EQUILIBRIUM COMMODITY PRICES WITH IRREVERSIBLE INVESTMENT AND NON-LINEAR TECHNOLOGIES

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Abstract

We model oil price dynamics in a general equilibrium production economy with two goods: a consumption good and oil. Production of the consumption good requires drawing from oil reserves. Investment necessary to replenish oil reserves is costly and irreversible. We solve for the optimal consumption, production and oil reserves policy for a representative agent. We analyze the equilibrium price of oil, as well as the term structure of oil futures prices. Because investment in oil reserves is irreversible and costly, the optimal investment in new oil reserves is periodic and lumpy. Investment occurs when the crude oil is relatively scarce in the economy. This generates an equilibrium oil price process that has distinct behavior across two regions (characterized by the abundance/scarcity of oil). We undertake three empirical tests suggested by our model. First, we estimate key parameters using SMM to match moments of oil price futures as well as other macro-economic properties of the data. Second, we estimate an affine regime switching model of the oil price, which captures the main features of our equilibrium model and preserves the tractability of reduced-form models. Lastly, we compare the risk premium in oil futures implied by our model to the data.

Keywords: Commodity prices, Futures prices, Convenience yield, Scarcity, Investment, Irreversibility, General equilibrium, Simulated Method of Moments (SMM), Regime-switching model, risk premium

JEL Classification: C0, G12, G13, D51, D81, E2.
1 Introduction

Oil prices, and energy costs in general, play an important role in the economy. Figure 1 plots the price of oil, US GDP growth, and the NBER recession periods for the post-war period. Hamilton (2003, 2005) documents that nine out of ten of the U.S. recessions since World War II were preceded by a spike up in oil prices. The “oil-shocks” of the 1970’s are the most dramatic example. More interesting, Hamilton (2003) documents that the connection between oil and economic activity is not simple. Oil expenditures account for roughly 4 percent of US GDP. However, the 7.8 percent reduction in world oil production in 1973 is associated with a large 3.2 percent drop in real US GDP. In contrast, the 8.8 percent drop in production that occurred at the time of the 1991 Persian Gulf War lead only a 0.1 percent drop on GDP. Many papers have studied the role of oil in real-business-cycle (RBC) models, which turns out to be a difficult task. In this paper we focus on a related question. We investigate the properties of the price of oil (and oil futures) in a general equilibrium model where oil is an important input to the production of the consumption good, and where oil production is characterized by significant adjustment costs and irreversibility.

Oil prices display a number of interesting characteristics. First, there is tremendous variation in the spot price of oil. Since 1947 the real spot price has fluctuated between $10 and $90 per barrel (see Figure 1). Even in the more recent period of 1990 to 2008, the spot price of oil ranged from $10 to $140 per barrel (Figure 2). This large volatility is not just present in the spot price. Figure 2 plots sample futures curves for oil in the 1990 to 2008 period. Although, the volatility is smaller for longer-horizon delivery futures contracts (the “Samuelson Effect”), even three-year futures contracts exhibit substantial volatility. More generally, there is substantial variation in the slope of the futures curve. Many of the futures curves, 63% in our sample, have a negative slope (called “backwardation”). Other times, the futures curve has a positive slope (“contango”) or has a single hump. Note that at different times the spot price is the same, but the properties of the rest of the futures curve differ. The fact that the spot price is not a sufficient state variable to characterize futures prices is the reason many commodity derivative models begin with a two-factor specification. Typically, the variation in the slope of the forward curve is usually captured with a factor called “convenience yield.” While oil prices are volatile and futures prices suggest some interesting dynamic properties, the evidence about whether oil-price risk commands a risk premium is unclear. Gorton and Rouwenhorst (2006) document that there is no risk-premium to holding commodity risk while Erb and Harvey (2006) come to the opposite conclusion. Specifically, the evidence is inconclusive as to whether the futures price of oil, on average, is equal to the future spot price. Understanding the theoretical connection between futures prices and the price of risk will shed some light on this ambiguous evidence.

