Dynamic Portfolio Choice with Frictions: Appendix

Nicolae Gârleanu and Lasse Heje Pedersen*

June 16, 2016

*Gârleanu is at the Haas School of Business, University of California, Berkeley, NBER, and CEPR; e-mail: garleanu@haas.berkeley.edu. Pedersen is at Copenhagen Business School, New York University, AQR Capital Management, CEPR, and NBER, http://www.lhpedersen.com/.
A Proofs

Proof of Proposition 1. This proposition is a special case of Proposition 2, but its simplicity allows for the relaxation of the constraints placed on other parameters to obtain a well-behaved problem. Specifically, it is immediate that the objective (5) is strictly concave in \( \{\tau_s\}_s \). Furthermore, its value is bounded above by the objective of the friction-less problem, given by \( \Lambda = 0 \), which is finite under appropriate conditions on \( f \).

The Hamilton-Jacoby-Bellman (HJB) equation is
\[
\rho V = \sup_{\tau} \left\{ x^\top B f - \frac{\gamma}{2} x^\top \Sigma x - \frac{1}{2} \tau^\top \Lambda \tau + \frac{\partial V}{\partial x} \tau + \frac{\partial V}{\partial f} \mu_f + V_f \right\},
\]
where the term \( V_f \) accounts for the diffusion and jump components.

Maximizing this expression with respect to the trading intensity results in
\[
\tau = \Lambda^{-1} \frac{\partial V}{\partial x}.
\]

Given the conjectured form (6) of the value function, the optimal choice \( \tau \) equals
\[
\tau_t = -\Lambda^{-1} A_{xx} x_t + \Lambda^{-1} A_x (f_t).
\]

Once this expression is inserted in the HJB equation, it results in the following equations defining the value-function coefficients (using the symmetry of \( A_{xx} \)):
\[
\begin{align*}
-\rho A_{xx} &= A_{xx} \Lambda^{-1} A_{xx} - \gamma \Sigma \\
\rho A_x (f) &= -A_{xx} \Lambda^{-1} A_x (f) + \mathcal{D} A_x (f) + B f \\
\rho A (f) &= A_x (f)^\top \Lambda^{-1} A_x (f) + \mathcal{D} A (f).
\end{align*}
\]
We use throughout the notation \( \mathcal{D} g (X) \) to designate \( \mathbb{E}_t [dg (X_t)] \) for a (twice-continuously differentiable) function \( g \) and jump diffusion \( X \).

Pre- and post-multiplying (A.2) by \( \Lambda^{-\frac{1}{2}} \), we obtain
\[
-\rho Z = Z^2 + \frac{\rho^2}{4} I - U,
\]
that is,
\[
\left( Z + \frac{\rho}{2} I \right)^2 = U,
\]
where
\[
\begin{align*}
Z &= \Lambda^{-\frac{1}{2}} A_{xx} \Lambda^{-\frac{1}{2}} \\
U &= \gamma \Lambda^{-\frac{1}{2}} \Sigma \Lambda^{-\frac{1}{2}} + \frac{\rho^2}{4} I.
\end{align*}
\]
This leads to
\[ Z = -\frac{\rho}{2}I + U^\frac{1}{2} \geq 0, \quad (A.9) \]

implying that
\[ A_{xx} = -\frac{\rho}{2}\Lambda + \Lambda^\frac{1}{2} \left( \gamma\Lambda^{-\frac{1}{2}}\Sigma\Lambda^{-\frac{1}{2}} + \frac{\rho^2}{4} \right)^\frac{1}{2} \Lambda^\frac{1}{2}. \quad (A.10) \]

The value of \( A_x \) follows as a solution to the ODE (A.3). Note that, using the Feynman-Kac formula, \( A_x \) can be written as
\[ A_x(f) = \mathbb{E} \left[ \int_0^\infty e^{-\left(\rho + A_{xx}\Lambda^{-1}\right)t} B f_t \, dt | f_0 = f \right] \]
\[ = \int_0^\infty e^{-\left(\gamma\Sigma\right)A_{xx}^{-1}t} \mathbb{E} [M_t | f_0 = f] \, dt \]
\[ = \int_0^\infty \left( \gamma\Sigma \right) e^{-A_{xx}^{-1}\left(\gamma\Sigma\right)t} \mathbb{E} [M_t | f_0 = f] \, dt, \quad (A.11) \]

where the second equality holds because of (A.2).

If \( \mu(f) = -\Phi f \), then \( A_x(f) = A_{xf} f \) and \( A(f) = f^\top A_{ff} f \), and (A.3)–(A.4) become
\[ \rho A_{xf} = -A_{xx}\Lambda^{-1}A_{xf} - A_{xf}\Phi + B \quad (A.14) \]
\[ \rho A_{ff} = A_{xf}^\top\Lambda^{-1}A_{xf} - 2A_{ff}\Phi. \quad (A.15) \]

The solution for \( A_{xf} \) follows from Equation (A.14), using the general rule that \( \text{vec}(XYZ) = (Z^\top \otimes X) \text{vec}(Y) \):
\[ \text{vec}(A_{xf}) = \left( \rho I + \Phi^\top \otimes I_K + I_S \otimes (A_{xx}\Lambda^{-1}) \right)^{-1} \text{vec}(B). \]

If \( \Lambda = \lambda \Sigma \), then \( A_{xx} = a\Sigma \) with
\[ -\rho a = a^2 \frac{1}{\lambda} - \gamma, \quad (A.16) \]

with solution
\[ a = -\frac{\rho}{2} \lambda + \sqrt{\gamma \lambda + \frac{\rho^2}{4} \lambda^2}. \quad (A.17) \]

In this case, (A.3) yields
\[ A_{xf} = B \left( \rho I + \frac{a}{\lambda} I + \Phi \right)^{-1} \]
\[ = B \left( \frac{\gamma}{a} I + \Phi \right)^{-1}, \]
where the last equality uses (A.16).

