The Paradox of Civilization
Pre-Institutional Sources of Security and Prosperity

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Abstract

The rise of civilizations involved the dual emergence of economies that could produce surplus (“prosperity”) and states that could protect surplus (“security”). But the joint achievement of security and prosperity had to escape a paradox: prosperity attracts predation, and higher insecurity discourages the investments that create prosperity. We study the trade-offs facing a proto-state on its path to civilization through a formal model informed by the anthropological and historical literatures on the origin of civilizations. We emphasize pre-institutional forces, such as physical aspects of the geographical environment, that shape productive and defense capabilities. The solution of the civilizational paradox relies on high defense capabilities, natural or man-made. We show that higher initial productivity and investments that yield prosperity exacerbate conflict when defense capability is fixed, but may allow for security and prosperity when defense capability is endogenous. Some economic shocks and military innovations deliver security and prosperity while others force societies back into a trap of conflict and stagnation. We illustrate the model by analyzing the rise of civilization in Sumeria and Egypt, the first two historical cases, and the civilizational collapse at the end of the Bronze Age.

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1 Introduction

Anatomically modern humans have lived in subsistence and stateless societies for roughly 97% of their 200,000-year-long history. If there is a Big Bang in human history, it occurred as recently as 5,000 years ago, when the first civilizations emerged. Civilization meant fundamental transformations: systematic surplus production, urbanization, public architecture, writing, and states. Although the rise of civilization is arguably more of a qualitative change than the Industrial Revolution, modern political economy has paid much less attention to it.

According to an influential view in archaeology, the rise of civilizations is primarily driven by an exceptional potential for food production, both in terms of endowments and technology. For V. Gordon Childe, the key features of Lower Mesopotamia, the “cradle of civilization,” were an extremely fertile alluvial soil, an abundance of edible animals, and irrigation technology. Identical factors were emphasized for the rise of Egypt, the first pristine civilization after Sumer. Both in Lower Mesopotamia and Egypt “irrigation agriculture could generate a surplus far greater than that known to populations on rain-watered soil” and “as productivity grew, so too did civilization” (Mann 1986: 80, 108).

Without a substantial surplus, it was not possible to fund the tangible components of civilizations. However, surplus production was only a necessary condition for civilization, not a sufficient one. In fact, prosperity could be self-defeating. Primitive food producers were surrounded by nomadic tribes for whom agricultural surpluses were a most tempting target for looting. The resulting clash is a primordial conflict shaping the civilizational process. According to McNeill (1979, p. 71), “Soon after cities first arose ... the relatively enormous wealth that resulted from [their economic activity] made such cities worthwhile objects of attack by armed outsiders.” For anthropologists, intergroup violence had been prevalent since before civilization (Keeley 1996), but the emergence of large surpluses intensified the potential for conflict. According to Michael Mann, “the greater the surplus generated, the more desirable it was to preying outsiders” (1986: 48).

Since civilization entailed the joint achievement of prosperity and security, its emergence is a fundamental paradox. Primitive societies that held production close to subsistence levels could hope to mitigate predation, but stagnation would foreclose the civilization process. To reach civilization, primitive societies with the capacity for surplus production had to
overcome the dangers of self-defeating prosperity without relying on the relative safety of stagnation. A proper balance was needed between surplus production and surplus protection.

The contextual conditions allowing for such a balance are rare, as evidenced by the fact that, out of thousands of primitive societies, only a handful could develop independent civilizations, starting with Sumer and Egypt. In this paper we develop a model to identify the logical conditions for successful civilization, and examine for the first time the historical record for the rise of Sumer and Egypt under the perspective of the civilizational paradox. The historical cases illustrate the logic of the model, and the model allows for a richer interpretation of the cases. Sumer and Egypt provide evidence that the potential for surplus emphasized by archaeologists and geographers was only half of the story of successful civilization. The other half was surplus protection. In addition to their historical preeminence as cases achieving the right balance between prosperity and security, Sumer and Egypt illustrate that protection occurs in two contrasting ways—defense can be natural as in Egypt, or man-made as in Sumer.

The rise of civilization, and its intrinsic paradox, can be usefully compared to the rise of the modern state in the post-Westphalia context, another major turning point in history. The rise of the modern state also involves a paradox. European rulers striving for a monopoly of violence were able to reach unprecedented levels of power but in the process they undermined their own ability to credibly commit to respecting private rights. Unstable property rights in turn diminished the capacity of the underlying society to grow, and ultimately damaged the ruler’s own power. The standard insight is that the solution for the modern state was “institutions” understood as rules of the political game: checks and balances, as well as the expansion of political rights, helped the ruler to solve its credibility problem either vis-à-vis society at large or vis-à-vis competing factions within the elite (North and Weingast 1989, Acemoglu and Robinson 2005, Lizzeri and Persico 2004).

In contrast to the solution to the paradox of the modern state, the solution to the paradox of civilization in our approach does not involve institutions. The joint achievement of order and prosperity in the context of pristine civilizations is a pre-institutional process, involving tangible assets and technologies, of either economic or military nature. Pristine civilizations emerged in areas with exceptional natural endowments for food production, and the man-made contributions to the civilizational breakthrough were not political rules, as institutional theories would emphasize, but productive and defense equipment. Two massive
engineering accomplishments are the mark of the Ancient Near East: irrigation infrastructure in both Egypt and Sumer, and perimetral walls in cities throughout Mesopotamia and the Levant. Each public good had a single, well-defined mission: surplus production and surplus protection. The prominence of the two types of public works reflects the centrality of the production-protection tension in the process of civilization building.

Our pre-institutional theory on the joint achievement of security and prosperity can help improve our understanding of the rise of the first civilizations, and shed light on the problem of state formation more generally. A broader goal is to generate insights for a wide class of development trajectories in which a potentially prosperous region, being surrounded by predatory threats, may fall in the traps of security-enhancing stagnation or self-defeating prosperity. This class includes the interaction between a large number of proto-cities and barbarian invaders from the steppes across the Eurasian continent throughout the Middle Ages; the long struggle in 19th century Latin America between elites from port-cities engaged in nation-building and rural warlords, “caudillos,” dominating the periphery; as well as contemporary state-building efforts in failed states of Sub-Saharan Africa and the Middle East, in which international economic aid, if not coupled with military buildup, may have counter-productive effects by inducing voracity among neighbors. Echoing concerns in history and anthropology about the reversibility of gains in social complexity that lead to statehood, our theory provides an account for civilization collapse, and more generally for economic or military reversals in societies that had achieved prosperity and security. For illustration, we will use the model to account for the End of the Bronze Age, a much-debated process in which dozens of civilization centers collapsed rather quickly throughout the Eastern Mediterranean, ushering in the first “dark ages” in the historical record.

1.1 Overview of the model

In our model, a population in control of an economy with the potential to create surplus (the “incumbent”) faces potential attacks by a predatory group (the “challenger”). The incumbent has the opportunity to invest and grow future income, which would lead to “prosperity;” however, the possibility of attacks may recommend spending resources in consumption and defense instead. Three key parameters are the initial productivity (or income), and the rates at which consumption can be turned into defense (defense capability) and future income
(growth capability). Higher initial productivity helps finance more defense, but it also attracts stronger predation. If sufficient defense can be financed, the challenger is deterred ("security" is attained). Productive investment may also intensify predatory challenges by raising future income. The result is a tradeoff between investment-led growth and security. The key formal question is whether some combination of parameter values allows for productive investment and deterrence to yield a civilizational breakthrough.

In the first part of our analysis the incumbent’s defense capability is exogenous, and in the second part defense capability can be improved. Both parts help rationalize different modalities in the rise of ancient civilizational states. The analysis in the first part characterizes the unique equilibrium of the game. The key result is that the parameter space is partitioned into four regions corresponding to the four possible “prosperity/security” combinations.

When both defense and growth capabilities are low, neither prosperity nor security are possible, and societies remain locked in the situation of economic stagnation and conflict characterized by Keeley (1996), which corresponds to the Hobbesian “state of nature.” If growth capability is high relative to defense capability, prosperity becomes possible even in the face of attacks. Although anti-Hobbesian, the possibility of growth despite predation is consistent with a widespread occurrence in the history of humanity, like the Chinese with the Mongolians and the Saxons with the Vikings in the 10th century. Lastly, when both defense and growth capabilities are high and "balanced," the incumbent can grow and also deter predators. The latter two cases are the key to our explanation for the emergence of civilizations. Civilizations occur in cases where high enough returns to productive investment allow the economy to grow, and where the incumbent manages to deter attacks by the challenger, or, if attacks occur, to repel them with reasonably high probability.

The case of Egypt can be explained in terms of natural endowments for both growth and (exogenous) defense. Growth capabilities were given by rich alluvial soils that could be quickly improved through productive investments, and defense was provided by the surrounding deserts, which protected dwellers along the Nile from most types of attack (Bradford 2001).

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1 The relationship between security and prosperity has been a perennial concern in the social sciences. A dominant view, inspired by Hobbesian philosophy, is that state-provided security is a precondition for prosperity (Lane 1958; Olson 2000; Bates 2001; see Boix 2015 for a contrasting approach). But the state itself has to be explained and the Hobbesian view provides no clear message on whether state formation requires a modicum of prosperity in the first place.
The model allows us to study how shocks (natural, or policy-originated) to defense and growth capabilities can generate transitions from one area to another in the security-prosperity outcome space. Such shocks can account not only for the rise of civilizations, but also for their fall and the loss of security and prosperity, ushering in “dark ages.” We illustrate this case with a study of the end of the Bronze Age around 1200BC. Our model also shows that enhanced defense capabilities are a necessary condition for achieving security and prosperity, but expanding growth capability, while valuable, is not strictly necessary. Moreover, under certain conditions, improved growth capabilities may worsen outcomes.

The rise of civilization in Southern Mesopotamia poses a challenge to our model with exogenous defense capability, because the Sumerian settlements, in contrast with Egypt, did not have natural protection. Rather, as widely attested in the archaeological record, they faced challenges from various pastoralist groups. Then, how could the Sumerian city-states ever emerge? According to the anthropological literature, the settled groups that formed pristine states exploited an agrarian “staple finance”, which, being highly rewarding, would fund their defense (Johnson and Earle 2000, p. 305-306). These groups had a material advantage that could be turned into a military one, by relying on walls, weaponry, and numbers, all of which could be used to deter or defeat their enemies. This process of endogenous improvement of defense capabilities can be accounted for in our extended model, where the incumbent can make investments to upgrade defense. We show that when initial productivity is high enough the incumbent can fund its way out of the parametric region without security or prosperity into a region with high levels of both.

While higher initial productivity always exacerbates conflict in the model with exogenous defense capability, in the model with endogenous defense it may pave the way to security and prosperity. That this should happen is not obvious, since improvements in defense capability are an investment, and as such they are discouraged by the insecurity associated with higher initial productivity. When stronger defense capability is put in place, a Hobbesian effect is observed: the enhanced security yields a higher effective return to productive investment and it fosters prosperity.
1.2 Plan for the paper

In the next section we relate our contribution to existing literature. In section 3 we present the model with exogenous defense capability, and use it in section 4 to analyze the rise of Egypt and the end of the Bronze Age. In section 5 we extend the model to allow for endogenous defense capability, and in section 6 we use the extended model to account for the rise of Sumeria. We conclude in section 7.

2 Related Literature

Archaeologists like V. Gordon Childe (1936), who first conceptualized the advent of the Neolithic era as an “agricultural revolution,” focused on the innovations in the means and relations of production while abstracting from the necessary accompanying innovations in military protection. On the other hand, several archaeologists have noted the paramount role of investments in protection, such as fortifications, walls, and moats, in the erection of the first cities (Service 1975, 299). According to Near Eastern archaeologist Volkmar Fritz, “in the Jordan Valley, settlements were surrounded by a wall even before it is possible to speak of the city proper” (1997 II: 19). Other authors, like Mann (1986) and McNeill (1979), explicitly connected food production with protection needs, as mentioned earlier. However, we are not aware of any account that has explicitly focused on the interplay of surplus production and surplus protection to point out a solution to the civilizational paradox. As we will show, the interplay is subtle and perhaps profitably analyzed through a formal model.

Our approach builds on, but departs from, historical accounts that emphasize the geographic sources of economic prosperity. The approaches emphasizing the availability of domesticable plants and animals to explain why some regions generated surpluses while others did not (e.g., Diamond 1999) contribute a necessary building block for understanding the prosperity of the first settled societies. However, a purely geographic approach is incomplete, for it misses the role of incentives and strategic action that is at the core of any viable sociopolitical account of the origin of civilizations. Our approach incorporates both strategic actors and geographic factors such as food production potential or protective terrain.

Our investigation comes at the cost of abstracting from some aspects that have been
considered in anthropological theories of the state. For example, an influential view in anthropology is that the state emerged to sustain and expand economic inequality (Fried 1960, 728). We abstract from social hierarchy not because we think political stratification is unimportant, but because it helps to focus attention on the incumbent-challenger interaction. For Carneiro (1970), states originated in fertile areas surrounded by less productive land. Growing populations would contest fertile land and losers, unable to flee, would accept domination. The ensuing political stratification is the basis of the state. The Nile valley, surrounded by deserts, is a good example of circumscribed productive land. Our model generates a similar empirical implication; however, it is not driven by exploitation but by the fact that low quality surrounding land can protect against challengers. Unlike Carneiro’s theory, our model does not appeal to population pressure, an assumption that has been challenged by some writers (see Allen 1997).

It is customary in the social sciences to view the state as the monopoly on violence. Adapting from Weber, we define the state not in binary terms but as a matter of degrees (Weber 1978: ch. I, s. 16), so that state formation involves higher degrees of protection from attacks. We focus on the state as “sovereignty,” defense from threats, and abstract from “rulership,” the creation of a political hierarchy and institutions within a society, which even critics of the institutional approach include in their definition (see Boix 2015: 66-77). The exclusion of both rulership and political institutions from our model helps identify a minimalist view of early civilizations as the intersection of surplus production and statehood seen as sufficient surplus protection.

Our work is related to both theories of state formation (Tilly 1975, 1992, Spruyt 1996) and theories of the political sources of prosperity (North and Weingast 1989; Olson 2000, Bates 2001; Boix 2015). In contrast to our model, theories of state formation do not place the state in the context of the “security-prosperity” tradeoff, and theories of the political sources of prosperity focus on rules of the political game once the state is already in place rather than on pre-state forces.

Our work has important complementarities with that by Mayshar, Moav and Neeman.

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2For a review of anthropological theories of early states see for example Claessen and Skalnik (1978).
3We share with Boix (2015) an interest in mechanisms of state formation extending back into prehistoric times, as well as in “hard” causes related to the physical environment. Although Boix finds sources of pre-institutional cooperation under conditions of anarchy (absence of state), he conceives of state formation as the selection of either republican or monarchic institutional settings. By contrast, we focus on the conditions for state formation that allow for investment and security before political institutions become central.
(2013), and Mayshar, Moav, Neeman and Pascali (2015). They also combine a focus on early states, an emphasis on geographic drivers, and the use of formal theory. For us, geography matters because it defines both productive and defense capabilities, while for them it determines the observability of production (the former paper) or its appropriability (the latter). Mayshar, Moav and Neeman (2013) use a principal-agent model to show how monitoring capabilities shape the extent of political centralization, and account for contrasting trajectories in Sumeria and Egypt, where observability of the Nile allowed for a more unified and lasting state. Our focus is not on the form of states, but on the conditions for their emergence. This is also the focus of Mayshar, Moav, Neeman and Pascali (2015), who focus on crop appropriability. They equate the state with the political hierarchy that results from appropriability and assume it results in the full prevention of conflict. We abstract from appropriability and internal hierarchy, and investigate whether it is true that conflict can be reduced or eliminated.

A recent literature studies the incentives to make investments in state capacity in situations where the ruler may lose control of the polity to a competing faction (Besley and Persson 2011), or to a foreign power (Gennaioli and Voth 2015). Gennaioli and Voth (2015) formalize and investigate empirically Tilly’s (1990) argument that modern European states formed as a result of the competitive pressures of military conflict, which created a need to centralize fiscal control. There are some differences in terms of modeling: unlike in Besley and Persson’s model, investments in our model can augment the virulence of challenges, and we abstract from the competitive dynamics between states that anchor Gennaioli and Voth’s analysis. There are differences in substantive focus as well. In the state capacity literature there is a pre-existing state, while we focus in pre-state societies that move towards statehood by attaining some degree of deterrence. Our paper is also related to models of state consolidation (Powell 2012, 2013); the key difference is that in our model consolidation is studied in relation to investment and growth.

Our work is obviously related to the literature on conflict, which is too extensive to review here, in particular the family of papers investigating the tradeoff between “guns” and “butter” (see Garfinkel and Skaperdas 2007 for a review), the closest of which is perhaps the paper by Grossman and Kim (1995), where agents choose between production and contestation. Our model differs at least in two respects: we consider a dynamic environment where investments and growth affect future consumption and means for defense, and we endogenize
the productivity of war efforts.

3 The Basic Model

3.1 Setup

Our baseline model features the incentives to raise an army to protect wealth from usurpers at the cost of resources for consumption or productive investment. Later on we introduce the decision to invest in defense capability.

Players An “incumbent” lives for two periods \( t = 1, 2 \), and controls a productive asset that yields a nonstorable flow \( v_t > 0 \), in each period. The asset can be a piece of land, a port, or any bundle of productive resources including people. The initial level \( v_1 \) tracks properties of the environment (e.g., weather, quality of the soil, topography) that affect the quantity and value of goods that the economy can produce, i.e., productivity. The incumbent faces a “challenger” in each period, who is interested in wresting control of the productive asset away from the incumbent.\(^4\) This interaction captures the large class of cases of inchoate urban centers (agricultural settlements, city-ports, markets at the crossing of interior roads) where food-producing populations or trading elites face the threat of predatory attacks by nomadic tribes or plundering warlords.

Actions, resources and technology In each period \( t \) the incumbent can spend its flow \( v_t \) on consumption, productive investment \( i_t \) or mobilizing resources to defend its asset. One dollar of productive investment \( i_t \) costs one dollar of consumption and it adds \( \rho > 1 \) dollars to the yield of the productive asset in the future.\(^5\) That is, productivity evolves according to the relation \( v_{t+1} = v_t + \rho i_t \); we abstract from depreciation and discounting for simplicity. \( \rho \) captures anything that affects the returns to productive investments in the asset controlled by the incumbent. For example, \( \rho \) could, like \( v_1 \), reflect various conditions of the physical environment such as climatic conditions and soil fertility, or economic factors such as the price of goods sold.\(^6\)

\(^4\)In the basic model it does not matter whether the challenger is different across periods or not. In the model with endogenous defense capability we assume the challenger is different each period.

\(^5\)Given that for \( \rho < 1 \) investment is never worthwhile, failure to obtain it in equilibrium is obvious and uninteresting. Hence our assumption \( \rho > 1 \) which makes investment at least a possibility.