To understand the properties of the oil price we consider a two-good economy. The consumption

\footnote{Some authors have extended the standard RBC model (e.g. Kydland and Prescott (1982)) to study the response of the economy to exogenous energy shocks. See Kim and Loungani (1992), Finn (1995), Rotemberg and Woodford (1996) and Leduc and Sill (2004) among others.}

\footnote{A barrel of oil is 42 US gallons, 35 imperial gallons, or 159 liters.}
good is used for consumption and investment. Production of this consumption good requires as inputs both consumption good and a second good, oil (which could more generally be seen as energy). Oil used in production flows from existing oil reserves. New oil can be added to reserves when needed. But this requires a costly and irreversible investment of the consumption good. We solve for the optimal consumption, investment, and oil reserves policy of a representative agent. From this we derive the equilibrium price of oil as well as the term structure of oil futures prices. The two key assumptions that drive our main results are that investment in new oil reserves is irreversible and has a fixed-cost component. Given these two assumptions, the optimal investment in new oil reserves is periodic and lumpy. This feature of investment generates variation in the important state variable, the ratio of oil-reserves to consumption good. This variation carries over to the equilibrium oil price dynamics.

The optimal consumption and investment is characterized by the state variable of the ratio of the stock of oil reserves to the stock of the consumption good (capital). We define the price of oil as the marginal value of an additional unit of oil reserves. Given this characterization, the central feature of our model is that the lumpy, periodic investment in new oil implies that the spot price of oil is non-monotonic in the state variable. When oil is plentiful (oil/capital ratio is high), investment in new oil reserves is a long way off. As oil is used in production the ratio of oil-to-capital decreases and the price of oil increases. In contrast, when oil is scarce (oil-to-capital ratio is low), we are near the investment boundary. In this region, as oil is used and the oil-to-capital ratio declines, the price of oil falls. Despite the fact the current quantity of oil is smaller, the marginal value of a unit of oil is smaller since the near-term investment will increase the stock of oil. The result is that the dynamic properties of the oil price (and hence futures prices) are notably different in the abundance and scarcity regions. In effect, the price of oil is governed by a process with two regimes.

Our model is similar to partial equilibrium commodity models that focus on the role of commodity storage. These models typically assume an inverse (net) demand function that maps the quantity of the commodity and a shock into a price. The fact that inventories must be non-negative produces an asymmetry in the spot price behavior at zero inventory (stock-out). This generates interesting dynamic properties for the commodity spot price and the related futures prices. Storage models are similar to our general equilibrium model in that they produce a two-regime process for the spot price. Our general equilibrium model adds to the storage approach along a few dimensions. First the zero-inventory or stock-out event that is so central to inventory models is infrequently observed in oil markets. For example, the oil-futures curve is downward sloping 63% of the time. It is hard to see zero (or even near-zero) inventory levels anywhere near this frequency. Of course,

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3 The U.S. crude oil and natural gas drilling activity is a result of investment in these energy sectors. Since 1974, the two-year count of rotary rigs in operation has spiked twice by more than 80% (in the 1980-1982 and 2000-2002 periods). We use this as evidence of lumpiness in commodity investments.

4 There are many papers that take this approach. For example, Gustafson (1958), Newbery and Stiglitz (1981), Wright and Williams (1982), Scheinkman and Schechtman (1983), Williams and Wright (1991), Deaton and Laroque (1992), Chambers and Bailey (1996), Routledge, Seppi, and Spatt (2000), and Bobenrieth, Bobenrieth, and Wright (2002).
this fact by itself may not be troubling as inventories are difficult to measure and the storage technology in these models does not capture the complexities of inventory. More troubling, however, is the difficulty inventory models have explaining longer-horizon dynamics. In order to explain frequent backwardation, models must be calibrated so inventory levels hit zero frequently. As a result, the inventory state variable has little impact over price levels beyond the short horizon of inventory. Routledge, Seppi, and Spatt (2000), for example, normalize oil-futures prices so that the twelve-month futures price has zero variance. Lastly, due to computational limitations, this literature typically assume risk-neutrality and hence do not have implications for commodity risk premia.

Our paper is also related to other equilibrium models that study the behavior of commodity prices. Carlson, Khokher, and Titman (2007) propose an equilibrium model of natural resources and study the effect of adjustment costs in the forward price dynamics. However, in contrast to our paper, they assume risk-neutrality, an exogenous demand function for commodity, and (the main friction in their model) that commodity is exhaustible, whereas in our paper commodity is essentially present in the ground in infinite supply but is costly to extract. Also, Kogan, Livdan, and Yaron (2008) identify a new pattern of futures volatility term structure that is inconsistent with standard storage models. This new pattern can be explained within their industry equilibrium model that exhibits investment constraints and irreversibility. Unlike our model, they take the demand side and risk-premia as exogenous and focus mainly on the implications for the volatility curve.