Then, we have
\[
\tau_t = \frac{a}{\lambda} \left[ \Sigma^{-1} B (a \Phi + \gamma I)^{-1} f_t - x_t \right]. \tag{A.18}
\]

It is clear from (A.17) that \( \frac{a}{\lambda} \) decreases in \( \lambda \) and increases in \( \gamma \). 

Proof of Lemma 1. The trader’s utility dependence on \( \tau \) is given by
\[
\int_0^\infty e^{-\rho s} \left( x_s^\top (B f_s - (r + R) D_s + C \tau_s) - \frac{\gamma}{2} x_s^\top \Sigma x_s - \frac{1}{2} \tau_s^\top \Lambda \tau_s \right) ds, \tag{A.19}
\]
which is clearly concave if
\[
\int_0^\infty e^{-\rho s} \left( x_s^\top (- (r + R) D_s + C \tau_s) - \frac{\gamma}{2} x_s^\top \Sigma x_s \right) ds \tag{A.20}
\]
is. To see the latter fact, we start by evaluating
\[
1 - \int_0^\infty e^{-\rho s} x_s^\top (r + R) D_s ds = - \int_0^\infty e^{-\rho s} \left( \int_0^s \tau_t^\top dt (r + R) \int_0^s e^{-R(s-u)} C \tau_u du \right) ds \tag{A.21}
\]
and break down (A.21) in two terms depending on whether \( u < t \) or vice-versa. On the set \( u \geq t \) we obtain the integral
\[
\int_0^u \int_0^t \tau_t^\top e^{-\rho u} C \tau_u du dt = \int_0^\infty x_u^\top e^{-\rho u} C \tau_u du \tag{A.22}
\]
\[
= [x_u^\top e^{-\rho u} C x_u]_0^\infty - \int_0^\infty x_u^\top e^{-\rho u} C (\tau_u - \rho x_u) du \tag{A.23}
\]
\[
= \frac{1}{2} \int_0^\infty \rho e^{-\rho u} x_u^\top C x_u du + \frac{1}{2} \lim_{u \to \infty} e^{-\rho u} x_u^\top C x_u, \tag{A.24}
\]
where the second equality follows by integration by parts and the third by solving the equation implicit in the second.

The integral on the set \( u < t \) is similarly shown to equal
\[
\frac{1}{2} \lim_{t \to \infty} e^{-\rho t} Y_t^\top CY_t + \frac{1}{2} (\rho + 2R) \int_0^\infty e^{-\rho t} Y_t^\top CY_t dt, \tag{A.25}
\]
where \( Y_t \equiv \int_0^t e^{-R(t-u)} \tau_u du \). Since \( C \) is positive definite, this term, when multiplied

\footnote{We take \( x_0 = D_0 = 0 \) without loss of generality, since these values do not affect the concavity of the objective.}
by $-\frac{r+R}{\rho+R}$, is also concave — and bounded above (by zero).

To ensure the concavity of (A.20), it therefore suffices that

$$
\frac{1}{2} \left(1 - \frac{r + R}{\rho + R}\right) \int_0^\infty \rho e^{-\rho u} x_u^T C x_u du - \frac{\gamma}{2} \int_0^\infty \rho e^{-\rho u} x_u^T \Sigma x_u du
$$

is concave, or

$$
\gamma > \frac{\rho - r}{\rho + R} \left\| \Sigma^{-\frac{1}{2}} C \Sigma^{-\frac{1}{2}} \right\|.
$$

\[\text{Proof of Proposition 2.}\] In this case, the conjectured value function is

$$
V = -\frac{1}{2} x^T A xx x + x^T A x D D + \frac{1}{2} D^T A D D D + x^T A x(f) + D^T A D(f) + A(f).
$$

(A.28)

Given the HJB equation

$$
\rho V = \sup_{\tau} \left\{-\gamma x^T \Sigma x + x^T (B_f + C \tau) - \frac{1}{2} \tau^T \Lambda \tau + \tau^T V_x^T + (C \tau - RD)^T V_D^T + x^T D A x(f) + D^T D A_D(f) + D A(f)\right\},
$$

(A.29)

the optimal trade follows as

$$
\tau = \Lambda^{-1} \left(A x(f) + C^T A D(f) + (A x D + C^T A D D) D - (A x x - C^T - C^T A D x) x\right)

\equiv \bar{M}_{\text{rate}} \left(M_{\text{aim}}(f) + \bar{M}_{\text{aim}} D - x\right).
$$

(A.30)

Plugging (A.30) in (A.29) and then proceeding as in part (i), we obtain

$$
\left[A x(f)^T, A_D(f)^T\right]^T = E \left[\int_0^\infty e^{-N_1 t} [B^T \ 0]^T f_t | f_0 = f\right]