\(^6\)If the value of what the incumbent produces follows a standard price \( \times \) quantity formulation we can write \( v_1 = pq \), and \( v_2 = v_1 + \rho pi = pq + \rho pi \), where \( q \) and \( i \) are physical units. Then, changes in \( p \) will
The effectiveness of the incumbent’s defense (or “army”) is denoted $a_t$ and such an army costs the challenger an amount $\frac{a_t}{\kappa_t}$ where $\kappa_t \geq 0$ is the value of the incumbent’s defense capability. The higher the defense capability of the incumbent, the higher the “firepower” $a_t$ attained by a given conflict effort $\frac{a_t}{\kappa_t}$. $\kappa_t$ captures anything that affects the costs for the incumbent of producing defense or military firepower, such as a rugged terrain or better military technology or expertise. In this section $\kappa_t$ is fixed at an initial level $\kappa_1$ and we will derive implications for conflict and prosperity stemming from different values of $\kappa_1$. The extended version of the model in section 5 will be devoted to endogenizing $\kappa_t$.

In period $t$ the incumbent faces a budget constraint,

$$v_t - i_t - \frac{a_t}{\kappa_t} \geq 0. \quad (1)$$

The challenger observes the choices of $a_t$ and $i_t$ by the incumbent and chooses its own conflict effort $b_t$. If victorious in the first period the challenger captures the incumbent’s income in the second period; in a two period model it is immaterial whether expropriation involves the income flow or the asset itself. Whenever the challenger attacks ($b_t > 0$), it prevails with probability $\frac{b_t}{a_t+b_t}$ and it gains nothing with the complementary probability (i.e., we adopt the typical Tullock contest success function). If the incumbent is defeated it obtains an outside payoff normalized to zero. If the challenger selects $b_t = 0$ we say the incumbent has successfully deterred the challenger, and this lack of challenge to the authority of the incumbent results in full security. As we explain later, we associate the degree of security—be it in terms of internal order or external sovereignty—with the degree of statehood, so full security corresponds to full statehood.

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7 One interpretation is that the incumbent has a stock $\kappa_t$ (e.g., a wall or a catapult) that is complementary to current army effort $e_t$. They jointly produce an army power $a_t = \kappa_t e_t$ and so given $\kappa_t$, the cost of the effort expense $e$ is $\frac{a_t}{\kappa_t}$. To save on notation we obviate the term $e_t$.

8 Assuming that the challenger’s war expense is basically effort is equivalent to assuming that the challenger’s income is sufficient to finance the optimal war effort $b_t^*$. Because the effects of interest are not driven by a budget constraint on the challenger being binding, we follow the most parsimonious approach of not introducing a positive income nor a resource constraint for the challenger.

9 Generalized versions of the ratio-based contest success function exist but are less tractable. Hirshleifer (2001) explores some of the difficulties. The typical generalization is to consider functions of the type $\frac{a_t^\alpha}{b_t^{\alpha+\beta}}$. The most important feature of our model, which is the possibility of generating deterrence, will obtain for any function satisfying $\alpha \in [0, n/(n-1)]$ in a symmetric contest with $n$ players. For $\alpha > n/(n-1)$ pure strategy equilibria cease to exist.
**Timing** In each period the incumbent selects $a_t$ and $i_t$. After observing $(a_t, i_t)$ the challenger selects $b_t$. If $b_1 = 0$, the incumbent remains in place in period 2. If $b_1 > 0$, then there is conflict at the end of period 1. The winner obtains the incumbent’s income in period 2.\(^{10}\) The challenger obtains zero in the event of no attack.

**Payoffs** Both challenger and incumbent are risk neutral and care linearly about consumption, which equals income net of costs of investment and defense. The incumbent acts as a Stackelberg leader, choosing $a_1$ and $i_1$ to maximize the value $V_t$:

$$V_t = v_t - \frac{a_t}{\kappa_t} - i_t + \frac{a_t}{a_t + b_t} V_{t+1}.$$  
(2)

The challenger chooses $b_t$ to maximize the expression

$$W_t = \frac{b_t}{a_t + b_t} V_{t+1} - \frac{b_t}{\kappa_c},$$
(3)

where $V_{t+1} = v_t + \rho i_t$, and where $\kappa_c$ is the military capability of the challenger. We could also parametrize the challenger’s objective with a factor $h$ and write the expected benefit as $\frac{b_t}{a_t + b_t} h V_{t+1}$, so as to capture different levels of “hunger” by the challenger.\(^{11}\) The parameter $h$ would capture a different substantive aspect than $\kappa_c$, but it would be mathematically redundant since the challenger’s problem could be rewritten as involving a military capability of $h \kappa_c$ instead. Therefore we will abstract from the parameter $h$. To simplify notation, we will develop the model normalizing $\kappa_c = 1$, and will comment later on how the solution changes with variations in $\kappa_c$. An additional simplification is we do not consider here the realistic possibility that conflict destroys part of the asset. Our results do not change qualitatively if conflict generates some destruction.\(^{12}\)

\(^{10}\)In the two period model it makes no difference whether we assume that the new incumbent faces a new challenger in period 2, since in that period there are not incentives to fight. Historically both cases were observed: intermittent raids, and full invasion with “replacement,” such as when Sargon of Akkad invaded the Sumerian city-states, or the Mongols invaded China.

\(^{11}\)This parameter could also track the differential ability of the challenger at “operating” the asset under the interpretation that a successful challenge leads to replacement. One issue we do not take up here is the case where a challenger has a high valuation for the stream of production (as when looting animals and food) but a low valuation for the asset due to an inability to operate it. These are interesting variations that go into the finer issue of modes of challenge that may be costly to the incumbent but do not pose a replacement threat. The study of these variations is left for future research.

\(^{12}\)The model presented here represents the limit case of a more general model where a fraction $\sigma \in [0, 1]$ of the asset survives the war. The solution to the expanded model is similar and continuous in $\sigma$, so the solution we focus on remains qualitatively similar when $\sigma$ dips below 1 (proof available upon request).
We will solve for a Subgame Perfect Nash Equilibrium by backward induction.

### 3.2 Solution

**Second period** The rewards from success in conflict accrue in the next period if any, so the challenger does not fight in the second and last period and \( b_2 = 0 \). Anticipating this the incumbent chooses \( a_2 = 0 \) and selects \( i_2 \) to maximize the value of consumption in the second period \( V_2 = v_2 - i_2 \), yielding \( i_2 = 0 \) and \( V_2 = v_2 \).

**First period** The challenger observes the pair \( (a_1, i_1) \) and chooses \( b_1 \) to maximize \( W_1 \) as given by expression (3). Since the first order condition is \( \frac{a_1}{(a_1 + b_1)^2} v_2 = 1 \), and \( v_2 = v_1 + \rho i_t \), the best response function of the challenger is,

\[
b_1(a_1, V_2) = \begin{cases} \sqrt{a_1(v_1 + \rho i_1) - a_1} & \text{if } a_1 < V_2 \\ 0 & \text{otherwise} \end{cases}
\]

This expression exhibits a key trade-off of the model: productive investments \( i_1 \) raise the value of the productive asset. Thus, conditional on maintaining control of the asset, investment is a good idea for the incumbent since \( \rho > 1 \); however, the future control of the asset is not a forgone conclusion. Investment raises the incentives of the challenger to arm itself since it makes it more attractive to become the incumbent (formally, \( \frac{db_1}{di_1} > 0 \) if \( a_1 < V_2 \)). Therefore, while productive investments increase the value of future incumbency, they may lower the chance that the current incumbent gets to reap that value. This is the civilizational paradox: future prosperity raises insecurity, which in turn depresses incentives to invest and undermines the creation of that future prosperity.\(^{13}\) Against this backdrop, our task is to understand whether there are any parameter values \( v_1 \), \( \kappa_1 \), and \( \rho \) that map into security and prosperity. To answer this question we must study the problem of the incumbent.

The incumbent maximizes \( V_1 \) as given by (2) subject to the budget constraint (1) and anticipating the challenger’s best response in (4). The latter indicates that if \( a_1 \geq v_1 + \rho i_1 \) the challenger will choose not to fight, and therefore the incumbent would never choose

\(^{13}\)The civilizational paradox, involving as it does the incentives of a challenger, is related to Hirshleifer’s (1991) paradox of power, but differs from it. Hirshleifer’s paradox consists of the fact that the poorer contender can end up better off. The paradox we refer involves a contradiction loop: investments leading to prosperity reduce security and therefore the motivation to bring about that prosperity.
with $a_1$ beyond the point $v_1 + \rho i_1$, which attains deterrence. This can be incorporated into the incumbent’s problem as an additional, deterrence constraint. The incumbent’s problem in period 1 can then be written as,

$$
\max_{a_1, i_1} \left\{ v_1 - \frac{a_1}{\kappa_1} - i_1 + \frac{a_1}{a_1 + b_1} (v_1 + \rho i_1) \right\}
$$

subject to,

$$
\begin{align*}
\frac{a_1}{\kappa_1} - i_1 & \geq 0 \quad (BC) \\
v_1 + \rho i_1 - a_1 & \geq 0 \quad (DC) \\
a_1 & \geq 0 \\
i_1 & \geq 0,
\end{align*}
$$

where (BC) is the incumbent’s budget constraint and (DC) is the deterrence constraint. The Lagrangian, which expresses the expected utility of the incumbent, is:

$$
\mathcal{L} = v_1 - \frac{a_1}{\kappa_1} - i_1 + \frac{a_1}{a_1 + b_1} (v_1 + \rho i_1) + \lambda_{BC} (v_1 - \frac{a_1}{\kappa_1} - i_1) + \lambda_{DC} (v_1 + \rho i_1 - a_1) + \lambda_a a_1 + \lambda_i i_1,
$$

where $\lambda_{BC}$, $\lambda_{DC}$, $\lambda_a$ and $\lambda_i$ are the Lagrange multipliers for each constraint (6)-(9). We will characterize the solution $(a_1, i_1, \lambda_{BC}, \lambda_{DC}, \lambda_a, \lambda_i)$ to this problem for each parameter combination $(\rho, \kappa_1, v_1) \in (1, \infty) \times (0, \infty) \times \mathbb{R}_+$. The first order and complementary slackness conditions that characterize the optimum are given by,

$$
\frac{\partial \mathcal{L}}{\partial a_1} = \frac{1}{2} \sqrt{\frac{v_1 + \rho i_1}{a_1}} - \frac{1}{\kappa_1} - \frac{\lambda_{BC}}{\kappa_1} - \lambda_{DC} + \lambda_a = 0; a_1 \geq 0, \lambda_a \geq 0, \lambda_a a_1 = 0 \text{ c.s.} \quad (11)
$$

$$
\frac{\partial \mathcal{L}}{\partial i_1} = \frac{\rho}{2} \sqrt{\frac{a_1}{v_1 + \rho i_1}} - 1 - \lambda_{BC} + \lambda_{DC} \rho + \lambda_i = 0; i_1 \geq 0, \lambda_i \geq 0, \lambda_i i_1 = 0 \text{ c.s.} \quad (12)
$$

13
\[
\lambda_{BC}(v_1 - \frac{a_1}{\kappa_1} - i_1) = 0 \text{ c.s.}, \quad \lambda_{DC}(v_1 + \rho i_1 - a_1) = 0 \text{ c.s.}
\]  
(13)

Since the optimization involves an objective that is concave in the control variables \((a_1, i_1)\) and linear constraints, the conditions in (11)-(13) are necessary and sufficient for a maximum. Solving the program (10) requires checking which combinations of values for the endogenous variables \((a_1, i_1, \lambda_{BC}, \lambda_{DC}, \lambda_a, \lambda_i)\) constitute the optimum for different regions of the parameter space \((\kappa_1, \rho, v_1)\). Note from (11) that the marginal benefit of \(a_1\) goes to infinity as \(a_1\) goes to zero (a typical feature of contests), so the optimum must feature \(a_1 > 0\) and \(\lambda_a = 0\). Beyond this, the method for solving the problem is tedious: it requires checking which combinations of values for the endogenous variables are consistent with the constraints for each parametric region and also yield the highest value for the program. The following proposition summarizes the solution, the details of which can be found in the appendix.

**Proposition 1** Optimal behavior by the incumbent yields a partition of the parameter space \((\kappa_1, \rho, v_1)\) into four distinct regions:

- **Region 1 (R1):** \(\{ (\kappa_1, \rho, v_1) | \kappa_1 > \rho, \rho > \kappa_1/(\kappa_1 - 1), \kappa_1 > 1 \} \) Security and prosperity
  
  In R1 the solution is: \(a_1 = v_1 \frac{\kappa_1(1 + \rho)}{\kappa_1 + \rho}, i_1 = v_1 \frac{\kappa_1(1 - 1)}{\kappa_1 + \rho}, V_1 = v_1 \frac{\kappa_1(1 + \rho)}{\kappa_1 + \rho}\)

- **Region 2 (R2):** \(\{ (\kappa_1, \rho, v_1) | \rho > \kappa_1, \rho > 4/\kappa_1 \} \) Prosperity without security
  
  In R2 the solution is: \(a_1 = \frac{\kappa_1}{2} \left(1 + \frac{1}{\rho}\right), i_1 = \frac{v_1}{2} \left(1 - \frac{1}{\rho}\right), V_1 = \frac{v_1}{2} \left(1 + \frac{1}{\rho}\right) \sqrt{\rho \kappa_1}\)

- **Region 3 (R3):** \(\{ (\kappa_1, \rho, v_1) | 2 > \kappa_1, \rho < 4/\kappa_1 \} \) Neither prosperity nor security
  
  In R3 the solution is: \(a_1 = v_1 \frac{\kappa_1}{2}, i_1 = 0, V_1 = v_1 \left(1 + \frac{\kappa_1}{2}\right)\)

- **Region 4 (R4):** \(\{ (\kappa_1, \rho, v_1) | \kappa_1 > 2, \rho < \kappa_1/(\kappa_1 - 1) \} \) Security without prosperity
  
  In R4 the solution is: \(a_1 = v_1, i_1 = 0, V_1 = v_1 \left(2 - \frac{1}{\kappa_1}\right)\).

The following figure contains a graphical representation of the solution.

A convenient feature of this model is that the optimal decisions by the incumbent on defense \(a_1\) and productive investment \(i_1\) are invariant in \(v_1\). This feature greatly simplifies the characterization of emerging “regimes” with exogenous defense capability, as we can restrict attention to the bidimensional space \((\kappa_1, \rho)\).

The main feature of the solution is that all four combinations of security and prosperity can be observed depending on the values of the parameters \((\kappa_1, \rho)\). For low values of both defense capability and yield of investment, the incumbent will be stuck in a situation of
economic stagnation and conflict ($R3$). In $R3$ the prospect of conflict lowers the rate of return to investment, preventing investment and hence growth. If defense capability $\kappa_1$ is higher but investment returns $\rho$ are still low, the incumbent will be in region $R4$, where the challenger is deterred but there is no investment. In this area growth is foreclosed not by existing conflict but by the fact that if growth were attempted the challenger would become more aggressive, which would raise the costs of maintaining deterrence. If returns $\rho$ are relatively high and defense capability $\kappa_1$ relatively low—i.e., in $R2$—growth occurs despite the fact that full security is not attained. If, starting from $R2$ or $R4$, defense capability $\kappa_1$ were to become sufficiently higher, the incumbent would enter $R1$. Relative to $R2$, the added defense capability helps attain deterrence, which increases net investment returns and then expands growth. Relative to $R4$, the added defense capability makes it cheaper to attain deterrence and releases resources for growth. Note that in a situation of military parity ($\kappa_1 = \kappa_c = 1$) deterrence is impossible. Some degree of military superiority by the incumbent is needed for complete security.

14Productive investment is higher in $R1$ than in $R2$ whenever $v_1 \frac{\kappa_1-\rho}{\kappa_1+\rho} > \frac{\rho}{2} \left(1 - \frac{1}{\rho}\right)$, which is always the case for $\kappa_1 > \rho$, a condition characterizing $R1$. 

15
Inspection of Figure 1 yields the following,

**Remark 1** From a situation of no prosperity nor security (R3), a large enough increase in defense capability $\kappa_1$ is a necessary and sufficient condition for attaining both prosperity and security; in contrast,

**Remark 2** Increases in the growth capability $\rho$ are not necessary nor sufficient for attaining both security and prosperity.

Natural shocks could increase or decrease parameters like $\rho$ and $\kappa_1$. An incumbent that enjoys security and prosperity in R1 could, through a reduction in $\kappa_1$, be plunged into stagnation and conflict in R3. A reduction in $\kappa_1$ could be thought of as a negative shock to the incumbent’s defense technology.

As said earlier, the (security, prosperity) regimes characterized in Proposition 1 are invariant in initial income $v_1$; that is, whether investment $i_1$ and arming by the challenger $b_1$ are positive or zero does not depend on $v_1$. But changes in income $v_1$ do affect the particular values of all endogenous variables whenever positive. In particular, we have the following,

**Remark 3** Increases in initial income $v_1$ exacerbate conflict; that is, in regimes where (either or both) $a_1$ and $b_1$ are positive, they increase with $v_1$.

**Proof:** see appendix.

This result highlights one of the central forces in the prosperity-security paradox, namely the fact that a more productive incumbent that cannot fully deter its enemies will be engulfed in more virulent conflict.

In order to connect the model to the historical record, we now relate the regions in Figure 1 to the event of a civilization rising. We defined stateness as a relative high degree of security. Figure 2 displays contour plots of relevant equilibrium magnitudes. The continuous lines within each region represent level curves, and the lighter shades of color represent higher values of the respective magnitude. Figure 2(a) shows that the arming effort of the incumbent increases as defense capability is higher. And this contributes to increasing security, as seen in Figure 2(b), where security is proxied by the probability that the incumbent will prevail. This probability is 1 in R4 and R1, and it decreases in R2 as defense capability goes down or growth capability goes up (as this fires up the challenger). The areas in R2 that are
Figure 2: Values of endogenous magnitudes in equilibrium with exogenous defense capability. Assumption: $v_1 = 1$. 
sufficiently close to $R_1$ display very high levels of security (approaching full security where $R_2$ meets $R_1$) which in our approach can be interpreted as a high degree of stateness. In other words, we may consider $R_1$ and the safer parts of $R_2$ and $R_4$ as the parameter combinations that yield statehood. But civilization requires more than security; it also requires the creation of surplus, which in our model amounts to growth ($v_2 - v_1 = i_1 \rho$).

Figure 2(c) shows how there is no growth in $R_3$ and $R_4$ (since there is no investment) and that there is growth in $R_2$ and $R_1$. Growth increases with returns $\rho$ and in $R_1$ it also increases with defense capability, as a higher defense capability lowers the costs of arming and releases resources for investment. In $R_2$ growth is unresponsive to defense capability because any increase in $\kappa_1$ is met with a similar increase in $a_1$, which keeps the resources devoted to defense $\frac{a_1}{\kappa_1}$ and investment $i_1$ constant.

We defined civilization as the joint attainment of growth and security. This would leave out parts of $R_2$ to the North-West, bordering $R_3$, where growth can be high but security low. This is sensible if we consider that civilization requires to consolidate growth by defending production from attacks. A good proxy for civilization would then be the continuation value perceived by the incumbent in period 1, which reflects both growth and security. This is the expected future income of the incumbent resulting from investment and the probability that the incumbent prevails. This combination of the magnitudes in panels (b) and (c) yields the pattern in panel (d) of Figure 2. We observe that this “intersection” of growth and security increases with both defense and growth capabilities, and indicates areas in $R_1$ and $R_2$ near the 45 degree line as those that best escape the security-prosperity paradox, and therefore as good parametric candidates for representing the rise of civilizations.

