only on news, and expected market value is always offered). A complete analysis of the seller's optimal policy in this problem is found in Appendix A.2. Of relevance here is Lemma A.3, in which we derive a closed from expression for γ^0 and show that if $\gamma > \gamma^0$, there exists $\underline{z}_H < \overline{z}_H$ such that the optimal policy for the high type is to reject in any state $z \in (\underline{z}_H, \overline{z}_H)$ and accept in any state $z \notin (\underline{z}_H, \overline{z}_H)$.

Intuitively, when γ is high and beliefs are intermediate, the high type expects a large enough benefit from news to make it worth the wait. In any SBM-equilibrium of the true game, $F_H \geq \Psi$ (No Deals) and Q non-decreasing (Belief Monotonicity) imply that rejecting in a given state z is at least as profitable for the high type as doing so is in the naive-market game.

Proposition 5.3. *If the SLC does not hold and $\gamma > \gamma^0$, then in any SBM-equilibrium there exist $z_1, z_2 \in \mathbb{R}$, $z_1 < z_2$, such that no trade occurs in any state $z \in (z_1, z_2)$.*

In addition, as $\gamma \to \infty$, we obtain similar payoff and efficiency properties to those in Proposition 4.1.

Proposition 5.4. *If the SLC does not hold, as $\gamma \to \infty$, in any SBM-equilibrium*

1. $F_H \overset{pw}{\to} V_H$ [34]
2. $F_L \overset{pw}{\to} V_L$
3. $\mathcal{L} \overset{u}{\to} 0$

Clearly, when it exists, $\Xi(\alpha^*, \beta^*)$ satisfies the properties of Propositions 5.3 and 5.4, but it may no longer be the unique SBM-equilibrium. Without the SLC, we cannot immediately rule out that the high type trades when beliefs are low. In fact, for some parameters, there exists an SBM-equilibrium where she does exactly that.

Example 5.1. Let $\gamma > \gamma^0$, and consider the following profile:

- For all beliefs $z \notin (\underline{z}_H, \overline{z}_H)$, $w(z) = \Psi(z)$, and both types accept with probability one.
- For all beliefs $z \in (\underline{z}_H, \overline{z}_H)$ $w(z) = V_L$, and both types reject.

This profile constitutes an SBM-equilibrium, supported by the belief process $Z = \hat{Z}$, if and only if K_L is above some threshold $\underline{K}_L$ (proof in Appendix B).

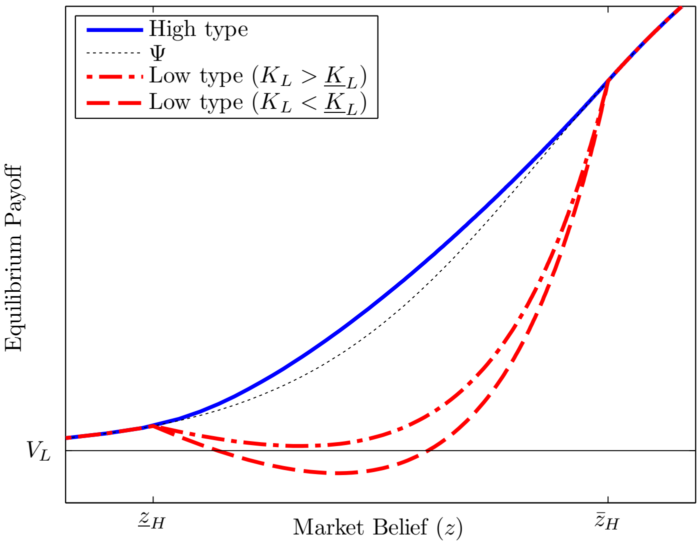

FIGURE 6 – Value functions and equilibrium conditions for Example 5.1.

The key difference between Example 5.1 and the Ξ-profile is that when beliefs are low, instead of a partial sell-off by the low type, there is trade with probability one by both types. Because the low type is always willing to accept Ψ, it is the high type's incentives that determine when it will be offered.[35] Consider the stopping problem for the high type if beliefs are currently low, they evolve based only on news, and $W_t = \Psi(Z_t)$ for all t. The expected increase in the probability the market assigns to $\theta = H$ and thus W in the near future is small, and so she has little to gain by waiting. Without the SLC, $K_H < \Psi(z)$ for all z, so the high type does best to accept.

There are parameters under which both $\Xi(\alpha^*, \beta^*)$ and Example 5.1 constitute SBM-equilibria. Again, the key is that if the market is not *expecting* a partial sell-off by the low type, rejecting when beliefs are low does not serve as any signal of high asset value. It is therefore belief consistent (and belief monotone) for buyers to update based only on the news, making the high type willing to accept Ψ and fulfilling the expectation that there is no partial sell-off by the low type.

Example 5.1 is not the only possible SBM-equilibrium other than $\Xi(\alpha^*, \beta^*)$. However, it does serve two purposes. First, the constructive proof of Proposition 5.1 shows that for any parameter values, if there does not exist an (α, β) such that $\Xi(\alpha, \beta)$ is an equilibrium, then either the "always immediately trade" profile of Proposition 5.2 or Example 5.1 constitutes

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[34] The convergence must be weakened from uniform in Proposition 4.1(3) to pointwise here—see Example 5.1 as an illustration.

[35] That is, the profile characterization is independent of K_L, and it constitutes an equilibrium so long as the low type is willing to endure the no-trade region $(\underline{z}_H, \overline{z}_H)$ rather than accept V_L (which determines the value of $\underline{K}_L$).

an SBM-equilibrium (establishing existence of the SBM-equilibrium).

Second, it is useful for motivating the next result. Within Example 5.1, let $z_m \equiv \arg\min_{z \in (\underline{z}_H, \overline{z}_H)} F_L(z)$. In Figure 6, notice that even for the value of $K_L > \underline{K}_L$, F_L is decreasing on $(\underline{z}_H, z_m)$. Between $\underline{z}_H$ and z_m beliefs evolve based only on news, so the low type is *hoping for bad news.* Put another way, if the game was modified such that μ_θ was only an upper-bound on the drift of the type-θ seller's news process, and the seller could continuously control her drift below μ_θ at no cost, then the low type would prefer to "sabotage" herself by choosing an arbitrarily large negative drift. Any equilibrium of this modified game would require non-decreasing value functions (NDVF). Thus, NDVF is a property that may be appealing in some settings.

We now demonstrate that, fixing all other parameters such that the SLC fails, as news quality increases, not only does the equilibrium from Example 5.1 fail NDVF, but $\Xi(\alpha^*, \beta^*)$ is the unique SBM-equilibrium satisfying NDVF.

Proposition 5.5. *If the SLC does not hold, then for all* γ *large enough,* $\Xi(\alpha^*, \beta^*)$ *is the essentially unique SBM-equilibrium satisfying NDVF.*

6 News Processes

We have modeled news via a Brownian diffusion process with type-dependent drift. This encompasses all continuous-time stochastic processes with (*i*) stationary, independent increments, and (*ii*) continuous sample paths almost surely. The first property is crucial for our analysis because our techniques rely on the stationary structure of the game, however, the second is not. Moreover, in many instances, information arrival is *not* continuous (e.g., using our first example, suppose the entrepreneur owns a small pharmaceutical firm that has developed a new drug awaiting FDA approval). In this section, we consider a simple alternative specification for a news process with lumpy information arrival. This helps to distill the key features of equilibrium trade dynamics in the presence of news.

Suppose that news arrives according to a compound Poisson process, $\{X_t, 0 \le t \le \infty\}$. Conditional on θ, X_t has a constant jump rate λ_θ and jump size distribution G_θ. For simplicity, assume that $supp(G_L) \cap supp(G_H) = \emptyset$. This specification implies that the first *arrival* (of a jump) fully reveals the seller's type at which point trade occurs immediately at V_θ. Let T denote the (random) time of the first arrival. Our discussion below will focus on equilibrium trade dynamics for $t < T$. Given a prior π_0 and seller strategies, at any time $t < T$ prior to trade the posterior is

$$\frac{\pi_0 e^{-\lambda_H t}(1 - S^H_{t^-})}{\pi_0 e^{-\lambda_H t}(1 - S^H_{t^-}) + (1 - \pi_0) e^{-\lambda_L t}(1 - S^L_{t^-})}$$

Again, taking the log-likelihood ratio we arrive at the equilibrium belief process

$$Z_t = Z_0 + (\lambda_L - \lambda_H)t + \ln\left(\frac{1 - S^H_{t^-}}{1 - S^L_{t^-}}\right), \qquad Z_0 = \ln\left(\frac{\pi_0}{1 - \pi_0}\right) \tag{25}$$

As in Section 2.3, let $\hat{Z}_t$ denote the process that updates only based on news

$$\hat{Z}_t = Z_0 + (\lambda_L - \lambda_H)t \tag{26}$$

and continue to let $Q^\alpha_t = \max\{\alpha - \inf_{s \le t} \hat{Z}_s, 0\}$ for any $\alpha \in \mathbb{R}$ (as in Definition 3.1).

From (26), if $\lambda_L > \lambda_H$, then $\hat{Z}$ drifts upward (matching the adage that "no news is good news"), whereas if $\lambda_L \le \lambda_H$, then $\hat{Z}$ drifts (weakly) downward ("no news is (weakly) bad news"). This distinction will play a key role in the equilibrium trade dynamics as demonstrated in Proposition 6.1.

Since the first arrival is fully revealing, the seller's outside option is to consume the flow payoff from the asset until the arrival at which point trade commences at a price of V_θ. Letting K'_θ denote the expected payoff of the outside option to a type-θ seller, we obtain

$$K'_\theta \equiv E^\theta\left[\int_0^T e^{-rt} r K_\theta dt + e^{-rT} V_\theta\right] = \frac{rK_\theta + \lambda_\theta V_\theta}{r + \lambda_\theta} \tag{27}$$

The high type will never rationally accept an offer less than K'_H. Therefore, the relevant condition for the potential existence of a standard adverse selection problem, denoted SLC$'$, is as follows.

Definition 6.1. The *SLC$'$* is satisfied if and only if $K'_H > V_L$.

Notice that the SLC$'$ is implied by the SLC, and if $\lambda_H = 0$, then $K'_H = K_H$ and the two conditions are equivalent.

Proposition 6.1. *Suppose the SLC$'$ holds.*

1. *If $\lambda_L > \lambda_H$ (no news is good news): for all $t < T$, $\Xi(\alpha, \beta)$ is an SBM-equilibrium for the uniquely specified (α, β) below.*[36]

$$\beta = \ln\left(\frac{1}{2q_H(\bar{V} - \bar{K}')}\left((1 - q_H)\bar{V} + 2q_H\bar{K}' + \sqrt{(1 - q_H)^2\bar{V} + 4q_H\bar{V}\bar{K}'}\right)\right) \tag{28}$$

$$\alpha = \beta - \frac{1}{q_L}\ln\left(\frac{\Psi(\beta) - K'_L}{V_L - K'_L}\right) \tag{29}$$

where $\bar{V} = V_H - V_L$, $\bar{K}' = K'_H - V_L$ and $q_\theta = \frac{r + \lambda_\theta}{\lambda_L - \lambda_H}$.

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[36] Where $\{S^{L,t}, S^{H,t}, Q^{\alpha,t}\}_{t \in \mathbb{R}_+}$ from $\Xi(\alpha, \beta)$ are extended as in Section 5.2.

2. *If $\lambda_L \leq \lambda_H$ (no news is (weakly) bad news): for all $t < T$, there exists an SBM-equilibrium such that $\{S^{L,t}, S^{H,t}\}_{t\in\mathbb{R}_+}$ and Z are as in $\Xi(\alpha,\beta)$, where (α,β) and W are uniquely specified below:*

$$\beta = \alpha = z^*, \textit{ where } z^* \textit{ is defined by } \Psi(z^*) = K'_H$$

$$W_t = \begin{cases} \Psi(Z_t) & \textit{if } Z_t > z^* \textit{ or both } Z_t = z^* \textit{ and } t \geq \mathfrak{t}' \\ V_L & \textit{otherwise} \end{cases}$$

where $\mathfrak{t}'$ is a Poisson random variable with arrival rate $\kappa = \frac{r(V_L - K_L)}{K'}$.

If $\lambda_L > \lambda_H$, then the description of the equilibrium (prior to T) is identical to the equilibrium in the Brownian-news model except for the exact location of the boundaries: there is a no-trade region with efficient trade above it and partial sell-off by the low type below it. However, there are important differences that arise because $(\hat{Z}_t)_{t<T}$ is deterministic and increasing. In equilibrium, $Z_t > \alpha$ for all $t \in (0,T)$, meaning the low type does not engage in any partial sell-off after time zero. Because the belief does not decrease, this aspect of behavior (essential in the Brownian-news model) is unnecessary. This further implies that β can be determined independent of α: when the high type is deciding whether or not she wants to accept Ψ in state z, equilibrium play in states $z' < z$ is irrelevant. Hence, β is pinned down completely by the high type's marginal considerations (notice that β as given by (28) does not depend on K_L or λ_L). Given β, α is then pinned down by the low type's indifference condition.

If, on the other hand, $\lambda_L \leq \lambda_H$, the equilibrium does not involve a no-trade region. In fact, it is quite similar to the $\gamma = 0$ equilibrium (Section 4.1). However, if $\lambda_L < \lambda_H$, the equilibrium recovers the partial sell-off feature: if $Z_0 < z^*$, because $(\hat{Z}_t)_{t<T}$ is deterministically decreasing, the low type is constantly engaging in a partial sell-off to exactly offset the news and maintain $Z_t = z^*$ for $t > 0$.[37] In this sense, the two cases isolate two salient features from the Brownian-news model. The potential for gradual good news creates a no-trade region, and the potential for gradual bad news mandates partial sell-offs by the low type to counteract sufficiently detrimental news. We conjecture that these two features prevail when news is modeled within in a broader class of Lévy processes.

Paralleling the analysis of the Brownian-news model, it is possible to show that in both cases the equilibrium in Proposition 6.1 is unique among SBM-equilibria (provided the SLC$'$ holds). When the SLC$'$ fails, if news quality is poor (i.e., λ_L, λ_H are small),

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[37] Clearly, if $\lambda_L = \lambda_H = 0$, the model and equilibrium are identical to those when $\gamma = 0$ in the Brownian-news setup (Section 4.1). If $\lambda_L = \lambda_H > 0$, the minimum state in which Ψ is offered and its (random) arrival time must be adjusted from $\underline{z}$ and $\mathfrak{t}$ to z^* and $\mathfrak{t}'$.

then immediate trade regardless of the state is the unique equilibrium outcome. Just as in the Brownian-news model, if we increase news quality, then equilibria matching those in Proposition 6.1 exist. The subtlety now is that news quality and the SLC$'$ both depend on λ_H. Define increasing news quality by the partial order: $(\lambda_L^1, \lambda_H^1)$ is *more informative* than $(\lambda_L^0, \lambda_H^0) \neq (\lambda_L^1, \lambda_H^1)$ if $(\lambda_L^1, \lambda_H^1) \geq (\lambda_L^0, \lambda_H^0)$. Thus, if $(\lambda_L^1, \lambda_H^1)$ is more informative than $(\lambda_L^0, \lambda_H^0)$, then $\max\{\lambda_L^1, \lambda_H^1\} \geq \max\{\lambda_L^0, \lambda_H^0\}$. It can be shown that if $\max\{\lambda_L, \lambda_H\}$ is large enough (which occurs if news quality is sufficiently high), then either the SLC$'$ holds (because λ_H is large enough), or an equilibrium with identical structure to (1) of Proposition 6.1 exists (because, just as in the Brownian-news model, the high type expects $\hat{Z}$ to increase fast enough to make it optimal to delay trade at intermediate beliefs).

7 Related Literature and Discussion

We have presented a continuous-time framework to analyze the effects of news in a dynamic market with asymmetric information. Our equilibrium of interest consists of three distinct regions: immediate trade, no trade, and partial sell-off. This paper encompasses environments in which a standard lemons problem may or may not exist. Further, we establish the strong connection between both of the environments and their equilibria. In this section, we discuss our contribution within the context of the literature.

7.1 Dynamic Adverse Selection and Dynamic Signaling

We start with the works of Akerlof (1970) and Spence (1973). There is a fundamental strategic difference between these two models. In Akerlof's model, the seller's choice is whether to trade *now or never*. Yet, in real markets, rejecting an offer today rarely prevents a seller from trading in the future. In Spence's model, prior to trade, the seller chooses a signaling action that carries a type-dependent cost. However, as pointed out by Weiss (1983) and Admati and Perry (1987), in many markets costly signaling takes time to materialize. Indeed the signaling action in Spence's primary application is the amount of time spent in school. The (implicit) assumption is that the student can *commit* to delay trade and ignore offers during this time. In a dynamic environment (without commitment and with durable assets) these two strategic settings are virtually identical. In both, privately-informed sellers choose whether to trade at each point in time based on the current offer, their expectations of future ones and any payoffs the asset endows in the interim. Their sole difference is whether there is a static adverse selection problem or not.

Among dynamic models, motivated by Spence's static model, one strand of literature assumes there is no static adverse selection problem. The papers in this strand each study a discrete-time model in which the seller receives offers from multiple buyers in each period, and the length of time between offers is small. Nöldeke and van Damme (1990) show that when buyers' offers are publicly observable, the unique equilibrium satisfying the never-a-weak-best-response refinement (Kohlberg and Mertens, 1986) closely approximates the least-cost-separating outcome.[38] Swinkels (1999) argues that their result hinges on the combination of public offers and the refinement. He shows that when offers are kept private, all types trade immediately in the unique sequential equilibrium outcome. When the SLC is not satisfied and news is completely uninformative, our model is the (two-type) continuous-time analog of Swinkels (1999). We confirm that trade is immediate and demonstrate how this result changes with the quality of the news.

Kremer and Skrzypacz (2007) introduce exogenous information into a dynamic signaling model with private offers. In their model, a grade is revealed at some fixed time, provided that trade has not already occurred. In contrast to Swinkels (1999), trade is always delayed with positive probability. A key insight of their work is that noisy information causes an *endogenous* market for lemons to develop. In equilibrium, trade breaks down completely just prior to revelation of the grade. A similar result obtains in our model. However, in an infinite-horizon model with gradual information arrival, the region of breakdown depends on market beliefs rather than time.

Motivated by Akerlof's static model, a second strand of literature assumes there is a static adverse selection problem. Janssen and Roy (2002) take a Walrasian approach and show that every equilibrium involves a sequence of increasing prices and qualities traded over time. Trade is delayed and therefore the outcome is inefficient, but all goods are traded in finite time. As agents become arbitrarily patient (i.e, as the interest rate tends to zero), the inefficiency persists because the expected time to trade grows unboundedly at the same rate. The same result obtains in our model without news (see the characterization of the $\gamma = 0$ equilibrium in Section 4.1). However, with news (i.e., $\phi > 0$), as the interest rate goes to zero, γ goes to ∞ and the inefficiency disappears (Proposition 4.1). Hörner and Vieille (2009) address the issue of public versus private offers in a dynamic adverse selection model. They demonstrate that when there is a single potential buyer each period trade always (eventually) occurs when offers are private, but often ends at an impasse when offers are public. The stark difference between their results (with public offers) and those of other works mentioned above (as well as the discrete-time version of our model) hinges crucially

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[38] The least-cost-separating outcome (or Riley outcome) is the equilibrium in which each type separates from all types below it by expending the minimum amount on signaling necessary to do so (Riley, 1979). Under the standard single-crossing assumption, it is the unique equilibrium satisfying stability-based refinements (Cho and Kreps, 1987; Banks and Sobel, 1987).

on the assumption of a single buyer each period.[39,40]

Our main contribution to this literature is in analyzing the effect of exogenous news arrival on equilibrium dynamics. We do so in a model that pertains to both strands: the difference amounts to a parametric assumption (the SLC). We show that when news is sufficiently informative, the distinction becomes less important, although not irrelevant, for equilibrium behavior. Finally, from a methodological viewpoint, we introduce continuous-time techniques to this literature, which facilitates sharp predictions and analytic tractability.