= \int_0^\infty e^{-N_1 t} [\gamma \Sigma \ 0]^T E [M_t | f_0 = f] dt,
$$

(A.31)

where

$$
N_1 = \rho + \left[\begin{array}{cc} A_{xx} - A_{x D} C - C \\ -A_{x D}^T - A_{D D} C \end{array}\right] \Lambda^{-1} \left[\begin{array}{c} I \\ C^T\end{array}\right] + \left[\begin{array}{cc} 0 & 0 \\ 0 & R \end{array}\right].
$$

(A.32)
Equation (12) follows immediately with

\[ N_2 = (A_{xx} + C^\top + C^\top A_{Dx})^{-1} [I \ C^\top] \tag{A.33} \]
\[ N_3 = [\gamma \Sigma \ 0]^\top. \tag{A.34} \]

To write explicitly the Riccati equations for the constant value-function coefficients and the ODEs for the ones that depend (non-linearly) on \( f \), we match the coefficients in the HJB equation evaluated at the optimal \( \tau \).

Using the notation

\[
Q_x = -A_{xx} + C^\top A_{xD} + C^\top \\
Q_D = A_{xD} + C^\top A_{DD} \\
Q_f = A_x(f) + CA_D(f).
\]

the constant coefficient matrices solve the system

\[
-\rho A_{xx} = -\gamma \Sigma + Q_x^\top \Lambda^{-1} Q_x \\
\rho A_{xD} = Q_x^\top \Lambda^{-1} Q_D - RA_{xD} \\
\rho A_{DD} = Q_D^\top \Lambda^{-1} Q_D - 2RA_{DD},
\] (A.36)

while the ODE system is

\[
\rho A_x(f) = Bf - Q_x^\top \Lambda^{-1} Q_f + DA_x(f) \\
\rho A_D(f) = Q_D^\top \Lambda^{-1} Q_f - RA_Df + DA_D(f) \\
\rho A(f) = \frac{1}{2} Q_f^\top \Lambda^{-1} Q_f + DA(f).
\]

We note that the equations above have to be solved simultaneously for \( A_{xx}, A_{xD}, \) and \( A_{DD} \); there is no closed-form solution in general. The complication is due to the fact that current trading affects the persistent price component \( D \) (that is, \( C \neq 0 \)). Furthermore, the ODEs for \( A_x \) and \( A_D \) are coupled, but the solution can be written reasonably simply as (A.31) above.  

**Proof of Proposition 3.**

Let’s start with the complete problem:

\[
V(x_t, D_t, f_t) = E_t \int_t^\infty e^{-\rho(s-t)} \left( x_s^\top (Bf_s - (r+R)D_s) - \frac{\gamma}{2} x_s^\top \Sigma x_s \right) ds \tag{A.38} \]
\[ + E_t \int_t^\infty e^{-\rho(s-t)} x_s^\top Cdx_s + \frac{1}{2} E_t \int_t^\infty e^{-\rho(s-t)} d\left[ x_s, Cx_s \right]. \]

As is customary with such problems, we write and solve the HJB equation, then use the fact that it is satisfied to provide a so-called verification argument for the
We conjecture a quadratic form for the value function \( \hat{V} \):

\[
\hat{V}(D, f) = \frac{1}{2} D^0 \bar{A} D + \frac{1}{2} D^0 \bar{D} A(f) + A(f),
\]

which leads to the simplification

\[
0 = \sup_x \left\{ - \rho \hat{V} + \frac{1}{2} x^\top C x + x^\top B f - x^\top (r^f + R)(D^0 + C x) - \frac{\gamma}{2} x^\top \Sigma x - \hat{V}(D^0 + C f) + D^0 \bar{D} A(f) + \bar{D} A(f) \right\}.
\]
the perceived cost of the risk and the decay in the distortion.

In order to write down the solution, let

\[ J = \frac{1}{2}(J_0 + J_0^\top) \]  
(A.44)

\[ J_0 = \gamma \Sigma + (2R + 2r^f - \rho)C \]  
(A.45)

\[ j = Bf - RC^\top A_D(f) - (RC^\top A_{DD} + r^f + R)D^0. \]  
(A.46)

It follows that

\[ x = J^{-1}\left(Bf - (r^f + R)D^0 - RC^\top \hat{V}_D^\top\right) \]  
(A.47)

\[ = J^{-1}j \]  
(A.48)

and the HJB equation becomes

\[ \rho \hat{V} = \frac{1}{2}j^\top J^{-1}j - \left(D^0^\top A_{DD} + A_D(f)^\top\right)RD^0 + D^0^\top D_A(f) + D_A(f). \]  
(A.49)

The constant matrix \( A_{DD} \) is computed in the usual way:

\[ \rho A_{DD} = \left(RA_{DD}C + r^f + R\right)J^{-1}\left(RC^\top A_{DD} + r^f + R\right) - 2RA_{DD}. \]  
(A.50)

The coefficient function \( A_D(f) \) satisfies the (integro-)differential equation

\[ \rho A_D(f) = \left(RA_{DD}C + r^f + R\right)J^{-1}\left(RC^\top A_D(f) - Bf\right) - RA_D(f) - DA_D(f), \]  
(A.51)

and has the representation

\[ A_D(f) = -\int_0^\infty \hat{N}_2 e^{-\hat{N}_1 t} B \mathbb{E}[f_t | f_0 = f] \]  
(A.52)

\[ = -\int_0^\infty \hat{N}_2 e^{-\hat{N}_1 t} \hat{N}_3 \mathbb{E}[M_t | f_0 = f], \]  
(A.53)

with

\[ \hat{N}_1 = \rho + R - \left(RA_{DD}C + r^f + R\right)J^{-1} \]  
(A.54)

\[ \hat{N}_2 = R \left(RA_{DD}C + r^f + R\right)J^{-1}C^\top. \]  
(A.55)