### 3.3 Discussion of modeling choices

**Internal vs external conflict and social structure of incumbent polity** The modern distinction between national and international conflict is irrelevant in our model. The process of civilization emergence precedes such distinctions. That process ended with the incorporation of the formerly hostile populations in some cases and their exclusion in others. If challengers are internal actors (ex ante or ex post) our definition of security is about internal order and the classic monopoly of violence. If challengers are external actors, our definition of security is best matched with the notion of sovereignty. Second, there is no
distinction between ruler and subjects within the incumbent actor. The incumbent in our model can be taken to be either a representative agent in the civilized center, a perfectly benevolent ruler acting on behalf of that settled population, or a perfectly extractive ruler who is a residual claimant.\textsuperscript{15}

**Asymmetries** We have kept as many aspects as possible symmetric between the incumbent and the challenger, and only introduced asymmetries that we deemed necessary to analyze the type of interaction of interest. One asymmetry is that the incumbent acts as a Stackelberg leader. This helps make deterrence possible. Another asymmetry is that while defense effort costs the incumbent resources, it does not deplete a budget for the challenger. This is for tractability. It would be possible to include a budget constraint for the challenger, and the advantage of a wealthier incumbent at being able to finance higher defense effort, and then attaining deterrence, would operate in similar fashion. However, the reaction function of the challenger would hit its constraint eventually and the analysis would become less elegant as kinks in the reaction function would have to be taken into account. A third asymmetry is that the incumbent can accumulate and the challenger does not. This was both for tractability and to match the situation of historical interest, where one settled, food-producing, group has more room to accumulate than rival nomadic groups.

**Dynamic considerations** Our model has two periods. Two natural questions arise, namely (i) whether the same results can obtain in a static setting, and (ii) whether things would change qualitatively if more periods, including possibly infinitely many, were considered. We now consider each question in turn.

(i) A static setting would not allow for investment and growth, but related tensions can be explored between activities that yield no return (consumption) and others that yield some return (production) but are perhaps more vulnerable to expropriation. However, in a static model, either all resources are appropriable, or only production is. What a static model cannot yield is a situation where as in our model baseline resources \( v \) are not appropriable now, but are fully appropriable later if investment triggers an attack. This inherently dynamic aspect matters for the partition result in Proposition 1.

\textsuperscript{15}The latter interpretation is more suitable if growth in our model is interpreted in per capita terms, because extraction and concentration of surplus in a few hands would be a way in which an ancient society could escape Malthusian population adjustment. Alternatively we can interpret growth in our model in terms of per working capita, or at the level of groups, where wealth becomes population. We thank Oded Galor and David Weil for bringing these issues to our attention.
(ii) In the two period model, investment is deterred by the prospect of insecurity, and since there is no conflict in period 2, there is no incentive to invest in period 1 and then use the proceeds to mount a strong defense in period 2. This lack of stationarity is the essential aspect one would hope to resolve in an infinite horizon model. While extending our model to an infinite horizon has proven difficult, the key insight one would hope to obtain from such an extension can be explored in a three period version, which we have solved. This extension is a particular case of the model solved below for endogenous defense capabilities and is available upon request. The question can be recast thus: does the presence of a third period, and the projected need to fight in the future, eliminate the disincentives to invest that arise from projected prosperity attracting predation? The answer is no. The key tension between prosperity and security remains.

**The use of contests**  Like many authors before us, we use contests and abstract from bargaining and transfers. As is well known, even when transfers are possible inefficient conflict may occur, for example due to inconsistent priors across players, agency issues, commitment problems, or asymmetric information (see Jackson and Morelli (2011) for a survey of reasons for war). Every paper on conflict must take a stand on whether to microfound the breakout of conflict by reference to one of those phenomena or not. The cost of the added microfoundation structure is justified when a specific distortion responsible for conflict is particularly likely or germane given the problem under investigation. When the researcher remains agnostic about such connections, the more parsimonious approach that we take seems warranted.

4 Historical illustrations

4.1 Egypt and the birth of a state

Among the first civilizations, Egypt is the prototype of a “pristine territorial state,” the undisputed pioneer in attaining both security and prosperity over an extended territory. Although the Neolithic Revolution occurred in Egypt later than in Mesopotamia, the ensuing process of social and political development in Egypt was faster.\textsuperscript{16} In less than a thousand

\textsuperscript{16}During the Neolithic revolution, gathering and hunting were gradually replaced by the domestication of plants and animals for food production. The process began in the ancient Near East (Southern Levant) about 10,000 years ago. The Neolithic in Egypt developed much later, around 5500 BC. According to Bard
years, the outcome would be a state that not only managed a relatively wealthy economy but was also able to protect the surplus generated within it for long stretches of time. As Allen (1997: 135) put it, “the Egyptian state lasted longer and was more stable than most empires established elsewhere.”

Although the specific conditions underlying Egypt’s dual economic and political evolution are disputed, a strong consensus exists around the idea that geography played a key role, in the form of the Nile river and the surrounding desert. Our model can be used to assess their role in explaining the emergence of a successful state by reference to our three deep parameters, \( v_1, \rho \) and \( \kappa_1 \).

(1) **The Nile river as a fundamental driver of the Egyptian economy.** The Nile had at least two key properties: a yearly flood that fertilized the soil with rich silt, and a two-way navigability that facilitated exchange along the entire valley.17 “[T]he Nile was perfectly ordered—its current carried boats downstream, the wind blew them back upstream—and the Nile’s regular flooding renewed the fields and made farming so easy that in the Delta men had ‘only to throw out seeds to reap a crop.’” (Bradford 2001: 9). Both properties, natural fertility and easy exchange, map into a high \( v_1 \) in our model, whereby initial income is high even before investments are made.

(2) **The growth potential of artificial irrigation.** Egyptians could vastly increase their economic output by investing in water management, which in the Nile valley took the form of basin irrigation. Egyptians used a grid of basins to trap the floodwater and hold it for much longer than it would naturally stay, vastly increasing soil fertility before planting.18 Economic sociologists agree that in Egypt irrigation agriculture “could generate crop-to-seed yield of between 12:1 and 24:1 . . . but only at the cost of high capital investments” (Morris and Manning 2005: 141). For Michael Mann, artificial irrigation was one of the earliest

(1994, p. 267), “The beginning of the First Dynasty was only about 1000 years after the earliest farming villages appeared on the Nile, so the Predynastic period, during the 4th millennium B.C., was one of fairly rapid social and political evolution.”

17Most of the flow originated from monsoons in the Ethiopian highlands, and a smaller part came from the upper watershed of the White Nile around Lake Victoria. With impressive precision, the river began to rise in the South in early July and the flood got to the Northern end of the valley by mid September. The Tigris and Euphrates were not only less predictable in timing, but also more irregular in volume.

18According to a long scholarly tradition (Weber [1909] 2013, Wittfogel 1957), water management and state formation were closely linked in ancient societies. The thesis of “hydraulic empires,” which claims that irrigation was a public good with enormous fixed costs, and that pristine states formed precisely in order to provide them, has been discredited by evidence showing that irrigation was not preceded by the emergence of state administrations.
forms of substantial economic investment, which in Egypt was even more profitable than in Mesopotamia. Both in Egypt and Mesopotamia, irrigation agriculture could “generate a surplus far greater than that known to populations on rain-watered soil” (1986: 80). In Egypt, “the process was as in Mesopotamia, but squared,” and “as productivity grew, so too did civilization, stratification, and the state” (1986: 108). In our model, a high value of the parameter $\rho$ reflects an environment in which investments yield large increases productivity in the same way that the construction of irrigation systems resulted in major expansions of food production in Egypt.

(3) Territorial isolation as natural protection. The Nile basin is surrounded by deserts, which made invasions much less likely than in other food-producing centers. According to Bradford (2001: 9), “The sea to the north and the deserts west and east isolated the Egyptians from the rest of mankind, except for merchants, some infiltrators, and the occasional raid.” The desert provided two kinds of protection. It discouraged the emergence and settlement of hostile neighbors nearby, and acted as a barrier against distant rivals. In terms of our model, Egypt’s territorial isolation maps into a naturally high $\kappa_1$.

How do these conditions account for Egypt’s twin achievements of security and prosperity in the context of our model? A high level of $v_1$ has no effects in terms of which of the four security-prosperity outcomes will arise, except that rivals, if present and belligerent (as in $R_2$ and $R_3$), would become more aggressive. The implication is that the extraordinary soil fertility along the Nile was not a favorable factor per se. In fact, highly productive soils are not unique to Egypt. But the model does highlight that a combination of a high $\kappa_1$ and a high $\rho$ could help Egypt attain prosperity and security. As just argued, the geography afforded Egypt natural protection yielding a high $\kappa_1$, and the potential for productivity-enhancing irrigation offered a high $\rho$. In this context, Egyptian rulers had incentives to promote investments that would increase future surplus, and could also defend it.

The resulting picture is one where Egypt is located in a favorable section of region $R_2$, if not directly in $R_1$. The reason to place Egypt during the state formation period (end of the Naqada II period, around 3200BC) in a good part of $R_2$ is that Egypt did face occasional attacks, and perhaps the total absence of challenges that characterizes $R_1$ is better reserved to the heights of Egyptian power under the New Kingdom, when the Egyptian state was even more dominant than during its formative phase. A “good” part of $R_2$ is one near the frontier with $R_1$, where conditional on $\rho$, $\kappa_1$ is so high and the cost of defense effort by the
incumbent so small that victory is very likely. Thus, a society in such “good” part of R2 would grow and enjoy a relatively secure existence, because the probability of defeat is small, and the returns to investment are high.

4.2 The end of the Bronze Age

For a period of almost 400 years, multiple states emerged in the Eastern Mediterranean that improved their productive capacity and were capable—mainly due to fortified walls and chariots—of defending their wealth against “barbarian” populations. This set of thriving states included the city-ports of the Levant, the kingdoms of Anatolia, the Egyptian empire, and the city-states of Mesopotamia and Cyprus. But suddenly a collapse epidemic swept across the Eastern Mediterranean around 1200BC. As Eric Cline puts it (2014: 241), “...the world as they had known it for more than three centuries collapsed and essentially vanished”.

According to Drews (1993: 3), “Altogether the end of the Bronze Age was arguably the worst disaster in ancient history, even more calamitous than the collapse of the western Roman Empire.”

A long debate on the causes behind the end of the Bronze Age has hypothesized earthquakes (Schaeffer 1948), droughts and famines (Carpenter 1968), internal rebellions (Zuckerman 2007 and Carpenter 1968), or innovations in military technology (Drews 1993).

The hypothesis of earthquakes has been discredited in the face of new archaeological evidence showing that most urban destruction was caused not by natural forces but by human attack. Hittites and Egyptians left unequivocal testimonies of attacks by the “Sea Peoples,” as the Egyptians called them, a diverse array of intruders with different geographic and ethnic origins (Sandars 1987). The same evidence challenges a pure internal rebellion story. The possibility of invasions remains, but forces the question of what caused them in the first place. Two hypotheses consistent with available evidence are:

(1) A severe change in climate, which caused draught and famines, and compelled populations in the periphery to invade in search for food. Cities that were storehouses of grain fell victim to “a final resort to violence by a drought sicken people” (Carpenter 1968: 69).

(2) A revolution in the means of war, which tipped the military balance in favor of

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19 A recent paleobotany study confirms a substantial climate change around the time of the collapse that could have caused a famine (Langgut, Finkelstein, and Litt 2013). In the interpretation of these authors the shock may have caused internal rebellions rather than foreign invasions.
nomadic intruders. According to Drews (1998: 33), “the Catastrophe was the result of a new style of warfare that appeared toward the end of the thirteen century BC, [which] opened up new and frightening possibilities for various uncivilized populations that until that time had been no cause of concern to the cities and kingdoms of the eastern Mediterranean”. What were the changes introduced by the “uncivilized populations”? Chrissantos (2008: 11) summarizes them: “these tribes developed better and lighter body armor, [...] lighter and smaller round shields, [and] revolutionary longer, stronger swords [...] They also invented a new weapon, the javelin, which could be used as a missile to hurl at an enemy. They [managed to] overcome the civilizations’ chariot advantage [...] Once these tribes mastered sea travel, no shore was too far for an attack. The failure of the chariot in the face of this new warfare marks the beginning of the Bronze Age world’s collapse”.

Our incumbent-challenger model is compatible with both hypotheses (acting over most of the region or by causing invasions in critical sites). Consider the challenger’s valuation to be parametrized as $h v$, and the challenger’s military capability $\kappa_c$. We can then distinguish between two separate forces at play: the motivation behind invasion ($h$ for hunger) versus the effectiveness of the means to invade ($\kappa_c$). The historical debate has sometimes considered changes in motivation and effectiveness as rival explanations. While $h$ and $\kappa_c$ capture substantively different forces, as discussed in Section 3 they are mathematically equivalent: both affect the aggressiveness of the challenger at the margin. Therefore, studying the comparative statics of $\kappa_c$ can illuminate the role of both changes in the motivation and aggressiveness of barbarians.

The parameter $\kappa_c$ was assumed equal to 1 in the baseline model. We now consider a move to $\kappa_c > 1$. How will the incumbent fare when facing a tougher challenger? In other words, how does a higher $\kappa_c$ affect the partition of the parameter space derived in Proposition 1? The following proposition yields the answer.

**Proposition 2** Optimal behavior by the incumbent yields a partition of the parameter space $(\rho, \kappa_1, v_1) \in (1, \infty) \times (0, \infty) \times \mathbb{R}_+$ into four distinct regions:

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20 A climate shock and military innovation are not mutually incompatible causes and can be combined under the form of a “Perfect Storm” (Cline 2014: Chapter 5). Another recent explanation builds on the idea of “System Collapse:” since late Bronze Age societies were tightly connected through commerce, the fall of a few of them (for whatever cause) could set off “domino” effects. This allows for the theoretical possibility that weather- and technology-induced invasions had devastated only a critical number of nodes in the interconnected Eastern Mediterranean, but eventually provoked a general collapse.
**Region 1** (R1): \((\kappa_1, \rho, v_1) | \rho < \kappa_1/\kappa_c, \rho > \kappa_1/(\kappa_1 - \kappa_c), \kappa_1 > \kappa_c \} \) Security and prosperity

In **R1** the solution is: 
\[ \{ a_1 = v_1 \frac{\kappa_1 \kappa_c (1 + \rho)}{\kappa_1 + \kappa_c \rho}, i_1 = v_1 \frac{(\kappa_1 - \kappa_c)}{\kappa_1 + \kappa_c \rho}, V_1 = v_1 \frac{\kappa_1 (1 + \rho)}{\kappa_1 + \kappa_c \rho} \} \]

**Region 2** (R2): \((\kappa_1, \rho, v_1) | \rho > \kappa_1/\kappa_c, \rho > 4\kappa_c/\kappa_1 \) Prosperity without security

In **R2** the solution is: 
\[ \{ a_1 = \frac{\kappa_1 v_1}{2} \left(1 + \frac{1}{\rho} \right), i_1 = \frac{\kappa_1}{2} \left(1 - \frac{1}{\rho} \right), V_1 = \frac{\kappa_1}{2} \left(1 + \frac{1}{\rho} \right) \sqrt{\rho \kappa_1} \} \]

**Region 3** (R3): \((\kappa_1, \rho, v_1) | 2\kappa_c > \kappa_1 \) Neither prosperity nor security

In **R3** the solution is: 
\[ \{ a_1 = v_1 \left(\frac{\kappa_1}{2}\right)^2, i_1 = 0, V_1 = v_1 \left(1 + \frac{\kappa_1}{4\kappa_c}\right) \} \]

**Region 4** (R4): \((\kappa_1, \rho, v_1) | \kappa_1 > 2\kappa_c, \rho < \kappa_1/(\kappa_1 - \kappa_c) \) Security without prosperity

In **R4** the solution is: 
\[ \{ a_1 = \kappa_c v_1, i_1 = 0, V_1 = v_1 \left(2 - \frac{\kappa_c}{\kappa_1}\right) \} \]

**Proof:** See appendix.

The comparison of the solutions in Propositions 1 (with \( \kappa_c = 1 \)) and 2 (with \( \kappa_c > 1 \)) is best appreciated in Figure 3 where the dashed lines show the partition of the parameter space with \( \kappa_c = 1 \) and the solid lines show the partition with \( \kappa_c > 1 \), reflecting a more aggressive challenger.

![Figure 3: Equilibrium partition of the parameter space with more aggressive challenger](image-url)
worsens outcomes in the following sense. For any point in the \((\kappa_1, \rho)\) space where either security or prosperity (or both) are attained, a higher \(\kappa_c\) implies that security, prosperity or both may be lost. A higher \(\kappa_c\) expands the area of all regions against \(R_1\) (where both security and prosperity obtain). In addition, \(R_3\) which combined insecurity and stagnation, grows at the expense of all others. A world with a more aggressive challenger is worse for the incumbent.

The historical victory of the “Sea Peoples” over the kingdoms of the Ancient Near East can be seen as a shift from the security-prosperity region \((R_1, \text{or good parts of } R_2)\) to the conflict-stagnation region \((R_3)\), as a result of positive shocks to the need that challengers had to seize the economic output of the incumbent (an increase in \(h\), in turn an effect of a climatic change) or to the technology of attack (an increase in \(\kappa_c\)).

We can use the model to delve deeper into the diverging fates of different regions that suffered attacks at the end of the Bronze Age. The extremes of that contrast are Egypt, which managed to repel the invasion, and cities near the Mediterranean coast of the Levant, like Ugarit, which were destroyed. In the case of Ugarit, surviving clay tablets provide textual evidence on the threat of the Sea Peoples and the fact that Ugarit was defenseless.\(^{21}\) In terms of our model, Ugarit’s vulnerability at the time of the invasions can be interpreted as either an initial location within \(R_1\) that was close to the vertex, and thereby not too far away from \(R_3\), or within a narrow strip along the \(R_2/R_3\) border on the side of \(R_2\). When the shocks that prompted the invasion occurred the effect was to push Ugarit deep into \(R_3\).

In the new location, Ugarit faced the prospects of attacks that were too strong for the city to resist. In the model, a deep position in \(R_3\) (closer to the \(\rho\) axis) entails a lower probability that the incumbent will prevail. In fact, for Ugarit, the attack resulted in destruction.

Egypt was also a victim of barbarian attacks but the outcome was very different. Since the end of the Second Intermediate Period, Egypt had developed a highly professional army and a formidable fleet. Before the shock, Egypt was located deep in \(R_1\) so that the worst effect of the shock could have been to relocate Egypt in a relatively safe neighborhood of

\(^{21}\)In the tablets, Ammurapi, the king of Ugarit makes desperate requests to his Hittite overlord, whom he addresses as his “father” and who was using Ugarit’s maritime fleet to defend other sections of the Hittite empire: “My father, behold, the enemy’s ships came (here); my cities(?) were burned, and they did evil things in my country. Does not my father know that all my troops and chariots(?) are in the Land of Hatti, and all my ships are in the Land of Lukka?...Thus, the country is abandoned to itself. May my father know it: the seven ships of the enemy that came here inflicted much damage upon us”. Letter RS 18.147 in Jean Nougaryol et al. 1968. Ugaritica V, 24: 87–90. (Note: question marks in the original.)
R2. Egypt became susceptible to challenges, but it could prevail in the battlefield with high probability. Pictorial inscriptions on the walls of the Karnak Temple attest to the threats posed by the Sea Peoples at roughly the same time they invaded the Levant. But, in contrast to Ugarit’s tablets, the Egyptian inscriptions actually honor king Merneptah’s success in subduing the invaders. The contrasting cases of Ugarit and Egypt correspond respectively to the points A or C and B in Figure 3.