* * *

Thinking more broadly about the implications of our work, it is natural to think that news should reduce the asymmetry between buyers and sellers and mitigate inefficiencies. We demonstrate that this is only partially true. The welfare results have policy implications. For example, would a social planner ever suppress or censor informative news? Are markets more efficient when information is revealed gradually, or all at once? What is the optimal way to reveal information to the market? Further investigation of these questions seems a promising direction for future research.

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[39] The force underlying Diamond's paradox (Diamond, 1971) plays an important role in the impasse result of Hörner and Vieille (2009) and would be eliminated by intra-temporal competition among buyers.

[40] There are also a number of papers focusing on the relationship between specific market institutions and trade dynamics: see Taylor (1999), Hendel and Lizzeri (1999, 2002), Johnson and Waldman (2003), Hendel et al. (2005). It is now well understood that markets can, and have, overcome certain inefficiencies through a variety of innovations such as quality inspection, certification intermediaries and rental contracts.

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A Technical Preliminaries

A.1 Useful Formulas

In this section we derive several formulas which will be useful for a number of proofs in Appendix B. For arbitrary $\alpha < \beta$, let $T(\beta) \equiv \inf\{t : Z_t \geq \beta\}$, where Z is the stochastic process given by (7). Let $g_\theta(z|\alpha,\beta) \equiv E_z^\theta[e^{-rT(\beta)}]$ where E_z^θ denotes the expectation with respect to the probability law of the process Z starting at $Z_0 = z$ and conditional on θ ($\mathcal{Q}_z^\theta$). We refer to g_θ as the *expected discount factor*.

Fact A.1. *For any* $z \geq \beta$, $g_\theta(z|\alpha,\beta) = 1$. *For any* $z < \beta$,

$$g_\theta(z|\alpha,\beta) = E_z^\theta\left[e^{-rT(\beta)}\right] = \frac{1}{q_1^\theta e^{q_2^\theta(\beta-\alpha)} - q_2^\theta e^{q_1^\theta(\beta-\alpha)}}\left(q_1^\theta e^{q_2^\theta(z-\alpha)^+} - q_2^\theta e^{q_1^\theta(z-\alpha)^+}\right) \tag{30}$$

where $(q_1^L, q_2^L) = \frac{1}{2}\left(1 \pm \sqrt{1+8/\gamma}\right)$, $(q_1^H, q_2^H) = \frac{1}{2}\left(-1 \pm \sqrt{1+8/\gamma}\right)$, *and* $(x)^+ = \max\{x, 0\}$.

Proof: Let $\theta = L$. Using standard techniques (e.g., Section 3.1), for all $z \in (\alpha,\beta)$, g_L solves

$$g_L = -\frac{\gamma}{2}g_L' + \frac{\gamma}{2}g_L'' \tag{31}$$

which has a unique solution of the form $g_L(z) = D_1 e^{q_1^L z} + D_2 e^{q_2^L z}$. Solving for D_1, D_2, using the boundary conditions $g_L'(\alpha) = 0$ (reflection) and $g_L(\beta) = 1$ (value-matching), gives (30) for all $z \in (\alpha,\beta)$. For any $z < \alpha$, the expression follows from the fact that beliefs jump instantaneously to α. For any $z < \alpha$, $g_L(z|\alpha,\beta) = g_L(\alpha|\alpha,\beta)$. The derivation for $\theta = H$ is analogous. *Q.E.D.*

Several additional formulas follow from (30).

- For any $z \in (\alpha,\beta)$,

$$\lim_{\gamma\to 0} g_\theta(z|\alpha,\beta) = 0 \tag{32}$$

$$\lim_{\gamma\to\infty} g_\theta(z|\alpha,\beta) = 1 \tag{33}$$

- Starting from $z = \alpha$, the expected discount factor can be simplified to

$$g_\theta(\alpha|\alpha,\beta) = \frac{q_1^\theta - q_2^\theta}{q_1^\theta e^{q_2^\theta(\beta-\alpha)} - q_2^\theta e^{q_1^\theta(\beta-\alpha)}} \tag{34}$$

- Taking the limit of (30) as $\alpha \to -\infty$ gives the expected discount factor for a process that evolves according to $\hat{Z}$ (i.e., without a reflecting barrier). Let $g_\theta^\ell(z|\beta)$ denote the expected discount factor without a reflecting barrier. For all $z \leq \beta$, we have that

$$g_\theta^\ell(z|\beta) = \lim_{\alpha\to-\infty} g_\theta(z|\alpha,\beta) = e^{q_1^\theta(z-\beta)} \tag{35}$$

Finally, for any z, let $\hat{F}_\theta(z|\alpha,\beta)$ denote the value to the type-θ seller who, starting from z, rejects all offers until Z is weakly above β (i.e., plays according to $T(\beta)$), at which point the offer is $\Psi(\max\{\beta, z\})$. Then

$$\hat{F}_\theta(z|\alpha,\beta) = K_\theta + g_\theta(z|\alpha,\beta)(\Psi(\max\{\beta,z\}) - K_\theta) \tag{36}$$

Therefore, in $\Xi(\alpha^*, \beta^*)$, $F_\theta(z) = \hat{F}_\theta(z|\alpha^*, \beta^*)$. The statement is obvious for the high type. For the low type, it holds because in equilibrium she is indifferent between accepting at $z = \alpha^*$, meaning always rejecting at α^* must yield her equilibrium payoff.

A.2 The Naive-Market Game

In this section, we introduce and solve a pair of optimal stopping problems. The results demonstrated here will be used in several of the proofs in Appendix B. The stopping problems are as follows. Suppose that the type-θ seller faces a "naive" market that updates beliefs only based on news, and offers $\Psi(z)$ at every (t, ω) such that $\hat{Z}_t(\omega) = z$. Define $h_\theta(t, z)$ to be the payoff to a type-θ seller from accepting (i.e., stopping) at time t in state z:

$$h_\theta(t, z) = \int_0^t rK_\theta e^{-rs} ds + e^{-rt}\Psi(z) = K_\theta + e^{-rt}(\Psi(z) - K_\theta)$$

The problem facing the seller is to find an optimal policy (a stopping time) τ_θ to maximize her expected payoff given any initial state.

$$N^*_\theta(z) = \sup_{\tau \geq 0} E^\theta_z \left[h_\theta(\tau, \hat{Z}_\tau) \right] \quad \text{for all } z \tag{NMG$_\theta$}$$

We refer to this as the *naive-market game* and denote the type-θ seller's problem as NMG$_\theta$. Note that h_θ is bounded, in C^2, and has $\lim_{t\to\infty} h_\theta(t, z) = K_\theta$ for all z. These conditions are sufficient to ensure that an optimal stopping time (possibly infinite) exists and that N^*_θ is lower semi-continuous (Shiryaev, 1978, Theorems 3.1,3.3).[41]

Lemma A.1. *The optimal policy in NMG$_L$ is to accept $\Psi(z)$ immediately for all z. That is, $\tau_L = 0$.*

Proof: First, the expected payoff to the seller in NMG$_L$ starting from an initial state $\hat{Z}_0 = z$ and under an arbitrary stopping rule τ such that $E^L_z[\tau] < \infty$ is, by Dynkin's formula,

$$E^L_z[h_L(\tau, \hat{Z}_\tau)] = h_L(0, z) + E^L_z \left[\int_0^\tau \mathcal{A}^L h_L(s, \hat{Z}_s) ds \right] \tag{37}$$

where $\mathcal{A}^L$ is the characteristic operator for the process $Y_t = (t, \hat{Z}_t)$ under $\mathcal{Q}^L_z$. For all s, z,

$$\begin{aligned} \mathcal{A}^L h_L(s, z) &= -\frac{\phi^2}{2}\frac{\partial}{\partial z} h_L(s, z) + \frac{\phi^2}{2}\frac{\partial^2}{\partial z^2} h_L(s, z) + \frac{\partial}{\partial t} h_L(s, z) \\ &= e^{-rs}\left(-\frac{\phi^2}{2}(\Psi'(z) - \Psi''(z)) - r(\Psi(z) - K_L) \right) < 0 \end{aligned}$$

where the inequality follows from $\Psi'(z) > \max\{0, \Psi''(z)\}$ for all z. Second, if $E^L_z[\tau] = \infty$, then $E^L_z[h^L(\tau, \hat{Z}_\tau)] = K_L$. Therefore, for all stopping rules τ, $E^L_z[h_L(\tau, \hat{Z}_\tau)] \leq h_L(0, z) = \Psi(z)$. Taking the supremum over all τ gives $N^*_L(z) \leq \Psi(z)$. Since $\tau = 0$ is feasible, $N^*_L(z) \geq h_L(0, z)$, and we conclude that $N^*_L(z) = \Psi(z)$. The optimal policy in NMG$_L$ is to stop immediately (i.e., $\tau_L = 0$) for all z. *Q.E.D.*

The form of the optimal policy in NMG$_H$ depends crucially on whether the SLC holds. As in (37), the expected payoff to the high type under an arbitrary stopping rule τ, satisfying $E^H_z[\tau] < \infty$,

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[41] The problem has been posed only for initial states such that $t = 0$. Since the problem is stationary, the optimal policy will be time invariant so this formulation is without loss.

is

$$E_z^H[h_H(\tau,\hat{Z}_\tau)] = h_H(0,z) + E_z^H\left[\int_0^\tau \mathcal{A}^H h_H(s,\hat{Z}_s)ds\right]$$

where

$$\mathcal{A}^H h_H(s,z) = e^{-rs}\left(\frac{\phi^2}{2}(\Psi'(z)+\Psi''(z)) - r(\Psi(z)-K_H)\right) \tag{38}$$

The expression inside the outermost parentheses on the RHS of (38) can be interpreted as the marginal benefit of waiting for an instant of time before accepting. The first term, $\frac{\phi^2}{2}(\Psi'(z)+\Psi''(z))$, represents the expected marginal increase in the offer an instant later. This term is positive since the high type expects good news. Further, it is single-peaked since information has its largest effect on the posterior for intermediate priors and tends to zero in both limits (as beliefs become degenerate). The second term, $r(\Psi(z)-K_H)$ represents the (opportunity) cost associated with delaying trade, which is strictly increasing in z. Let $MB_H(z) \equiv \frac{\phi^2}{2}(\Psi'(z)+\Psi''(z)) - r(\Psi(z)-K_H)$ and $U_H \equiv \{z : MB_H(z) > 0\}$. Naturally, stopping at any $z \in U_H$ cannot be optimal (Oksendal, 2007, p. 216).

Under the SLC, there exists z^+ such that $U_H = \{z : z < z^+\}$ and MB_H is strictly decreasing above z^+. Hence, a cutoff policy of accepting above some threshold and rejecting below it is a natural candidate.

Lemma A.2. *If the SLC holds, there exists a unique $z_H^* \in \mathbb{R}$, given by (41), such that the optimal policy in NMG_H is to accept Ψ for all $z \geq z_H^*$ and reject at all $z < z_H^*$: $\tau_H = \inf\{t : \hat{Z}_t \geq z_H^*\}$.*

PROOF: We first demonstrate that any optimal policy must have the stated form. Let $D = \{z : N_H^*(z) > \Psi(z)\}$ be the open set (open by lower semi-continuity of N_θ^* and continuity of Ψ) representing the optimal rejection region, and note that $N_H^*(z) = \Psi(z)$ for all $z \in \mathbb{R}\backslash D$. As already established, $U_H = (-\infty, z^+) \subset D$. Rejecting at *all* z cannot be optimal since in that case $E_z^H[e^{-r\tau}] = \lim_{\beta\to\infty} g_H^\ell(z|\beta) = 0$ (from (35)), and $E_z^H[h_H(\tau,\hat{Z}_\tau)] = K_H$, which is inferior to the policy that stops at all $z > \underline{z}$. Hence, there must exist some $z > z^+$ such that acceptance at z is optimal. Let $z_H^* = \inf\{z : N_H(z) = \Psi(z)\}$.

To see that acceptance must be optimal for all $z > z_H^*$, suppose there exists some interval (z_1, z_2), $z_1 \geq z_H^*$, such that $(z_1,z_2) \in D$. Clearly it must be that $z_2 < \infty$ (otherwise for z large enough $E_z^H[h_H(\tau,\hat{Z}_\tau)]$ is a convex combination of K_H and $\Psi(z_1)$, which is strictly less than $\Psi(z)$). Starting from any $z \in (z_1,z_2)$, the stopping time $\tau = \inf\{t : \hat{Z}_t \notin (z_1,z_2)\} > 0$, $\mathcal{Q}_z^H$ almost surely, and $E_z^H[\tau] < \infty$. Therefore, the expected payoff is $h_H(0,z) + E_z^H\left[\int_0^\tau e^{-rs} MB_H(\hat{Z}_s)ds\right] < \Psi(z)$ violating $z \in D$, since $MB_H(z) < 0$ for all $z > z^+$ and $z_1 \geq z^+$. Hence no such interval can exist and the optimal policy must be of the stated form.

We now construct the value function and solve for the unique z_H^* consistent with optimality. For any z_0, let $\tau_0 = \inf\{t : \hat{Z}_t \geq z_0\}$ and $N_H(z) = E_z^H[h_H(\tau_0, Z_{\tau_0})]$. Using standard arguments, N_H solves

$$-N_H(z) + \frac{\gamma}{2}N_H'(z) + \frac{\gamma}{2}N_H''(z) + K_H = 0 \qquad \text{for } z < z_0 \tag{39}$$

$$N_H(z_0) = \Psi(z_0) \tag{40}$$

which has a solution of the form $N_H(z) = C_1 e^{q_1^H z} + C_2 e^{q_2^H z} + K_H$, where C_1, C_2 are arbitrary constants and $(q_1^H, q_2^H) = \frac{1}{2}\left(-1 \pm \sqrt{1 + 8/\gamma}\right)$. Note that $q_2^H < 0 < q_1^H$. Since N_H is bounded below by K_H as $z \to -\infty$, it must be that $C_2 = 0$. Using the boundary condition (40) gives $C_1 = (\Psi(z_0) - K_H)e^{-q_1^H z_0}$.

Since $h_H \in C^2$, a necessary condition for optimality is that $N_H \in C^1$ (Peskir and Shiryaev, 2006, p. 151). Thus, if τ_0 is optimal then $N_H'(z_0^-) = \Psi'(z_0)$, equivalently $\Psi'(z_0) - q_1^H(\Psi(z_0) - K_H) = 0$, which has a unique solution yielding the optimal cutoff

$$z_H^* = \ln\left(\frac{1}{2q_1^H(\bar{V} - \bar{K})}\left((1 - q_1^H)\bar{V} + 2q_1^H\bar{K} + \sqrt{\bar{V}^2(1 - q_1^H)^2 + 4q_1^H\bar{K}\bar{V}}\right)\right) \tag{41}$$

where $\bar{V} \equiv V_H - V_L$ and $\bar{K} \equiv K_H - V_L$. Since an optimal policy exists and must be of the stated form, and z_H^* is the unique cutoff satisfying necessary conditions for optimality, the proof is now complete. *Q.E.D.*

When the SLC does not hold, the form of the optimal policy depends on the quality of the news.

Lemma A.3. *If the SLC does not hold, there exists a γ^0, given by (42), such that:*

(i) If $\gamma \leq \gamma^0$, then the optimal policy in NMG_H is to accept Ψ immediately for all z: $\tau_H = 0$.

(ii) If $\gamma > \gamma^0$, then there exists a unique pair $\underline{z}_H < \overline{z}_H$, such that the optimal policy in NMG_H is to accept Ψ for all $z \notin (\underline{z}_H, \overline{z}_H)$ and reject for all $z \in (\underline{z}_H, \overline{z}_H)$: $\tau_H = \inf\{t : \hat{Z}_t \notin (\underline{z}_H, \overline{z}_H)\}$.

Proof: Note that $\operatorname{sgn}(MB_H(z)) = \operatorname{sgn}\left(\Psi' + \Psi'' - \frac{2}{\gamma}(\Psi - K_H)\right)$. Since Ψ', Ψ'' are bounded, if γ is small enough, then $U_H = \emptyset$. Define $\gamma^0 = \sup\{\gamma : U_H = \emptyset\}$. To obtain a closed-form expression for γ^0, maximize MB_H over z to get $z^{**} = \ln\left(-(1+\gamma) + \sqrt{3\gamma(1+\gamma)}\right)$. Setting $MB_H(z^{**}) = 0$ and solving for γ gives

$$\gamma^0 = \frac{8\bar{V}^2}{\left(8\bar{V}^2 - 36\bar{K}\bar{V} + 27\bar{V}^2 - \sqrt{\bar{K}}(9\bar{K} - 8\bar{V})^{\frac{3}{2}}\right)} \tag{42}$$

where $\bar{V} \equiv V_H - V_L$ and $\bar{K} \equiv K_H - V_L$. For all $\gamma \leq \gamma^0$, because $U_H = \emptyset$, $\mathcal{A}^H h_H(s, z) < 0$ for all s, z. Just as in the proof of Lemma A.1, this implies that the optimal policy is to accept immediately.

For $\gamma > \gamma^0$, $U_H = (z^-, z^+)$ for some $z^- < z^+$ and MB_H is increasing below z^- and decreasing above z^+. Let $K_1 = (-\infty, z^-)$, $K_2 = (z^+, \infty)$ and $D = \{z : N_H^*(z) > \Psi(z)\}$. Since $U_H \subset D$, there is at least one interval in $\mathbb{R}$ where it is optimal to reject. Rejecting at all $z \in K_1$ is not optimal since in that case $\lim_{z \to -\infty} E_z^H[h_H(\tau, \hat{Z}_\tau)] = K_H < \lim_{z \to -\infty} \Psi(z) = V_L$. Hence, acceptance must be optimal for at least some $z \in K_1$, so let $\underline{z}_H = \sup\{z \in K_1 : N_H^*(z) = \Psi(z)\}$. Similarly, rejecting at all $z \in K_2$ cannot be optimal, so let $\overline{z}_H = \inf\{z \in K_2 : N_H^*(z) = \Psi(z)\}$. Using the same arguments as in the proof of Lemma A.2, there cannot exist a rejection region below $\underline{z}_H$ or above $\overline{z}_H$ (since $MB_H(z) < 0$ for all such z). Therefore, any optimal policy must have the stated form. *Q.E.D.*

B Proofs

B.3 Proofs for Section 3

Given the analysis in Sections 3.1–3.2, we now restate the system of boundary conditions (16)–(21) using the closed-form expressions for value functions derived in (14)–(15). The three boundary conditions on the low type's value function are

$$C_1^L e^{q_1^L \alpha} + C_2^L e^{q_2^L \alpha} + K_L = V_L \tag{43}$$

$$C_1^L e^{q_1^L \beta} + C_2^L e^{q_2^L \beta} + K_L = \Psi(\beta) \tag{44}$$

$$q_1^L C_1^L e^{q_1^L \alpha} + q_2^L C_2^L e^{q_2^L \alpha} = 0 \tag{45}$$

Similarly, the boundary conditions on the high type's value function are

$$C_1^H e^{q_1^H \beta} + C_2^H e^{q_2^H \beta} + K_H = \Psi(\beta) \tag{46}$$

$$q_1^H C_1^H e^{q_1^H \alpha} + q_2^H C_2^H e^{q_2^H \alpha} = 0 \tag{47}$$

$$q_1^H C_1^H e^{q_1^H \beta} + q_2^H C_2^H e^{q_2^H \beta} = \Psi'(\beta) \tag{48}$$

Definition B.1. Let $\mathcal{B}_L : \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondence such that (α, β), $\alpha \leq \beta$ satisfies (43)–(45) if and only if $\beta \in \mathcal{B}_L(\alpha)$. Let $\mathcal{B}_H : \mathbb{R} \rightrightarrows \mathbb{R}$ be the correspondence such that (α, β), $\alpha \leq \beta$ satisfies (46)–(48) if and only if $\beta \in \mathcal{B}_H(\alpha)$.