To prove that the proposed solution does, indeed, solve the trader’s problem, we follow a verification argument. Let \( \hat{x} \) be an arbitrary trading strategy (satisfying technical transversality conditions) and \( V \) quadratic, defined by (22) and the coefficients \( A \). We note that similar calculations to the ones used in the proof of Lemma 1 show that, under restrictions on the matrix \( C \), the objective is bounded above.
Since it holds generally that
\[
e^{-\rho T} V(\hat{x}_t, \hat{D}_t, f_t) = e^{-\rho t} V(\hat{D}_t - C\hat{x}_t, f_t) - \frac{1}{2} e^{-\rho t} \hat{x}_t^\top C\hat{x}_t \\
= e^{-\rho T} V(\hat{D}_T - C\hat{x}_T, f_T) - \frac{1}{2} e^{-\rho T} \hat{x}_T^\top C\hat{x}_T \\
- \int_t^T \! \! d \left( e^{-\rho s} V(\hat{D}_s - C\hat{x}_s, f_s) - \frac{1}{2} e^{-\rho s} \hat{x}_s^\top C\hat{x}_s \right),
\]
the following three facts would be sufficient: (i) for any \( T > t \),
\[
E_t \left[ - \int_t^T \! \! d \left( e^{-\rho s} V(\hat{D}_s - C\hat{x}_s, f_s) - \frac{1}{2} e^{-\rho s} \hat{x}_s^\top C\hat{x}_s \right) \right] \\
\geq E_t \int_t^T \! \! e^{-\rho s} \left( \hat{x}_s^\top \left( B f_s - (r + R)\hat{D}_s \right) - \frac{\gamma}{2} \hat{x}_s^\top \Sigma \hat{x}_s \right) ds \\
+ E_t \int_t^T \! \! e^{-\rho s} \hat{x}_s^\top C d\hat{x}_s + \frac{1}{2} E_t \int_t^T \! \! e^{-\rho s} d [\hat{x}_s, C\hat{x}_s], \tag{A.56}
\]
(ii) the inequality holds with equality at the conjectured optimum control, and (iii):
\[
\lim_{T \to \infty} e^{-\rho T} \left( E_T \int_T^\infty \! \! e^{-\rho(s-T)} \left( \hat{x}_s^\top \left( \alpha_s - (r^f + R)\hat{D}_s \right) - \frac{\gamma}{2} \hat{x}_s^\top \Sigma \hat{x}_s \right) ds \right. \\
+ E_T \int_T^\infty \! \! e^{-\rho(s-T)} \hat{x}_s^\top C d\hat{x}_s + \frac{1}{2} E_T \int_T^\infty \! \! e^{-\rho(s-T)} d [\hat{x}_s, C\hat{x}_s] \bigg) = 0. \tag{A.57}
\]
Fact (iii) follows from the fact that the objective is bounded above — basically, from the fact that the frictions are such that they worsen trader’s objective. For fact (i), we use Ito’s lemma to write
\[
d \left( \hat{V}(\hat{D}_s - C\hat{x}_s, f_s) - \frac{1}{2} \hat{x}_s^\top C\hat{x}_s \right) \\
= \hat{V}_d (d\hat{D}_s - C d\hat{x}_s) + \hat{V}_f (df_s - \Delta f_s) - \hat{x}_s^\top C d\hat{x}_s - \frac{1}{2} d [\hat{x}_s, C\hat{x}_s] + \frac{1}{2} d[f_s, \hat{V}_f f_s] - \frac{1}{2} \Delta f_s^\top \hat{V}_{ff} \Delta f_s + \hat{V}(\hat{D}_s - C\hat{x}_s, f_s) - \hat{V}(\hat{D}_s - C\hat{x}_s, f_{s-}). \tag{A.58}
\]
Taking conditional expectations of (A.58), one gets
\[
-\hat{V}_D R\hat{D}_s - \hat{x}_s^\top C E_s \frac{1}{ds}[d\hat{x}_s] - \frac{1}{2} \frac{1}{ds} d [\hat{x}_s, C\hat{x}_s] + (\hat{D}_s - C\hat{x}_s)^\top \mathcal{D} A_D(f_s) + \mathcal{D} A(f_s),
\]
the negative of which we wish to be larger than
\[ \frac{\rho}{2} \hat{x}_s^\top C \hat{x}_s - \rho \hat{V} + \hat{x}_s^\top \left( B f_s - (r + R) \hat{D}_s \right) - \frac{\gamma}{2} \hat{x}_s^\top \Sigma \hat{x}_s + \hat{x}_s^\top C \frac{1}{ds} E_s [d\hat{x}_s] + \frac{1}{2} \frac{1}{ds} E_s d [\hat{x}_s, C \hat{x}_s]. \]

This outcome is ensured by the HJB equation (A.43), which is satisfied by \( \hat{V} \) and \( \hat{x} \), for any value of \( \hat{D}^0 \).

Fact (ii) follows automatically, given how \( \hat{V} \) and \( \hat{x} \) were jointly determined.