The more general point resulting from our use of the model is to relate changes in deep military and economic fundamentals to the arguments made by social theorists that the evolution of political complexity is not unilinear, but plagued by dead ends and reversals. According to our model a shock to the fundamentals behind prosperity and security could cause societies to lose either or both. The end of the Bronze Age involved state de-consolidation and a regression to lower income levels—a Dark Age—, as in the region of conflict and stagnation, R3, in our model.

5 Endogenous defense capability and the transition to security and prosperity

5.1 Setup

We will now allow the incumbent to spend resources in one period to increase its defense capability in the next period. We introduce a period 0 before the periods 1 and 2, with the incumbent facing a different challenger in each period. Since the challenger will never fight in period 2, the incumbent will never spend in expanding defense capability in period 1. Thus, the decision to augment defense capability will be relevant only in period 0. In period 0 the incumbent has a defense capability $\kappa_0$, and can spend an amount $m_0$ that will take defense capability in the next period to $\kappa_1 = \kappa_0 + \gamma m_0$, where $\gamma$ captures the purchasing power of income in terms of defense means. We assume $\gamma \in \left(0, \frac{16}{4+\kappa_0}\right)$ where the upper bound is a technical assumption to guarantee the possibility of partial transitions (where either prosperity or security are attained but not both). To make things interesting, we assume $(\kappa_0, \rho)$ are such that if things were left unchanged, in period 1 the incumbent would find himself in region R3, which means he cannot expect growth nor security. In particular, we impose the following,
**Assumption 1** \( \rho \kappa_0 < 4 \) and \( \kappa_0 < 2 \).

All other aspects of the interaction between challenger and incumbent remain as before, and for simplicity we return to the case where \( \kappa_c = 1 \).

**Timing** In period 0, the incumbent starts by selecting \( m_0 \). Then, in each period \( t = 0, 1, 2 \) the incumbent selects \( a_t \) and \( i_t \). After observing \((m_0, a_t, i_t)\) the challenger selects \( b_t \). If \( b_t = 0 \), the incumbent retains his position in the next period. If \( b_t > 0 \), then there is a war at the end of period \( t \). The winner becomes the incumbent in the next period, and faces a new challenger then.

**Payoffs** The fact that there is a new type of expenditure changes the incumbent’s budget constraint to

\[
 v_0 - m_0 - \frac{a_0}{\kappa_0} - i_0 \geq 0.
\]

As before, we solve the model through backward induction. The solution for periods 1 and 2 is given by our analysis in the previous section. That analysis tells us the expected payoff for being an incumbent in period 1 is,

\[
 V_1(i_0, m_0) = (v_0 + \rho i_0) \times \begin{cases} 
 (\kappa_0 + \gamma m_0, \rho) \in R1 \\
 \sqrt{\frac{(\kappa_0 + \gamma m_0)(1+\rho)}{\rho}} \quad (\kappa_0 + \gamma m_0, \rho) \in R2 \\
 (1 + \frac{\kappa_0 + \gamma m_0}{4}) \quad (\kappa_0 + \gamma m_0, \rho) \in R3 \\
 (2 - \frac{1}{\kappa_0 + \gamma m_0}) \quad (\kappa_0 + \gamma m_0, \rho) \in R4
\end{cases}
\]

Given this continuation value, we can solve for decisions in period 0. After the incumbent has selected \( m_0, a_0 \) and \( i_0 \), the challenger decides whether to arm himself. Using the same logic as in the previous section, we see that the challenger’s best response function is,

\[
 b_0(a_0, m_0, i_0) = \begin{cases} 
 \sqrt{a_0 V_1(i_0, m_0)} - a_0 & \text{if } a_0 < V_1(i_0, m_0) \\
 0 & \text{if } a_0 \geq V_1(i_0, m_0)
\end{cases}
\]

This notation embeds the four regions over which \( V_1(i_0, m_0) \) is defined into the calculus of the challenger. Given this best response function, the incumbent has to choose \( a_0, i_0 \) after he chose \( m_0 \) such that the incumbent maximizes his expected utility:

---

\(^{22}\)The assumption that \( m_0 \) is decided before \( a_0 \) and \( i_0 \) is just to simplify the exposition. It is equivalent to assume that the incumbent selects all three variables simultaneously since the challenger does not move until the incumbent has selected all of his actions. What is of course important is that the incumbent makes his choices before the challenger.
subject to the nonnegativity constraints \(a_0 \geq 0, i_0 \geq 0\), the budget constraint \(v_0 - m_0 - \frac{a_0}{\kappa_0} - i_0 \geq 0\) (BC) and the deterrence constraint \((v_0 + \rho i_0)S(m_0) - a_0 \geq 0\) (DC).

Notice this problem in period 1 is similar to the one with two periods in the previous section, except now the continuation value depends explicitly on \(m_0\) (which is fixed at this stage, given the convention that it was selected before \(a_0 \) and \(i_0\)) through \(S(m_0)\). The objective function is differentiable and concave in \(a_0 \) and \(i_0\), the constraints are linear, so the first order and complementary slackness conditions that are necessary and sufficient for a maximum are,

\[
\begin{align*}
    a_0 & : \frac{1}{2} \sqrt{\frac{(v_0 + \rho i_0)S(m_0)}{a_0}} - \frac{1}{\kappa_0} - \frac{\lambda_{BC}}{\kappa_0} - \lambda_{ND} + \lambda_a = 0 \quad (14) \\
    i_0 & : \frac{\rho}{2} \sqrt{\frac{a_0}{(v_0 + \rho i_0)S(m_0)}} - 1 - \lambda_{BC} + \lambda_{ND} \rho S(m_0) + \lambda_i = 0 \quad (15)
\end{align*}
\]

\[
\begin{align*}
\lambda_{BC}(v_0 - m_0 - \frac{a_0}{\kappa_0} - i_0) = 0, \quad \lambda_{ND} ((v_0 + \rho i_0)S(m_0) - a_0) = 0, \quad \lambda_a a_0 = 0, \quad \lambda_i i_0 = 0 \quad (16)
\end{align*}
\]

As before, \(\lambda_{BC}, \lambda_{ND}\) are the Lagrange multipliers for the budget constraint and deterrence constraints, and \(\lambda_a, \lambda_i\) are the multipliers for the non-negativity constraints for the control variables.

5.2 Solution

Again the infinite marginal utility of \(a_0\) at zero implies \(a_0 > 0\) and \(\lambda_a = 0\), so there are in principle eight possible cases depending on whether the remaining three Lagrange multipliers are positive or zero. A technical lemma that we prove as part of the next proposition demonstrates that under our Assumption 1 there are only two feasible cases in period 0, neither of which features productive investment. With this result, we can compute the incumbent’s expected utility for any value of \(m_0\), and study his incentives to make changes in defense capability.
To preview, the effect of $m_0$ on the incumbent’s utility depends on the initial conditions in period 0. If the maximum utility is attained for extremely low $m_0$, then the incumbent will remain stuck with insecurity and no prosperity ($R3$) in period 1. On the contrary, if the optimal $m_0$ is sufficiently high, security and prosperity will obtain in period 1. In other words, the level of $m_0$ can induce transitions from $R3$ to other regions in the next period, as well as shift the optimal choices of $a_0$ and $i_0$. The key difficulty is that these shifts cause the objective not to be everywhere differentiable nor concave in $m_0$. Characterizing the optimum requires examining the expected utility levels that obtain in the different cases (this lengthy proof is relegated to the online appendix). We now establish,

**Proposition 3** (a) Under Assumption 1 and provided that $\rho < 2$, there exist cutoffs $\tau_L, \tau_M$ and $\tau_H$, $\tau_L < \tau_M < \tau_H$ such that:

1. If $\gamma v_0 < \tau_L$, the polity stays in $R3$ (stagnation without security);
2. If $\tau_H < \gamma v_0$, the polity moves to $R1$ (attains security and prosperity); and
3. If $\tau_M < \gamma v_0 < \tau_H$, the polity moves to $R4$ (attains security without prosperity).

(b) Under Assumption 1 and provided that $\rho \geq 2$, there exist cutoffs $\sigma_L, \sigma_{M1}, \sigma_{M2}$ and $\sigma_H$, satisfying $\sigma_L < \sigma_{M1} < \sigma_H$ and $\sigma_L < \sigma_{M2} < \sigma_H$ such that:

1. If $\gamma v_0 < \sigma_L$, the polity stays in $R3$ (stagnation and conflict);
2. If $\sigma_H < \gamma v_0$, the polity moves to $R1$ (attains security and prosperity); and
3. If $\sigma_{M1} < \gamma v_0 < \sigma_{M2}$, the polity moves to $R2$ (attains prosperity without security).

This proposition tells us that, given the initial military capacity $\kappa_0$ and the productivity of investment $\rho$, the transitions followed by the polity will be very different depending on the initial income $\gamma v_0$ in terms of defense capability purchasing power. If $\gamma v_0$ is very low, the polity will remain trapped without security or prosperity. If $\gamma v_0$ lies in an intermediate region, the polity can move into a region of partial achievement. If $\rho < 2$, the transition is to $R4$ where it will attain peace but will not grow. The reason is that even though it attains a higher defense capability $\kappa_1$ in the next period, which gives the incumbent the ability to fend off attacks at a lower cost, the benefit from consumption will still be higher than the present value from investing. If $\rho > 2$, the “hybrid” transition is to $R2$ where it will grow without attaining full security (proving the existence of bounds $\sigma_{M1}, \sigma_{M2}$ for this transition makes use of our technical assumption $\gamma \in \left(0, \frac{16}{4 + \kappa_0}\right)$). If $\gamma v_0$ is very high, however, the subsequent
military capacity $\kappa_1$ will allow the incumbent to free resources for both a deterrent army and a large-scale investment at $R_1$.

To summarize, while large enough initial income (or cheap enough defense capability) guarantees security and prosperity through sufficient accumulation of defense capability, intermediate levels may only allow to attain either security or prosperity. The Hobbesian argument that security is a precondition for prosperity is qualified in this model. Baseline prosperity can buy defense capability, and only then can the ensuing security promote more prosperity.

6 Historical Illustration: Sumeria and the origin of civilization

Southern Mesopotamia was the source of the first substantial surplus in human evolution, and the area where the first city-states formed, giving rise to the first major civilization. A riverine valley along the Tigris and Euphrates, exceptionally endowed for alluvial agriculture, was the key for economic prosperity. Like in Egypt, Southern Mesopotamians made massive investments in the creation of irrigation infrastructure, securing extraordinary returns. According to Mann (1986: 78), “If [the alluvium] can be diverted onto a broad area of existing land, then much higher crop yields can be expected. This is the significance of irrigation in the ancient world: the spreading of water and silt over the land. Rain-watered soils gave lower yields.” Liverani (2005, p. 5) offers an idea of the increase in yields that could be obtained: “The agricultural production of barley underwent a notable, possibly tenfold, increase thanks to the construction of water reservoirs and irrigation canals, of long fields adjacent to the canals watered by them, and thanks to the use of the plow, of animal power, of carts, of threshing sledges, of clay sickles, and of improved storage facilities.”

These high returns to productive investment help place Sumeria in the parameter space of our model as a case of high $\rho$. But what about the other parameter, the defense capability $\kappa_1$? In contrast to Egypt, geography did not afford the Sumerians natural protection against attacks. On the contrary, the natural landscape exposed Sumeria to numerous threats. As Bradford (1993: 4) puts it, “Their neighbors to the west, the Amorites, nomads of the desert, infiltrated Mesopotamia... The neighbors to the east, who dwelled in the mountains, were the
Gutians and the Elamites. The Gutians and, to a lesser extent, the Elamites considered Sumer and Akkad a treasurehouse to be raided”. Finer (1997: 102) also emphasized the porousness of the Sumerian frontiers.

In terms of our model, the vulnerability of Sumeria to invaders means that defense capability $\kappa_0$ was low. Given a low $\kappa_0$, Sumeria’s trajectory must have begun in the conflict-stagnation region, $R_3$. But if output was so insecure, how could the first human civilization emerge at all? That is, how did Sumerians solve the problem of protecting surpluses from raiders and encouraging investments in productive infrastructure?

The extended model featuring endogenous defense capability provides an answer. Our proposition 3 states that an incumbent that is initially in $R_3$ due to a low defense capability $\kappa_0$, may invest in defense capability in order to attain sufficient security against the challenger. The key condition for this investment to be undertaken is for initial productivity $v_0$ to be high enough. The archaeological record suggests Sumeria was well placed to meet that requirement. The availability of alluvial agriculture combined with an unparalleled initial endowment of plant and animal domesticates furnished the entire Fertile Crescent with exceptional advantages. It is well known that due to altitude and climatic variation, the Fertile Crescent hosted a wide variety of plants with high potential for food production as well as domesticable animals.\(^{23}\) According to Trigger (2003: 281), domesticated animals afforded large gains in agricultural labor productivity, and may help explain why Sumeria and Egypt were the first areas in the world to develop civilization. Given the natural advantages, Diamond (1997: 135) claims that “any attempt to understand the origins of the modern world must come to grips with the question why the Fertile Crescent’s domesticate plants and animals gave it such a potent head start”. In terms of our model, this combination of initial advantages can be captured by a high $v_0$.

What is delicate about the role of the Fertile Crescent’s initial advantages is that a high $v_0$ can encourage predation by outside challengers. However, a high $v_0$ could also help finance the investments in defense that were needed to escape the conflict-stagnation trap represented by region $R_3$. It is by no means obvious that the “defense-financing” force should dominate the predation force. This tension has a resolution in our model: for $v_0$ low

\(^{23}\)Diamond (1997: Ch. 8) highlights that all eight founder crops in the Neolithic were present in the area; in addition, out of the five most important domesticated animals, four were available in the Fertile Crescent (pigs, cows, sheep and goats).
enough, no escape from \textbf{R3} is possible. For \( v_0 \) high enough, the incentive to finance defensive capabilities dominates. When defense is upgraded, security rises and ushers in productive investments and growth. Figure A1 in the Online Appendix shows the comparison of the cases of Egypt and Sumeria in terms of our parameter space \((\rho, \kappa_1)\): Egypt started in a good region with relatively high levels of both parameters. Sumeria started with a low level of \( \kappa_0 \) and it was through investment that it raised its defense capability to a higher \( \kappa_1 \) that could deliver sufficient security.

What is the evidence of endogenous defense capabilities in Sumeria? The archaeological record offers evidence of large and generalized investments to improve defense in the form of protective perimeter walls, which made Sumerian cities large-scale fortifications. Figure A2 in the Online Appendix includes illustrations of four Sumerian cities. All of them had walls. In fact, virtually every city in ancient history had walls. Walls were the endogenous, artificial substitute for the missing natural protection that was present in Egypt (where cities did not have walls).\textsuperscript{24}

7 Conclusion

We build a model to investigate the combined operation of productive and defense capabilities in a society where an incumbent seeks to consolidate security and grow the economy. The components of the model are chosen by reference to the anthropological and historical literature, in order to capture relevant environmental parameters and the minimalistic strategic dilemma facing proto-civilizations. States and civilizations arose together, and therefore stateness, defined as a high degree of security, had to emerge in association with the prosperity that tended to undermine it.

In elucidating the requirements for escaping the paradox, our model helps evaluate claims about the relative role of security and prosperity that are central to classic theories of state formation. We show that all four combinations involving the presence or absence of security

\textsuperscript{24}According to van de Mieroop’s (1997) study of Mesopotamian cities, “\textit{Perhaps the presence of walls was the main characteristic of a city in the eyes of an ancient Mesopotamian}.” The archaeological record substantiates not only the generalized presence of defense investments in rising city-states, but also their costliness, which would have been prohibitive to societies with low initial productivity. Both walls and the often complementary moats have been estimated to involve large investments (e.g., the cost estimate for the moat in the Babylonian city of Dur-Jakin is ten thousand men working for three and a half months (Van de Mieroop, p. 76)).
and prosperity are possible, preventing simple characterizations of security or prosperity being necessary or sufficient conditions for one another. These combinations match various historical experiences of societies that attained neither, one, or both objectives. The key implication of our theory is that a balance between productive and defense capabilities is important in order to prevent a security breakdown and promote prosperity. History offers examples of how naturally occurring high levels of productivity and defense, as in Egypt, enabled the emergence of civilization. But it also offers examples of negative shocks that destroyed civilizations, as in the end of the Bronze Age, which the model helps rationalize. The extended model with endogenous defense capabilities explains how areas without natural defense could, if productive enough, transition into security with prosperity and resolve the civilizational paradox. The possibility of improving defense helps create the conditions where productive investments can eventually be made without triggering predatory challenges. This result may also help rationalize historical experiences where a temporary economic boom allows the state to consolidate its power and usher in a phase of more sustained growth. Isolating formally the pivotal role of defense capability to the civilizational process contributes to the demanding enterprise of discerning how economic shocks can hinder or help state formation and political stability more generally.

8 Appendix

Proof of Proposition 1: This is a particular case of the model with general values of \( \kappa_c \), which is studied in Proposition 2.

Proof of Remark 3: That \( a_1 \) increases in \( v_1 \) in all regimes follows directly from inspection of the solution for \( a_1 \) in each regime in Proposition 1. To see that whenever positive \( b_1 \) also increases in \( v_1 \), take the value of \( b_1 \) from the best response expression (4), and substitute in the values of \( i_1, a_1 \) in regions \( R2 \) and \( R3 \). This yields respectively,

\[
b_{1,R2} = v_1 \left\{ \frac{n_1}{2} \left( 1 + \frac{1}{\rho} \right) \left( 1 + \rho \frac{n_1-1}{n_1+1+\rho} \right) - \frac{n_1}{2} \left( 1 + \frac{1}{\rho} \right) \right\} \quad \text{and} \quad b_{1,R3} = v_1 \left( \frac{n_1}{2} \right) \left\{ 1 - \frac{n_1}{2} \right\},
\]

which are both positive and increasing in \( v_1 \).
Proof of Proposition 2: The problem is to maximize,

\[ L = v_1 - \frac{a_1}{\kappa_1} - i_1 + \frac{a_1}{a_1 + b_1}(v_1 + \rho i_1) \]

\[ + \lambda_{BC}(v_1 - \frac{a_1}{\kappa_1} - i_1) + \lambda_{DC}(\kappa_c v_1 - a_1 + \kappa_c \rho i_1) + \lambda_a a_1 + \lambda_i i_1. \]  

(17)

We characterize the solution \((a_1, i_1, \lambda_{BC}, \lambda_{DC}, \lambda_a, \lambda_i)\) to this problem for each parameter combination \((\rho, \kappa_1, \kappa_c, v_1)\). To save on notation, let us define \(PS = \{(\kappa_1, \kappa_c, \rho, v_1)|\kappa_1, \kappa_c > 0, \rho > 1, v_1 > 0\}\) the parameter space we consider throughout the proof. The first order and complementary slackness conditions that characterize the optimum are given by,

\[ \frac{\partial L}{\partial a_1} = \frac{1}{2\sqrt{\kappa_c}} \sqrt{\frac{v_1 + \rho i_1}{a_1} - \frac{1}{\kappa_1} - \frac{\lambda_{BC}}{\kappa_1} - \lambda_{DC} + \lambda_a = 0; a_1 \geq 0, \lambda_a \geq 0, \lambda_a a_1 = 0 \text{ c.s.} } \quad (18) \]

\[ \frac{\partial L}{\partial i_1} = \frac{\rho}{2\sqrt{\kappa_c}} \sqrt{\frac{a_1}{v_1 + \rho i_1} - 1 - \lambda_{BC} + \lambda_{DC} \kappa_c \rho + \lambda_i = 0; i_1 \geq 0, \lambda_i \geq 0, \lambda_i i_1 = 0 \text{ c.s.} } \quad (19) \]

\[ \lambda_{BC}(v_1 - \frac{a_1}{\kappa_1} - i_1) = 0 \text{ c.s., } \lambda_{DC}(\kappa_c v_1 - a_1 + \kappa_c \rho i_1) = 0 \text{ c.s.} \quad (20) \]

Since \(\lambda_a = 0\), there are eight possible cases to be analyzed, given by whether the remaining Lagrange multipliers \(\lambda_{BC}, \lambda_{DC}\), and \(\lambda_i\) are zero or positive. We will assume in each case that the conditions defining it hold, and then determine which part if any of the parameter space supports a solution. When the case implies conditions for the parameters that are mutually exclusive, the case will be deemed infeasible. When the case implies that the solution can be supported for knife-edge combinations of the parameter values (i.e., combinations that, under a suitable measure, would have measure zero) we consider the case to be non-generic and also drop it from further consideration.