Definition B.2. For $\theta = L, H$ and for all $\alpha \in \mathbb{R}$, let $B_\theta(\alpha) = \max \mathcal{B}_\theta(\alpha)$.

Lemma B.1. *For all $\alpha \in \mathbb{R}$, $\mathcal{B}_L(\alpha)$ is singleton, and B_L is increasing and continuously differentiable with* $\lim_{\alpha \to -\infty} B_L(\alpha) = -\infty$.

Proof: Given a lower boundary α, solve (43) and (45) to get C_1^L, C_2^L:

$$C_1^L(\alpha) = A e^{-q_1^L \alpha} \tag{49}$$

$$C_2^L(\alpha) = B e^{-q_2^L \alpha} \tag{50}$$

where $A \equiv \frac{-q_2^L (V_L - K_L)}{\left(q_1^L - q_2^L\right)}$ and $B \equiv \frac{q_1^L (V_L - K_L)}{(q_1^L - q_2^L)}$. Using boundary condition (44) gives an implicit expression for any value of $\beta \in \mathcal{B}_L(\alpha)$

$$A e^{q_1^L(\beta - \alpha)} + B e^{q_2^L(\beta - \alpha)} + K_L - \Psi(\beta) = 0 \tag{51}$$

The function on the LHS of (51) is continuously differentiable in α and β. For any fixed α, as $\beta \to \infty$, the LHS of (51) becomes arbitrarily large and thus is strictly positive for β sufficiently large. Further, at $\beta = \alpha$, the LHS of (51) is equal to $V_L - \Psi(\beta) < 0$. Therefore, $\mathcal{B}_L(\alpha)$ is non-empty for all α by the intermediate value theorem.

We now apply the implicit function theorem. In order to do so, it suffices to show that for any (α, β) such that $\beta \in \mathcal{B}_L(\alpha)$, the derivative of the LHS of (51) w.r.t. β is strictly positive. Differentiating (51) w.r.t. β gives

$$q_1^L A e^{q_1^L(\beta - \alpha)} + q_2^L B e^{q_2^L(\beta - \alpha)} - \Psi'(\beta) \tag{52}$$

Using (44), we have that $q_1^L A e^{q_1^L(\beta-\alpha)} = q_1^L\left(\Psi(\beta) - K_L - Be^{q_2^L(\beta-\alpha)}\right)$ and therefore

$$\begin{aligned} q_1^L A e^{q_1^L(\beta-\alpha)} + q_2^L B e^{q_2^L(\beta-\alpha)} &= q_1^L\left(\Psi(\beta) - K_L - Be^{q_2^L(\beta-\alpha)}\right) + q_2^L B e^{q_2^L(\beta-\alpha)} \\ &= q_1^L(\Psi(\beta) - K_L) - (q_1^L - q_2^L)Be^{q_2^L(\beta-\alpha)} \\ &= q_1^L\left(\Psi(\beta) - K_L - q_1^L(V_L - K_L)e^{q_2^L(\beta-\alpha)}\right) \\ &= q_1^L\left(\frac{e^\beta}{1+e^\beta}\bar{V} + (V_L - K_L)(1 - e^{q_2^L(\beta-\alpha)})\right) \\ &> q_1^L\frac{e^\beta}{1+e^\beta}\bar{V} > \frac{e^\beta}{(1+e^\beta)^2}\bar{V} = \Psi'(\beta) \end{aligned}$$

Hence, (52) is strictly positive for all such (α,β), implying (by the implicit function theorem) that $\mathcal{B}_L(\alpha)$ is singleton and $B_L(\alpha)$ is continuously differentiable. Since the LHS of (51) grows unboundedly large as $\alpha \to -\infty$, it must be that $\lim_{\alpha\to-\infty} B_L(\alpha) = -\infty$.

To see that $B_L(\alpha)$ is increasing, by implicit differentiation we have that:

$$\left(B_L'(\alpha) - 1\right)\left(q_1^L A e^{q_1^L(\beta-\alpha)} + q_2^L B e^{q_2^L(\beta-\alpha)}\right) - \Psi'(\beta)B_L'(\alpha) = 0$$

Rearranging and using the fact that $q_1^L A = -q_2^L B$ gives

$$B_L'(\alpha) = \frac{q_1^L A(e^{q_1^L(\beta-\alpha)} - e^{q_2^L(\beta-\alpha)})}{q_1^L A e^{q_1^L(\beta-\alpha)} + q_2^L B e^{q_2^L(\beta-\alpha)} - \Psi'(\beta)} \tag{53}$$

The numerator is strictly positive since $q_1^L A > 0$ and $q_1^L > q_2^L$. The denominator is the same expression as in (52), which we already demonstrated was strictly positive. Thus, we conclude that B_L is increasing, completing the proof. *Q.E.D.*

Lemma B.2. *Let either the SLC hold or $\gamma > \underline{\gamma}$. B_H is a well-defined, increasing, continuously differentiable function with $\lim_{\alpha\to-\infty} B_H(\alpha) \equiv \underline{\beta}_H > -\infty$. Further, if the SLC holds, then for all $\alpha \in \mathbb{R}$, $\mathcal{B}_H(\alpha)$ is singleton.*

PROOF: For a given β, solve (46) and (48) to get

$$C_1^H(\beta) = \frac{\Psi'(\beta) + q_2^H(K_H - \Psi(\beta))}{q_1^H - q_2^H}e^{-q_1^H\beta} \tag{54}$$

$$C_2^H(\beta) = \frac{-(\Psi'(\beta) + q_1^H(K_H - \Psi(\beta)))}{q_1^H - q_2^H}e^{-q_2^H\beta} \tag{55}$$

Using (47), we arrive at the correspondence

$$\mathcal{B}_H(\alpha) = \left\{\beta \in \mathbb{R} : \beta \geq \alpha,\ C_1^H(\beta)q_1^H e^{q_1^H\alpha} + C_2^H(\beta)\, q_2^H e^{q_2^H\alpha} = 0\right\}$$

or equivalently $\mathcal{B}_H(\alpha) = \{\beta \in \mathbb{R} : \beta \geq \alpha, \alpha = A_H(\beta)\}$, where $A_H(\beta) \equiv \frac{1}{q_1^H - q_2^H}\ln\left(-\frac{q_2^H}{q_1^H}\frac{C_2^H(\beta)}{C_1^H(\beta)}\right)$.

First, suppose that the SLC holds. Then $A_H(\beta)$ is real-valued and weakly less than β if and only if $\beta > \underline{\beta}_H \equiv \inf\{\beta : C_2^H(x) > 0\ \forall x > \beta\}$. Note that $C_2^H(x) < 0$ for all $x < \underline{\beta}_H$. Thus, $\mathcal{B}_H(\alpha) \subset [\underline{\beta}_H, \infty)$. On $[\underline{\beta}_H, \infty)$, A_H is strictly increasing and continuously differentiable with

$\lim_{\beta \to \underline{\beta}_H} A_H(\beta) = -\infty$ and $\lim_{\beta \to \infty} A_H(\beta) = \infty$. Thus $\mathcal{B}_H(\alpha)$ is non-empty and singleton for all $\alpha \in \mathbb{R}$, further $B_H = \max A_H^{-1}$, is also increasing and continuously differentiable by the inverse function theorem.

Next, suppose that the SLC fails and notice that this implies that $C_1^H(\beta) > 0$ for all $\beta \in \mathbb{R}$. Since $q_1^H > 0 > q_2^H$, (47) requires that $\text{sgn}(C_1^H) = \text{sgn}(C_2^H)$. Thus, $C_2^H(\beta) > 0$ is necessary for any $\beta \in \mathcal{B}_H(\alpha)$. Note that $\text{sgn}(C_2^H(\beta)) = \text{sgn}(\Upsilon(\beta))$, where $\Upsilon(\beta) \equiv q_1^H(\Psi(\beta) - K_H) - \Psi'(\beta)$. Fixing β, $\Upsilon(\beta)$ is decreasing in γ (recall q_1^H decreases with γ). For γ sufficiently small, q_1^H gets arbitrarily large and hence $\Upsilon(\beta)$ is everywhere positive. Define $\gamma_2 \equiv \min_{\gamma \geq 0}\{\Upsilon(\beta) \geq 0 \; \forall \beta \in \mathbb{R}\}$. Υ is minimized at $\beta_m \equiv \ln\left(\frac{1-q_1^H}{1+q_1^H}\right)$ provided $q_1^H < 1$, setting $\Upsilon(\beta_m) = 0$ and solving for γ gives $\gamma_2 = \frac{2}{\underline{q}_1(1+\underline{q}_1)}$, where $\underline{q}_1 \equiv \frac{V_H + V_L - 2K_H - 2\sqrt{(K_H - V_L)(K_H - V_H)}}{V_H - V_L} < 1$ since $K_H < V_L$. Notice that $\gamma_2 = \underline{\gamma}$. Now fix any $\gamma > \underline{\gamma}$. $\Upsilon(\beta)$ has two real roots, in between which it is negative. Let $\underline{\beta}_H$ denote the upper root and note that as before $\underline{\beta}_H \equiv \inf\{\beta : C_2^H(x) > 0 \; \forall x > \beta\}$. For all $\beta > \underline{\beta}_H$, $A_H(\beta)$ is real-valued, weakly less than β, strictly increasing and continuously differentiable with $\lim_{\beta \to \underline{\beta}_H} A_H(\beta) = -\infty$ and $\lim_{\beta \to \infty} A_H(\infty) = \infty$. The rest of the proof follows from the case where the SLC holds. *Q.E.D.*

Proof of Lemma 3.1 For existence and uniqueness of a solution, we first express the intersection as the root of a continuous function, denoted Λ, of the upper boundary β. From the lower bound on B_H derived in Lemma B.2, we can restrict attention to looking for (α, β) with $\beta \geq \underline{\beta}_H$. We demonstrate that Λ is a differentiable and strictly decreasing function on $[\underline{\beta}_H, \infty)$ that is positive as $\beta \to \underline{\beta}_H$ and negative as $\beta \to \infty$.

For analytical convenience, we make a change of variables from log-likelihood space to likelihood space. We use $\tilde{z}$ to denote e^z (similarly for $\tilde{\alpha}, \tilde{\beta}$) and $\tilde{\Psi}(y) \equiv \Psi(\ln(y))$ so that $\tilde{\Psi}(\tilde{z}) = \Psi(z)$. Let $\tilde{\underline{\beta}}_H \equiv \exp(\underline{\beta}_H)$. Making the change of variables gives the following expression for $\tilde{\alpha}$ in terms of $\tilde{\beta}$ that solves the high type's equations ((46)–(48).

$$\tilde{\alpha} = \tilde{A}_H(\tilde{\beta}) \equiv \tilde{\beta}\left(\frac{q_2^H}{q_1^H}\left(\frac{\tilde{\Psi}'(\tilde{\beta}) + q_1^H(K_H - \tilde{\Psi}(\tilde{\beta}))}{\tilde{\Psi}'(\tilde{\beta}) + q_2^H(K_H - \tilde{\Psi}(\tilde{\beta}))}\right)\right)^{\frac{1}{q_1^H - q_2^H}} \tag{56}$$

Let $f(y) \equiv \frac{y}{\tilde{A}_H(y)}$ and note that

$$\begin{aligned}\text{sgn}(f'(y)) &= \text{sign}\left(\frac{d}{dy}\left(\frac{q_2^H(\tilde{\Psi} - K_H) - \tilde{\Psi}'}{\tilde{\Psi}' + q_1^H(K_H - \tilde{\Psi})}\right)\right) \\ &= \text{sgn}\left((q_2^H - q_1^H)(\tilde{\Psi}''(K_H - \tilde{\Psi}) + (\tilde{\Psi}')^2)\right) < 0, \; \forall y > \tilde{\underline{\beta}}_H\end{aligned}$$

Making the change of variables and plugging $C_1^L(\alpha)$ and $C_2^L(\alpha)$ from (49) and (50) into (51) gives

$$A\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)^{q_1^L} + B\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)^{q_2^L} + K_L - \tilde{\Psi}(\tilde{\beta}) = 0 \tag{57}$$

We have reduced the problem to solving two equations: (56) and (57). Substituting f for $\frac{\tilde{\beta}}{\tilde{\alpha}}$ into (57) gives a function, denoted by Λ. Any real root of Λ is a solution to the six boundary conditions.

$$\Lambda(y) \equiv A(f(y))^{q_1^L} + B(f(y))^{q_2^L} + K_L - \tilde{\Psi}(y) \tag{58}$$

Note that Λ is continuous for all $y > \tilde{\underline{\beta}}_H$. As $y \to \tilde{\underline{\beta}}_H$, $f(y) \to \infty$ and since $A > 0$, $\Lambda \to \infty$.

On the other hand, $f(y) \to 1$ as $y \to \infty$ and so $\lim_{y\to\infty} \Lambda(y) = A + B + K_L - V_H = V_L - V_H < 0$. Therefore, Λ has at least one root greater than $\tilde{\underline{\beta}}_H$ implying existence of a solution. To prove its uniqueness, we show that Λ is decreasing for all $y > \tilde{\underline{\beta}}_H$. Taking the derivative and simplifying gives

$$\Lambda' = f' f^{-1}\left(A q_1^L f^{q_1^L} + B q_2^L f^{q_2^L}\right) - \tilde{\Psi}'$$

Since $f' < 0$, $f > 0$ and $\tilde{\Psi}' > 0$, it is sufficient to show that the term inside the parentheses is positive. Since $A q_1^L = -B q_2^L > 0$, this requires only that $f^{q_1^L} - f^{q_2^L} > 0$ which follows from $f > 1$, $q_1^L > 0 > q_2^L$. Finally, from Lemmas B.1 and B.2, it is immediate that, given that the curves intersect exactly once, B_L intersects B_H from below. *Q.E.D.*

PROOF OF PROPOSITION 3.1. Let $Z_0 = \alpha^*$. From (9) we have that for small t, $E_{\alpha^*}^L[1 - e^{-Q_t}] \approx E_{\alpha^*}^L[Q_t] = E_0^L[-\inf_{0\le s\le t} \hat{Z}_s]$, which is approximately $\phi\sqrt{2t/\pi}$ as we now demonstrate. Let $M_t \equiv -\inf_{0\le s\le t} \hat{Z}_s^L$ for $\hat{Z}_0^L = 0$, and note that $M_t = \sup_{0\le s\le t} -\hat{Z}_s^L$. Thus, the density of M_t for all $y > 0$ is

$$f_{M_t}(y) = \frac{2}{\sqrt{2\pi\phi^2 t}} e^{-\frac{1}{2\phi^2 t}(y - \frac{1}{2}\phi^2 t)^2} - e^y \Phi\left(\frac{-y - \frac{1}{2}\phi^2 t}{\phi\sqrt{t}}\right) \tag{59}$$

which can be obtained from (Shreve, 2004, p. 114), where $\Phi(\cdot)$ denotes the standard normal CDF. Given the density of M_t, taking the limit gives $\lim_{t\to 0} \frac{E[M_t]}{\sqrt{t}} = \frac{\int_0^\infty y f_{M_t}(y)dy}{\sqrt{t}} = \phi\sqrt{2/\pi}$. *Q.E.D.*

PROOF OF LEMMA 3.2 Twice differentiating F_H gives $F_H''(z) = (q_1^H)^2 C_1^H e^{q_1^H z} + (q_2^H)^2 C_2^H e^{q_2^H z} > 0$ because both terms are strictly positive. F_H convex and $F_H'(\alpha^*) = 0$ implies $F_H'(z) > 0$ for all $z \in (\alpha^*, \beta^*)$. To see that $F_H(z) > \Psi(z)$ for all $z \in (\alpha^*, \beta^*)$, take any $\beta \ge \underline{\beta}_H$ and solve (46) and (48) for C_1^H and C_2^H. By direct calculation, the resulting function $C_1^H e^{q_1^H z} + C_2^H e^{q_2^H z} + K_H > \Psi(z)$ for all $z < \beta$. Therefore, the same property must hold for $\beta = \beta^*$. To prove the second part of the lemma, take the second derivative of F_L to get $F_L''(z) = (q_1^L)^2 C_1^L e^{q_1^L z} + (q_2^L)^2 C_2^L e^{q_2^L z} > 0$. Therefore F_L is also convex and increasing on (α^*, β^*) because $F_L'(\alpha^*) = 0$. The result then follows since $F_L(\alpha^*) = V_L$. *Q.E.D.*

The proof of Lemma 3.3 requires the following step:

Lemma B.3. *Let either the SLC hold or* $\gamma > \underline{\gamma}$. *Then* $\beta^* > z^+$ *and hence* $MB_H(z) < 0$ *for all* $z > \beta^*$, *where* z^+ *and* $MB_H(z)$ *are as defined in Appendix A.2.*

PROOF: First under the SLC, from the proofs of Lemmas A.2 and B.2, observe that $\underline{\beta}_H = z_H^*$. Since $\beta^* > \underline{\beta}_H$ (Lemma B.2) and $z_H^* > z^+$ (Lemma A.2), we have that $\beta^* > z^+$. When the SLC fails, recall that $\gamma > \underline{\gamma}$ implies that $q_1^H < \underline{q}_1$ and therefore $\Upsilon(\beta_m) < 0$ (Lemma B.2). Further,

$$\Upsilon'(z) = q_1^H \Psi'(z) - \Psi''(z) = \bar{V}\frac{e^z}{1 + e^z}(e^z(1 + q_1^H) - 1 + q_1^H) > 0, \ \forall\, z > \beta_m \tag{60}$$

In addition, $\Upsilon(\underline{\beta}_H) = 0$ and

$$\Upsilon(z) > 0 \implies q_1^H(\Psi(z) - K_H) > \Psi'(z), \ \forall\, z > \underline{\beta}_H \tag{61}$$