**Proof of Proposition 4.** Consider evaluating the objective (33) at \( x_n^{(\Delta t)} \) and letting \( \Delta t \) go to zero. The only non-standard terms in the objective are the ones involving transaction costs:
\[
E_0 \left[ \sum_n - (1 - \rho \Delta t)^{n+1} x_n^\top \left( R + r^f \right) (D_n + C \Delta x_n) \Delta t 
+ (1 - \rho \Delta t)^t \left( -\frac{1}{2} \Delta x_n^\top \Lambda(\Delta t) \Delta x_n + x_n^\top C \Delta x_n + \frac{1}{2} \Delta x_n^\top C \Delta x_n \right) \right].
\] (A.59)

Under regularity conditions, the sum of the terms not involving \( \Lambda \) tends to
\[
E_0 \int_0^\infty -e^{-\rho t} (x_t^\top (r^f + R)D_t) \, dt 
+ E_0 \int_0^\infty e^{-\rho t} x_t^\top C dx_t + \frac{1}{2} E_0 \int_0^\infty e^{-\rho t} d \left[ x, C x \right]_t.
\] (A.60)

If we write \( \Lambda(\Delta t) = (\Lambda + \Lambda_1(\Delta t))s(\Delta t) \) for some scalar function \( s \) and matrix \( \Lambda_1(\Delta t) \) such that \( \lim_{\Delta t \to 0} \Lambda_1(\Delta t) = 0 \), and let \( \tau_t = \frac{1}{\rho} E_t [d\hat{x}_t] \), then we can express the remaining limit term as \(-\frac{1}{2}\) times
\[
E_0 \int_0^\infty e^{-\rho t} \tau_t^\top \Lambda \tau_t dt \times \lim_{\Delta t \to 0} (s(\Delta t) \Delta t) + E_0 \int_0^\infty e^{-\rho t} d \left[ x, \Lambda x \right]_t \times \lim_{\Delta t \to 0} s(\Delta t). \] (A.61)

We note that, under the assumption of part (i), the first term is finite while the second is infinite if \( x \) has non-zero quadratic variation. Under the assumption of part (ii), both terms are zero, so that the objective coincides with (21). Finally, in the special case (iii) the first term is zero, while the second is finite and non-zero if \( x \) has quadratic variation.

**Proof of Proposition 5.** The proof is the same as in Gărleanu and Pedersen (2013), and it follows the standard procedure of assuming the functional form (in this case, quadratic) of the value function, computing the optimal control conditional on this function, and then calculating the coefficients that yield the value function as a

\footnote{This is possible in the three cases of the proposition, though not in complete generality.}
fixed point.
Letting $\bar{r}^f \Delta t = e^{r^f \Delta t} - 1$, $\bar{\rho} = 1 - \rho \Delta t$ and $\bar{\Lambda} = \bar{\rho}^{-1} \Lambda(\Delta t)$, recalling $y_t = (f_t, D_t)$, and making the additional definitions

\[ \Pi = \begin{bmatrix} \Phi & 0 \\ 0 & R I_s \end{bmatrix} \Delta t, \]
\[ \tilde{C} = (1 - R \Delta t) \begin{bmatrix} 0 \\ C \end{bmatrix}, \] (A.62)
\[ \tilde{B} = \begin{bmatrix} B - (R + \bar{r}^f) \Delta t \end{bmatrix}, \]

the unknown matrices have to satisfy the system of equations

\[ -\bar{\rho}^{-1} A_{xx} = S^T_x J^{-1} S_x - \bar{\Lambda} - \bar{\rho}^{-1} C + \tilde{C}^T A_{yy} \tilde{C} \] (A.63)
\[ \bar{\rho}^{-1} A_{xy} = S^T_x J^{-1} S_y - \tilde{C}^T A_{yy} (I - \Pi) \] (A.64)
\[ \bar{\rho}^{-1} A_{yy} = S^T_y J^{-1} S_y + (I - \Pi)^T A_{yy} (I - \Pi), \] (A.65)

where the matrices $J$, $S_x$, and $S_y$ are explicit functions of the unknown coefficients:

\[ J = \gamma \Sigma \Delta t + \bar{\Lambda} + (2(R + \bar{r}^f) \Delta t - \bar{\rho}^{-1}) C + A_{xx} - 2A_{xy} \tilde{C} - \tilde{C}^T A_{yy} \tilde{C} \] (A.66)
\[ S_x = \bar{\Lambda} + (R + \bar{r}^f) \Delta t C - A_{xy} \tilde{C} - \tilde{C}^T A_{yy} \tilde{C} \] (A.67)
\[ S_y = \tilde{B} + A_{xy} (I - \Pi) + \tilde{C}^T A_{yy} (I - \Pi). \] (A.68)

The optimal position $x_t$ can be written as

\[ x_t = x_{t-1} + (I - J^{-1} S_x) \left( (I - J^{-1} S_x)^{-1} (J^{-1} S_y) y_t - x_{t-1} \right), \] (A.69)

so that

\[ M^{rate} = I - J^{-1} S_x \] (A.70)
\[ M^{aim} = (I - J^{-1} S_x)^{-1} (J^{-1} S_y). \] (A.71)

Proof of Proposition 6. Proposition 4 establishes that, for any continuous-time trading strategy, the discrete-time objective evaluated at the discretely sampled strategy tends to the continuous-time objective for that strategy. All that we need to prove, therefore, is that the optimal trade in the discrete-time model has a well-defined continuous-time limit; for if the limit exists, it has to be optimal, or else its discretely sampled counterpart can be improved upon, at least for $\Delta t$ small enough.