1. Case \(\lambda_{BC} > 0\) **(BC binds)**, \(\lambda_{DC} > 0\) **(DC binds, consolidation)**, and \(\lambda_i = 0\) \((i_1 > 0)\) Since \(a_1\) is always positive, and in this case \(i_1\) is also positive, the FOCs in (18), (19) must hold with equality. Because this case involves binding BC and DC constraints,
they also hold with equality. This is,

\[ a_1 : \frac{1}{2\sqrt{\kappa_c}} \sqrt{\frac{T}{\kappa_c}} - \frac{1}{\kappa_1} - \lambda_{BC} \kappa_1 - \lambda_{DC} = 0; \quad i_1 : \frac{\rho}{2\sqrt{\kappa_c}} \sqrt{\kappa_c - 1} - \lambda_{BC} + \lambda_{DC} \rho \kappa_c = 0 \]

\[ DC : \kappa_c v_1 - a_1 + \kappa_c \rho i_1 = 0; \quad BC : v_1 - \frac{a_1}{\kappa_1} - i_1 = 0, \]

implying that investment and army are \( i_1 = v_1 \frac{(\kappa_1 - \kappa_c)}{(\kappa_1 + \kappa_c \rho)} \) and \( a_1 = v_1 \frac{\kappa_1 \kappa_c (1 + \rho)}{(\kappa_1 + \kappa_c \rho)} \), respectively.

As a result \( \lambda_i = 0 \) (or \( i_1 > 0 \)) is supported by \( \kappa_1 > \kappa_c \). Using the FOCs, \( \lambda_{DC} > 0 \iff \kappa_1 > \rho \kappa_c \) and \( \lambda_{BC} > 0 \iff \rho > \kappa_1/(\kappa_1 - \kappa_c) \). The parameter set supporting this solution is \( R1 = \{(\kappa_1, \rho, v_1) \in PS|\rho < \kappa_1/\kappa_c, \rho > \kappa_1/(\kappa_1 - \kappa_c), \kappa_1 > \kappa_c \} \) and in this region there is investment and deterrence. The expected utility is \( V_1 = v_1 \frac{\kappa_1 (1 + \rho)}{(\kappa_1 + \kappa_c \rho)} \). When \( \kappa_c > \kappa_1 \), this case would be infeasible, as it contradicts \( i_1 > 0 \).

2. Case \( \lambda_{BC} > 0 \) (BC binds), \( \lambda_{DC} = 0 \) (DC does not bind, conflict), and \( \lambda_i = 0 \) \( (i > 0) \) The first order and complementary slackness conditions yield

\[ a_1 : \frac{1}{2\sqrt{\kappa_c}} \sqrt{\frac{v_1 + \rho i_1}{a_1}} - \frac{1}{\kappa_1} - \lambda_{BC} \kappa_1 = 0; \quad i_1 : \frac{\rho}{2\sqrt{\kappa_c}} \sqrt{\frac{a_1}{v_1 + \rho i_1}} - 1 - \lambda_{BC} = 0 \]

\[ BC : v_1 - \frac{a_1}{\kappa_1} - i_1 = 0 \]

Investment and army solutions are respectively \( i_1 = \frac{\rho v_1}{2} \left(1 - \frac{1}{\rho} \right) \) and \( a_1 = \frac{\rho v_1}{2} \left(1 + \frac{1}{\rho} \right) \).

This solution is consistent with \( \lambda_{DC} = 0 \) (DC holds with strict inequality) and \( \lambda_i = 0 \iff \rho > \frac{\kappa_1}{\kappa_c} \) and \( \rho > 1 \) respectively. The solution is consistent with \( \lambda_{BC} > 0 \iff \rho > \frac{4\kappa_c}{\kappa_1} \) (from checking \( \lambda_{BC} > 0 \) in the FOCs). As a result, the parameter set supporting this case is \( R2 = \{(\kappa_1, \rho, v_1) \in PS|\rho > \kappa_1/\kappa_c, \rho > 4\kappa_c/\kappa_1 \} \). Expected utility for the incumbent is \( V_1 = \frac{v_1}{2} \left(1 + \frac{1}{\rho} \right) \sqrt{\rho \kappa_1} \).

3. Case \( \lambda_{BC} = 0 \) (BC does not bind), \( \lambda_{DC} = 0 \) (DC does not bind, conflict), and \( \lambda_i = 0 \) \( (i_1 > 0) \) Non-generic, since it is consistent for subset of the space \( (\rho, \kappa_1, v_1) \) that has measure zero. This follows from (18) and (19), so when \( \lambda_{BC}, \lambda_{DC}, \lambda_i = 0, a_1 : \frac{1}{2\sqrt{\kappa_c}} \sqrt{\frac{v_1 + \rho i_1}{a_1}} - \frac{1}{\kappa_1} = 0 \) and \( i_1 : \frac{\rho}{2\sqrt{\kappa_c}} \sqrt{\frac{a_1}{v_1 + \rho i_1}} - 1 = 0 \). The first FOC implies \( \frac{\sqrt{v_1 + \rho i_1}}{a_1} = \frac{2\sqrt{\kappa_c}}{\kappa_1} \) and substituting into the second FOC, we get \( \frac{\rho}{2\sqrt{\kappa_c}} \frac{\kappa_1^2}{2\sqrt{\kappa_c}} = 1 \) or \( \frac{\rho}{2\kappa_1} = 1 \), which implies this holds for a non-generic parameter set.

4. Case \( \lambda_{BC} = 0 \) (BC does not bind), \( \lambda_{DC} > 0 \) (DC binds, consolidation), and
\[ \lambda_i = 0 \ (i_1 > 0) \] The FOCs are,
\[
a_1 : \frac{1}{2\sqrt{\kappa}} \sqrt{\frac{v_1 + \rho i_1}{a_1}} - \frac{1}{\kappa_1} - \lambda_{DC} = 0; \ i_1 : \frac{\rho}{2\sqrt{\kappa}} \sqrt{\frac{a_1}{v_1 + \rho i_1}} - 1 + \kappa_\rho \lambda_{DC} = 0,
\]
where \( a_1 = \kappa_c(v_1 + \rho i_1) \) indicating that \( \lambda_{DC} \) must simultaneously equal \( \frac{1}{2\kappa_c} - \frac{1}{\kappa_1} \) and \( \frac{(\frac{\rho}{\kappa} - 1)}{\kappa_c \rho} \), which forces the equality \( \rho = \frac{a_1}{\kappa_c} \), which is non-generic.

5. **Case \( \lambda_{BC} > 0 \) (BC binds), \( \lambda_{DC} > 0 \) (DC binds, deterrence), and \( \lambda_i > 0 \) (i_1 = 0)** Because \( i_1 = 0 \), BC binding implies that \( a_1 = \kappa_1 v_1 \), but DC binding implies that \( a_1 = \kappa_c v_1 \), so \( \kappa_1 = \kappa_c \) which is non-generic.

6. **Case \( \lambda_{BC} > 0 \) (BC binds), \( \lambda_{DC} = 0 \) (DC does not bind, conflict), and \( \lambda_i > 0 \) (i_1 = 0)** The BC binding and \( i_1 = 0 \) yield \( a_1 = v_1 \kappa_1 \). The BC binds iff \( \kappa_1 > 4\kappa_c \). The DC not binding, however, implies \( v_1 \kappa_c - v_1 \kappa_1 > 0 \Leftrightarrow \kappa_c > \kappa_1 \) which violates the condition for BC to bind \( \kappa_1 > 4\kappa_c \), making this case infeasible.

7. **Case \( \lambda_{BC} = 0 \) (BC does not bind), \( \lambda_{DC} = 0 \) (DC does not bind, conflict), and \( \lambda_i > 0 \) (i_1 = 0)** In this case \( a_1 = v_1 \kappa_1^2/(4\kappa_c) \) and \( i_1 = 0 \). This solution is consistent with \( \lambda_{BC} = 0 \) and \( \lambda_{DC} = 0 \Leftrightarrow \kappa_1 < 2\kappa_c \). Also for \( \lambda_i > 0 \) we need \( 1 - \rho \kappa_1/(4\kappa_c) > 0 \) (from the FOC of \( i_1 \)). Thus, this holds for any triple \( (\rho, \kappa_1, v_1) \in PS \) such that \( \kappa_1 < 2\kappa_c \) and \( \rho < 4\kappa_c/\kappa_1 \). In other words, the parameter set for which this region contains the solution to the incumbent’s problem is \( R3 = \{ (\kappa_1, \rho, v_1) \in PS | 2\kappa_c > \kappa_1, \rho < 4\kappa_c/\kappa_1 \} \). Expected utility in this case is given by \( V_1 = v_1 \left( 1 + \frac{a_1}{4\kappa_c} \right) \).

8. **Case \( \lambda_{BC} = 0 \) (BC does not bind), \( \lambda_{DC} > 0 \) (DC binds, consolidation), and \( \lambda_i > 0 \) (i_1 = 0)** In this case the system of conditions is given by \( a_1 : \frac{1}{2\kappa_c} - \frac{1}{\kappa_1} - \lambda_{DC} = 0; \ i_1 : \frac{\rho}{2} - 1 + \lambda_{DC} \rho \kappa_c + \lambda_1 = 0; \) and \( BC : \kappa_c v_1 - a_1 = 0 \). Since \( i_1 = 0 \), the DC yields \( a_1 = \kappa_c v_1 \). For this to be consistent with \( \lambda_{DC} > 0 \), we must have from the first equation that \( \kappa_1 > 2\kappa_c \), and to be consistent with \( \lambda_i > 0 \) we need \( \rho < \kappa_1/(\kappa_1 - \kappa_c) \), yielding \( R4 = \{ (\kappa_1, \rho, v_1) \in PS | \kappa_1 > 2\kappa_c, \rho < \kappa_1/(\kappa_1 - \kappa_c) \} \). The expected utility is \( V_1 = v_1 \left( 2 - \frac{\kappa_c}{\kappa_1} \right) \).

**References**


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Figure A.1: The different parameters of Egypt and Sumeria

Nippur


Eridu


Uruk

Source: J. Jordan (1931).

Figure A.2: Plans of Sumerian cities displaying perimeter walls
B Proofs

Proof of Proposition 3: Before analyzing the choices of \( m_0 \), we narrow down the possible equilibrium choices of \( a_0 \) and \( i_0 \). The following Lemma shows that given our Assumption 1 there are only two feasible cases in period 0.

**Lemma 1** If Assumption 1 holds, then in period 0 the incumbent chooses:

i) \( i_0 = 0 \) and \( a_0 = \frac{\kappa_0^2}{4} v_0 S(m_0) \) when \( \frac{v_0 S(m_0)}{v_0 - m_0} < \frac{4}{\kappa_0} \); or

ii) \( i_0 = 0 \) and \( a_0 = \kappa_0 (v_0 - m_0) \) when \( \frac{v_0 S(m_0)}{v_0 - m_0} > \frac{4}{\kappa_0} \).

**Proof of lemma 1:** There are two cases that constitute a solution out of eight possible ones. We will show that the assumption 1 implies that the first case holds when \( \frac{v_0 S(m_0)}{v_0 - m_0} < \frac{4}{\kappa_0} \) and the second case holds when \( \frac{v_0 S(m_0)}{v_0 - m_0} > \frac{4}{\kappa_0} \). The remaining six cases can be shown to be either inconsistent with any set of parameter values or consistent only with a non-generic set.

1. Case \( \lambda_{BC} = 0 \) (BC does not bind), \( \lambda_{ND} = 0 \) (DC does not bind, conflict), and \( \lambda_i > 0 \) (\( i_0 = 0 \))

The FOCs are,

\[
a_0 : \frac{1}{2} \sqrt{\frac{v_0 S(m_0)}{a_0}} - \frac{1}{\kappa_0} = 0 \quad (1)
\]

\[
i_0 : \frac{\rho}{2} \sqrt{\frac{a_0}{v_0 S(m_0)}} - 1 + \lambda_i = 0 \quad (2)
\]

This implies,

\[
a_0 = \frac{\kappa_0^2}{4} v_0 S(m_0)
\]
\[ \lambda_i = 1 - \frac{\rho \kappa_0}{4} \]

The necessary and sufficient conditions for this case to hold are,

\[
\begin{align*}
\lambda_{BC} &= 0 \iff \frac{v_0 S(m_0)}{v_0 - m_0} < \frac{4}{\kappa_0} \quad \text{(from the BC not binding)} \\
\lambda_{DC} &= 0 \iff 1 > \frac{\kappa_0^2}{4} \iff \kappa_0 < 2 \quad \text{(from the DC not binding)} \\
\lambda_i &> 0 \iff 1 - \frac{\rho \kappa_0}{4} > 0 \iff \rho < \frac{4}{\kappa_0} \quad \text{(from the FOC for } i_0) 
\end{align*}
\]

The first inequality holds for values of \( m_0 \) low enough given \( \kappa_0 < 2 \), and the second and third hold by assumption 1.

2. Case \( \lambda_{BC} > 0 \) (BC binds), \( \lambda_{DC} = 0 \) (DC does not bind, conflict), and \( \lambda_i > 0 \) \( (i_0 = 0) \)

The FOCs are,

\[
\begin{align*}
a_0 : & \quad 1 \sqrt{\frac{v_0 S(m_0)}{a_0}} - \frac{1}{\kappa_0} - \frac{\lambda_{BC}}{\kappa_0} = 0 \quad (3) \\
\end{align*}
\]

\[
\begin{align*}
i_0 &= \frac{\rho}{2} \sqrt{\frac{a_0}{v_0 S(m_0)}} - 1 - \lambda_{BC} + \lambda_i = 0 \quad (4) 
\end{align*}
\]

\( \lambda_{BC} > 0 \) implies \( \kappa_0 (v_0 - m_0) = a_0 \).

From the FOCs, we obtain,

\[
\begin{align*}
\lambda_{BC} &= \frac{\kappa_0}{2} \sqrt{\frac{v_0 S(m_0)}{a_0}} - 1 \\
\lambda_i &= \frac{\kappa_0}{2} \sqrt{\frac{v_0 S(m_0)}{a_0}} - \frac{\rho}{2} \sqrt{\frac{a_0}{v_0 S(m_0)}}.
\end{align*}
\]
The parameter conditions for this to be a solution are,

\begin{align*}
\lambda_{BC} &= \frac{\kappa_0}{2} \sqrt{\frac{v_0S(m_0)}{a_0}} - 1 > 0 \iff \frac{v_0S(m_0)}{v_0 - m_0} > \frac{4}{\kappa_0} \\
\lambda_{DC} &= 0 \iff \frac{v_0S(m_0)}{\kappa_0(v_0 - m_0)} > 1 \\
\lambda_i &= \frac{\kappa_0}{2} \sqrt{\frac{v_0S(m_0)}{\kappa_0(v_0 - m_0)}} - \frac{\rho}{2} \sqrt{\frac{\kappa_0(v_0 - m_0)}{v_0S(m_0)}} > 0 \iff \frac{v_0S(m_0)}{\rho(v_0 - m_0)} > 1.
\end{align*}

The inequalities \( \frac{v_0S(m_0)}{\kappa_0(v_0 - m_0)} > 1 \) and \( \frac{v_0S(m_0)}{\rho(v_0 - m_0)} > 1 \) are both implied by the condition \( \frac{v_0S(m_0)}{v_0 - m_0} > \frac{4}{\kappa_0} \).

We now cover the cases that are inconsistent with any set of parameter values or consistent with only a non-generic set.

3. Case \( \lambda_{BC} > 0 \) (BC binds), \( \lambda_{DC} > 0 \) (DC binds), and \( \lambda_i = 0 \)

If \( \lambda_{DC} > 0 \) then

\[(v_0 + \rho i_0)S(m_0) = a_0,\]

FOC

\begin{align*}
\frac{\kappa_0}{2} - 1 - \lambda_{BC} - \lambda_{DC} \kappa_0 &= 0 \quad (5) \\
\frac{\rho}{2} - 1 - \lambda_{BC} + \lambda_{DC} \rho S(m_0) &= 0 \quad (6)
\end{align*}

If \( \lambda_{BC} > 0 \) then

\[v_0 - m_0 - \frac{a_0}{\kappa_0} - i_0 = 0\]
We have four equations and four unknowns. Therefore

\[
\lambda_{DC} = \frac{1}{2} \left( \frac{\kappa_0 - \rho}{\kappa_0 + \rho S(m_0)} \right)
\]

\[
\lambda_{BC} = \frac{\rho}{2} \left( \frac{\kappa_0 + \kappa_0 S(m_0)}{\kappa_0 + \rho S(m_0)} \right) - 1
\]

\[
i_0 = \frac{v_0 \kappa_0 - m_0 \kappa_0 - v_0 S(m_0)}{\kappa_0 + \rho S(m_0)}
\]

\[
a_0 = \left( \frac{v_0 \kappa_0 (1 + \rho) - \rho m_0 \kappa_0}{\kappa_0 + \rho S(m_0)} \right) S(m_0)
\]

We need to establish the conditions on the parameters and \(m_0\) such that those parameters support this equilibrium.

\[
\lambda_{DC} = \frac{1}{2} \left( \frac{\kappa_0 - \rho}{\kappa_0 + \rho S(m_0)} \right) > 0 \iff \kappa_0 > \rho
\]

\[
\lambda_{BC} = \frac{\rho \kappa_0}{2} \left( \frac{1 + S(m_0)}{\kappa_0 + \rho S(m_0)} \right) - 1 > 0 \iff \rho > \frac{2 \kappa_0}{\kappa_0 (1 + S(m_0)) - 2 S(m_0)}
\]

\[
\iff S(m_0) \left( \frac{\kappa_0}{2} - 1 \right) > \frac{\kappa_0}{\rho} - \frac{\kappa_0}{2}.
\]

The LHS of this inequality is positive since from above \(\lambda_{DC} > 0 \iff \kappa_0 > \rho\) and by assumption \(\kappa_0 < 2\). This last assumption also implies the RHS is negative, so the inequality can never hold and this case can never occur.