Recall that $\mathrm{sgn}(MB_H(z)) = \mathrm{sgn}(\Psi'(z) + \Psi''(z) - \frac{2}{\gamma}(\Psi(z) - K_H))$. For any $z > \underline{\beta}_H$, we have that

$$\begin{aligned}\Psi'(z) + \Psi''(z) - \frac{2}{\gamma}(\Psi(z) - K_H) &< (1 + q_1^H)\Psi' - \frac{2}{\gamma}(\Psi(z) - K_H) \\ &< (\Psi(z) - K_H)\left((1 + q_1^H)q_1^H - \frac{2}{\gamma}\right) \\ &= 0\end{aligned}$$

Where the first inequality is from (60) and the second from (61) and rearranging. The result then follows since $\beta^* > \underline{\beta}_H$ (Lemma 3.1). *Q.E.D.*

PROOF OF LEMMA 3.3 It suffices to show that $G_\theta^*(z) \leq F_\theta(z)$. By Lemma 3.2, $\max\{F_\theta(z), w(z)\} = F_\theta(z)$. Therefore $G_\theta^*(z) = \sup_{\tau \geq 0} E_z^\theta\left[f_\theta(\tau, Z_\tau)\right]$ where $f_\theta(t, z) \equiv (1 - e^{rt})K_\theta + e^{-rt}F_\theta(z)$. Start with $\theta = H$ and note that f_H is C^2 on $U \equiv \mathbb{R}\backslash\{\alpha^*, \beta^*\}$. For any $Z_0 = z \geq \alpha^*$, by Ito's formula

$$f_H(t, Z_t) = f_H(0, Z_0) + \int_0^t \mathcal{A}^H f_H(s, Z_s) I(Z_s \in U) ds + \int_0^t \phi e^{-rs} F_H'(Z_s) dB_s + \int_0^t e^{-rs} F_H'(\alpha^*) dQ_s^{\alpha^*}$$

From (13), $\mathcal{A}^H f_H(t, z) = 0$ for all $z \in (\alpha^*, \beta^*)$ and $\mathcal{A}^H f_H(t, z) = e^{-rs} MB_H(z) < 0$ for all $z > \beta^*$ (Lemma B.3). Therefore, $\mathcal{A}^H f_H \leq 0$ everywhere on U. Since $\mathcal{Q}_z^H(Z_s \in U) = 1$ for all z, s such that $s > 0$, and $F_H'(\alpha^*) = 0$ we have that

$$f_H(t, Z_t) \leq f_H(0, Z_0) + M_t = F_H(z) + M_t \tag{62}$$

where M is a martingale given by $M_t = \int_0^t \phi e^{-rs} F_H'(Z_s) dB_s$ (using $F_H'(z) \leq \max_z \Psi'(z) = \frac{V_H - V_L}{4}$, it is easily verified that M is a martingale). For every stopping time τ, we have by (62) that $f_H(\tau, Z_\tau) \leq F_H(z) + M_\tau$. Taking the Q_z^H-expectation and using the optional stopping theorem, we have that $E_z^H[f_H(\tau, Z_\tau)] \leq F_H(z)$. Taking the supremum over all τ, we conclude that $G_H^*(z) \leq F_H(z)$ for all $z \geq \alpha^*$. For any $Z_0 < \alpha^*$, rejecting gives $G_H^*(\alpha) \leq F_H(\alpha) = F_H(Z_0)$ and accepting gives $F_H(Z_0) = F_H(\alpha)$. In both cases, the payoff is bounded above by $F_H(Z_0)$. Thus, $G_H^*(z) \leq F_H(z)$ for all z as desired.

The proof for $\theta = L$ follows a similar argument where we use the fact that (12) implies that $\mathcal{A}^L f_L = 0$ for all $z \in (\alpha^*, \beta^*)$ and $\mathcal{A}^L f_L < 0$ for all $z > \beta^*$ (Lemma A.1). *Q.E.D.*

PROOF OF THEOREM 3.1 By construction, the belief process satisfies Belief Consistency. To see this, note that if $S_{t^-}^L \cdot S_{t^-}^H < 1$, then $S_{t^-}^H = 0$ and $S_{t^-}^L = 1 - e^{-Q_t^{\alpha^*}}$. Therefore, $Z_t = \hat{Z}_t + Q_t^{\alpha^*}$ satisfies equation (3). The Zero Profit condition is immediate since only the low type trades with positive probability for $z \leq \alpha^*$ where the offer is V_L, and both types trade with probability one for $z \geq \beta^*$ where the offer is $\Psi(z)$. No Deals follows from Lemma 3.2 and the argument given in Section 3.3.

For Seller Optimality, we have from Lemma 3.3 that $G_\theta^*(z) \leq F_\theta(z)$. Since $F_\theta^*(z) \leq G_\theta^*(z)$, we conclude that $F_\theta^*(z) \leq F_\theta(z)$. We are left to show that S^θ obtains F_θ. For the high type, let $T(\beta^*) = \inf\{t : Z_t \geq \beta^*\}$ and observe that $\mathcal{S}^H = \{T(\beta^*)\}$. By construction, $F_H(z) = E_z^H[(1 - e^{-rT(\beta^*)})K_H + e^{-rT(\beta^*)}\Psi(Z_{T(\beta^*)})]$. Since $T(\beta^*)$ is feasible, we conclude that $F_\theta^*(z) = F_\theta(z)$ and S^H solves (SP_H). For the low type, it suffices to show that both: (1) $T(\beta^*)$ and (2) $\tau_L = \inf\{t : Z_t \notin (\alpha^*, \beta^*)\}$ achieve an expected payoff equal to $F_L(z)$ starting from any initial $Z_0 = z$. Let $F_{L,i}(z)$ denote the expected payoff from playing according to the pure strategy (i) for $i = 1, 2$ starting from $Z_0 = z$. For $z \in (\alpha^*, \beta^*)$, $F_{L,i}$ must solve (12) and therefore is of the form (14). To pin down the constants, note that $F_{L,1}$ must satisfy value-matching at β^* (44) and reflection at α^* (45), which are sufficient to imply that $F_{L,1}(z) = F_L(z)$ for all $z \in (\alpha^*, \beta^*)$. Similarly, $F_{L,2}$ must satisfy value-matching at both α^* (43) and β^* (44), implying that $F_{L,2}(z) = F_L(z)$ for all $z \in (\alpha^*, \beta^*)$. Verifying that $F_{L,i}(z) = F_L(z)$ for $z \notin (\alpha^*, \beta^*)$, $i = 1, 2$, is immediate, completing the proof that S^L solves (SP_L). *Q.E.D.*

B.4 Proofs for Section 4

Proof of Proposition 4.1 For (1), from Lemma 3.1, for all $\gamma > \underline{\gamma}$, β^* exists and $\beta^* \geq \underline{\beta}_H$. From the proof of Lemma B.2, the expression for $C_2^H(\beta)$ (i.e., (55)), the upper root of which determines $\underline{\beta}_H$, does not depend on whether the SLC holds. Thus for $\gamma > \underline{\gamma}$, we have

$$\underline{\beta}_H = \ln\left(\frac{1}{2q_1^H(\bar{V} - \bar{K})}\left((1 - q_1^H)\bar{V} + 2q_1^H\bar{K} + \sqrt{\bar{V}^2(1 - q_1^H)^2 + 4q_1^H\bar{K}\bar{V}}\right)\right) \tag{63}$$

where $q_1^H = \frac{1}{2}\left(-1 + \sqrt{1 + 8/\gamma}\right)$, $\bar{V} = V_H - V_L$ and $\bar{K} = K_H - V_L$. Since, $\bar{V} > \max\{\bar{K}, 0\}$, as $\gamma \to \infty$, $q_1^H \to 0$, and $\underline{\beta}_H \to \infty$, establishing the claim.

For (2), we proceed in two main steps and, as in the proof of Lemma 3.1, we conduct our analysis in likelihood space. We first establish that $\lim_{\gamma\to\infty}(\tilde{\beta}^* - \tilde{\underline{\beta}}_H) = \frac{V_L - K_L}{V_H - K_H}$. Having established the rate at which $\tilde{\beta}^* \to \infty$, we use (56) to find $\lim_{\gamma\to\infty}\tilde{\alpha}^*$.

For the first step, suppose first that $\lim_{\gamma\to\infty}(\tilde{\beta}^* - \tilde{\underline{\beta}}_H)$ exists. Let $\delta = \lim_{\gamma\to\infty}(\tilde{\beta}^* - \tilde{\underline{\beta}}_H)$. From (58), we have that

$$\lim_{\gamma\to\infty}\Lambda(\tilde{\underline{\beta}}_H + \delta) = \frac{(V_H - V_L)(K_L - V_L - \delta(K_H + V_H))}{\delta(K_H - V_H)} \tag{64}$$

Therefore, if $\lim_{\gamma\to\infty}(\tilde{\beta}^* - \tilde{\underline{\beta}}_H)$ exists, then (64) must equal zero, and it must be that $\delta = \delta^* \equiv \frac{V_L - K_L}{V_H - K_H}$. To see that the limit must exist, suppose it did not. Then, either $\limsup_{\gamma\to\infty}(\tilde{\beta}^* - \tilde{\underline{\beta}}_H) > \delta^*$ or $\liminf_{\gamma\to\infty}(\tilde{\beta}^* - \tilde{\underline{\beta}}_H) < \delta^*$. Suppose it is the former, so $\limsup_{\gamma\to\infty}(\tilde{\beta}^* - \tilde{\underline{\beta}}_H) = \delta^* + \varepsilon$ for some $\varepsilon > 0$. Then there exists an infinite sequence $\{\gamma_1, \gamma_2, ...\}$, such that $\lim_{n\to\infty}\gamma_n = \infty$, and $(\tilde{\underline{\beta}}_H(\gamma_n) + \delta^* + \varepsilon/2) \in \left(\tilde{\underline{\beta}}_H(\gamma_n) + \delta^*, \tilde{\beta}^*(\gamma_n)\right)$ for all n. Λ decreasing (from proof of Lemma 3.1) implies that, for all n,

$$\Lambda(\tilde{\underline{\beta}}_H(\gamma_n) + \delta^* + \varepsilon/2) \in \left(\Lambda(\tilde{\beta}^*(\gamma_n)), \Lambda(\tilde{\underline{\beta}}_H(\gamma_n) + \delta^*)\right)$$

By the squeeze theorem

$$\lim_{n\to\infty}\Lambda(\tilde{\underline{\beta}}_H(\gamma_n) + \delta^* + \varepsilon/2) = 0$$

which contradicts (64). Hence, $\limsup_{\gamma\to\infty}(\tilde{\beta}^* - \tilde{\underline{\beta}}_H) \leq \delta^*$. An identical argument shows that $\liminf_{\gamma\to\infty}(\tilde{\beta}^* - \tilde{\underline{\beta}}_H) \geq \delta^*$, implying $\lim_{\gamma\to\infty}(\tilde{\beta}^* - \tilde{\underline{\beta}}_H) = \delta^* = \frac{V_L - K_L}{V_H - K_H}$.

Next, from (56), notice that $\tilde{A}_H$ is continuous, so if two sequences $\{x_n\}, \{y_n\}$ converge to each other, then $\lim_{n\to\infty}\tilde{A}_H(x_n) = \lim_{n\to\infty}\tilde{A}_H(y_n)$. We have just shown that, as $\gamma \to \infty$, $(\tilde{\underline{\beta}}_H + \delta^*) \to \tilde{\beta}^*$, so $\lim_{\gamma\to\infty}\tilde{A}_H(\tilde{\beta}^*) = \lim_{\gamma\to\infty}\tilde{A}_H(\tilde{\underline{\beta}}_H + \delta^*)$. Using the closed-form expressions for $\tilde{\underline{\beta}}_H$ and δ^*, the latter is easily obtained as $\frac{V_L - K_L}{V_H - K_H}$. Transforming back to log-likelihood space gives the result.

For (3), the proof of Lemma 3.2 establishes that F_H is increasing on (α^*, β^*), so $\min_{z\in\mathbb{R}} F_H(z) = F_H(\alpha^*)$. $F_H \leq V_H$ by Zero Profit, so it is sufficient to show that $\lim_{\gamma\to\infty} F_H(\alpha^*) = V_H$. From (36), recall that $\hat{F}_H(z|\alpha, \beta)$ denotes the value to the type-θ seller who rejects all offers until Z, as given by (7), reaches β and $w(\beta) = \Psi(\beta)$. From the proof of (1) and (2) above, for all γ large enough $\alpha^* < \underline{\beta}_H$, and further: $F_H(\alpha^*) = \hat{F}_H(\alpha^*|\alpha^*, \beta^*) \geq \hat{F}_H(\alpha^*|\alpha^*, \underline{\beta}_H)$, where the equality is by definition and the inequality is by No Deals and Seller Optimality. By $\hat{F}_H(z|\alpha, \beta)$ continuous in α, and (2), $\lim_{\gamma\to\infty} \hat{F}_H(\alpha^*|\alpha^*, \underline{\beta}_H) = \lim_{\gamma\to\infty} \hat{F}_H(\alpha_\infty|\alpha_\infty, \underline{\beta}_H)$, where $\alpha_\infty = \ln\left(\frac{V_L - K_L}{V_H - K_H}\right)$. Using the closed-form expression for $\underline{\beta}_H$ and (34), the latter is easily obtained as V_H.

For (4), given any state z, because of common knowledge of gains from trade, the maximum total value in this game is $\Psi(z)$. In $\Xi(\alpha^*, \beta^*)$ the expected payoff to the seller is $p(z)F_H(z) + (1 - p(z))F_L(z)$. The Zero Profit condition implies that the expected payoff to the buyer side of the market is zero. Therefore $p(z)F_H(z) + (1 - p(z))F_L(z) \leq \Psi(z)$. From (3) above, for all z, $\lim_{\gamma\to\infty} F_H(z) = V_H$. Hence, for all z, $\lim_{\gamma\to\infty} F_L(z) \leq V_L$. For all z, $\lim_{\gamma\to\infty} F_L(z) \geq V_L$ by No Deals, giving the result.

For (5), the denominator of $\mathcal{L}$ is always positive, so we need to show that $(\Pi^*(z) - \Pi^S(z)) \overset{u}{\to} 0$. Zero Profit and No Deals imply that for all $\gamma > \underline{\gamma}$ and z,

$$0 \leq \Pi^*(z) - \Pi^S(z) = \Psi(z) - (p(z)F_H(z) + (1 - p(z))F_L(z)) \leq p(z)(V_H - F_H(z))$$

(3) above implies that $p(z)(V_H - F_H(z)) \overset{u}{\to} 0$, giving the result. *Q.E.D.*

Proof of Proposition 4.2 For (1) and (2), we will first establish that $\lim_{\gamma\to 0}(\beta^* - \alpha^*) = 0$, and then that their common limit is $\underline{z}$. First, suppose that $\lim_{\gamma\to 0}(\beta^* - \alpha^*) > 0$. Then, by (32), (36), and continuity of g_L with respect to α, β,

$$F_L\left((\alpha^* + \beta^*)/2\right) = K_L + g_L\left((\alpha^* + \beta^*)/2|\alpha^*, \beta^*\right)\left(\Psi(\beta^*) - K_L\right) \to K_L < V_L$$

violating No Deals. Hence, $\lim_{\gamma\to 0}(\beta^* - \alpha^*) = 0$.

To see that their common limit is $\underline{z}$, note first that $\beta^* \geq \underline{z}$ by Seller Optimality (since the high type always has the option to retain the asset forever and receive K_H). To demonstrate $\beta^* \leq \underline{z}$, we claim that it is sufficient to prove that $\lim_{\gamma\to 0} g_H(\alpha^*|\alpha^*, \beta^*) < 1$. To see this notice that for any $\gamma > 0$

$$F_H(\alpha^*) = K_H + g_H(\alpha^*|\alpha^*, \beta^*)(\Psi(\beta^*) - K_H) \geq \Psi(\alpha^*)$$

where the inequality follows from No Deals and is equivalent to

$$K_H(1 - g_H(\alpha^*|\alpha^*, \beta^*)) \geq \Psi(\alpha^*) - g_H(\alpha^*|\alpha^*, \beta^*)\Psi(\beta^*)$$

If $\lim_{\gamma\to 0} g_H(\alpha^*|\alpha^*, \beta^*) < 1$, then, for all γ small enough $g_H(\alpha^*|\alpha^*, \beta^*) < 1$, so we can divide through by $(1 - g_H(\alpha^*|\alpha^*, \beta^*))$, rearrange and obtain

$$K_H \geq \Psi(\beta^*) + \frac{\Psi(\alpha^*) - \Psi(\beta^*)}{1 - g_H(\alpha^*|\alpha^*, \beta^*)}$$

Since $\lim_{\gamma\to 0}(\beta^* - \alpha^*) = 0$ and Ψ is continuous, it must be that $\lim_{\gamma\to 0} \Psi(\beta^*) \leq K_H$, implying $\lim_{\gamma\to 0} \beta^* \leq \underline{z}$.

We are left only to establish our premise that $\lim_{\gamma\to 0} g_H(\alpha^*|\alpha^*, \beta^*) < 1$. Recall that

$$F_L(\alpha^*) = K_L + g_L(\alpha^*|\alpha^*, \beta^*)(\Psi(\beta^*) - K_L) = V_L$$

so, for all $\gamma > 0$, $g_L(\alpha^*|\alpha^*, \beta^*) = \frac{V_L - K_L}{\Psi(\beta^*) - K_L} < 1$. Finally, notice that *i)* for any fixed values of α, β, taking $\gamma \to 0$, and *ii)* for any fixed γ, taking $(\beta - \alpha) \to 0$ both result in $(g_H(\alpha|\alpha, \beta) - g_L(\alpha|\alpha, \beta)) \to$

0. Combining this with $(g_H(\alpha|\alpha,\beta) - g_L(\alpha|\alpha,\beta))$ continuous in both γ and $(\beta-\alpha)$, and that $\lim_{\gamma\to 0}(\beta^* - \alpha^*) = 0$, yields $\lim_{\gamma\to 0} g_H(\alpha^*|\alpha^*,\beta^*) = \frac{V_L - K_L}{\Psi(\beta^*) - K_L} < 1$.

For (3), define the function $\underline{F}_H(z) = \max\{K_H, \Psi(z)\}$ for all z. Seller Optimality and No Deals imply that, for all z, $F_H(z) \geq \underline{F}_H(z)$. Next, for fixed $\gamma > 0$, F_H and Ψ both weakly increasing implies that $\max_z(F_H(z) - \underline{F}_H(z)) \leq \Psi(\beta^*) - K_H$. Continuity of Ψ and (1) then produce the result.