To illustrate the quantitative features and advantages of our general equilibrium model, we undertake three (related) empirical exercises. First we estimate our model using Simulated Method of Moments (SMM) estimation proposed by Lee and Ingram (1991) and Duffie and Singleton (1993). We estimate the parameters of our model to match the sample average term-structure of the oil futures price and the sample volatility of the term-structure along with some aggregate macroeconomic moments. With (arguably) sensible parameters, we match the term-structure of futures prices and volatilities reasonably well. The model generates backwardation (negative slope in the futures curve) on average. The model also captures the level and shape of the term-structure of volatilities pretty well. In theory, the dynamic properties of the oil price are different in the scarcity and abundance regions. This is only relevant empirically if the (endogenous) state variable has a distribution that accesses these regions frequently enough to matter. Under our estimated model parameters, this is indeed the case. The economy is scarcity of oil 14% of the time and in abundance 86% of the time.

One of the disadvantages of a general equilibrium model is that it is challenging to compute. In practice, commodity pricing is often implemented using an exogenously specified process of the spot price and a convenience yield (a second factor that captures the dynamics of the futures curve slope). In this setting, pricing futures contracts and other derivatives is straightforward.\footnote{See for example, Gibson and Schwartz (1990), Brennan (1991), Ross (1997), Schwartz (1997), Schwartz and Smith (2000) and Casassus and Collin-Dufresne (2005).}
This approach is very effective since it is tractable enough to price complex commodity-contingent claims, but often suffers from the lack of theoretical justification for the underlying dynamics chosen for the state variables.

Our second empirical exercise is motivated by trying to link both approaches. In the model, the equilibrium process for the oil price differs across the scarcity and abundance regions. However, it turns out that within those regions the spot price behavior is simple to characterize. In particular, within a regime, a linear approximation to the drift and volatility functions is a reasonable approximation of our equilibrium price. We use quasi-maximum likelihood technique of Hamilton (1989) to estimate a two-regime model with crude oil price data. The resulting parameter estimates on the drift and volatility are in line with the calibrated model. Moreover, since the stochastic process is, up to the two-regimes, affine, we expect the reduced-form approximation of our model to be useful for derivative pricing. In addition, we estimate the smoothed inference about whether the state of the economy is in the scarcity or the abundance regions (see Kim (1994)). This is a helpful diagnostic on the model since we can confirm that the data are consistent with a two-regime process that is economically meaningful (both regimes are visited frequently).

One of the advantages of a model in general equilibrium is that we can investigate the risk premium properties of the oil commodity. Interestingly, in our model the risk-premium on oil is state-dependent. In the abundance region, a long position in oil (e.g., an oil futures) is risky and commands a risk premium. In the scarcity region, a long position in oil is a hedge and the risk premium is negative. Our third empirical exercise is to investigate the conditional risk premium in oil-futures data. In regressing oil price return on the market return we find that the beta is significantly negative in the estimated scarcity regime and positive (though not statistically significant) in the other regime. The difference across regimes is consistent with our model. More importantly, variation in the commodity risk premium makes unconditional tests of the risk-premium more difficult to interpret (e.g., Gorton and Rouwenhorst (2006) and Erb and Harvey (2006)).

Our paper proceeds with the model developed next in Section 2. The equilibrium properties are characterized in Section 3 (with most of the proofs contained in the appendix). Section 4 contains the empirical implementations of our model where we estimate with SMM, characterize a two-regime affine model, and explore the properties of the commodity risk premium. Finally, Section 5 concludes.

2 The Model

We consider an infinite horizon production economy with two goods. The model extends the Cox, Ingersoll Jr., and Ross (1985) production economy to the case where the production technology requires a second input which we interpret as oil. The second commodity is not directly consumed.

Sundaresan (1984) first extended the CIR economy to non-linear production technology with two goods (and no frictions) to study the term structure of interest rates.
but is a required input for the production of the consumption (numeraire) good. We solve the representative agent’s optimal consumption/investment policy and use this to characterize equilibrium prices.