**4. Case** \(\lambda_{BC} > 0\) (BC binds), \(\lambda_{DC} = 0\) (DC does not bind, conflict), and \(\lambda_i = 0\) \((i_0 > 0)\)
FOC

\[ \frac{\kappa_0}{2} \sqrt{\frac{(v_0 + \rho i_0) S(m_0)}{a_0}} - 1 - \lambda_{BC} = 0 \]

\[ \frac{\rho}{2} \sqrt{\frac{a_0}{(v_0 + \rho i_0) S(m_0)}} - 1 - \lambda_{BC} = 0 \]

If \( \lambda_{BC} > 0 \) then

\[ v_0 - m_0 - \frac{a_0}{\kappa_0} = i_0 \]

Equating the two FOCs after solving for \( \lambda_{BC} \) yields

\[ \frac{\kappa_0}{\rho} \left( \frac{v_0 + \rho (v_0 - m_0)}{1+S(m_0)} \right) S(m_0) = a_0, \]

and using this into the investment equation above we get

\[ i_0 = \frac{v_0 \left(1 - \frac{S(m_0)}{\rho} \right)}{1+S(m_0)} \]

which then yields

\[ a_0 = \frac{\kappa_0}{\rho} \left( \frac{v_0 + \rho (v_0 - m_0)}{1+S(m_0)} \right) S(m_0) > 0. \]

Using these expressions for \( a_0 \) and \( i_0 \) we can write,

\[ \lambda_{BC} = \frac{\kappa_0}{2} \sqrt{\frac{\left( \frac{v_0 + \rho (v_0 - m_0)}{1+S(m_0)} \right) S(m_0)}{\frac{\kappa_0}{\rho} \left( \frac{v_0 + \rho (v_0 - m_0)}{1+S(m_0)} \right) S(m_0)}} - 1 = \frac{1}{2} \sqrt{\frac{\kappa_0}{\rho}} - 1. \]

Since \( \lambda_{BC} > 0 \), it follows that \( \sqrt{\frac{\kappa_0}{\rho}} > 2 \), or \( \kappa_0 \rho > 4 \), which violates the assumption placing the polity in R3.

**5. Case \( \lambda_{BC} = 0 \) (BC does not bind), \( \lambda_{DC} = 0 \) (DC does not bind), and \( \lambda_i = 0 \) (\( i_0 > 0 \))**

The FOCs are,

\[ a_0 : \frac{\kappa_0}{2} \sqrt{\frac{(v_0 + \rho i_0) S(m_0)}{a_0}} - 1 = 0 \] (7)

\[ i_0 : \frac{\rho}{2} \sqrt{\frac{a_0}{(v_0 + \rho i_0) S(m_0)}} - 1 = 0. \] (8)
The first FOC yields $\sqrt{\frac{(v_0 + \rho_0)S(m_0)}{a_0}} = \frac{2}{\kappa_0}$, and substituting into the second FOC we get

$$\rho \kappa_0 = 4$$

which violates the assumption $\rho \kappa_0 < 4$, making this an infeasible case.

6. Case $\lambda_{BC} = 0$ (BC does not bind, $\lambda_{DC} > 0$ (DC binds, deterrence), and $\lambda_i = 0$ ($i_0 > 0$)

The FOCs are,

$$a_0 \ : \ 1 - \frac{1}{\kappa_0} - \lambda_{DC} = 0$$
$$i_0 \ : \ \frac{\rho}{2} - 1 + \lambda_{DC} \rho S(m_0) = 0.$$

From the first FOC, $\lambda_{DC} = \frac{1}{2} - \frac{1}{\kappa_0}$ and substituting into the second FOC we get,

$$S(m_0) = \frac{\frac{\rho}{2} - \frac{1}{\kappa_0}}{\frac{1}{2} - \frac{1}{\kappa_0}}.$$

Now we show this cannot hold. $S(m_0)$ is increasing in $m_0$. The lowest it can be is when staying in R3, yielding $S(m_0) = 1 + \frac{\kappa_0}{4}$. We show that $\frac{\frac{\rho}{2} - \frac{1}{\kappa_0}}{\frac{1}{2} - \frac{1}{\kappa_0}} < 1 + \frac{\kappa_0}{4}$, implying the equality $S(m_0) = \frac{\frac{\rho}{2} - \frac{1}{\kappa_0}}{\frac{1}{2} - \frac{1}{\kappa_0}}$ can never hold. This will require $\frac{\frac{\rho}{2} - \frac{1}{\kappa_0}}{\frac{1}{2} - \frac{1}{\kappa_0}} = \kappa_0 \left(1 - \frac{1}{\rho}\right) < 1 + \frac{\kappa_0}{4}$, or $\frac{3\kappa_0}{4} - 1 < \frac{\kappa_0}{\rho}$. If $\frac{3\kappa_0}{4} - 1 < 0$, then the inequality $\frac{3\kappa_0}{4} - 1 < \frac{\kappa_0}{\rho}$ must always hold, making the equality $S(m_0) = \frac{\frac{\rho}{2} - \frac{1}{\kappa_0}}{\frac{1}{2} - \frac{1}{\kappa_0}}$ impossible. If $\frac{3\kappa_0}{4} - 1 > 0$, then we need $\rho < \frac{\kappa_0}{\frac{3\kappa_0}{4} - 1}$. The RHS of this last inequality is decreasing in $\kappa_0$, hence it attains its lowest value at the highest permissible value of $\kappa_0$ keeping $\frac{3\kappa_0}{4} - 1 > 0$. This value is 2. Substituting that value into the RHS of $\rho < \frac{\kappa_0}{\frac{3\kappa_0}{4} - 1}$ we get $\rho < \frac{2}{\frac{3}{2} - 1} = 4$. Now because in this case $\kappa_0 = 2$ and by assumption $\rho \kappa_0 < 4$, then $\rho < 2$, guaranteeing that $\rho < 4$ and the equality $S(m_0) = \frac{\frac{\rho}{2} - \frac{1}{\kappa_0}}{\frac{1}{2} - \frac{1}{\kappa_0}}$ is impossible, rendering
this case infeasible.

7. Case $\lambda_{BC} > 0$ (BC binds), $\lambda_{DC} > 0$ (DC binds, deterrence), and $\lambda_i > 0$ ($i_0 = 0$)

The FOCs are,

$$a_0 : \frac{1}{2} - \frac{1}{\kappa_0} - \frac{\lambda_{BC}}{\kappa_0} - \lambda_{DC} = 0 \quad (9)$$

$$i_0 : \frac{\rho}{2} - 1 - \lambda_{BC} + \lambda_{DC}\rho S(m_0) = 0 \quad (10)$$

$$v_0 - m_0 - \frac{a_0}{\kappa_0} = 0, \quad v_0 S(m_0) - a_0 = 0. \quad (11)$$

Using $\lambda_{DC} = \frac{1}{2} - \frac{1}{\kappa_0} - \frac{\lambda_{BC}}{\kappa_0}$ into $\lambda_{BC} = \frac{\rho}{2} - 1 + \lambda_{DC}\rho S(m_0)$, we obtain $\lambda_{BC} = \frac{\rho}{2} - 1 + \left(\frac{1}{2} - \frac{1}{\kappa_0}\right)\rho S(m_0)$. For $\lambda_{BC} > 0$ we need $\frac{\rho}{2} - 1 + \left(\frac{1}{2} - \frac{1}{\kappa_0}\right)\rho S(m_0) > 0$ or,

$$\frac{1}{2} - \frac{1}{\kappa_0} > S(m_0),$$

which can never happen. We established that $S(m_0) > \frac{\rho - \frac{1}{2}}{\frac{1}{2} - \frac{1}{\kappa_0}}$ when analyzing the previous case, and since $\frac{1}{2} - \frac{1}{\kappa_0} = \frac{\rho - \frac{1}{2}}{\frac{1}{2} - \frac{1}{\kappa_0}}$, the inequality can never hold.

8. Case $\lambda_{BC} = 0$ (BC does not bind), $\lambda_{DC} > 0$ (DC binds, deterrence), and $\lambda_i > 0$ ($i_0 = 0$)

$$a_0 : \frac{1}{2} - \frac{1}{\kappa_0} - \lambda_{DC} = 0 \quad (12)$$

$$i_0 : \frac{\rho}{2} - 1 + \lambda_{DC}\rho S(m_0) + \lambda_i = 0 \quad (13)$$
\[ v_0 - m_0 - \frac{a_0}{\kappa_0} > 0, \quad v_0S(m_0) = a_0 \] (14)

From the first FOC,

\[ \lambda_{DC} = \frac{1}{2} - \frac{1}{\kappa_0} > 0 \]

which cannot happen because it requires \( \kappa_0 > 2 \), which violates the assumption \( \kappa_0 < 2 \).

\[ \blacksquare \]

Lemma 1 reveals that no productive investment occurs in period 0 and that the arming effort depends on the value of \( m_0 \) through its impact on the continuation payoff. With this result, we are now equipped to study the incentives of the incumbent to make changes in defense capability. We can trace how those changes will affect period 0 army decisions for both incumbent and challenger, victory probabilities, and subsequent future investment, army sizes, and security and prosperity outcomes. The way to analyze whether the incumbent is interested in raising \( m_0 \) to exit R3 is to analyze the expected utility of doing so. This requires utilizing the function \( S(m_0) \) that corresponds to the parametric region where the polity will land in period 1. However, matters are complicated by the presence of two potential cases in period 0 as per Lemma 1, depending on whether \( \frac{v_0S(m_0)}{v_0-m_0} < \frac{4}{\kappa_0} \), or \( \frac{v_0S(m_0)}{v_0-m_0} > \frac{4}{\kappa_0} \). These inequalities show that which case should be considered to be in play in period 0—which will affect incentives to raise \( m_0 \)—depends on \( S(m_0) \), which depends on what parametric region the polity will land on in period 1, which in turn depends on whether the incentives are present to make the necessary investments in the first place. Therefore this proof proceeds by checking which combinations of cases in period 0 can obtain for the different possible plans to expand defense and land in each of the possible alternative parametric regions in period 1.
The first step to this analysis is to compute the expected utility in period \( t = 0 \) for each \( m_0 \) fixing all the other parameters for the two cases highlighted in Lemma 1:

1. **Case** \( \lambda_{BC} = 0 \) (**BC not binding**), \( \lambda_{ND} = 0 \) (**DC not binding, conflict**), and \( \lambda_i > 0 \) \((i_0 = 0)\)

   The proof to Lemma 1 showed that in this case the Lagrange multiplier conditions defining the case respectively imply \( \frac{v_0 S(m_0)}{v_0 - m_0} < \frac{4}{\kappa_0} \), \( \kappa_0 < 2 \), and \( \rho < \frac{4}{\kappa_0} \), and expected utility is,

\[
EU = v_0 - m_0 + \frac{\kappa_0}{4} v_0 S(m_0).
\]

2. **Case** \( \lambda_{BC} > 0 \) (**BC binds**), \( \lambda_{ND} = 0 \) (**DC not binding, conflict**) and \( \lambda_i > 0 \) \((i_0 = 0)\)

   The Lagrange multiplier conditions imply, \( \frac{v_0 S(m_0)}{v_0 - m_0} > \frac{4}{\kappa_0} \), \( \frac{v_0 S(m_0)}{v_0 - m_0} > \kappa_0 \), \( \frac{v_0 S(m_0)}{v_0 - m_0} > \rho \), and expected utility is,

\[
EU = \sqrt{\frac{\kappa_0}{4} (v_0 - m_0) v_0 S(m_0)}.
\]

The second step is to note that there are critical values of investment in military capacity that shift the regimes the polity is in both in period 0 and period 1. Denote with \( \bar{m} \) the value of \( m_0 \) that satisfies \( \frac{v_0 S(m_0)}{v_0 - m_0} = \frac{4}{\kappa_0} \) and which makes the polity switch from case 1 to case 2 in Lemma 1 in period 0.

**Part (a) \((\rho < 2)\):**

Denote with \( m_{R3|R4} \) and \( m_{R4|R1} \) the values of \( m_0 \) such that regimes change in period 1 from R3 to R4 and from R4 to R1 respectively: \( m_{R3|R4} = \frac{2 - \kappa_0}{\gamma} \) and \( m_{R4|R1} = \frac{1}{\gamma} \left( \frac{\rho}{\rho - 1} - \kappa_0 \right) \), \( m_{R3|R4} < m_{R4|R1} \). Because \( \bar{m} \) is an implicit function of \( S(.) \), we need to compute the conditions on the parameters when \( \bar{m} \) lies below and above \( m_{R3|R4} \) and above \( m_{R4|R1} \). The reason it is important to know where \( \bar{m} \) lies relative to \( m_{R3|R4} \) and \( m_{R4|R1} \) is that it will
indicate which expected utility expression to use to evaluate choices of \(m_0\). If, for example, \(\bar{m} > m_{R4|R1}\) then we know the payoff from choosing an \(m_0\) that keeps the polity in \(R3\), moves it to \(R4\) or an early part of \(R1\) can be evaluated with a single expected utility expression, namely that in case 1 from Lemma 1.

Before proving part (a) of Proposition 3 we need a technical result. The following lemma establishes the conditions of the parameters that determine the value of \(\bar{m}\) relative to \(m_{R3|R4}\) and \(m_{R4|R1}\).

**Lemma 2** Under assumption 1,

i) If \(0 < \gamma v_0 < \frac{8(2-\kappa_0)}{8-3\kappa_0}\), then \(\bar{m} < m_{R3|R4}\).

ii) If \(\frac{8(2-\kappa_0)}{8-3\kappa_0} < \gamma v_0 < \frac{4}{\kappa_0} \left(\frac{\rho-1-\kappa_0}{(1+\rho)}\right)\), then \(m_{R3|R4} < \bar{m} < m_{R4|R1}\).

iii) If \(\frac{4}{\kappa_0} \left(\frac{\rho-1-\kappa_0}{(1+\rho)}\right) < \gamma v_0\), then \(m_{R4|R1} < \bar{m}\).

**Proof of lemma 2:** To determine whether \(\bar{m}\) lies within \([0, m_{R3|R4}], [m_{R3|R4}, m_{R4|R1}]\) or \([m_{R4|R1}, \infty]\) first notice that \(\frac{v_0 S(m_0)}{v_0 - m_0}\) is increasing in \(m_0\). Therefore, the conditions on the parameters for each of these cases to hold are:

For \(\bar{m} < m_{R3|R4}\) This is the case if \(\frac{v_0 S(m_{R3|R4})}{v_0 - m_{R3|R4}} > \frac{4}{\kappa_0} \iff v_0\gamma < \frac{8(2-\kappa_0)}{8-3\kappa_0}\).

For \(m_{R3|R4} < \bar{m} < m_{R4|R1}\) From above \(m_{R3|R4} < \bar{m} \iff \frac{8(2-\kappa_0)}{8-3\kappa_0} < v_0\gamma\). Now we need to find the condition for \(\bar{m} < m_{R4|R1}\). This requires \(\frac{v_0 S(m_{R4|R1})}{v_0 - m_{R4|R1}} > \frac{4}{\kappa_0}\), and this follows iff \(v_0\gamma < \frac{\frac{4}{\kappa_0} \left(\frac{\rho-1-\kappa_0}{(1+\rho)}\right)}{\kappa_0}\). Therefore, this case occurs when

\[
\frac{8(2-\kappa_0)}{8-3\kappa_0} < v_0\gamma < \frac{4}{\kappa_0} \left(\frac{\rho-1-\kappa_0}{(1+\rho)}\right)
\]

For \(m_{R4|R1} < \bar{m}\) It follows directly from before
\[
\frac{4}{\kappa_0} \left( \frac{\rho}{\rho - 1} - \kappa_0 \right) < v_0 \gamma
\]

Due to this lemma 2, we know how to write expected utility depending on the value of \( \gamma v_0 \), given all other parameters.

**Part (a)1:** We only need to find a cutoff in the space of possible values for \( \gamma v_0 \) such that the incumbent prefers to stay in \( R_3 \). We propose \( \tau_L \equiv \frac{8}{8-3\kappa_0} (2 - \kappa_0) \).

In this case, \( v_0 \gamma < \frac{8(2-\kappa_0)}{8-3\kappa_0} \) is equivalent to a regime described by \( \bar{m} < m_{R3|R4} \), which means we have to use two different EU expressions depending on whether \( m_0 < \bar{m} \), or \( m_0 > \bar{m} \). Let us analyze the expected utility in each of these situations. A useful fact will be that \( \frac{8(2-\kappa_0)}{8-3\kappa_0} \) is strictly decreasing in \( \kappa_0 \) so its maximum value is at \( \kappa_0 = 1 \) (since \( \kappa_0 > \kappa_c = 1 \)). In this case \( \frac{8(2-1)}{8-3\times1} = \frac{8}{5} < 2 \).

**Segment \([0, \bar{m}]\)** Expected utility in period \( t = 0 \) is

\[
EU = v_0 - m_0 + \frac{\kappa_0}{4} v_0 S(m_0) = v_0 - m_0 + \frac{\kappa_0}{4} v_0 (1 + \frac{\kappa_0 + \gamma m_0}{4}) = v_0(1 + \frac{\kappa_0}{4} + \left( \frac{\kappa_0}{4} \right)^2) + m_0 \left( \frac{\kappa_0 \gamma}{16} \right) - 1
\]

Since \( \bar{m} < m_{R3|R4} \iff v_0 \gamma < \frac{8(2-\kappa_0)}{8-3\kappa_0} \) then \( \frac{\kappa_0 \gamma}{16} - 1 < 0 \). To see why, replace \( v_0 \gamma = \frac{8(2-\kappa_0)}{8-3\kappa_0} \) in \( \frac{\kappa_0 \gamma}{16} \) so \( \frac{\kappa_0 \gamma}{16} = \frac{\kappa_0 \gamma}{16} \leq \frac{\kappa_0 \gamma}{16} < 1 \). This implies EU is decreasing in \( m_0 \) and the optimal choice is \( m_0 = 0 \).