For (4), the result follows immediately from (1), (2), and the definition of $\Xi(\alpha^*,\beta^*)$. For (5), the result follows immediately from (3) and (4). *Q.E.D.*

Proof of Proposition 4.3 From (36), the equilibrium value function can be written as $F_\theta(z) = K_\theta + g_\theta(z|\alpha^*,\beta^*)(\Psi(\beta^*) - K_\theta)$. For $\theta = L, H$, by construction of g_θ, $F_\theta(\beta^{*-}) = \Psi(\beta^*)$ and $F'_\theta(\alpha^{*+}) = 0$. Thus, the two remaining equilibrium boundary conditions are $F'_H(\beta^{*+}) = \Psi'(\beta^*)$ and $F_L(\alpha^{*+}) = V_L$. These can be written as

$$\frac{\partial}{\partial z} g_H(\beta^*|\alpha^*,\beta^*) = \frac{\Psi'(\beta^*)}{\Psi(\beta^*) - K_H}$$
$$g_L(\alpha^*|\alpha^*,\beta^*) = \frac{V_L - K_L}{\Psi(\beta^*) - K_L}$$

Using (30), note that both $\frac{\partial}{\partial z} g_H(\beta|\alpha,\beta)$ and $g_L(\alpha|\alpha,\beta)$ can be written as functions of only $\beta - \alpha$ and γ. For analytic convenience, define $\zeta_H(c,x), \zeta_L(c,x)$ such that $\frac{\partial}{\partial z} g_H(\beta|\alpha,\beta) = \zeta_H(c,x)$ and $g_L(\alpha|\alpha,\beta) = \zeta_L(c,x)$ for all $c \equiv \beta - \alpha$, $x \equiv \sqrt{1 + 8/\gamma}$. These can easily be derived as $\zeta_H(c,x) = \frac{(x^2-1)(e^{cx}-1)}{2(x-1+e^{cx}(1+x))}$ and $\zeta_L(c,x) = \frac{2xe^{\frac{1}{2}(c(x-1))}}{1+e^{cx}(x-1)+x}$. Letting $\beta^*(x)$ and $c^*(x)$ denote the equilibrium values as they depend on x, we have

$$\zeta_H(c^*(x),x) = \frac{\Psi'(\beta^*(x))}{\Psi(\beta^*(x)) - K_H}$$
$$\zeta_L(c^*(x),x) = \frac{V_L - K_L}{\Psi(\beta^*(x)) - K_L}$$

All the functions are continuously differentiable. Thus, by implicit differentiation:

$$\underbrace{\frac{\partial \zeta_H}{\partial c}}_{+} \cdot \frac{\partial c^*}{\partial x} + \underbrace{\frac{\partial \zeta_H}{\partial x}}_{+} = \underbrace{\frac{(\Psi(\beta^*) - K_H)\Psi''(\beta^*) - \Psi'(\beta^*)^2}{(\Psi(\beta^*) - K_H)^2}}_{-} \cdot \frac{\partial \beta^*}{\partial x} \tag{65}$$

$$\underbrace{\frac{\partial \zeta_L}{\partial c}}_{-} \cdot \frac{\partial c^*}{\partial x} + \underbrace{\frac{\partial \zeta_L}{\partial x}}_{-} = \underbrace{\frac{-(V_L - K_L)\Psi'(\beta^*)}{(\Psi(\beta^*) - K_L)^2}}_{-} \cdot \frac{\partial \beta^*}{\partial x} \tag{66}$$

The sign of the partial derivatives can be easily verified from the functional forms of ζ_θ and Ψ for $\beta^* > \underline{\beta}_H$. From (66), we conclude that if $\frac{\partial c^*}{\partial x} \geq 0 \Rightarrow \frac{\partial \beta^*}{\partial x} > 0$ and from (65), we conclude that if $\frac{\partial c^*}{\partial x} \geq 0 \Rightarrow \frac{\partial \beta^*}{\partial x} < 0$. Thus, it must be that $\frac{\partial c^*}{\partial x} < 0$, which implies that $\beta^* - \alpha^*$ is increasing in γ.

To prove that β^* is also increasing in γ, suppose it is not (i.e., suppose $\frac{\partial \beta^*}{\partial x} \geq 0$). This implies that $\frac{\partial \zeta_L}{\partial c}\frac{\partial c^*}{\partial x} + \frac{\partial \zeta_L}{\partial x} \leq 0$, or equivalently that $\frac{\partial c^*}{\partial x} \geq -\frac{\partial \zeta_L}{\partial x} / \frac{\partial \zeta_L}{\partial c}$. Note that the LHS of (65) is increasing in $\frac{\partial c^*}{\partial x}$. Using the lower bound just derived on $\frac{\partial c^*}{\partial x}$, we have that

$$\begin{aligned}\frac{\partial \zeta_H}{\partial c}\frac{\partial c^*}{\partial x}+\frac{\partial \zeta_H}{\partial x} &\geq \frac{\partial \zeta_H}{\partial c}\left(-\frac{\partial \zeta_L}{\partial x}/\frac{\partial \zeta_L}{\partial c}\right)+\frac{\partial \zeta_H}{\partial x}\\ &> 0 \qquad \forall (x,c)\in(1,\infty)\times(0,\infty)\end{aligned}$$

Where the strict inequality follows from the closed-form expressions for ζ_L, ζ_H (above). Therefore, $\frac{\partial \zeta_H}{\partial c}\frac{\partial c^*}{\partial x}+\frac{\partial \zeta_H}{\partial x} > 0$ implying from (65) that $\frac{\partial \beta^*}{\partial x} < 0$, contradicting the original supposition, and completing the proof that β^* is increasing in γ. *Q.E.D.*

The proof of Proposition 4.4 requires the following technical lemma.

Lemma B.4. *Consider any two news qualities $\gamma < \bar{\gamma}$. Let (α^*, β^*), $(\bar{\alpha}^*, \bar{\beta}^*)$ denote the corresponding equilibrium boundaries with value functions denoted $F_\theta, \bar{F}_\theta$. If $\alpha^* \leq \bar{\alpha}^*$, then $\bar{F}'_\theta(z) < F'_\theta(z)$ for all $z \in (\alpha^*, \beta^*)$, $\theta \in \{L, H\}$.*

Proof: From Proposition 4.3, $\bar{\beta}^* > \beta^*$. If $\bar{\alpha}^* \geq \beta^*$, then for all $z \in (\alpha^*, \beta^*)$, $\bar{F}'_L(z) = 0 < F'_L(z)$ (from proof of Lemma 3.2). Next, if $\bar{\alpha}^* < \beta^*$, then for any $z \in (\bar{\alpha}^*, \beta^*)$, by differentiating (14) and using the equations for C_1^L and C_2^L, ((49) and (50)) we have that:

$$\begin{aligned}F'_L(z) = C_1^L q_1^L e^{q_1^L z} + C_2^L q_2^L e^{q_2^L z} &= \frac{-q_2^L q_1^L (V_L - K_L)}{q_1^L - q_2^L} e^{q_1^L (z-\alpha^*)} + \frac{q_1^L q_2^L (V_L - K_L)}{q_1^L - q_2^L} e^{q_2^L (z-\alpha^*)}\\ &= \frac{\left(q_1^L - 1\right) q_1^L (V_L - K_L)}{2q_1^L - 1}\left(e^{q_1^L (z-\alpha^*)} - e^{q_2^L (z-\alpha^*)}\right)\\ &> \frac{\left(\bar{q}_1^L - 1\right) \bar{q}_1^L (V_L - K_L)}{2\bar{q}_1^L - 1}\left(e^{q_1^L (z-\alpha^*)} - e^{q_2^L (z-\alpha^*)}\right)\\ &\geq \frac{\left(\bar{q}_1^L - 1\right) \bar{q}_1^L (V_L - K_L)}{2\bar{q}_1^L - 1}\left(e^{\bar{q}_1^L (z-\bar{\alpha}^*)} - e^{\bar{q}_2^L (z-\bar{\alpha}^*)}\right) = \bar{F}'_L(z)\end{aligned}$$

The first inequality follows from $\frac{\left(q_1^L - 1\right) q_1^L (V_L - K_L)}{2q_1^L - 1}$ increasing in q_1^L, $q_1^L > \bar{q}_1^L$, and $e^{q_1^L (z-\alpha^*)} > e^{q_2^L (z-\alpha^*)}$. The second follows from $q_1^L > \bar{q}_1^L > 1$, $q_2^L < \bar{q}_2^L < 0$ and $z > \bar{\alpha}^* \geq \alpha^*$ for all $z \in (\bar{\alpha}^*, \beta^*)$. Finally, for any $z \in (\alpha^*, \bar{\alpha}^*]$ (if the set is non-empty) $\bar{F}'_L(z) = 0 < F'_L(z)$. The proof for $\theta = H$ follows an analogous method and is therefore omitted. *Q.E.D.*

Proof of Proposition 4.4. For (1), $F_H^1(z) > \Psi(z)$, $\forall z < \beta_1^*$ by Lemma 3.2, and $F_H^0(z) = \Psi(z)$, $\forall z \geq \beta_0^*$. Since $\beta_1^* > \beta_0^*$ (Proposition 4.3), we have $F_H^1(z) > F_H^0(z)$ $\forall z \in [\beta_0^*, \beta_1^*)$. Clearly, $F_H^1(z) = F_H^0(z) = \Psi(z)$, $\forall z \geq \beta_1^*$. All that is left to show is that $F_H^1(z) > F_H^0(z)$ $\forall z < \beta_0^*$.

If $\beta_0^* \leq \alpha_1^*$, then this is immediate since for all $z < \alpha_1^*$, $F_H^1(z) = F_H^1(\alpha_1^*) > \Psi(\alpha_1^*) = F_H^0(\alpha_1^*)$ and F_H^0 is increasing. If $\beta_0^* > \alpha_1^*$, letting g_θ^i correspond to (30) for $\gamma = \gamma_i$ we have that for any $z < \beta_0^*$

$$\begin{aligned}F_H^1(z) &= K_H + g_H^1(z|\alpha_1^*, \beta_0^*)(F_H^1(\beta_0^*) - K_H)\\ &\geq K_H + g_H^1(z|\alpha_0^*, \beta_0^*)(F_H^1(\beta_0^*) - K_H)\\ &> K_H + g_H^0(z|\alpha_0^*, \beta_0^*)(F_H^1(\beta_0^*) - K_H)\\ &> K_H + g_H^0(z|\alpha_0^*, \beta_0^*)(\Psi(\beta_0^*) - K_H) = F_H^0(z)\end{aligned}$$

where the first (weak) inequality follows from $\alpha_1^* \geq \alpha_0^*$, the second from g_H increasing in γ and the third from $F_H^1(\beta_0^*) > \Psi(\beta_0^*)$.

For (2), it is immediate that $F_L^0(z) = F_L^1(z)$, $\forall z \leq \alpha_0^*$ and $z \geq \beta_1^*$. For any $z \in (\alpha_0^*, \alpha_1^*]$ (if the set is non-empty), $F_L^1(z) = V_L$, while $F_L^0(z) > V_L$ by Lemma 3.2. For $z \in [\beta_0^*, \beta_1^*)$, $F_L^1(z) < \Psi(z) = F_L^0(z)$. If $\beta_0^* > \alpha_1^*$, then $F_L^1(z) < F_L^0(z)$, $\forall z \in (\alpha_1^*, \beta_0^*)$, follows from Lemma B.4.

From (1) and (2), $\mathcal{L}^1(z) < \mathcal{L}^0(z)$ for all $z \leq \alpha_0^*$ and $\mathcal{L}^1(\beta_0^*) > \mathcal{L}(\beta_0^*) = 0$. By continuity there exists some $z' \in (\alpha, \beta)$ such that $\mathcal{L}^1(z) < \mathcal{L}^0(z)$ for all $z < z'$ yielding (3). Similarly, there must exist a $z'' \in (\alpha_0^*, \beta_0^*)$ such that $\mathcal{L}^1(z) > \mathcal{L}^0(z)$ for all $z \in (z'', \beta_1^*)$. That $z' = z''$ follows from $\frac{d}{dz}F_H^0(z) > \frac{d}{dz}F_H^1(z)$ and $\frac{d}{dz}F_L^0(z) > \frac{d}{dz}F_L^1(z)$ for all $z \in (\alpha_0^*, \beta_0^*)$ (Lemma B.4), which implies single crossing of $\mathcal{L}^0$ and $\mathcal{L}^1$. *Q.E.D.*

B.5 Proofs for Section 5

Proof of Lemma 5.1. For (a), given any state z, because of common knowledge of gains from trade, the maximum total value in this game is $\Psi(z)$. The expected payoff to the seller is $p(z)F_H(z) + (1 - p(z))F_L(z)$, where $p(z) = e^z/(1 + e^z)$, and the Zero Profit condition implies that the expected payoff to the buyer side of the market is zero. Therefore $p(z)F_H(z) + (1 - p(z))F_L(z) \leq \Psi(z)$. Combining this with the No Deals condition gives (a).

For (b), fix a state z. If $w(z) > \Psi(z)$, then, by definition of value function, $F_L(z) \geq w(z) > \Psi(z)$ violating (a). If $w(z) < \Psi(z)$, the high type must reject with probability one by the No Deals condition. Hence, by Zero Profit, if trade occurs when $w(z) < \Psi(z)$, it must be at $w(z) = V_L$.

For (c), we will establish that if $w(z) = \Psi(z)$ for any z, then the low type accepts with probability one. Zero Profit then implies that the high type accepts with probability one as well. For the remainder of the argument fix the seller's type as L. Recall that the optimal policy in NMG$_L$ is to accept immediately in all states (Lemma A.1). We now draw the following connection to the seller's problem induced by Z and w in the true game. Fix any initial state $Z_0 = z$ at time $t = 0$. For an arbitrary $\epsilon > 0$, consider modified versions of both NMG$_L$ and of the seller's problem in the true game in which the seller is constrained to continue for all $t < \epsilon$. Let $\hat{\tau}, \tau \geq \epsilon$ be optimal stopping times in the modified versions of NMG$_L$ and the true game respectively. We wish to establish that, for every ϵ small enough,

$$E_z^L\left[\int_0^\tau e^{-rt} rK_L dt + e^{-r\tau} w(Z_\tau)\right] \leq E_z^L\left[\int_0^{\hat{\tau}} e^{-rt} rK_L dt + e^{-r\hat{\tau}} \Psi(\hat{Z}_{\hat{\tau}})\right] < \Psi(Z_0)$$

The second inequality follows from Lemma A.1; in fact, $\hat{\tau} = \epsilon$ uniquely. To see the first, we will separately consider the case where $Z_\epsilon \leq \hat{Z}_\epsilon$, and the case where $Z_\epsilon > \hat{Z}_\epsilon$. If $Z_\epsilon \leq \hat{Z}_\epsilon$, then by (a), $F_L(Z_\epsilon) \leq \Psi(\hat{Z}_\epsilon)$ so

$$\begin{aligned} E_z^L\left[\int_0^\tau e^{-rt} rK_L dt + e^{-r\tau} w(Z_\tau) \middle| Z_\epsilon \leq \hat{Z}_\epsilon\right] &\leq E_z^L\left[\int_0^\epsilon e^{-rt} rK_L dt + e^{-r\epsilon} F_L(Z_\epsilon) \middle| Z_\epsilon \leq \hat{Z}_\epsilon\right] \\ \leq E_z^L\left[\int_0^\epsilon e^{-rt} rK_L dt + e^{-r\epsilon} \Psi(\hat{Z}_\epsilon) \middle| Z_\epsilon \leq \hat{Z}_\epsilon\right] &= E_z^L\left[\int_0^{\hat{\tau}} e^{-rt} rK_L dt + e^{-r\hat{\tau}} \Psi(\hat{Z}_{\hat{\tau}}) \middle| Z_\epsilon \leq \hat{Z}_\epsilon\right] \end{aligned}$$

If $Z_\epsilon > \hat{Z}_\epsilon$ then (24) implies that $S_{\epsilon^-}^{L,0} > S_{\epsilon^-}^{H,0}$. Zero Profit and (b) above then imply that there must be positive probability that the low type accepts V_L at some time $t^* \in [0, \epsilon]$ in the true game.

Therefore

$$E_z^L\left[\int_0^{\tau} e^{-rt} rK_L dt + e^{-r\tau} w(Z_\tau)\middle| Z_\epsilon > \hat{Z}_\epsilon\right] \leq E_z^L\left[\int_0^{t^*} e^{-rt} rK_L dt + e^{-rt^*} V_L\middle| Z_\epsilon > \hat{Z}_\epsilon\right]$$

$$\leq E_z^L\left[\int_0^{\epsilon} e^{-rt} rK_L dt + e^{-r\epsilon} \Psi(\hat{Z}_\epsilon)\middle| Z_\epsilon > \hat{Z}_\epsilon\right] = E_z^L\left[\int_0^{\hat{\tau}} e^{-rt} rK_L dt + e^{-r\hat{\tau}} \Psi(\hat{Z}_{\hat{\tau}})\middle| Z_\epsilon > \hat{Z}_\epsilon\right]$$

where the second inequality comes follows from $K_L < \Psi(z)$ for all z, and ϵ small enough. Hence, it is uniquely optimal for the low type to accept $\Psi(Z_0)$ at $t = 0$ if it is offered. Because the seller's problem is stationary, the argument extends to any time $t \geq 0$.

For (d), suppose the claim was false for state z. Then if $Z_t = z$, by Belief Consistency, $Z_{t'} = \infty$ for all $t' > t$. It is immediate that $F_L(\infty) = V_H$, and that low type would then do better to reject V_L when $Z_t = z$, contradicting the premise. *Q.E.D.*

PROOF OF LEMMA 5.2. For the purpose of contradiction, suppose that there exists an SBM-equilibrium and a z_0 such that F_L is discontinuous at z_0. This rules out that z_0 is an element of a no-trade region: from Section 3.1, in any no-trade region $F_L = C_1^L e^{q_1^L z} + C_2^L e^{q_2^L z} + K_L$, which is continuous for any values of the constants C_1^L, C_2^L. Therefore, by Lemma 5.1(b), for any $\varepsilon > 0$, there must exist $z_1 \neq z_2$ such that $F_L(z_1) = V_L$, $F_L(z_2) = \Psi(z_2)$, $|z_1 - z_2| < \varepsilon$, and either $z_1 \leq z_0 \leq z_2$ or $z_2 \leq z_0 \leq z_1$.

As we established in the proof of Lemma 5.1(a), $p(z)F_H(z) + (1 - p(z))F_L(z) \leq \Psi(z)$ for all z. Hence for any Z_{t_0} such that $F_L(Z_{t_0}) = \Psi(Z_{t_0})$, No Deals implies that $F_H(Z_{t_0}) = \Psi(Z_{t_0})$. We will establish that if $F_H(Z_{t_0}) = \Psi(Z_{t_0})$, then Z is continuous at time t_0 with probability one, according to both $\mathcal{Q}_{Z_{t_0}}^H$ and $\mathcal{Q}_{Z_{t_0}}^L$, and that Z_{t_0} is *not* a lower reflecting barrier of Z. We then show this rules out the existence of a z_1, z_2 described above, proving the lemma. Because Z is a time-homogenous Markov, it is without loss to normalize $t_0 = 0$.