We know that the discrete-time optimal trade is given as a quadratic function of the exogenous process $\{f_t\}_t$, so the only claim to prove is that the sequence of matrix

\[ \text{We omit notational dependence on } \Delta t \text{ for simplicity.} \]
tuples \((A_{xx}, A_{xy}, A_{yy})\), which generate the coefficients of this function, has a (finite) limit.

We can achieve this goal through direct manipulation of the Riccati equations defining the discrete-time coefficients and taking the limit. The precise details depend on the case of the proposition and the coefficient, but the general idea is the same. We illustrate for the matrix \(A_{xx}\) under the assumption \(\Lambda(\Delta t) = \Lambda/\Delta t\).

We work with the characterization of solutions provided in the proof of Proposition 5. We show that, as \(\Delta t \to 0\), Equation (A.63) tends to its counterpart in (A.36), which implies that the solutions also do.

We first rewrite this equation as

\[-A_{xx} = \bar{\rho} S_x^T J^{-1} S_x - \Lambda(\Delta t) - C + \bar{\rho} \tilde{C}^T A_{yy} \tilde{C}\]

and then rearrange it, using (A.66) and (A.67), as

\[-A_{xx} (1 - \bar{\rho}) = \bar{\rho} (S_x - J)^T J^{-1} (S_x - J) - \bar{\rho} \gamma \Sigma \Delta t.\]  

(A.73)

Dividing through by \(\Delta t\) and ignoring terms in \(\Delta t\) in \(S_x - J\) and \(J\Delta t\) — note that \(\tilde{\Lambda}(\Delta t)\Delta t \to \Lambda\) as \(\Delta t \to 0\) — we obtain

\[-\rho A_{xx} = -\gamma \Sigma + \left(A_{xx} - A_{xy} \tilde{C} - C\right) \Lambda^{-1} \left(A_{xx} - \tilde{C}^T A_{xy}^T - C^T\right),\]  

(A.74)

the same as in continuous time — see the systems of equations (A.35) and (A.36).

Once the value-function coefficients in discrete time are established to have as limit their counterparts in continuous time, one proceeds by noting that, when letting \(\Delta t\) go to 0, the rate term \(M^{rate}/\Delta t\) is given by

\[\lim_{\Delta t \to 0} \frac{M^{rate}}{\Delta t} = \lim_{\Delta t \to 0} (J \Delta t)^{-1} (J - S_x) = \Lambda^{-1} \left(A_{xx} - C - \tilde{C} A_{xy}^T\right),\]  

(A.75)

while the aim term \(M^{aim}\) by

\[\lim_{\Delta t \to 0} M^{aim} = (A_{xx} - C^T A_{xy}^T - C^T)^{-1} \left(A_{xy} + \tilde{C}^T A_{yy}\right).\]  

(A.76)

These expressions are the same as obtained in continuous time, as seen in equation (A.30). ■

**Proof of Proposition 7.** According to the market maker’s beliefs, we have

\[\hat{E}_t [p_{t+h}] = e^{r/\rho} p_t,\]
so that the maximization problem (37) becomes
\[
\max_q \left\{ q(p_t - \hat{p}_t) - e^{-r' h} \frac{\gamma^M}{2} \text{Var}_t [qp_{t+h}] \right\}. \tag{A.77}
\]

The variance of \(h\)-periods-ahead prices (denoted by \(V_h\)) can be easily calculated since \(p\) is exogenous and Gaussian.\(^4\) The price \(\hat{p}\) is set so as to satisfy the market-clearing condition
\[
0 = \Delta x_t + q \frac{\Delta t}{h}, \tag{A.78}
\]
yielding
\[
\hat{p}_t = p_t + e^{-r' h} \gamma^M V_h \frac{\Delta x_t}{\Delta t} h. \tag{A.79}
\]

Consequently, if the trader trades an amount \(\Delta x_t\), he trades at the unit price of \(p_t\) and pays an additional transaction cost of
\[
e^{-r' h} \gamma^M \Delta x_t^\top V_h \frac{\Delta x_t}{\Delta t} h =: \frac{1}{2} \Delta x_t^\top \Lambda(\Delta t) \Delta x_t, \tag{A.80}
\]
which has the quadratic form posited in Section 2.1. In particular, the transaction-cost parameter \(\Lambda(\Delta t)\) emerges as proportional to \(e^{-r' h} V_h h / \Delta t\).

We now consider the two cases. In case (i), \(h\) is constant, then \(\Lambda(\Delta t)\) is inversely proportional to \(\Delta t\), to the leading term in \(\Delta t\). In case (ii), \(V_h = V_{\Delta t}\) has leading term of order \(\Delta t\), and therefore \(\Lambda(\Delta t)\) is of order \(O(\Delta t)\).