### 2.1 Representative Agent Characterization

There is a representative agent with time-separable, constant relative risk aversion expected utility preferences over lifetime consumption of a single numeraire good.

\[
E_0 \int_0^\infty e^{-\rho s} \frac{C_s^{1-\gamma}}{1-\gamma} ds
\]

The rate of time preference is \( \rho > 0 \) and the level of relative risk aversion is \( \gamma > 0 \) (with the usual understanding that \( \gamma = 1 \) is the log risk aggregator). In particular, note that only one good is consumed. The single consumption good in our economy has stock denoted \( K_t \). There is a second good in our economy, oil, that is necessary for consumption good production. The stock oil reserves is \( Q_t \).\(^7\) As oil is used in production its stock is depleted. For simplicity, we assume that oil reserves can be used at a constant flow rate \( \bar{i} \).\(^8\) This captures the physical difficulty in adjusting rapidly the extraction flow of known oil reserves.\(^9\) The stock of oil also depreciates at rate \( \delta \) if not used. New oil reserves can be created by investing the consumption good. Specifically, \( X_t \) units of new oil capacity can be created at \( t \) by investing \( \beta(X_t, Q_t, K_t) \) units of the consumption good. Given the oil investment cost structure we describe below, investment in new oil will be periodic and we denote \( dI_t = 1 \) if there is investment at \( t \) and 0 otherwise. The dynamics of the consumption good stock, \( K_t \) and oil capacity \( Q_t \) are given by

\[
\begin{align*}
dK_t &= \left( \alpha K_t^{1-\eta} (\bar{i}Q_t)^\eta - C_t \right) dt - \beta(X_t, Q_t, K_t) dI_t + \sigma_K K_t dw_{K,t} \\
dQ_t &= -(\bar{i} + \delta) Q_t dt + X_t dI_t + \sigma_Q Q_t dw_{Q,t}
\end{align*}
\]

The growth rate of the stock of consumption good (capital) is determined by capital and the quantity of oil used in production. The two input goods follow a Cobb-Douglas production function with the parameter \( \alpha \) as the productivity of capital and \( \eta \) determining the importance of oil in the economy. Since oil flows at rate \( \bar{i} \), the amount of oil used in production at \( t \) is proportional to the stock of oil reserves \( Q_t \).

\(^7\)One could also think of \( Q_t \) as the number of oil wells, each of which produces a constant flow of oil \( \bar{i} \).

\(^8\) We can extend the model to allow the rate of flow to be optimally chosen. However, such an extension makes the model inconsistent with the degrees of backwardation observed in the data. The forward prices would rise at the riskless interest rate if the agents can make costless demand adjustments (this is an implication from Hotelling (1931)). Carlson, Khokher, and Titman (2007) discusses this point for the case where producers of the commodity can make costless supply adjustments.

\(^9\)Anecdotal stories describe oil wells in Siberia where the flow cannot be interrupted to avoid freezing. Generically, modulating the speed at which oil can be extracted from existing wells and delivered to end-users is a costly and inflexible process.
Investment in new oil capacity is irreversible \((X_t \geq 0)\). The cost of an investment to generate new oil reserves includes a fixed cost. Specifically, we assume that building \(X_t\) oil reserves costs \(\beta(X_t; Q_t, K_t)\) where
\[
\beta(X_t; Q_t, K_t) = \beta_K K_t + \beta_Q Q_t + \beta_X X_t
\]
where \(\beta_K, \beta_Q, \beta_X > 0\). The fixed cost component is \(\beta_K K_t + \beta_Q Q_t\). This cost is scaled by the size of the economy \(K_t\) and \(Q_t\) to maintain the homotheticity of the model. Practically, if the fixed cost were not scaled its influence over investment behavior would vanish as the economy grows. The importance, of course, of the fixed-cost assumption is that the creation of new oil is an ‘impulse control’ optimization problem, where the optimal investment decision occurs only at discretely dates (it is lumpy). In order for oil-capacity investment to be feasible, we assume \(\beta_K < 1\) and \(\beta_Q < \beta_X\).

Our model abstracts from technological improvements in the oil sector by assuming that importance of oil to production \(\eta\) and investment-cost parameters \(\beta_K, \beta_Q, \text{and} \beta_X\) are constant. This assumption delivers a homothetic (hence, tractable) model. However, they ignore technological improvement in oil use and exploration (e.g., advances in miles-per-gallon and the changes in oil exploration technology). This is surely an important aspect but we leave it to future research.