**Segment \([\bar{m}, m_{R3|R4}]\)** Expected utility in period \( t = 0 \) is

\[
EU = \sqrt{\kappa_0 (v_0 - m_0) v_0 S(m_0)} = \sqrt{\kappa_0 (v_0 - m_0) v_0 (1 + \frac{\kappa_0 + \gamma m_0}{4})}, \text{ and we now show this to decrease in } m_0. \text{ Note,}
\]

\[
\frac{dEU}{dm} = \frac{1}{2} \left[ \kappa_0 (v_0 - m_0) v_0 (1 + \frac{\kappa_0 + \gamma m_0}{4}) \right]^{-\frac{1}{2}} \left( -\kappa_0 v_0 (1 + \frac{\kappa_0 + \gamma m_0}{4}) + \gamma v_0 (v_0 - m_0) v_0 \right) \text{ and,}
\]

\[
\frac{dEU}{dm} < 0 \iff \frac{\gamma}{4} \kappa_0 (v_0 - m_0) v_0 < \kappa_0 v_0 (1 + \frac{\kappa_0 + \gamma m_0}{4}), \text{ or, iff } \gamma v_0 < 4 + \kappa_0 + 2\gamma m_0.
If $4 + \kappa_0 + 2\gamma m_0$ is higher than $\frac{8(2-\kappa_0)}{8-3\kappa_0}$, our condition to be in this scenario $\bar{m} < m_{R3|R4}$ ($\iff v_0\gamma < \frac{8(2-\kappa_0)}{8-3\kappa_0}$) is a sufficient condition for $EU$ in this segment to be decreasing. So, it is sufficient to show that $4 + \kappa_0 + 2\gamma m_0 > \frac{8(2-\kappa_0)}{8-3\kappa_0}$. Because the right hand side is decreasing in $\kappa_0$, it attains a maximum at $\kappa_0 = 1$ and it is equal to $8/5$ which is smaller than any feasible value of the expression in the left hand side, which is at least $4$. Therefore, in this segment utility is maximized at $m_0 = \bar{m}$, and equals $EU = \sqrt{\kappa_0 (v_0 - m_0) v_0 S(m_0)} = 4(\bar{v} - \bar{m})^2 = 2(\bar{v} - \bar{m})$.

Segment $[m_{R3|R4}, m_{R4|R1}]$ Since now $m_0$ can only be larger than $\bar{m}$, we know expected utility is $\sqrt{\kappa_0 (v_0 - m_0) v_0 S(m_0)}$. In $R4$, we have $EU = \sqrt{\kappa_0 (v_0 - m_0) v_0 \left(2 - \frac{1}{\kappa_0 + \gamma m_0}\right)}$. Computing the first derivative with respect to $m_0$, we get,

$$\frac{dEU}{dm_0} = \left(\kappa_0 (v_0 - m_0) v_0 \left(2 - \frac{1}{\kappa_0 + \gamma m_0}\right)\right)^{\frac{1}{2}} \left[ -\kappa_0 v_0 \left(2 - \frac{1}{\kappa_0 + \gamma m_0}\right) + \kappa_0 \frac{(v_0 - m_0) v_0 \gamma}{(\kappa_0 + \gamma m_0)^2}\right]$$

which is negative whenever $2 \left(\kappa_0 + \gamma m_0\right) - 1 > \frac{(v_0 - m_0) \gamma}{\kappa_0 + \gamma m_0}$, or, $2 \left(\kappa_0 + \gamma m_0\right)^2 - \kappa_0 > v_0 \gamma$. Note $2 \left(\kappa_0 + \gamma m_{R3|R4}\right)^2 - \kappa_0 = 8 - \kappa_0$. Now note $8 - \kappa_0 > \frac{8(2-\kappa_0)}{8-3\kappa_0} \equiv \tau_L$, since the LHS is at least $6$ and the RHS is at most $\frac{8}{5}$. Thus, $\frac{dEU}{dm_0} < 0$ and utility would be maximized at $m_{R3|R4}$ in this segment.

Segment $[m_{R4|R1}, \infty]$ Here, $EU = \sqrt{\kappa_0 (v_0 - m_0) v_0 \frac{(\kappa_0 + \gamma m_0)(1+\rho)}{\kappa_0 + \gamma m_0 + \rho}}$ and

$$\frac{dEU}{dm_0} = \left(\kappa_0 (v_0 - m_0) v_0 \frac{(\kappa_0 + \gamma m_0)(1+\rho)}{\kappa_0 + \gamma m_0 + \rho}\right)^{\frac{1}{2}} \left[ -\kappa_0 v_0 \frac{(\kappa_0 + \gamma m_0)(1+\rho)}{\kappa_0 + \gamma m_0 + \rho} \right. + \kappa_0 \left( v_0 - m_0 \right) v_0 \gamma \frac{(1+\rho)(\kappa_0 + \gamma m_0 + \rho) - (\kappa_0 + \gamma m_0)(1+\rho) \gamma}{(\kappa_0 + \gamma m_0 + \rho)^2} \right].$$

Note $\frac{dEU}{dm_0} < 0$ whenever $\gamma v_0 < \frac{(\kappa_0 + \gamma m_0 + \rho)(\kappa_0 + \gamma m_0)}{\rho} + \gamma m_0$. The right hand side of this expression is increasing in $m_0$, so the minimum is attained at $m_0 = m_{R4|R1}$ and it equals $\frac{\rho}{(\rho - 1)^2} + 2 \frac{\rho}{(\rho - 1)} - \kappa_0$. The highest possible value of $\gamma v_0$, $\tau_L = \frac{8(2-\kappa_0)}{8-3\kappa_0}$ is smaller than $\frac{8}{5}$ which, in turn, is always smaller than $\frac{\rho}{(\rho - 1)^2} + 2 \frac{\rho}{(\rho - 1)} - \kappa_0$ given that $\rho < 2$. Therefore the maximum
of $EU$ in this segment is attained at $m_0 = m_{R4|R1}$.

Considering all of the segments together, we now show that the global maximum in this case is $m_0 = 0$. This follows from the just demonstrated fact that the maximum within each segment of the support is at the minimum value, and from the fact that $EU$ is continuous. $S(.)$ is continuous for all $m_0$ and $EU$ in period $t = 0$ is also continuous at $\bar{m}$. In $t = 0$, in segment $[0, \bar{m}]$ $EU$ evaluated at $\bar{m}$ is $2(v_0 - \bar{m})$ which is equal to the $EU$ in segment $[\bar{m}, m_{R3|R4}]$ evaluated at $\bar{m}$. This can be shown noticing that $\frac{\kappa_0 v_0 S(\bar{m})}{4} = v_0 - \bar{m}$, and reeplacing in $EU$ in segment $[0, \bar{m}]$. Thus, the polity will stay at $R3$ in period 1.

Part (a)(2-3) For these parts of Proposition 3 (the existence of cutoffs such that the polity will move away from $R3$ in period 1), we consider the case in which $\frac{4}{\kappa_0} \frac{(\rho - 1 - \kappa_0)}{\rho - 1} < v_0 \gamma$ (as we only need to focus on a sufficient condition for the polity to exit $R3$ and move respectively into $R4$ or $R1$). In this case $m_{R4|R1} < \bar{m}$ by Lemma 2. We proceed by analyzing the optimal decision of $m_0$ under the different segments of the domain of $m_0$:

**Segment** $[0, m_{R3|R4}]$ Expected utility in period $t = 0$ is

$$EU = v_0 - m_0 + \frac{\kappa_0}{4} v_0 S(m_0) = v_0 - m_0 + \frac{\kappa_0}{4} v_0 \left(1 + \frac{\kappa_0 + \gamma m_0}{4}\right) = v_0 \left(1 + \frac{\kappa_0}{4} + \left(\frac{\kappa_0}{4}\right)^2\right) + m_0 \left(\frac{\kappa_0 \gamma}{16} - 1\right)$$. In this case, $\frac{4}{\kappa_0} \frac{(\rho - 1 - \kappa_0)}{\rho} > \frac{16}{\kappa_0}$ does not always hold under Assumption 1, so marginal utility is not necessarily positive. Therefore, a sufficient condition for the polity to exit $R3$ is that $\gamma v_0$ be higher than $\tau_M \equiv \max \left\{ \frac{4}{\kappa_0} \frac{(\rho - 1 - \kappa_0)}{\rho}, \frac{16}{\kappa_0} \right\}$. Next, we show that it may either stay in $R4$ or move to $R1$.

**Segment** $[m_{R3|R4}, m_{R4|R1}]$ Expected utility in period $t = 0$ is $EU = v_0 - m_0 + \frac{\kappa_0}{4} v_0 S(m_0) = v_0 - m_0 + \frac{\kappa_0}{4} v_0 \left(2 - \frac{1}{\kappa_0 + \gamma m_0}\right)$ (recall that since $\frac{4}{\kappa_0} \frac{(\rho - 1 - \kappa_0)}{\rho} < v_0 \gamma$, $\tilde{m} > m_{R4|R1}$, so in the interval $[m_{R3|R4}, m_{R4|R1}]$ we are considering $m_0 < \tilde{m}$).

The marginal utility of $m_0$ is: $-1 + \frac{\kappa_0 v_0}{4} \frac{\gamma}{(\kappa_0 + \gamma m_0)^2}$. The value of $m_0$ that maximizes $EU$
is $m_0 = \frac{1}{\gamma} \left( \sqrt{\frac{\kappa_0 \gamma v_0}{4}} - \kappa_0 \right)$. For the optimum to fall in the segment $[m_{R3|R4}, m_{R4|R1}]$ we need, in addition to $\gamma v_0 > \tau_M$, that,

$$m_{R3|R4} < \frac{1}{\gamma} \left( \sqrt{\frac{\kappa_0 \gamma v_0}{4}} - \kappa_0 \right) < m_{R4|R1},$$

The first inequality requires $4 \left( \frac{2 - \kappa_0}{\kappa_0} \right)^2 < v_0 \gamma$. Since $\tau_M \equiv \max \left\{ \frac{4}{\kappa_0} \left( \frac{\rho - 1 - \kappa_0}{\rho} \right), \frac{16}{\kappa_0} \right\}$ and $\frac{16}{\kappa_0} > 4 \left( \frac{2 - \kappa_0}{\kappa_0} \right)^2$, a sufficient condition for the first inequality is that $\gamma v_0 > \tau_M$. The second inequality requires $v_0 \gamma < \frac{4}{\kappa_0} \left( \frac{\rho - 1}{\rho} \right)^2$. Note that $\frac{4}{\kappa_0} \left( \frac{\rho - 1}{\rho} \right)^2 > \tau_M$ (because simple algebra shows that $\frac{4}{\kappa_0} \left( \frac{\rho - 1 - \kappa_0}{\rho} \right)^2 > \frac{16}{\kappa_0} \left( \frac{\rho - 1}{\rho} \right)^2$). Defining $\tau_H \equiv \left( \frac{\rho}{\rho - 1} \right)^2 \frac{4}{\kappa_0}$, we conclude that whenever $\tau_M < \gamma v_0 < \tau_H$ the policy reaches $R4$. If $\tau_H < \gamma v_0$ then the policy must reach $R1$. Now we explore if it is possible to move into the interior of $R1$.

**Segment** $[m_{R4|R1}, \bar{m}]$ Expected utility is $EU = v_0 - m_0 + \frac{\kappa_0}{4} v_0 S(m_0) = v_0 - m_0 + \frac{\kappa_0}{4} v_0 \left( \frac{(\kappa_0 + \gamma m_0)(1 + \rho)}{\kappa_0 + \gamma m_0 + \rho} \right)$. This is a concave function with a maximum at $m_0 = \frac{1}{\gamma} \left( \sqrt{\frac{\kappa_0 \gamma v_0}{4}} - \kappa_0 - \rho \right)$. Note if $\gamma v_0 > \tau_H = \left( \frac{\rho}{\rho - 1} \right)^2 \frac{4}{\kappa_0}$ then that maximum is an interior optimum and larger than $R4|R1$. To see this, rewrite the inequality $m_0 > R4|R1$ as $\sqrt{\frac{\kappa_0 \gamma v_0}{4}} (1 + \rho) - \kappa_0 - \rho > \frac{\rho}{\rho - 1} - \kappa_0$ then plug $\tau_H$ into the LHSto obtain $1 + \rho > \rho$, which always holds. Therefore for $\gamma v_0 > \tau_H$ the policy will be in the interior of $R1$ in period 1.

**Part (b)** ($\rho \geq 2$): As in the proof of part (a), $\bar{m}$ satisfies the equation: $\frac{v_0 S(\bar{m})}{v_0 - \bar{m}} = \frac{1}{\gamma}$, and represents the value of $m_0$ at which the polity switches from case 1 to case 2 from Lemma 1 in period 0.

Let us call $m_{R3|R2} = \frac{1}{\gamma} \left( \frac{4}{\rho} - \kappa_0 \right)$ and $m_{R2|R1} = \frac{1}{\gamma} (\rho - \kappa_0)$ the values in which regimes change in period 1. The following lemma shows the conditions on the parameters such that for any given $m_0$ we can fully describe the $EU$ in period 0.
Lemma 3 Under assumption 1,

i) If \(0 < \gamma v_0 < \frac{16 - 4\kappa_0 \rho}{4\rho - (1 + \rho)\kappa_0}\) then \(\bar{m} < m_{R3|R2}\)

ii) If \(\frac{16 - 4\kappa_0 \rho}{4\rho - (1 + \rho)\kappa_0} < \gamma v_0 < \frac{8(\rho - \kappa_0)}{8 - (1 + \rho)\kappa_0}\) then \(m_{R3|R2} < \bar{m} < m_{R2|R1}\)

iii) If \(\frac{8(\rho - \kappa_0)}{8 - (1 + \rho)\kappa_0} < \gamma v_0\) then \(m_{R2|R1} < \bar{m}\)

Proof of lemma 3: It follows from replacing the definitions of \(\bar{m}, m_{R3|R2}, m_{R2|R1}\) and following steps analogous to Lemma 2, as follows. To determine whether \(\bar{m}\) lies within 
\([0, m_{R3|R2}], [m_{R3|R2}, m_{R2|R1}]\) or \([m_{R2|R1}, \infty]\) recall that \(\frac{v_0 S(m_0)}{v_0 - m_0}\) is increasing in \(m_0\). Therefore, the conditions on the parameters for each of these cases to hold are:

For \(\bar{m} < m_{R3|R2}\) This is the case when \(\frac{v_0 S(m_{R3|R2})}{v_0 - m_{R3|R2}} > \frac{4}{\kappa_0} \iff v_0 \gamma < \frac{16 - 4\kappa_0 \rho}{4\rho - (1 + \rho)\kappa_0}\).

For \(m_{R3|R2} < \bar{m} < m_{R2|R1}\) From above \(m_{R3|R2} < \bar{m} \iff \frac{16 - 4\kappa_0 \rho}{4\rho - (1 + \rho)\kappa_0} < v_0 \gamma\). Now we need to find the condition for \(\bar{m} < m_{R2|R1}\). This requires \(\frac{v_0 S(m_{R2|R1})}{v_0 - m_{R2|R1}} > \frac{4}{\kappa_0}\), or equivalently \(v_0 \gamma < \frac{8(\rho - \kappa_0)}{8 - (1 + \rho)\kappa_0}\). Therefore, this case occurs when

\[
\frac{16 - 4\kappa_0 \rho}{4\rho - (1 + \rho)\kappa_0} < v_0 \gamma < \frac{8(\rho - \kappa_0)}{8 - (1 + \rho)\kappa_0} \tag{15}
\]

For \(m_{R2|R1} < \bar{m}\) It follows directly from before

\[
\frac{8(\rho - \kappa_0)}{8 - (1 + \rho)\kappa_0} < v_0 \gamma.
\]

Due to this lemma 3, we know how to write expected utility depending on the value of \(\gamma v_0\), given all other parameters.

Part (b)1 We need to find a cutoff such that the polity stays in \(R3\). We propose \(\sigma_L \equiv \frac{16 - 4\kappa_0 \rho}{4\rho - (1 + \rho)\kappa_0}\). In this case, \(v_0 \gamma < \sigma_L\) is equivalent to a regime in which \(\bar{m} < m_{R3|R2}\) which means
we have to use two different EU expressions depending on whether $m_0 < \bar{m}$ or $m_0 > \bar{m}$.

Let us analyze the expected utility in each of these situations. A useful fact will be that $\sigma_L = \frac{16 - 4\kappa_0 \rho}{4\rho - (1 + \rho)\kappa_0}$ is decreasing in both $\rho$ and $\kappa_0$. Thus, its maximum value is at $\rho = 2$ and $\kappa_0 = 1$. Then $\sigma_L = \frac{16 - 4 \times 1 \times 2}{4 \times 2 - (1 + 2) \times 1} = \frac{8}{5}$.

**Segment $[0, \bar{m}]$** Expected utility in period 0 is

$$EU = v_0 - m_0 + \frac{\kappa_0}{4} v_0 S(m_0) = v_0 - m_0 + \frac{\kappa_0}{4} v_0 (1 + \frac{\kappa_0 + \gamma m_0}{4}) = v_0 (1 + \frac{\kappa_0}{4} + (\frac{\kappa_0}{4})^2) + m_0 (\frac{\kappa_0 \rho \gamma}{16} - 1).$$

Since $\bar{m} < m_{R3|R2} \iff v_0 \gamma < \sigma_L$ then it follows that $\frac{\kappa_0 \sigma_L}{16} > \frac{\kappa_0 v_0 \gamma}{16}$. Therefore, if $\frac{\kappa_0 \sigma_L}{16} < 1$, it must follow that $\frac{\kappa_0 \sigma_L}{16} - 1 < 0$, implying the optimal choice is $m_0 = 0$. To see that $\frac{\kappa_0 \sigma_L}{16} < 1$, note that $\sigma_L$ is at at most $\frac{8}{5}$ and $\kappa_0 < 2$.

**Segment $[\bar{m}, m_{R3|R2}]$** Expected utility in period 0 is

$$EU = \sqrt{\kappa_0 v_0 (v_0 - m_0)} (1 + \frac{\kappa_0 + \gamma m_0}{4}).$$  To show this payoff is decreasing in $m_0$ note that,

$$\frac{dEU}{dm_0} = (\kappa_0 v_0 (v_0 - m_0) (1 + \frac{\kappa_0 + \gamma m_0}{4}))^{-\frac{1}{2}} (-\kappa_0 v_0 (1 + \frac{\kappa_0 + \gamma m_0}{4}) + \frac{\gamma}{4} \kappa_0 v_0 (v_0 - m_0)), $$

which must be negative because $\frac{\gamma}{4} (v_0 - m_0) < 1 + \frac{\kappa_0 + \gamma m_0}{4} \iff \gamma v_0 < 4 + \kappa_0 + 2 \gamma m_0$ which must hold since $\gamma v_0 < \sigma_L < 4$. As a result, the maximum is attained at $m_0 = 0$ in the interval $[0, m_{R3|R2}]$.

**Segment $[m_{R3|R2}, m_{R2|R1}]$** Expected utility in period 0 is,

$$EU = \sqrt{\kappa_0 v_0 (v_0 - m_0)} \left( \sqrt{\frac{\kappa_0 + \gamma m_0}{\rho} \frac{(1 + \rho)}{2}} \right).$$  Marginal expected utility is,

$$\frac{dEU}{dm_0} = \frac{\sqrt{\kappa_0 v_0}}{2} \left( (v_0 - m_0) \left( \sqrt{\frac{\kappa_0 + \gamma m_0}{\rho} \frac{(1 + \rho)}{2}} \right) \right)^{-\frac{1}{2}} \left( - \left( \sqrt{\frac{\kappa_0 + \gamma m_0}{\rho} \frac{(1 + \rho)}{2}} \right) \right)$$

and this is negative whenever $(v_0 - m_0) \frac{1}{2} \left( \frac{\kappa_0 + \gamma m_0}{\rho} \right)^{-\frac{1}{2}} \frac{\gamma}{\rho} < \sqrt{\frac{\kappa_0 + \gamma m_0}{\rho}}$, or, $\gamma v_0 < 2 \kappa_0 + 3 \gamma m_0$. The RHS of this expression is higher than $\frac{8}{5}$, and since $\gamma v_0 < \sigma_L \leq \frac{8}{5}$, we must conclude that $\frac{dEU}{dm_0} < 0$. In this segment EU would be maximized at $m_0 = m_{R3|R2}$. 