**Proof of Proposition 8.** The first-order condition with respect to \(q_t\) is
\[
0 = \hat{E}_t \sum_{s = t+N\Delta t} e^{-r'(s-t)} \left( \psi p_s^\top - \gamma^M I_s^\top \frac{V_{\Delta t}}{\Delta t} \right) \frac{\partial I_{s-\Delta t}}{\partial q_t} \Delta t - \hat{p}_t^\top. \tag{A.81}
\]

Using the fact that \(\frac{\partial I_s}{\partial q_t} = (1 - \psi \Delta t)^{s-t} \Delta t\) for \(s \geq t\), the first-order condition yields
\[
\hat{p}_t = \hat{E}_t \sum_{s = t+N\Delta t} e^{-r'(s-t)}(1 - \psi \Delta t)^{s-t} \Delta t \left( \psi p_s - \gamma^M V_{\Delta t} I_{s-\Delta t} \right) \Delta t. \tag{A.82}
\]

Using the facts that \(\hat{E}_t[e^{-r(s-t)} p_s] = p_t\) and \(\hat{E}_t[I_s] = (1 - \psi \Delta t)^{s-t} I_t\), we obtain
\[
\hat{p}_t = p_t - \kappa_t I_t \tag{A.83}
\]

\(^4\)The resulting value is \(V_h = \frac{\Delta t}{e^{r' h} - 1} \left( e^{r' h} - 1 \right) \Sigma\). Note that the fraction tends to \(1/r'\) as \(\Delta t\) tends to zero.
for a matrix

$$
\kappa_I = \sum_{s=N\Delta t}^{\infty} e^{-r'(s+\Delta t)} (1 - \psi \Delta t)^2 \mathbf{z}^\top \gamma^M \frac{V_{\Delta t}}{\Delta t} \Delta t.
$$

(A.84)

We note that, as $\Delta t$ goes to zero, $\kappa_I$ tends to $\frac{2^M}{r' + 2\psi} \Sigma$, a constant. Since we are focusing on the limit, we omit notationally the dependence of $\kappa_I$ on $\Delta t$.

The price $\hat{p}_t$ is only the price at the end of trading date $t$ — the price at which the last unit of the $q_t$ shares is traded. We assume that, during the trading date, orders of infinitesimal size come to market sequentially and the market makers’ expectation is that the remainder of date-$t$ order flow aggregates to zero — thus, the order flow is a martingale. It follows that the price paid for the $k^{th}$ percentile of the order flow $q_t$ is $p_t - \kappa_I((1 - \psi \Delta t)I_{t-\Delta t} + kq_t)$. This mechanism ensures that round-trip trades over very short intervals do not have transaction-cost implications. We discuss in Remark 1 the effect of alternative assumptions.

This price specification is the same as in Section 2.1, with $\Lambda = 0$ and $D_t = -\kappa_I(1 - \psi \Delta t)I_{t-\Delta t}$, $(1 - \psi \Delta t)I_{t-\Delta t}$ being the inventory remaining at the beginning of trading at $t$, and $\kappa_I$ its price impact:

$$
D_{t+\Delta t} = -\kappa_I(1 - \psi \Delta t)I_t
= -\kappa_I(1 - \psi \Delta t)((1 - \psi \Delta t)I_{t-\Delta t} + \Delta x_t)
= -(1 - \kappa_I \psi \kappa_I^{-1} \Delta t)(\kappa_I(1 - \psi \Delta t)I_{t-\Delta t} + \kappa_I \Delta x_t)
\equiv (I - R(\Delta t)) (D_t + C \Delta x_t).
$$

(A.85)

To see that, under the definition of $D_t$ above, the specification $\hat{p}_t = p_t + D_t$ yields the same objective for the trader as (33), we note that, intra-period at $t$, the agent gains $-\kappa_I q_t = \kappa_I \Delta x_t = C \Delta x_t$ on the holding $x_{t-\Delta t}$ already in place, and $\frac{1}{2} \Delta x_t^\top \kappa_I \Delta x_t = \frac{1}{2} \Delta x_t^\top C \Delta x_t$ on the newly acquired shares. The marking-to-market on the entire portfolio $x_t$ between the end of period $t$ and the beginning of period $t + \Delta t$ is indeed

$$
p_{t+\Delta t} - e^{r' \Delta t} p_t + D_{t+\Delta t} - e^{r' \Delta t} (D_t + C \Delta x_t)
$$

(A.86)
as required, since the closing price at $t$ is $p_t + D_t + C \Delta x_t$, while the opening price at $t + \Delta t$ is $p_t + D_{t+\Delta t}$.

**Remark 1** Suppose that trading at $t$ occurs in its entirety at price $\hat{p}_t$. Then the trader no longer makes the gain $\frac{1}{2} \Delta x_t^\top C \Delta x_t$ on the newly acquired shares, so that he faces in effect an additional transitory cost equal to $\frac{1}{2} \Delta x_t^\top C \Delta x_t$.

Since the leading term of $C$ is constant as $\Delta t$ varies, this term falls under case (iii) of Proposition 4. In particular, as $\Delta t$ goes to zero, it gives rise to a quadratic variation term. The agent can achieve any utility value that is arbitrarily close to, yet lower than, the value function without such a term.
Transitory and Persistent Transaction Costs. Here we provide a formal exposition and result pertaining to costs that share both features. As written is the main text, there are two types of dealers.

The first type is as in the sub-section on transitory costs (leading to Proposition 7). Instead of selling to end users, \( h \) units of time later, at price \( p_{t+h} \), they sell to the second type of dealers, at an endogenous price \( \tilde{p}_{t+h} \). The second type of dealers are as in the subsection on persistent costs (leading to Proposition 8); in particular, they deplete at every time \( s \) a proportion \( \psi \Delta t \) of their time-\( s \) inventory with end users at price \( p_s \).