First, suppose that Z is not continuous with probability one at $t = 0$ under $\mathcal{Q}_{Z_0}^H$. Then there exists a $\delta > 0$ and $P \in (0, 1)$ such that for all $\epsilon > 0$, $\mathcal{Q}_{Z_0}^H(|Z_\epsilon - Z_0| < \delta) < P$. Belief Monotonicity implies that Z_ϵ weakly first-order stochastically dominates $\hat{Z}_\epsilon$ under $\mathcal{Q}_{Z_0}^H$. Hence, as $\epsilon \to 0$, $\mathcal{Q}_{Z_0}^H(Z_\epsilon > Z_0 + \delta) > 1 - P > 0$. It then follows that for small enough ϵ

$$\Psi(Z_0) < E_{Z_0}^H\left[\int_0^\epsilon e^{-rt} rK_H dt + e^{-r\epsilon}\Psi(Z_\epsilon)\right] \leq E_{Z_0}^H\left[\int_0^\epsilon e^{-rt} rK_H dt + e^{-r\epsilon} F_H(Z_\epsilon)\right]$$

where the first inequality is due to positive probability of a jump and the second follows from No Deals. Hence, the high type has a profitable deviation from acceptance at Z_0, so Z must be continuous at time 0 with probability one under $\mathcal{Q}_{Z_0}^H$. Now, Belief Consistency requires

$$Z_{0^+} = Z_0 + \underbrace{\lim_{t \downarrow 0} \ln\left(\frac{f_t^H(X_t)}{f_t^L(X_t)}\right)}_{= 0 \text{ w.p.1 under } \mathcal{Q}_{Z_0}^L \text{or } \mathcal{Q}_{Z_0}^H} + \ln\left(\frac{1 - S_0^{H,0}}{1 - S_0^{L,0}}\right)$$

Finally, notice that the distribution of $\ln\left(\frac{1-S_0^{H,0}}{1-S_0^{L,0}}\right)$ is degenerate regardless of the θ. We have just shown that if $\theta = H$, it is equal to 0, hence it must also be 0 if $\theta = L$. Therefore, Z must be continuous at time 0 with probability one under $\mathcal{Q}_{Z_0}^L$.

Second, suppose Z_0 is a lower reflecting barrier of Z. Let $\tau_\epsilon = \inf\{t : Z_t \geq Z_0 + \epsilon\}$, let τ_H denote an arbitrary element of $\mathcal{S}^{H,\tau_\epsilon}$, where $\mathcal{S}^{H,t} \equiv supp(S^{\theta,t})$, and consider the stopping time $\hat{\tau} = \tau_\epsilon \vee \tau_H$. Let $f(Z_0, \epsilon)$ denote the expected payoff to the high type from playing according to $\hat{\tau}$ starting from Z_0 and note that Belief Monotonicity implies that $f(Z_0,\epsilon) \geq K_H + g_H(Z_0|Z_0, Z_0+\epsilon)(F_H(Z_0+\epsilon) - K_H)$. By hypothesis, $F_H(Z_0) = \Psi(Z_0) \geq f(Z_0,\epsilon)$ and No Deals requires $F_H(Z_0+\epsilon) \geq \Psi(Z_0+\epsilon)$. Therefore we have that $F_H(Z_0+\epsilon) - f(Z_0,\epsilon) \geq \Psi(Z_0+\epsilon) - \Psi(Z_0)$. Using the lower bound above for f, dividing both sides by ϵ and taking the limit as $\epsilon \to 0$ gives $0 \geq \Psi'(Z_0)$, contradicting the hypothesis and implying that Z_0 cannot be a reflecting barrier.

Finally, for small $\delta > 0$, let τ_L denote an arbitrary element of $\mathcal{S}^{H,\delta}$, and $\tau_{z_2} = \inf\{t : Z_t = z_2\}$. Starting from $Z_0 = z_1$, if the low type plays according to the stopping rule: $\tau = \tau_{z_2} 1_{\{\tau_{z_2} \leq \delta\}} + \tau_L 1_{\{\tau_{z_2} > \delta\}}$, then her payoff is

$$\mathcal{Q}^L_{z_1}(\tau = \tau_{z_2}) E^L_{z_1}\left[\int_0^{\tau_{z_2}} e^{-rt} r K_L dt + e^{-r\tau_{z_2}}\Psi(z_2) | \tau_{z_2} \leq \delta\right] + \mathcal{Q}^L_{z_1}(\tau = \tau_L) E^L_{z_1}\left[\int_0^{\delta} e^{-rt} r K_L dt + e^{-r\delta} F_L(Z_\delta) | \tau_{z_2} > \delta\right]$$

For small $\delta > 0$ this payoff is greater than V_L because the $\mathcal{Q}^L_{z_1}(\tau = \tau_{z_2})$ is strictly positive and $F_L(Z_\delta) \geq V_L$ by No Deals. This contradicts the supposition that $F_L(z_1) = V_L$ and completes the proof. *Q.E.D.*

The proof of Proposition 5.1 requires the following technical lemma.

Lemma B.5. *Suppose the SLC does not hold, and let $\gamma > \gamma^0$ (but not necessarily $\gamma \geq \underline{\gamma}$). Define $\underline{\alpha} \equiv A_H(\overline{z}_H)$, where A_H is defined in the proof of Lemma B.2 and $\overline{z}_H$ in Lemma A.3.*

- *(i)* $\underline{\alpha} < \underline{z}_H$
- *(ii)* *B_H is continuous for all $z > \underline{\alpha}$*
- *(iii)* *There exists $\bar{\alpha}$ such that $B_L(\alpha) > B_H(\alpha)$ for all $\alpha > \bar{\alpha}$*

PROOF: For (i), the boundary conditions required at $\overline{z}_H$ in NMG$_H$ are identical to those required at β (i.e., (46) and (48)). The differential equation the value function in NMG$_H$ satisfies is also the same (i.e., (13)). Let $\vartheta_H(z)$ denote the function which satisfies (13), (46), and (48) at $\beta = \bar{z}_H$. Note that

$$\vartheta_H(z) = C_1^H(\overline{z}_H) e^{q_1^H z} + C_2^H(\overline{z}_H) e^{q_2^H z} + K_H$$

Without a lower reflecting barrier, the lower boundary $\underline{z}_H$ in NMG$_H$ must solve $\vartheta'_H(\underline{z}_H) = \Psi'(\underline{z}_H) > 0$, whereas $\underline{\alpha}$ solves $\vartheta'_H(\underline{\alpha}) = 0$. Since $C_1^H(\overline{z}_H), C_2^H(\overline{z}_H) > 0$ (Lemma B.2), $\vartheta''_H(z) > 0$ and therefore $\vartheta'_H(\underline{z}_H) > \vartheta'_H(\underline{\alpha}) \Rightarrow \underline{z}_H > \underline{\alpha}$.

For (ii), note that for $\gamma \in (\gamma^0, \underline{\gamma})$, $C_2^H(\beta) > 0$ for all $\beta \in \mathbb{R}$. Hence,

$$A_H(\beta) = \frac{1}{q_1^H - q_2^H} \ln\left(-\frac{q_2^H}{q_1^H}\frac{C_2^H(\beta)}{C_1^H(\beta)}\right) = \beta - \frac{1}{q_1^H - q_2^H} \ln\left(-\frac{q_1^H}{q_2^H} \cdot \frac{\Psi'(\beta) + q_2^H(K_H - \Psi(\beta))}{q_1^H(K_H - \Psi(\beta)) - \Psi'(\beta)}\right) \quad (67)$$

is real-valued, differentiable and weakly less than β for all $\beta \in \mathbb{R}$. Further, differentiating A_H gives $\text{sgn}(A'_H(z)) = \text{sgn}(-MB_H(z))$, and so A_H is increasing over $\mathbb{R}\backslash(z^-, z^+)$. This implies that B_H has exactly one point of discontinuity which occurs at $A_H(z^+)$. From Lemma A.3, $z^+ < \bar{z}_H$, and since A_H is increasing above z^+, we conclude that $A_H(z^+) < A_H(\bar{z}_H) = \underline{\alpha}$.

For *(iii)*, since A_H is increasing above z^+, it is invertible and $B_H(z) = A_H^{-1}(z)$ for all $z > \underline{\alpha}$. From (67), $\lim_{\beta\to\infty} \beta - A_H(\beta) = 0$, which implies that $\lim_{\alpha\to\infty} B_H(\alpha) - \alpha = 0$. From Lemma B.1, $B_L(\alpha) \geq \alpha$ for all α and B_L is continuous. Therefore, $\lim_{\alpha\to\infty} B_L(\alpha) - \alpha \geq 0$. Suppose that $\lim_{\alpha\to\infty} B_L(\alpha) - \alpha = 0$, then the LHS of (51) becomes $A + B + K_L - V_H = V_L - V_H < 0$, a contradiction. Hence, it must be that $\lim_{\alpha\to\infty} B_L(\alpha) - \alpha > 0$ and for α sufficiently large, $B_L(\alpha) > B_H(\alpha)$. *Q.E.D.*

Proof of Proposition 5.1. The argument proceeds by partitioning the parameter space into four subsets and then identifying an SBM-equilibrium that exists for each subset. Our first three cases are covered by other results.

(i) If the SLC holds, then $\Xi(\alpha^*, \beta^*)$ exists and is an SBM-equilibrium (Theorem 5.1).

(ii) If the SLC fails and $\gamma \leq \gamma^0$, then $w = \Psi$, $Z = \hat{Z}$, and $S_{t'}^{H,t} = S_{t'}^{L,t} = 1$ for all $0 \leq t \leq t'$ constitutes an SBM-equilibrium (see the first paragraph of the proof of Proposition 5.2).

(iii) If the SLC fails, $\gamma > \gamma^0$, and $K_L \geq \underline{K}_L$, then the SBM-equilibrium of Example 5.1 exists.

In the fourth and final parameter subset the SLC fails, $\gamma > \gamma^0$, and $K_L < \underline{K}_L$. We now argue that these conditions are sufficient to ensure that there exists an (α_0, β_0) such that $\Xi(\alpha_0, \beta_0)$ constitutes an SBM-equilibrium.

The argument relies on properties of B_L and B_H. Lemma B.5 shows that there exists a unique $\underline{\alpha}$ such that $B_H(\underline{\alpha}) = \overline{z}_H$, that $\underline{\alpha} < \underline{z}_H$, and that B_H is continuous at all $\alpha > \underline{\alpha}$. In addition, Lemma B.1 shows that B_L is continuous and strictly increasing. Finally, Lemma B.5 shows that there exists $\bar{\alpha}$ such that $B_L(\alpha) > B_H(\alpha)$ for all $\alpha > \bar{\alpha}$. Putting all of this together leads to: if $B_L(\underline{\alpha}) < B_H(\underline{\alpha})$, then the curves must intersect at some $\alpha_0 \in (\underline{\alpha}, \bar{\alpha}]$. We now show that the hypothesis holds. Consider the resulting low type's value function in $(\underline{z}_H, \overline{z}_H)$ if the interval is a no-trade region with value-matching boundary conditions $F_L(\underline{z}_H) = \Psi(\underline{z}_H)$ and $F_L(\overline{z}_H) = \Psi(\overline{z}_H)$. Because $K_L < \underline{K}_L$, there exists $(z_1, z_2) \subset (\underline{z}_H, \overline{z}_H)$ such that $F_L(z) < V_L$ for all $z \in (z_1, z_2)$. Notice that both pairs $\{z_1, \overline{z}_H\}$ and $\{z_2, \overline{z}_H\}$ satisfy boundary conditions (43) and (44). However, using $\{z_1, \overline{z}_H\}$ fails (45) because $F_L'(z_1^+) < 0$, and using $\{z_2, \overline{z}_H\}$ fails (45) because $F_L'(z_1^+) > 0$. It follows that $B_L^{-1}(\overline{z}_H) > z_1 > \underline{z}_H > \underline{\alpha}$. B_L increasing completes the argument, implying the existence of an intersection α_0, β_0. As $\Xi(\alpha_0, \beta_0)$ is clearly stationary and belief monotone, the verification that it is an SBM-equilibrium is identical to the verification in proof of Theorem 3.1. *Q.E.D.*

Proof of Theorem 5.1 Outline: the proof proceeds using six steps, most of which establish necessary properties of SBM-equilibrium value functions:

1. In any SBM-equilibrium, there must exist a state z_0 where $F_L(z_0) = V_L$.
2. For any such z_0, conditional on rejection, the equilibrium belief instantaneously transitions to some $\alpha_0 \geq z_0$.
3. The state α_0 is the lower bound of a no-trade region (α_0, β_0), that must satisfy boundary conditions (43)–(48). Therefore, $(\alpha_0, \beta_0) = (\alpha^*, \beta^*)$.
4. $F_L(z) = F_H(z)$ for all states $z \geq \beta^*$.
5. $F_L(z) = V_L$ for all states $z \leq \alpha^*$.
6. The only SBM-equilibrium consistent with the established value-function properties is $\Xi(\alpha^*, \beta^*)$.

Step 1. In any SBM-equilibrium, there exists $z_0 \in \mathbb{R}$ such that $F_L(z_0) = V_L$.

Proof: For the purpose of contradiction, suppose the claim was false. Then, by No Deals, $F_L(z) > V_L$ for all $z \in \mathbb{R}$. Recall that $\underline{z}$ is defined implicitly by $\Psi(\underline{z}) = K_H$. Therefore, neither $\Psi(z)$ nor V_L are executable trade prices in any state $z < \underline{z}$, so no trade occurs when $z < \underline{z}$ (Lemma 5.1(b)). Now fix $F_L(\underline{z})$ at any value in the low type's payoff bounds $(V_L, \Psi(\underline{z})]$. Let $\underline{\tau} = \inf\{t : \hat{Z}_t \geq \underline{z}\}$. Then from (35), for any $z < \underline{z}$, $F_L(z) = K_L + g^\ell(z|\underline{z})(F_L(\underline{z}) - K_L)$. As $z \to -\infty$, $g^\ell \to 0$, so $F_L(z) \to K_L < V_L$, violating No Deals and contradicting the supposition. *Q.E.D.*

Step 2. In any SBM-equilibrium, for any z_0 such that $F_L(z_0) = V_L$, there exists an $\alpha_0 \in [z_0, \infty)$ such that

a. for any t_0 such that $Z_{t_0} = z_0$, $\mathcal{Q}^\theta_{Z_{t_0}}(Z_{t_0^+} = \alpha_0) = 1$, for $\theta = L, H$

b. $F_L(\alpha_0) = V_L$

Proof: (a) First, if $F_L(z_0) = V_L$, then $w(z_0) \leq V_L$. If $w(z_0) < V_L$, then by assumption $F_L(z_0) > w(z_0)$ so $S^{L,t_0}_{t_0} = 0$, and if $w(z_0) = V_L$, then Lemma 5.1(d) implies that $S^{L,t_0}_{t_0} < 1$. In both cases, rejection is an on-path event, and therefore $Z_{t_0^+}$ must satisfy (24):

$$\alpha_0 \equiv Z_{t_0^+} = Z_{t_0} + \underbrace{\lim_{t_1 \downarrow t_0} \ln\left(\frac{f^H_{t_1-t_0}(X_{t_1} - X_{t_0})}{f^L_{t_1-t_0}(X_{t_1} - X_{t_0})}\right)}_{= 0 \text{ with prob. 1 under } \mathcal{Q}^L_{Z_{t_0}} \text{ or } \mathcal{Q}^H_{Z_{t_0}}} + \ln\left(\frac{1 - S^{H,t_0}_{t_0}}{1 - S^{L,t_0}_{t_0}}\right) \geq Z_{t_0} = z_0$$

where the inequality is an implication of (on-path) Belief Monotonicity.

(b) If $\alpha_0 = z_0$, then $F_L(\alpha_0) = V_L$ by assumption. If $\alpha_0 > z_0$, from the Belief Consistency condition, the discontinuity is caused by an atom of acceptance by the low type in state z_0. Because $\alpha_0 < \infty$, the low type is mixing, so must be indifferent. Therefore, $F_L(\alpha_0) = F_L(z_0) = V_L$. *Q.E.D.*

Step 3. In any SBM-equilibrium, for any z_0 such that $F_L(z_0) = V_L$ and corresponding α_0 from Step 2, there exists finite $\beta_0 \equiv \min\{z > \alpha_0 : F_L(z) = \Psi(z)\}$. Further,

a. For all z in (α_0, β_0), $F_L(z)$ and $F_H(z)$ must be of the form derived in (14) and (15) for some unknown constants C^θ_i.

b. Boundary conditions (43)–(48) must hold at (α_0, β_0).

Proof: First, given that $\alpha_0 < \infty$, there must exists $z_1 > \alpha_0$ such that $F_L(z_1) = \Psi(z_1)$. Suppose not; then by Lemma 5.1(b), the high type does not trade in any state $z > \alpha_0$. An argument analogous to the one used in the proof of Step 1 applies. Fix $F_H(\alpha_0)$ at any value in the high type's payoff bounds $[\Psi(\alpha_0), V_H]$. Let $\tau_0 = \inf\{t : \hat{Z}_t = \alpha_0\}$. Belief Monotonicity implies that the increments of Z weakly first-order stochastically dominate the increments of $\hat{Z}$, therefore for any $z > \alpha_0$, $F_H(z) \leq E^H_z\left[\int_0^{\tau_0} e^{-rt} r K_H dt + e^{-r\tau_0} F_H(\alpha_0)\right]$. It is immediate that as $z \to \infty$, $\mathcal{Q}^H_z(\tau_0 > T) \to 1$ for any $T > 0$, and $\lim_{z\to\infty} F_H(z) \leq K_H < \Psi(z)$, which contradicts No Deals. Hence, such a z_1 does exist, and β_0 is simply the infimum of such states, which is the same as their minimum given that F_L is continuous (Lemma 5.2).

(a) Let $\alpha_1 = \max\{z \in [\alpha_0, \beta_0] : F_L(z) = V_L\}$. Hence, $F_L(z) \in (V_L, \Psi(z))$ for all $z \in (\alpha_1, \beta_0)$. Lemma 5.1(b) then implies that (α_1, β_0) is a no-trade region. From Section 3.1, F_L and F_H must be of the form given by (14) and (15), for some constants C^θ_i, for $z \in (\alpha_1, \beta_0)$. It is therefore sufficient to show that $\alpha_0 = \alpha_1$.

Suppose $\alpha_0 < \alpha_1$. Because $F_L(z) < \Psi(z)$ for all $z \in [\alpha_0, \alpha_1]$, and $F_L(\alpha_0) = F_L(\alpha_1) = V_L$, it must be that $F_L(z) = V_L$ for all $z \in [\alpha_0, \alpha_1]$. But then, for any $z \in (\alpha_0, \alpha_1)$ and any $\epsilon > 0$, Z must have positive probability of a jump discontinuity at some $z' \in (z-\epsilon, z+\epsilon)$ under $\mathcal{Q}_z^L$. To see this, suppose not. Then, by Lemma 5.1(d), for $\epsilon > 0$, but small enough such that $(z-\epsilon, z+\epsilon) \subset (\alpha_0, \alpha_1)$, $F_L(z) = K_L + E_z^L[e^{-r\tau^\epsilon}](V_L - K_L)$, where $\tau^\epsilon = \inf\{t : t \notin (z-\epsilon, z+\epsilon)\}$. Given that $\epsilon > 0$, and Z is continuous by our supposition, $E_z^L[e^{-r\tau^\epsilon}] < 1$, so $F_L(z) < V_L$ violating No Deals. Further, Belief Monotonicity implies that these jump discontinuities must increase Z. However, recalling the definition of α_0 (as it corresponds to z_0), this implies that, starting from z_0 at time t_0, $\mathcal{Q}^\theta_{Z_{t_0}}(Z_{t_0^+} = \alpha_0) \neq 1$, contradicting Step 2(a). Hence, $\alpha_0 = \alpha_1$.

(b) The necessity of the value-matching conditions (43), (44), (46) are immediate. We now argue the necessity of the remaining boundary conditions.