Finally, uncertainty in our economy is captured by the Brownian motions \(w_{Q,t}\) and \(w_{K,t}\) which drive the diffusion terms in equations (1) and (2). We allow the shocks to be correlated, denoted \(\rho_{KQ}\). As is standard, we assume that there exists an underlying probability space \((\Omega, \mathcal{F}, P)\) satisfying the usual conditions, and where \(F = \{\mathcal{F}\}_{t \geq 0}\) is the natural filtration generated by the Brownian motions.

### 2.2 Optimal Consumption and Investment with Fixed Costs and Irreversibility

Given the fixed cost of irreversible investment, it is natural to seek an oil investment policy of the form \(\{(X_t, T_t)\}_{i=0,1,...} \in \mathcal{A}\) where \(\{T_i\}_{i=0,...}\) is a sequence of stopping times of the filtration \(\mathcal{F}\) such that \(I_t = 1_{\{T_i \leq t\}}\) and the \(X_{T_i}\) are \(\mathcal{F}_{T_i}\)-measurable random variables. The set of admissible strategies \(\mathcal{A}\) ensure the consumption good stock process is positive \((K_t > 0\ a.s.)\). Lastly, the set of allowable consumption policies \(\mathcal{C}\) must be positive integrable \(\mathcal{F}\) adapted processes. The optimal consumption-investment policy of the representative agent has the ‘current’ value function
\[
J(K_t, Q_t) = \sup_{C \in \mathcal{A}} E_t \left[ \int_t^\infty e^{-\rho(s-t)} \frac{C_s^{1-\gamma}}{1-\gamma} \, ds \right]
\]
and subject to the laws-of-motion in equations (1) and (2).

Given that investment in new oil is irreversible \((X_t \geq 0)\) and the presence of fixed costs, the optimal investment will be infrequent and ‘lumpy’ (e.g., Dumas (1991)). This defines two zones of the state space \(\{K_t, Q_t\}\): A no-investment region where \(dI_t = 0\) and an investment region where
$dI_t = 1$. This is analogous to the shipping cone in Dumas (1992), but with only one boundary because investment is irreversible. Appendix A establishes sufficient conditions on the parameters for a solution to the problem in (4) to exist. We also present there a closed-form solution to the problem when oil investment is perfectly reversible.

2.2.1 Optimal Consumption Strategy in the No-Investment Region

When the state variables $\{K_t, Q_t\}$ are in the no-investment region, a portion of the consumption good, $K$, can be consumed with the remainder used in consumption-good production (along with oil). This region is defined by the condition $J(K_t - \beta(X_t; Q_t, K_t), Q_t + X_t) < J(K_t, Q_t)$ since it is not optimal to use the consumption good to invest in new oil reserves. In this region, the solution of the problem in equation (4) is determined by the following the Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{\{C \geq 0\}} \left\{ -\rho J + \frac{C^{1-\gamma}}{1-\gamma} + DJ \right\} = 0 \quad (5)$$

where $D$ is the Itô operator

$$DJ(K, Q) \equiv \left( \alpha K_t^{1-\eta} (\hat{\theta} Q_t)^\eta - C \right) J_K - (\hat{\theta} + \delta) Q J_Q$$

$$+ \frac{1}{2} \sigma_K^2 K^2 J_{KK} + \frac{1}{2} \sigma_Q^2 Q^2 J_{QQ} + \rho_{KQ} \sigma_K \sigma_Q K Q J_{KQ} \quad (6)$$

with $J_K$ and $J_Q$ representing the marginal value of an additional unit of consumption good and oil respectively. $J_{KK}$ is the second derivative with respect to $K$.

To characterize optimal consumption we derive the first order conditions for equation (5)

$$C_t^* = J_K^{-\frac{1}{\gamma}}. \quad (7)$$

At the optimum, the marginal value of consumption is equal to the marginal value of an additional unit of the consumption good.

2.2.2 Optimal Investment Strategy

It is optimal to invest in new oil reserves when the benefit of additional oil reserves exceeds its cost. This condition is $J(K_t - \beta(X_t; Q_t, K_t), Q_t + X_t) \geq J(K_t, Q_t)$.\footnote{Without loss of generality we assume that the initial capital stocks $\{K_0, Q_0\}$ are in the no-investment region. If this is not the case, there is an initial lumpy investment that takes the state variables into the no-investment zone.} Investment occurs when crude oil is relatively scarce in the economy. Let $J_1 = J(K_t^*, Q_t^*)$ be the value function immediately before investment and $J_2 = J(K_t^* - \beta(X_t^*, Q_t^*, K_t^*), Q_t^* + X_t^*)$ be the value function immediately after the
investment is made. The investment zone is defined by the value matching condition

\[ J_1 = J_2 \]  \hspace{1cm} (8)