The specification of beliefs is as follows. For the first market maker,

\[
\hat{E}_t^{(1)} [\Delta x_{t+k\Delta t}] = 0 \quad \text{(A.87)}
\]
\[
\hat{E}_t^{(1)} [\Delta x_{t+k\Delta t}^\top \Delta x_{t+k\Delta t}] = v \quad \text{(A.88)}
\]

for all \( k > 0 \). More important, this market maker knows \( I_t \) as well as the history of trades \( \Delta x_{t-k\Delta t} \), and can therefore compute, at time \( t \), \( I_{t+h} \) precisely.

For the second market maker,

\[
\hat{E}_t^{(2)} [\Delta x_{t-h+k\Delta t}] = 0 \quad \text{(A.89)}
\]
\[
\hat{E}_t^{(2)} [\Delta x_{t-h+k\Delta t}^\top \Delta x_{t+k\Delta t}] = v \quad \text{(A.90)}
\]

for all \( k > 0 \). The important distinction is that the second market maker, who has just traded with a market maker of the first type, cannot forecast the next trade with a first-type market maker. This trade is determined by the position taken earlier by first-type market makers, but the second-type market maker is assumed not to observe those trades.

Under these assumptions, we have:

**Proposition A.1 (Transitory and Persistent Transaction Costs)** Suppose that the trader trades serially with two types of market makers as described above. Then the trader is faced with a price \( \bar{p}_t = p_t + D_t \), with \( D_t \) of the form (29), as well as transitory quadratic transactions costs.

**Proof of Proposition A.1.** We proceed by backward induction. The behavior of the second type of market-maker is described by Proposition 8. In particular, the first market-maker knows, when transacting at time \( t \) with the trader, that at \( t+h \), he will liquidate the last unit of his inventory \( q \) at the price \( p_{t+h} - \kappa_I ( (1 - \psi \Delta t) I_{t+h-\Delta t} + q \) .

The proceeds from the sale are

\[
q^\top \left( p_{t+h} - \kappa_I \left( (1 - \psi \Delta t) I_{t+h-\Delta t} + \frac{1}{2} q \right) \right). \quad \text{(A.91)}
\]

We note that the first market maker has observed all trades until date \( t \), and therefore knows exactly \( I_{t+h-\Delta t} \).
Based on (A.91), the transaction price $\bar{p}_t$ is determined so that the optimal choice $q$ of the first market-maker cancel the trader’s trade $\Delta x_t$. The analysis is similar to that leading to Proposition 7. Solving

$$\max_q \mathbb{E}^{(1)}_t \left[ e^{-r_h t} q^\top \left( p_{t+h} - \kappa I \left( (1 - \psi \Delta t)I_{t+h-\Delta t} + \frac{1}{2} q \right) \right) - q^\top \bar{p}_t \right]$$

(A.92)

and imposing the market-clearing condition $q = -\Delta x_t$, we obtain

$$\bar{p}_t = p_t - k_1 I_{t+h-\Delta t} + k_2 \Delta x_t$$

(A.93)

for positive constants $k_1$ and $k_2$ that follow easily. We let $D_t = -k_1 I_{t+h-\Delta t}$, while $k_2 \Delta x_t$ gives the transitory transaction cost. $\blacksquare$

**Proof of Proposition 9.** Note first that the aggregate noise-trader holding $z_t$ satisfies

$$dz_t = \kappa \left( \sum_{l=1}^L f^l_t - z_t \right) dt.$$ 

(A.94)

Given the definition of $f$, the mean-reversion matrix $\Phi$ is given by

$$\Phi = \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\kappa & -\kappa & \cdots & \kappa \end{pmatrix}.$$  

(A.95)

Supposing that $E_t[d p_t - r^f t p_t dt] = B f_t dt$, we use the results in Proposition 1 together with the market-clearing conditions to derive

$$\frac{a}{\lambda} \sigma^{-2} B (a \Phi + \gamma I)^{-1} + \frac{a}{\lambda} e_{L+1} = -\kappa (1 - 2 e_{L+1}),$$

(A.96)

where $e_{L+1} = (0, \cdots, 0, 1) \in \mathbb{R}^{L+1}$ and $1 = (1, \cdots, 1) \in \mathbb{R}^{L+1}$. It consequently follows that, if the investor is to hold $-z_t = -f^L_t$ at time $t$ for all $t$, then the factor loadings must be given by

$$B = \sigma^2 \left[ -\frac{\lambda}{a} \kappa (1 - 2 e_{L+1} - e_{L+1}) (a \Phi + \gamma I) \right].$$

(A.97)
For $l \leq L$, we calculate $B_l$ further as

\begin{align*}
B_l &= -\sigma^2 \kappa (\lambda \psi_l + \lambda \gamma a^{-1} + \lambda \kappa - a) \\
 &= -\lambda \sigma^2 \kappa (\psi_l + \rho + \kappa),
\end{align*}

(A.98)

while

\begin{align*}
B_{L+1} &= \sigma^2 (\rho \lambda \kappa + \lambda \kappa^2 - \gamma).
\end{align*}

(A.99)

We have thus shown that a (unique) matrix $B$ exists such that, if $E_t[dp_t - r' p_t \, dt] = B f_t \, dt$, then the market is in equilibrium. The comparative-static results are immediate. \hfill \blacksquare