(47): We first need to establish the following: there exists an $\epsilon > 0$ such that $F_L(z) = V_L$, $\forall$ $z \in (\alpha_0 - \epsilon, \alpha_0]$. Suppose not, and let $z_1 = \max\{z < \alpha_0 : F_L(z) = V_L \text{ or } F_L(z) = \Psi(z)\}$. By our supposition and continuity of F_L (Lemma 5.2), z_1 is bounded away from α_0. Therefore, by Lemma 5.1(b), (z_1, α_0) is (part of) a no-trade region. It must be that $F_L(z_1) = \Psi(z_1)$, otherwise $F_L(z)$ would fall below V_L for all $z \in (z_1, \alpha_0)$, in violation of No Deals. We have already established that (α_0, β_0) is a no-trade region, and Step 2 implies that there is zero probability of trade when $z = \alpha_0$. Putting all of this together: our supposition implies that α_0 is an element of a no-trade region (z_1, β_0), where $F_L = \Psi$ at both boundaries. Recall that $F_L(z) = \Psi(z) \Rightarrow F_H(z) = \Psi(z)$. From Appendix A.2, we know that such a no-trade region can exist only if $MB_H(z_1), MB_H(\beta_0) < 0$ and $MB_H(z) > 0$ for some $z \in (z_1, \beta_0)$. However, also from Appendix A.2, if the SLC holds then $U_H = \{z : z < z^+\}$, meaning no such pair exists. Hence, there exists an open neighborhood below α_0 wherein $F_L = V_L$. Now, just as in (a), Z must experience (increasing) jump discontinues in this neighborhood. The only possibility commensurate with this and Step 2(a), is for α_0 to be a lower reflecting barrier of Z. Finally, Harrison (1985, Chap. 5) shows that $F_H'(\alpha_0^+) = 0$ is a necessary condition for the high type's solution to her seller's problem given that α_0 is a reflecting barrier.

(45): In order for the α_0 to be a lower reflecting barrier of Z, Belief Consistency requires the low type to play a mixed strategy at $z = \alpha_0$. Hence, she must be indifferent. Let $F_{L,a}(z)$, $F_{L,r}(z)$ denote the value functions from following the strategy of always accept (reject) at α_0. Obviously, these two functions must be identical for mixing to be optimal. Since Z reflects conditional on rejection $F_{L,r}'(\alpha_0^+) = 0$ (Harrison, 1985, Chap. 5). If $F_{L,a}'(\alpha_0^+) \neq 0$, indifference is violated. Thus $F_L'(\alpha_0^+) = 0$ is necessary.

(48): No Deals and $F_H(\beta_0) = \Psi(\beta_0)$ immediately imply that $F_H'(\beta_0^-) \leq \Psi'(\beta_0)$. Suppose that $F_H'(\beta_0^-) < \Psi'(\beta_0)$. Consider a deviation that rejects all $z \in [\beta_0, \beta_0 + \epsilon)$ for ϵ sufficiently small. By Belief Monotonicity, such a deviation must be at least as profitable as it is if $Z = \hat{Z}$, where it is strictly profitable (see Dixit and Pindyck, 1994, p. 130-132).

Q.E.D.

Corollary B.1. *In any SBM-equilibrium, for any z_0 such that $F_L(z_0) = V_L$, the corresponding α_0, β_0 are α^*, β^**

PROOF: Immediate implication of Step 3 and Lemma 3.1. *Q.E.D.*

Step 4. In any SBM-equilibrium, for all $z \geq \beta^*$, $F_L(z) = F_H(z) = \Psi(z)$.

Proof: From Corollary B.1, for all $z > \beta^*$, $F_L(z) > V_L$. By Lemma 5.1(b,c), if trade occurs in states $z > \beta^*$, then both type's trade with probability one. Hence, if the claim were false, then there would exist $z_2 > z_1 \geq \beta^*$ such that no-trade occurs in (z_1, z_2), and $F_H(z_i) = \Psi(z_i)$ for $i = 1, 2$. However, the same argument used in the proof of the necessity of boundary condition (47) in Step 3 establishes that this cannot occur in equilibrium. *Q.E.D.*

Step 5. In any SBM-equilibrium, for all $z \leq \alpha^*$, $F_L(z) = V_L$.

Proof: Suppose that the claim is false. Then as we argued in the proof of the necessity of boundary condition (47) in Step 3, there must exist a $z < \alpha^*$ such that $F_L(z) = \Psi(z)$. Let $\hat{\beta} \equiv \min\{z : F_L(z) = \Psi(z)\}$. Since $F_L(z) = \Psi(z)$ only if $F_H(z) = \Psi(z)$, $\hat{\beta} \geq \underline{z}$. Using the same argument given in the proof of Step 1, there must exist a largest $\hat{\alpha} < \hat{\beta}$ such that $F_L(\hat{\alpha}) = V_L$. In addition, the continuity of F_L (Lemma 5.2) and Lemma 5.1(b), imply that $(\hat{\alpha}, \hat{\beta})$ is a no-trade region, so F_L and F_H must follow the forms in Step 3 for some unknown constants $C_1^L, C_2^L, C_1^H, C_2^H$.

The necessary boundary conditions for $(\hat{\alpha}, \hat{\beta})$ are weaker than those given in Step 3. (43), (44), (46), and (48) are still necessary for the same reasons given in the proof of Step 3. However, because $\hat{\alpha}$ need not be a reflecting barrier, (47) and (45) can be weakened as follows.

- $F_H'(\hat{\alpha}^+) \leq 0$: By the same argument given for the necessity of boundary condition (47) in Step 3, $\hat{\alpha}$ must be a lower barrier of Z. However, because it does not have the constraint from Step 2(a) as α_0 did, it may not necessarily be a reflecting barrier. The only other possibility is that Z experiences a jump discontinuity at $\hat{\alpha}$. If so, we proceed in a similar manner to that of Step 2, with $\hat{\alpha}$ playing the role of z_0: there exists $\alpha_j \in (\hat{\alpha}, \infty)$ such that if $Z_{t_0} = \hat{\alpha}$, then $Z_{t_0^+} = \alpha_j$. Further, since $\alpha_j < \infty$ the low type is mixing, and therefore indifferent, so $F_L(\alpha_j) = V_L$. Because $F_L(z) > V_L$ for all $z \in (\hat{\alpha}, \hat{\beta})$, $\alpha_j > \hat{\beta}$. Because the jump is instantaneous $F_H(\hat{\alpha}) = F_H(\alpha_j)$, and by No Deals, $F_H(\alpha_j) \geq \Psi(\alpha_j) > \Psi(\hat{\beta}) = F_H(\hat{\beta}) > K_H$. Therefore, $\hat{\alpha} = \arg\max_{z \in [\hat{\alpha}, \hat{\beta}]} F_H(z)$, and $F_H'(\hat{\alpha}^+) \leq 0$.
- $F_L'(\hat{\alpha}^+) \geq 0$: This follows immediately from $F_L(\hat{\alpha}) = V_L$ and No Deals.

We now establish that the only solution to these boundary conditions, and the constraint that $\hat{\beta} \leq \beta^*$ is (α^*, β^*), implying that $F_L(z) < \Psi(z)$ for all $z \leq \beta^*$, and completing the proof. We do this by establishing two facts. Fact B.1 establishes that any pair (z_1, z_2) that satisfies the necessary conditions for F_H lies weakly above the curve B_H (Lemma B.2). Similarly, Fact B.2 establishes that any pair (z_1, z_2) that satisfies the necessary conditions for F_L lies weakly below the curve B_L (Lemma B.1). From the last statement in the proof of Lemma 3.1, for all $z < \alpha^*$, $B_L(z) < B_H(z)$, establishing $(\hat{\alpha}, \hat{\beta}) = (\alpha^*, \beta^*)$. *Q.E.D.*

Fact B.1. *Fix any $z_1 \in \mathbb{R}$ and F_H of the form in (15) for all $z < z_1$ that solves (46) and (48) for $\beta = z_1$:*

1. *If $z_1 \leq z_H^*$ then $F_H'(z) > 0$ for all $z < z_H^*$.*
2. *If $z_1 > z_H^*$ then for any $z < B_H^{-1}(z_1)$, $F_H'(z) < 0$ and for any $z > B_H^{-1}(z_1)$, $F_H'(z) > 0$.*

Proof: First take $z_1 < z_H^*$. From Lemma B.2, this implies that solving (46) and (48) for C_1^H and C_2^H results in $C_1^H > 0$ and $C_2^H < 0$. Hence $F_H'(z) = C_1^H q_1^H e^{q_1^H z} + C_2^H q_2^H e^{q_2^H z} > 0$ for all $z \in \mathbb{R}$. Next take any $z_1 > z_H^*$. Again from Lemma B.2, solving (46) and (48) gives $C_1^H, C_2^H > 0$. Hence $F_H''(z) = (q_1^H)^2 C_1^H e^{q_1^H z} + (q_2^H)^2 C_2^H e^{q_2^H z} > 0$ for all z. Since $F_H'(z)$ is increasing and $F_H'(z)|_{z=B_H^{-1}(z_1)} = 0$: if $z < B_H^{-1}(z_1)$ then $F_H'(z) < 0$ and if $z > B_H^{-1}(z_1)$ then $F_H'(z) > 0$. *Q.E.D.*

Fact B.2. *Take any* $z_0 < z_1$*,* $(z_0, z_1) \in \mathbb{R}^2$ *and* F_L *of the form in (14) for all* $z \in (z_0, z_1)$ *that solves (43) for* $\alpha = z_0$ *and (44) for* $\beta = z_1$*. If* $z_0 < B_L^{-1}(z_1)$ *then* $F_L'(z_0) < 0$*, and if* $z_0 > B_L^{-1}(z_1)$ *then* $F_L'(z_0) > 0$*.*

Proof: For analytical convenience and without loss of generality, normalize $V_H = 1$ and $V_L = 0$, so $K_L < 0$. Given the functional form of (14), equations (43) and (44) are linear in C_1^L, C_2^L. Fix z_1 and for any $z_0 < z_1$, denote the solution by $C_1^L(z_0|z_1), C_2^L(z_0|z_1)$. Given these constants, the slope of the low type's value function at z_0 is given by $q_1^L C_1^L(z_0|z_1)e^{q_1^L z_0} + q_2^L C_2^L(z_0|z_1)e^{q_2^L z_0}$. By definition, this expression is equal to zero at any (z_0, z_1) such that $z_0 = B_L^{-1}(z_1)$. Hence, it is sufficient to show that it is increasing in z_0. The derivative is equal to

$$M\left[q_1^L e^{(1+q_1^L)z_1+q_2^L z_0} + (-q_2^L)e^{(1+q_2^L)b+q_1^L z_0} - K_L(1+e^{z_1})\left(q_1^L(e^{q_1^L z_1+q_2^L z_0} - e^{z_1}) + q_2^L(e^{q_1^L z_0+q_2^L z_1} - e^{z_1})\right)\right]$$

where $M \equiv \frac{e^{(q_1^L+q_2^L)z_0}}{(1+e^{z_1})(e^{q_1^L z_1+q_2^L z_0}-e^{q_1^L z_0+q_2^L z_1})^2}(q_1^L - q_2^L) > 0$. The first two terms inside the square brackets are strictly positive. Since $-K_L(1 + e^{z_1}) > 0$, it is sufficient to show that $q_1^L(e^{q_1^L z_1+q_2^L z_0} - e^{z_1}) - q_2^L(e^{q_1^L z_0+q_2^L z_1} - e^{z_1})$ is strictly positive for all $z_0 < z_1$. We first show that this term is strictly decreasing in z_0:

$$\frac{\partial}{\partial z_0}\left(q_1^L(e^{q_1^L z_1+q_2^L z_0} - e^{z_1}) - q_2^L(e^{q_1^L z_0+q_2^L z_1} - e^{z_1})\right) = q_1^L q_2^L\left(e^{q_1^L z_1+q_2^L z_0} - e^{q_1^L z_0+q_2^L z_1}\right) < 0$$

Moreover, as $z_0 \to z_1$, $q_1^L(e^{q_1^L z_1+q_2^L z_0} - e^{z_1}) - q_2^L(e^{q_1^L z_0+q_2^L z_1} - e^{z_1}) \to 0$ and hence it is strictly positive for all $z_0 < z_1$. *Q.E.D.*

Step 6. The only SBM-equilibrium consistent with the value-function properties established in Steps 1–5 is $\Xi(\alpha^*, \beta^*)$.

Proof: In states $z \geq \beta^*$, $F_L(z) = F_H(z) = \Psi(z)$. Given Lemma 5.1, the only behavior consistent with this is $w(z) = \Psi(z)$ and both types accepting with probability one. In states $z \in (\alpha^*, \beta^*)$, $F_L(z) \in (V_L, \Psi(z))$, therefore, by Lemma 5.1(b), this must be a no-trade region as described in Definition 3.1. Finally, for states $z \leq \alpha^*$, Step 2(a) establishes that Z jumps from z to α^* immediately following a rejection. Belief Consistency implies that the low type must be accepting V_L with the probability given in $\Xi(\alpha^*, \beta^*)$. *Q.E.D.*

This completes the proof of Theorem 5.1. *Q.E.D.*

Proof of Propisition 5.2 Given the seller strategies, verification that the given candidate satisfies conditions 2–6 of Definition 5.2 is immediate. Condition 1, Seller Optimality, follows from Lemmas A.1 and A.3 for all $\gamma \leq \gamma^0$.

For uniqueness, we first establish that if γ is low enough there cannot exist a no-trade region: (z_1, z_2) such that no trade occurs in any state $z \in (z_1, z_2)$. The same argument used in the proof of Step 3 establishes that, in any SBM-equilibrium, for any z, there must exist $z' > z$ such that $F_H(z') = \Psi(z')$ (otherwise F_H would fall below Ψ for high values of z, violating No Deals). Hence, if there exists one (or more) no-trade region(s) in a given SBM-equilibrium, there must exist a no-trade region (z_1, z_2) such that either *i)* $F_H(z_i) = \Psi(z_i)$ for $i = 1, 2$, or *ii)* $F_L(z_1) = V_L$, $F_L(z_2) = F_H(z_2) = \Psi(z_2)$, and z_1 is a reflecting barrier conditional on rejection.

We now show that neither is possible if γ is sufficiently small. The necessary conditions for a no-trade region corresponding to (i) are studied in Appendix A.2, Lemma A.3. Such a no-trade region can exist only if $\gamma > \gamma^0 > 0$. For (ii), recall that such a no-trade can exist only if $\Lambda(\tilde{z}_2) = 0$. However, when the SLC fails, for all $y > 0$, $\lim_{\gamma \to 0} \Lambda'(y) = -\tilde{\Psi}'(y) < 0$. Because $\Lambda(0) = 0$ for all γ, no such no-trade region can exist.

Having established that there cannot be a no-trade region, Lemma 5.1 implies that for any z either $F_L(z) = V_L$ or $F_L(z) = \Psi(z)$. Since there must exist at least one state z such that $F_L(z) = \Psi(z)$, continuity of F_L (Lemma 5.2) then implies that $F_L(z) = \Psi(z)$ for all z. As argued in the proof of Lemma 5.1(a), $F_L(z) = \Psi(z)$ implies that $F_H(z) = \Psi(z)$. It is immediate that the only strategy profile and consistent on-path beliefs generating these value functions are those given in the Proposition. *Q.E.D.*

Proof of Proposition 5.3. Fix $\gamma > \gamma^0$. Then in NMG$_H$ there exists $\underline{z}_H < \overline{z}_H$ where the high type rejects in all states $z \in (\underline{z}_H, \overline{z}_H)$ (Lemma A.3). In any SBM-equilibrium, Belief Monotonicity implies that Z_t weakly first-order stochastically dominates $\hat{Z}_t$ under $\mathcal{Q}_z^H$ for all t, and No Deals implies that $F_H \geq \Psi$, so it must still be optimal for the high type to reject in all states $z \in (\underline{z}_H, \overline{z}_H)$. In addition, the same argument used in the proof of Step 3 establishes that there must exist at least one $z_0 \in \mathbb{R}$ such that the high type accepts Ψ in state z_0. From Lemma 5.1, $F_H(z_0) = F_L(z_0) = \Psi(z_0)$.

Now suppose there does not exist a no-trade region. Then the low type must be willing to accept in all states z where the high type rejects, meaning $F_L(z) = V_L$ by Zero Profit. Hence, $\mathbb{R}$ is partitioned into two non-empty sets: $\{z \in \mathbb{R} : F_L(z) = \Psi(z)\}$ and $\{z \in \mathbb{R} : F_L(z) = V_L\}$. This violates the continuity of F_L established in Lemma 5.2. *Q.E.D.*

Proof of Proposition 5.4 For (1), Belief Monotonicity and No Deals imply that in any SBM-equilibrium, for all z and $\gamma > \underline{\gamma}$, $V_H \geq F_H(z) \geq N_H^*(z)$ (recall that N_H^* is the seller's value function in NMG$_H$ from Appendix A.2). Hence, it is sufficient to show that $N_H^*(z) \overset{pw}{\to} V_H$. In NMG$_H$, consider paying according to the stopping rule: $T(\underline{\beta}_H) = \inf\{t : \hat{Z}_t \geq \underline{\beta}_H\}$. Since this is a viable strategy, from (35), for all $z < \underline{\beta}_H$

$$N_H^*(z) \geq E_z^H[h_H(T(\underline{\beta}_H), \underline{\beta}_H)] = K_H + e^{q_1^H (z - \underline{\beta}_H)}(\Psi(\underline{\beta}_H) - K_H)$$

Taking the limit as $\gamma \to \infty$ gives $\underline{\beta}_H \to \infty$ and $E_z^H[h_H(T(\underline{\beta}_H), \underline{\beta}_H)] \to V_H$ for all z, completing the proof.

For (2), the proof of Proposition 4.1(4) relies only on pointwise convergence of F_H to V_H (i.e., it does not rely on uniform convergence, nor any other feature specific to value functions under $\Xi(\alpha^*, \beta^*)$), and hence applies here as well.

For (3), fix an equilibrium endowing value functions F_H, F_L. We will show that for any $\varepsilon > 0$, there exists a γ_ε such that for all $\gamma > \gamma_\varepsilon$, $\mathcal{L}(z) < \varepsilon$ for all z. Fix an arbitrary $\varepsilon > 0$. Notice that

$$\mathcal{L}(z) = \frac{\Pi^*(z) - \Pi^S(z)}{\Pi^*(z)} = \frac{p(z)(V_H - F_H(z)) + (1 - p(z))(V_L - F_L(z))}{\Pi^*(z)} \leq \frac{p(z)(V_H - F_H(z))}{\Pi^*(z)}$$

since $F_L \geq V_L$ by No Deals. Next, $F_H(z)$ bounded from below by $\Psi(z)$, $\Pi^*(z)$ bounded away from 0, and $\lim_{z \to -\infty} p(z) = 0$, imply that there exists a z_ε such that, for all $z < z_\varepsilon$,

$$\varepsilon > \frac{p(z)(V_H - F_H(z))}{\Pi^*(z)} \geq \mathcal{L}(z)$$

regardless of γ. Therefore, it is sufficient to show that $F_H \overset{u}{\to} V_H$ on the domain $[z_\varepsilon, \infty)$. Recall from the proof of (1) that for all z

$$V_H \geq N_H^*(z) \geq E_z^H[h_H(T(\underline{\beta}_H), \underline{\beta}_H)] = K_H + e^{q_1^H(z-\underline{\beta}_H)}(\Psi(\underline{\beta}_H) - K_H)$$

Notice that the final term is increasing in z. Therefore, that $E_{z_\epsilon}^H[h_H(T(\underline{\beta}_H), \underline{\beta}_H)] \to V_H$ as $\gamma \to \infty$ (see the proof of (1)) establishes the result. *Q.E.D.*

VERIFICATION OF EXAMPLE 5.1. It is immediate that, from Definition 5.2, Belief Consistency, Stationarity, and Belief Monotonicity are all satisfied. To verify Zero Profit, notice that for $z \notin (\underline{z}_H, \overline{z}_H)$ at time t, $w(z) = \Psi(z) = E[V_\theta|\mathcal{H}_t] = E[V_\theta|\mathcal{H}_t, \tau^* = t]$, where the last equality follows from the fact that both type's accept with probability one. For $z \in (\underline{z}_H, \overline{z}_H)$, there is zero probability of trade, so Zero Profit has no implication. To verify Seller Optimality and No Deals for $\theta = H$, notice that there is no meaningful distinction between the high-type's seller's problem in the candidate equilibrium and NMG$_H$: the belief process is the same, and the offer only differs by being *lower* in states where high type rejects in NMG$_H$. Lemma A.3 ensures the conditions are satisfied.