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Segment \([m_{R2|R1}, \infty)\] Expected utility in period 0 is,

\[
EU = \sqrt{\kappa_0 v_0 (v_0 - m_0) \left( \frac{(\kappa_0 + \gamma m_0)(1+\rho)}{\kappa_0 + \gamma m_0 + \rho} \right)}.
\]

Marginal expected utility is,

\[
\frac{dEU}{dm_0} = \sqrt{\kappa_0 v_0 \left( \frac{(\kappa_0 + \gamma m_0)(1+\rho)}{\kappa_0 + \gamma m_0 + \rho} \right)} \cdot \left( \frac{\rho}{\kappa_0 + \gamma m_0 + \rho} \right) \cdot \left( v_0 - m_0 \right) (1 + \rho) \left( \frac{\rho}{(\kappa_0 + \gamma m_0 + \rho)^2} \right).
\]

This is negative whenever \((v_0 - m_0) \frac{\rho}{\kappa_0 + \gamma m_0 + \rho} < (\kappa_0 + \gamma m_0)\), or whenever,

\[
v_0 \gamma < \frac{1}{\rho} (\kappa_0 + \gamma m_0)^2 + (\kappa_0 + \gamma m_0) + \gamma m_0.
\]

The right hand side of this expression is increasing in \(m_0\), so the minimum of this expression is attained at \(m_0 = m_{R2|R1}\) and it equals \(3\rho - \kappa_0\). Since \(\gamma v_0 < \frac{8}{\kappa_0} < 3\rho - \kappa_0\) the result follows.

In sum, the global maximum when \(\bar{m} < m_{R3|R2}\) \((\iff v_0 \gamma < \frac{16 - 4\kappa_0 \rho}{1\rho - (1+\rho)\kappa_0} = \sigma_L)\) is \(m_0 = 0\). This follows from the just demonstrated fact that the maximum within each segment of the support is at the minimum point, and from the fact that \(EU\) is continuous.

\(S(.)\) is continuous for all \(m_0\) and hence \(EU\) in period 0 is also continuous at \(\bar{m}\). In period 0 in segment \([0, \bar{m}]\), \(EU\) evaluated at \(\bar{m}\) is \(2(v_0 - \bar{m})\) which is equal to \(EU\) in segment \([\bar{m}, m_{R3|R2}]\) evaluated at \(\bar{m}\). This can be shown noticing that \(\frac{\kappa_0 v_0 S(\bar{m})}{4} = v_0 - \bar{m}\), and substituting into the expression for \(EU\) in segment \([0, \bar{m}]\). Thus, the polity will stay at \(R3\) in period 1 whenever \(\gamma v_0 < \sigma_L\). Next, we show that there exist \(\sigma_H = 16/\kappa_0 > \sigma_M2\), such that the polity moves to \(R1\).

Part (b)(2) To prove the existence of values for \(\gamma v_0\) high enough that the polity will move away from \(R3\) into \(R1\), consider the case in which \(\frac{16}{\kappa_0} < \gamma v_0\). Note that \(\frac{8(\rho - \kappa_0)}{8 - (1+\rho)\kappa_0} < \frac{16}{\kappa_0}\), so \(m_{R2|R1} < \bar{m}\). We proceed by analyzing the optimal \(m_0\) in each segment in what follows.

Segment \([0, m_{R3|R2}]\] Expected utility in period \(t = 0\) is,

\[
EU = v_0 - m_0 + \frac{\kappa_0}{4} v_0 S(m_0) = v_0 - m_0 + \frac{\kappa_0}{4} v_0 (1 + \frac{\kappa_0 + \gamma m_0}{4}) = v_0 (1 + \frac{\kappa_0}{4} + \left( \frac{\kappa_0}{4} \right)^2) + m_0 \left( \frac{\kappa_0 v_0 \gamma}{16} - 1 \right).
\]

Marginal utility is positive iff \(\gamma v_0 > \frac{16}{\kappa_0}\), so the polity moves to \(R2\), or
advances to R1.

**Segment** \([m_{R3|R2}, m_{R2|R1}]\) The (concave) expected utility is, 
\[
EU = v_0 - m_0 + \frac{\kappa_0 v_0 (1 + \rho)}{8} \sqrt{\frac{\kappa_0 \gamma m_0}{\rho}}.
\]

The marginal utility of \(m_0\) is: 
\[
-1 + \frac{\kappa_0 v_0 (1 + \rho)}{8} \frac{1}{2} \sqrt{\frac{\rho}{\kappa_0 + \gamma m_0 \rho}}.
\]

This marginal utility may be either negative, zero or positive in \([m_{R3|R2}, m_{R2|R1}]\). Given concavity, it is negative iff the value at which the marginal utility is zero, 
\[
m_0 = \frac{\frac{\kappa_0 v_0 (1 + \rho)}{16} \sqrt{\frac{\rho}{\gamma}}}{\gamma} - \kappa_0,
\]

is smaller than \(m_{R3|R2}\):
\[
\left(\frac{4}{\rho} - \kappa_0\right) \gamma < m_{R3|R2} = \frac{\left(\frac{4}{\rho} - \kappa_0\right)}{\gamma}.
\]

\(\iff v_0 \gamma < \left(\frac{16}{\kappa_0}\right) \frac{2}{1 + \rho}\). In this case, the maximum \(EU\) in this segment is at \(m_0 = m_{R3|R2}\).

The solution is interior in \(R2\) if 
\[
m_0 = \frac{\kappa_0 v_0 (1 + \rho)}{16} \sqrt{\frac{\rho}{\gamma}} - \kappa_0
\]
is in between \(m_{R3|R2}\) and \(m_{R2|R1}\):
\[
\left(\frac{4}{\rho} - \kappa_0\right) \gamma = m_{R3|R2} < \frac{\kappa_0 v_0 (1 + \rho)}{16} \sqrt{\frac{\rho}{\gamma}} - \kappa_0 < m_{R2|R1} = \frac{\left(\frac{4}{\rho} - \kappa_0\right)}{\gamma}
\]
\(\iff \frac{16}{\kappa_0} \frac{2}{1 + \rho} < v_0 \gamma < \frac{16}{\kappa_0} \frac{\rho}{1 + \rho}\). But since \(\frac{16}{\kappa_0} \frac{\rho}{1 + \rho} < \frac{16}{\kappa_0} \frac{\rho}{1 + \rho}\), then \(v_0 \gamma > \frac{16}{\kappa_0} \frac{\rho}{1 + \rho}\) and the incumbent moves to \(R1\). The following analysis shows the incumbent moves to the interior of \(R1\) in this case.

**Segment** \([m_{R2|R1}, \bar{m}]\) Expected utility is 
\[
EU = v_0 - m_0 + \frac{\kappa_0 v_0 S(m_0)}{4} = v_0 - m_0 + \frac{\kappa_0 v_0}{4} \left(\frac{(\kappa_0 + \gamma m_0)(1 + \rho)}{\kappa_0 + \gamma m_0 + \rho}\right),
\]
and marginal utility is 
\[
\frac{dEU}{dm_0} = \frac{\kappa_0 v_0}{4} \left(\frac{(\kappa_0 + \gamma m_0 + \rho)(1 + \rho) - (\kappa_0 + \gamma m_0)(1 + \rho)}{(\kappa_0 + \gamma m_0 + \rho)^2}\right) = \frac{\kappa_0 v_0}{4} \frac{\gamma \rho (1 + \rho)}{(\kappa_0 + \gamma m_0 + \rho)} - 1,
\]
so the interior optimum is 
\[
m_0 = \frac{1}{\gamma} \left(\sqrt{\frac{\kappa_0 v_0}{4} \rho (1 + \rho)} - \kappa_0 - \rho\right).
\]
It is straightforward (using arguments analogous to the case in segment \([m_{R3|R2}, m_{R2|R1}]\)) to show that if \(\frac{16}{\kappa_0} \frac{\rho}{1 + \rho} < \gamma v_0\) then we also have 
\[
m_0 = \frac{1}{\gamma} \left(\sqrt{\frac{\kappa_0 v_0}{4} \rho (1 + \rho)} - \kappa_0 - \rho\right) > m_{R2|R1}.
\]
This implies that for \(\gamma v_0 > \frac{16}{\kappa_0} = \sigma_H\) the polity must move to the interior of \(R1\) in period 1.

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In sum, whenever \( \sigma_H < \gamma v_0 \) the polity moves to \( R1 \). In what follows, we show that there exist

\[
\sigma_{M1} = \max \left\{ \frac{16 - 4\kappa_0 \rho}{4\rho - (1 + \rho)\kappa_0}, \frac{1}{\kappa_0} - \sqrt{\frac{1}{4\kappa_0} - 3 \times \frac{(1 + \rho)^2}{\rho} \left( \frac{\kappa_0}{4} \right)} \right\}
\]

\[
\sigma_{M2} = \min \left\{ \frac{8(\rho - \kappa_0)}{8 - (1 + \rho)\kappa_0}, \frac{4}{\kappa_0} + \sqrt{\frac{1}{4\kappa_0} - 3 \times \frac{(1 + \rho)^2}{\rho} \left( \frac{\kappa_0}{4} \right)} \right\}, 3\rho - \kappa_0 \right\},
\]

with \( \sigma_L < \sigma_{M1} < \sigma_H \) and \( \sigma_L < \sigma_{M2} < \sigma_H \), such that the polity moves to \( R2 \) whenever \( \sigma_{M1} < \gamma v_0 < \sigma_{M2} \).

**Part (b)(3)** To prove the existence of values of \( \gamma v_0 \) such that the polity will move into \( R2 \), consider the case in which (15) holds, so \( m_{R3|R2} < m < m_{R2|R1} \). We characterize the set of parameters \((\kappa_0, \rho)\) such that the optimal point lies in \( R2 \).

**Segment \([0, m_{R3|R2}]\)** Expected utility in period \( t = 0 \) is, \( EU = v_0 - m_0 + \frac{\kappa_0}{4} v_0 S(m_0) = v_0 - m_0 + \frac{\kappa_0}{4} v_0 (1 + \frac{\kappa_0 + \gamma m_0}{4}) = v_0 (1 + \frac{\kappa_0}{4} + \left( \frac{\kappa_0}{4} \right)^2) + m_0 \left( \frac{\kappa_0 \gamma m_0}{16} - 1 \right) \). Marginal utility is positive iff \( \gamma v_0 > \frac{16}{\kappa_0} \) and negative iff \( \gamma v_0 < \frac{16}{\kappa_0} \). Algebra shows that under Assumption 1 \( \frac{16 - 4\kappa_0 \rho}{4\rho - (1 + \rho)\kappa_0} < \frac{8(\rho - \kappa_0)}{8 - (1 + \rho)\kappa_0} < \frac{16}{\kappa_0} \). Thus, \( m_0 = 0 \) is optimal in \([0, m_{R3|R2}]\) and the polity will stay in \( R3 \). \( EU \) evaluated at \( m_0 = 0 \) is \( v_0 (1 + \frac{\kappa_0}{4} + \left( \frac{\kappa_0}{4} \right)^2) \), and we later show that this value is lower than \( EU \) at the optimum in \( R2 \) whenever \( \sigma_{M1} < \gamma v_0 < \sigma_{M2} \).

**Segment \([m_{R3|R2}, \bar{m}]\)** There are two possibilities for an optimum in \( R2 \). It could lie in \([m_{R3|R2}, \bar{m}]\) or in \([\bar{m}, m_{R2|R1}]\). We only need to show the result for one of the two cases so we focus on the first. The (concave) expected utility is,

\[ EU = v_0 - m_0 + \frac{\kappa_0 v_0 (1 + \rho)}{8} \sqrt{\frac{\kappa_0 + \gamma m_0}{\rho}} \]. The marginal utility of \( m_0 \) is: \(-1 + \frac{\kappa_0 v_0 (1 + \rho)}{8} \frac{1}{2} \sqrt{\frac{\rho}{\kappa_0 + \gamma m_0}} \).

Hence, the optimum point is given by \( m_{Int|R2} = \frac{\left( \frac{\kappa_0 v_0 (1 + \rho)}{8} \right)^2 - \kappa_0}{\gamma} \). This point is less than \( \bar{m} \) if
and only if \( \frac{v_0 S(m_{IntR2})}{v_0 - m_{IntR2}} < \frac{4}{\kappa_0} \), which is equivalent to,

\[
\frac{4}{\kappa_0} - \frac{\sqrt{\left( \frac{4}{\kappa_0} \right)^2 - 3 \times \left( \frac{1 + \rho}{\rho} \right)^2 \left( \frac{\kappa_0}{4} \right) \rho}}{2 \times \left( \frac{1 + \rho}{\rho} \right)^2 \left( \frac{\kappa_0}{16} \times \frac{3}{4} \right)} < \gamma v_0 < \frac{4}{\kappa_0} + \frac{\sqrt{\left( \frac{4}{\kappa_0} \right)^2 - 3 \times \left( \frac{1 + \rho}{\rho} \right)^2 \left( \frac{\kappa_0}{4} \right) \rho}}{2 \times \left( \frac{1 + \rho}{\rho} \right)^2 \left( \frac{\kappa_0}{16} \times \frac{3}{4} \right)}.
\]  

(16)

Then \( EU(m_{IntR2}) \) is \( \frac{1}{2} \left( \gamma v_0 \left( 1 + v_0 \gamma \left( \frac{\kappa_0}{16} \sqrt{\frac{\kappa_0}{\rho}} \right)^2 \right) + \kappa_0 \right) \). Hence, the optimal value lies in \([m_{R3|R2}, \bar{m}]\) if \( EU(m_{IntR2}) > EU(m_0 = 0) \). This inequality implies (after some algebra) that \( \left( \frac{\kappa_0}{4} \left( 1 + \frac{\kappa_0}{4} \right) - \frac{\kappa_0}{\gamma} \right) \left( \frac{16 \sqrt{\rho}}{\kappa_0 (1 + \rho)} \right)^2 < v_0 \gamma \). Note the LHS of this expression is negative as long as \( \gamma < 16/(4 + \kappa_0) \), the technical assumption introduced in the text.

Our strategy for this part of the proof is to find the combinations of \((\kappa_0, \rho)\) for which \( EU(m_{IntR2}) > EU(m_0 = 0) \). This is a sufficient condition for the optimal point to lie in \( R_2 \) as long as \( EU \) is decreasing in \( m_0 \) over \( R_1 \).

### Segment \([m_{R2|R1}, \infty)\)

Expected utility is

\[
EU = \sqrt{\kappa_0 v_0 (v_0 - m_0) S(m_0)} = \sqrt{\kappa_0 v_0 (v_0 - m_0) \left( \frac{\kappa_0 + \gamma m_0 (1 + \rho)}{\kappa_0 + \gamma m_0 + \rho} \right)}.
\]

Marginal utility is,

\[
\frac{dEU}{dm_0} = \sqrt{\kappa_0 v_0} \left[ - (v_0 - m_0)^{-\frac{1}{2}} \sqrt{\frac{\kappa_0 + \gamma m_0 (1 + \rho)}{\kappa_0 + \gamma m_0 + \rho}} + \sqrt{(v_0 - m_0) \left( \frac{\kappa_0 + \gamma m_0 (1 + \rho)}{\kappa_0 + \gamma m_0 + \rho} \right)^{-\frac{1}{2}} \frac{\gamma \rho (1 + \rho)}{(\kappa_0 + \gamma m_0 + \rho)^2} \right].
\]

Note \( \frac{dEU}{dm_0} < 0 \) iff \(- (\kappa_0 + \gamma m_0)(\kappa_0 + \gamma m_0 + \rho) + \gamma \rho (v_0 - m_0) < 0 \). The LHS of this expression is decreasing in \( m_0 \) so if evaluated at \( m_{R2|R1} = \frac{\rho - \kappa_0}{\gamma} \) such LHS is negative, then \( EU \) is decreasing in \( m_0 \) within \( R_1 \). Evaluating at \( m_{R2|R1} = \frac{\rho - \kappa_0}{\gamma} \) yields,

\[
\gamma v_0 < 3 \rho - \kappa_0.
\]  

(17)

Combining (15), (16), and (17), we define \( \sigma_{M1} = \max \left\{ \frac{16 - 4 \kappa_0 \rho}{4 \rho - (1 + \rho) \kappa_0}, \frac{4}{\rho} - \frac{\sqrt{\left( \frac{4}{\rho} \right)^2 - 3 \times \left( \frac{1 + \rho}{\rho} \right)^2 \left( \frac{\kappa_0}{4} \right) \rho}}{2 \times \left( \frac{1 + \rho}{\rho} \right)^2 \left( \frac{\kappa_0}{16} \times \frac{3}{4} \right)} \right\} \)

and \( \sigma_{M2} = \min \left\{ \frac{8 (\rho - \kappa_0)}{8 - (1 + \rho) \kappa_0}, \frac{4}{\rho} + \frac{\sqrt{\left( \frac{4}{\rho} \right)^2 - 3 \times \left( \frac{1 + \rho}{\rho} \right)^2 \left( \frac{\kappa_0}{4} \right) \rho}}{2 \times \left( \frac{1 + \rho}{\rho} \right)^2 \left( \frac{\kappa_0}{16} \times \frac{3}{4} \right)}, 3 \rho - \kappa_0 \right\} \). As a result, if \( \sigma_{M1} < \gamma v_0 < $

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\( \sigma_{M2} \) then the polity moves to \( \mathbb{R}^2 \). Note the set over the real line bounded by \( \sigma_{M1} \) and \( \sigma_{M2} \) is non-empty for a measurable set of \((\kappa_0, \rho)\). For example, fix \( \rho = 4 \kappa_0 = 1 \). In this case, the conditions (15), (16), and (17) become respectively \( 0 < \gamma v_0 < 8, \frac{2}{16} \sqrt{\left(\frac{5}{16}\right)^2 - 4\left(\frac{5}{16} - 2\right)^2} < 0 < \gamma v_0 < \frac{32}{5} \) and \( \gamma v_0 < 11 \). Hence, \( \sigma_{M1} = 0 \) and \( \sigma_{M2} = 32/5 \) and for \( 0 < \gamma v_0 < 32/5 \) the polity moves to \( \mathbb{R}^2 \). As these inequalities are strict, pairs \((\rho, \kappa_0)\) in a neighborhood of \((4, 1)\) also yield a transition to \( \mathbb{R}^2 \) for a measurable set of \( \gamma v_0 \).

In sum, there exist \( \sigma_L, \sigma_{M1}, \sigma_{M2} \) and \( \sigma_H \) defined above, \( \sigma_L < \sigma_{M1} < \sigma_H \) and \( \sigma_L < \sigma_{M2} < \sigma_H \), (simple algebra shows that \( \sigma_{M1} < \sigma_H \) and \( \sigma_{M2} < \sigma_H \)) such that if \( \gamma v_0 < \sigma_L \) the polity stays in \( \mathbb{R}^3 \); if \( \sigma_{M1} < \gamma v_0 < \sigma_{M2} \) the polity moves from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \); and if \( \gamma v_0 > \sigma_H \) the polity moves from \( \mathbb{R}^3 \) to \( \mathbb{R}^1 \). ■