The key conditions to check are Seller Optimality and No Deals for $\theta = L$. Recall that in NMG$_L$ the low type's optimal policy is to accept immediately. In Example 5.1, $Z = \hat{Z}$, but $w(z) \leq \Psi(z)$ for all z, meaning it is still optimal for the low type to accept Ψ whenever it is offered. However, if she adheres to the candidate equilibrium prescription, for $z \in (\underline{z}_H, \overline{z}_H)$,

$$F_L(z) = \left(1 - E_z^L[e^{-r\tau_H}]\right) K_L + E_z^L\left[e^{-r\tau_H}\Psi(Z_{\tau_H})\right]$$

where $\tau_H = \inf\{t : Z_t \notin (\underline{z}_H, \overline{z}_H)\}$. Note that $\underline{z}_H < \overline{z}_H$ for all $\gamma > \gamma^0$ and $(\underline{z}_H, \overline{z}_H)$ are independent of K_L (Lemma A.3). Hence, for $z \in (\underline{z}_H, \overline{z}_H)$, $E_z^L[e^{-r\tau_H}] < 1$ and $F_L(z)$ is linearly increasing in K_L with some cutoff value $\underline{K}_L(z)$ such that $F_L(z) = V_L$ if $K_L = \underline{K}_L(z)$. Now define $\underline{K}_L = \max_{z \in (\underline{z}_H, \overline{z}_H)} \underline{K}_L(z)$, and the proposition is established. *Q.E.D.*

PROOF OF PROPOSITION 5.5. Let $\gamma > \max\{\gamma^0, \underline{\gamma}\}$, and fix an SBM-equilibrium (existence of which is guaranteed by Proposition 5.1). Proposition 5.3 implies there exists a no-trade region. Define $\beta \equiv \inf\{z : F_L(z) = \Psi(z)\}$, whose existence is implied by the same argument used in the proof of Step 3. For any SBM-equilibrium satisfying NDVF, $F_L(z) \geq \Psi(\beta) > V_L$ for all $z > \beta$. Therefore, if trade occurs at $z > \beta$, then it occurs with probability one and at a price of $\Psi(z)$ (Lemma 5.1(b,c)). Now, either (*i*) both types trade for all $z > \beta$, or (*ii*) there exists a no-trade region (z_1, z_2), $z_1 \geq \beta$.

If (*i*), then, $\beta > -\infty$, otherwise there would not exist a no-trade region. By definition of β, $F_L(z) < \Psi(z)$ for all $z < \beta$, which implies that the low type and, therefore, also the high type, are not trading at a price of $\Psi(z)$ for any $z < \beta$ (Lemma 5.1(b)). By continuity of F_L (Lemma 5.2), $\Psi(z) > F_L(z) > V_L$ for all z in an open neighborhood below β. Hence, there is a no-trade region whose upper boundary is β. Given that the high type does not trade at any $z < \beta$, the same argument given in the proof of Step 1 establishes that there must exist some $z_0 < \beta$ such that $F_L(z_0) = V_L$. Let $\alpha \equiv \sup\{z : F_L(z) = V_L\}$. By NDVF and No Deals, $F_L(z) = V_L$ for all $z < \alpha$. Therefore, (α, β) comprise a no-trade, below which $F_L = V_L$ and above which $F_L = \Psi$. The only SBM-equilibrium candidate consistent with this is $\Xi(\alpha, \beta)$. Because $\gamma > \underline{\gamma}$, Theorem 3.1 and Lemma B.5 establish that (α, β) must be (α^*, β^*) and that $\Xi(\alpha^*, \beta^*)$ is a valid SBM-equilibrium.

If instead (*ii*), then $F_\theta(z_i) = \Psi(z_i)$ for both $\theta = L, H$ and and $i = 1, 2$. Belief Monotonicity and No Deals imply that $(\underline{z}_H, \overline{z}_H) \subseteq (z_1, z_2)$. In addition, $(z_1, z_2) \neq (\underline{z}_H, \overline{z}_H)$, only if the evolution of Z differs from the evolution of $\hat{Z}$ for some states in (z_1, z_2). However, this fails Belief Consistency; hence $(z_1, z_2) = (\underline{z}_H, \overline{z}_H)$. As in the proof of Lemma 3.1, we now make the change of variables to likelihood space, letting $(\underline{y}_H, \overline{y}_H), (y^-, y^+), \tilde{F}_\theta, \tilde{\Psi}$ be the transformations of $(\underline{z}_H, \overline{z}_H), (z^-, z^+), F_\theta, \Psi$.

From the proof of Lemma A.3, $\underline{y}_H \leq y^-$ and $\operatorname{sgn}(MB_H(z)) = \operatorname{sgn}\left(\tilde{\Psi}' + \tilde{\Psi}'' - 2/\gamma\left(\tilde{\Psi} - K_H\right)\right)$. Let $\eta = 1/\gamma$. The high type strictly prefers to reject for all $y \in (y^-, y^+)$. To shorten analytic expressions, and WLOG, normalize $V_L = 0$, $V_H = 1$ (and hence $K_H < 0$). Straightforward calculation shows that $\lim_{\eta \to 0} \frac{y^-(\eta)}{\eta} = \frac{-K_H}{2}$. Therefore $\underline{y}_H$ must converge to zero at a rate at least proportional to η (i.e., $\underline{y}_H = O(\eta)$ as $\eta \to 0$). Recall that $\tilde{F}'_L(\underline{y}_H) = q_1^L C_1^L \underline{y}_H^{q_1^L} + q_2^L C_2^L \underline{y}_H^{q_2^L}$. As $\eta \to 0$: $q_1^L \to 1^+$, $C_1^L \to 0^+$ (at the same rate as $\sqrt{\eta}$), $\underline{y}_H^{q_1^L} \to 0^+$ and $\underline{y}_H^{q_1^L} = O(y^-(\eta)^{q_1^L}) = O(\eta^{q_1^L}) = o(\eta)$. This implies that $q_1^L C_1^L \underline{y}_H^{q_1^L}$ is $O(\eta^{3/2})$. In the second term: $q_2^L \to 0^-$ (at the same rate as η), $C_2^L \to (V_L - K_L)^-$, $\underline{y}_H^{q_2^L} \to 1^-$ implying that the second term is $O(\eta)$. The first term goes to zero faster and hence the second term dominates the sign of the derivative for small η, and since the second term is negative the derivative converges from below. To see this, note first that $\operatorname{sgn}(\tilde{F}'_L(\underline{y}_H)) = \operatorname{sgn}(\tilde{F}'_L(\underline{y}_H)/q_1^L C_1^L \underline{y}_H^{q_1^L})$ then take the limit as $\eta \to 0$, $\lim_{\eta \to 0} 1 + \frac{q_2^L C_2^L \underline{y}_H^{q_2^L}}{q_1^L C_1^L \underline{y}_H^{q_1^L}} \leq 1 + \frac{-O(\eta)}{O(\eta^{3/2})} = -\infty$. Hence for all η small enough (conversely, γ large enough), $\tilde{F}'_L(\underline{y}_H) < 0$ violating NDVF (given that the transformation to likelihood space is an increasing one). *Q.E.D.*

B.6 Proofs for Section 6

PROOF OF PROPOSITION 6.1 Checking conditions 2, 3, 5, and 6 of Definition 5.2 is straightforward in both cases. To verify Seller Optimality and No Deals, we will construct the equilibrium value functions. Because of Stationarity, whether or not these conditions are satisfied is history independent. Thus, it is without loss of generality to verify them for any $Z_0 = z$.

1. If $\lambda_L > \lambda_H$: For all $z \in (\alpha, \beta)$ and $t < T$, $dZ_t = (\lambda_L - \lambda_H)dt$. Thus, the differential equation for the value function in the no-trade region is

$$F_\theta(z) = K_\theta + \frac{\lambda_\theta}{r}(V_\theta - F_\theta(z)) + \frac{\lambda_L - \lambda_H}{r} F'_\theta(z), \qquad \forall\, z \in (\alpha, \beta) \tag{68}$$

which has a unique solution (Polyanin and Zaitsev, 2003, p. 4) of the form

$$F_\theta(z) = \frac{rK_\theta + \lambda_\theta V_\theta}{r + \lambda_\theta} + C_\theta e^{q_\theta z} \tag{69}$$

where $q_\theta = \frac{r + \lambda_\theta}{\lambda_L - \lambda_H}$ and C_θ is an arbitrary constant to be determined.

To determine β: given W, Z, the high type must be indifferent between accepting and rejecting at $z = \beta$ (see Section 3.2). Since $F_H(z) = \Psi(z)$ for all $z \geq \beta$, value matching and smooth-pasting require $F_H(\beta^-) = \Psi(\beta)$ and $F'_H(\beta^-) = \Psi'(\beta)$. Using the functional form in (69) and solving these two boundary conditions yields $C_H = (\Psi(\beta) - K'_H)e^{-q_H \beta}$ and (28) for β. Unlike when news arrives according to a diffusion, β can be determined independently of α. Note that β solves

$$\Psi'(z)(\lambda_L - \lambda_H) - (r + \lambda_H)(\Psi(z) - K'_H) = 0 \tag{70}$$

The LHS of (70), is analogous to MB$_H$ in NMG$_H$ (Section A.2). β is the unique real root, and the expression is positive (negative) for all $z < (>)\beta$. β also corresponds to the optimal cutoff at which the high type would stop in the analogous version of NMG$_H$ with Poisson information arrival.

Given β, two value-matching conditions on the low type's value function determine α: namely $F_L(\alpha^+) = V_L$ and $F_L(\beta^-) = \Psi(\beta)$. Solving these two (given β from (28)) yields $C_L = (\Psi(\beta) -$

$K'_H)e^{-q_L\beta}$ and (29) for α. To summarize, we have that,

$$F_L(z) = \begin{cases} V_L & z \leq \alpha \\ K'_L + e^{q_L(z-\beta)}(\Psi(\beta) - K'_L) & z \in (\alpha, \beta) \\ \Psi(z) & z \geq \beta \end{cases}$$

$$F_H(z) = \begin{cases} K'_H + e^{q_H(\alpha-\beta)}\left(\Psi(\beta) - K'_H\right) & z \leq \alpha \\ K'_H + e^{q_H(z-\beta)}\left(\Psi(\beta) - K'_H\right) & z \in (\alpha, \beta) \\ \Psi(z) & z \geq \beta \end{cases}$$

For No Deals, by inspection, $F_L(z) \geq V_L$ for all z since $F_L(\alpha) = V_L$ (by construction) and F_L is weakly increasing. To see that $F_H(z) > \Psi(z)$ for all $z < \beta$,

$$\begin{aligned} F_H(z) = K'_H + (\Psi(\beta) - K'_H)e^{q_H(z-\beta)} &> \Psi(z) \\ \Leftrightarrow (\Psi(\beta) - K'_H)e^{-q_H\beta} &> (\Psi(z) - K'_H)e^{-q_H z} \end{aligned}$$

Therefore, it suffices to show that $(\Psi(z) - K'_H)e^{-q_H z}$ is increasing for $z < \beta$. Taking the derivative gives $e^{-q_H z}\left(\Psi'(z) - q_H(\Psi(z) - K'_H)\right)$, which has the same sign as the LHS of (70) and hence strictly positive for all $z < \beta$.

For Seller Optimality, we proceed in a manner analogous to the proof of Lemma 3.3. First, note that the expected payoff from stopping at an arbitrary τ can be calculated as

$$\begin{aligned} &E^\theta_z\left[\left((1-e^{-r\tau})K_\theta + e^{-r\tau}V_\theta\right)1_{\{\tau \geq T\}} + \left((1-e^{-r\tau})K_\theta + e^{-r\tau}w(Z_\tau)\right)1_{\{\tau < T\}}\right] \\ &\quad = E^\theta_z\left[\int_0^\tau \left((1-e^{-rs})K_\theta + e^{-rs}V_\theta\right)\lambda_\theta e^{-\lambda_\theta s}ds + e^{-\lambda_\theta \tau}\left((1-e^{-r\tau})K'_\theta + e^{-r\tau}w(Z_\tau)\right)\right] \\ &\quad = E^\theta_z\left[\left(1 - e^{-(r+\lambda_\theta)\tau}\right)K'_\theta + e^{-(r+\lambda_\theta)\tau}w(Z_\tau)\right] \end{aligned}$$

Therefore, let $f_\theta(z,t) = \left(1 - e^{-(r+\lambda_\theta)t}\right)K'_\theta + e^{-(r+\lambda_\theta)t}w(z)$ and write the seller's problem as

$$F^*_\theta(z) = \sup_\tau E^\theta_z\left[f_\theta(Z_\tau, \tau)\right] \tag{71}$$

As in the proof of Lemma 3.3, consider the problem in which the type-θ seller can choose the maximum of $w(Z_\tau)$ and $F_\theta(Z_\tau)$ when stopping at time τ. Since $F_\theta(z) \geq w(z)$ (from No Deals above), is suffices to consider only policies that select F_θ upon stopping. Let $\jmath_\theta(z,t) = \left(1 - e^{-(r+\lambda_\theta)t}\right)K'_\theta + e^{-(r+\lambda_\theta)t}F_\theta(z)$ and consider the alternate stopping problem

$$J^*_\theta(z) = \sup_\tau E^\theta_z\left[\jmath_\theta(Z_\tau, \tau)\right] \tag{72}$$

From No Deals, it is immediate that $J^*_\theta(z) \geq F^*_\theta(z)$. Thus, for Seller Optimality, it suffices to show that $J^*_\theta(z) \leq F_\theta(z)$. Note that

$$\jmath_\theta(Z_\tau, \tau) = \jmath_\theta(Z_0, 0) + \int_0^\tau \mathcal{A}^\theta \jmath_\theta(Z_s, s)ds$$

where $\mathcal{A}^\theta \jmath_\theta(z,t) = \frac{\partial \jmath_\theta}{\partial t} + \frac{\partial \jmath_\theta}{\partial z}(\lambda_L - \lambda_H) = e^{-s(r+\lambda_\theta)}\left(F'_\theta(z)(\lambda_L - \lambda_H) - (r+\lambda_\theta)(F_\theta(z) - K'_\theta)\right)$. By (68), $\mathcal{A}^\theta \jmath_\theta = 0$ for all $s, z \in (\alpha, \beta)$. For any $z > \beta$,

$$\mathcal{A}^\theta \jmath_\theta(t,z) = e^{-(r+\lambda_\theta)t}\left(\Psi'(z)(\lambda_L - \lambda_H) - (r+\lambda_\theta)(\Psi(z) - K'_\theta)\right) \tag{73}$$

For $\theta = L$, the RHS of (73) is strictly negative for all z (analogous to $\mathrm{MB}_L(z) < 0$ for all z in NMG_L). For $\theta = H$, the RHS of (73) is positive for $z < \beta$, equal to zero at $z = \beta$ and negative for $z > \beta$ (see (70)). Hence, $\mathcal{A}^\theta g_\theta \leq 0$ for all z implying that $\jmath_\theta(Z_\tau, \tau) \leq \jmath_\theta(Z_0, 0)$, taking the supremum over all τ gives $J^*_\theta(z) \leq \jmath_\theta(Z_0, 0) = F_\theta(z)$ as desired.

<u>2. If $\lambda_L \leq \lambda_H$</u>: First, $F_H(z) = K'_H + (\Psi(z) - K'_H)1_{\{z \geq z^*\}} \geq \Psi(z)$ for all z since $K'_H = \Psi(z^*)$ by definition and Ψ increasing. Next, $F_L(z) = V_L + (\Psi(z) - V_L)1_{\{z > z^*\}} \geq V_L$ for all z (that $F_L(z^*) = V_L$ will be verified shortly) and so No Deals is satisfied. For Seller Optimality, we first claim that the low type is indifferent between accepting V_L and rejecting for any $z \leq z^*$. To see this, consider the payoff to a low type who upon reaching z^* rejects V_L for some arbitrary $\tau \in (0, T)$. Since z^* is an absorbing state for the equilibrium belief process, the value function from this strategy evolves according to

$$F_H(z^*) \approx K_L dt + e^{-rdt}\left(\lambda_L dt V_L + \kappa dt K'_H + (1 - \kappa dt - \lambda dt) F_H(z^*)\right)$$

Isolating the F_H terms, dividing by dt and taking the limit gives

$$F_L(z^*) = \frac{rK_L + \lambda_L V_L + \kappa K'_H}{r + \lambda_L + \kappa}$$

Inserting the expression for $\kappa = \frac{r(V_L - K_L)}{K'_H - VL}$ and solving yields $F_L(z^*) = V_L$, verifying the low type's indifference over any such τ. The same logic as used above following (73) implies that accepting $\Psi(z)$ whenever it is offered is optimal for the low type. Thus, the low type's strategy solves SP_L. For the high type, clearly it is optimal to reject for all $z < z^*$ and she is indifferent at $z = z^*$ conditional on $\Psi(z^*)$ being offered. Consider any $Z_0 > z^*$, the expected payoff to the high type from stopping at any arbitrary time $\tau \in (0, T)$ is

$$\begin{aligned}\int_0^\tau \left((1 - e^{-rs})K_H + e^{-rs}V_H\right)\lambda_H e^{-\lambda_H s} ds + e^{-\lambda_H \tau}\left((1 - e^{-r\tau})K'_H + e^{-r\tau} w(Z_\tau)\right) \\ = (1 - e^{-(r+\lambda_H)\tau})K'_H + e^{-(r+\lambda_H)\tau} w(Z_\tau)) \\ \leq (1 - e^{-(r+\lambda_H)\tau})K'_H + e^{-(r+\lambda_H)\tau} \Psi(Z_\tau) \\ < \Psi(Z_0)\end{aligned}$$

The third line is a convex combination of K'_H and $\Psi(Z_\tau)$. Hence, the strict inequality follows from the fact that conditional on $\tau < T$, (i) $Z_\tau \leq Z_0$ (i.e., no news is bad news) thus $\Psi(Z_\tau) \leq \Psi(Z_0)$, and (ii) $\Psi(Z_0) > K'_H$ since $Z_0 > z^*$ by supposition. The above holds for all $\tau > 0, Z_0 > z^*$. Therefore it is optimal for the high type to stop immediately for all $z > z^*$, which completes the verification of SP_H.

Q.E.D.