There are three optimality conditions that determine the investment threshold level of consumption good \( K^*_t \), the amount of oil \( Q^*_t \), and the size of the optimal oil investment \( X^*_t \) at the investment boundary. We follow Dumas (1991) to determine these super-contact (smooth pasting) conditions.\(^\text{11}\)

\[
\begin{align*}
J_{1K} &= (1 - \beta_k)J_{2K} \hspace{1cm} (9) \\
J_{1Q} &= -\beta_Q J_{2K} + J_{2Q} \hspace{1cm} (10) \\
0 &= -\beta_X J_{2K} + J_{2Q} \hspace{1cm} (11)
\end{align*}
\]

Together, these equations imply that

\[
(\beta_X - \beta_Q)J_{1K} - (1 - \beta_k)J_{1Q} = 0. \hspace{1cm} (12)
\]

### 2.2.3 Reduction of number of state variables

The consumption good production function is homogeneous of degree one and the utility function is homogeneous of degree \((1 - \gamma)\). The value function inherits this property. As a result, the ratio of oil reserves to the consumption good stock is sufficient to characterize the economy. Define \( j(z) \) as

\[
J(K, Q) = \frac{K^{1-\gamma}}{1-\gamma} j(z) \hspace{1cm} (13)
\]

where \( z \) is the log of the oil reserves to capital ratio

\[
z = \log \left( \frac{Q}{K} \right) \hspace{1cm} (14)
\]

and measures the relative abundance of crude oil in the economy. The dynamic process for \( z_t \) is obtained using a generalized version of Itô’s Lemma.

\[
dz_t = \mu z_t dt + \sigma_z dw_{z,t} + \Lambda_z dI^*_t \hspace{1cm} (15)
\]

The no-investment and investment regions are characterized solely by \( z_t \). Let \( z_1 = \log(Q^*_t) - \log(K^*_t) \) be the log oil to capital ratio just prior to investment. Similarly, define \( z_2 = \log(Q^*_t + X^*_t) - \log(K^*_t - \beta(X^*_t)) \) as the log ratio immediately after the optimal investment in oil occurs. When \( z_t > z_1 \) it

\(^{11}\)For a discussion of value-matching and super-contact (smooth-pasting) conditions, see Dumas (1991), Dixit (1991) and Dixit (1993). If \( \beta_K = \beta_Q = 0 \) in equation (3) then we face an Infinitesimal Control problem. In this case, the optimal investment is a continuous regulator (Harrison (1990)), so that oil stock before and after investment are the same. In this case, equations (9) to (12) result directly from equation (8) as can be checked via a Taylor series expansion (as shown in Dumas (1991)). To solve this case we consider two additional ‘super-contact’ conditions \(-J_{1QK} + \beta_X J_{1KK} = 0\) and \(-J_{1QQ} + \beta_X J_{1KQ} = 0\).
is optimal to postpone investment in new oil. If the state variable $z_t$ reaches $z_1$, an investment to increase oil stocks by $X_t^*$ is made. $z_1$ delineates the investment trigger which occurs when the relative scarcity of oil is at its highest level. The result is that the state variable jumps to $z_2$ which is inside the no-investment region, therefore $\Lambda_z = z_2 - z_1$. Given the investment cost structure in equation (3), the proportional addition to oil, $x_t$, is just a function of $z_1$ and $z_2$.

$$x_t^* \equiv \frac{X_t^*}{Q_t^*} = \frac{e^{-z_1} - e^{-z_2} - (\beta_K e^{-z_1} + \beta_Q)}{e^{-z_2} + \beta_X}$$

(16)

The jump in oil reserves is

$$\frac{Q_2}{Q_1} \equiv \frac{Q_t^* + X_t^*}{Q_t^*} = 1 + x_t^*$$

(17)

and, we can express the jump in the consumption good stock simply as:

$$\frac{K_2}{K_1} \equiv \frac{K_t^* - \beta(X_t^*)}{K_t^*} = 1 - (\beta_K + \beta_X x_t^* e^{z_1} + \beta_Q e^{z_1})$$

(18)

Finally, the optimal consumption from (7) can be rewritten in terms of $j(z_t)$ as:

$$c_t^* \equiv \frac{C_t^*}{K_t} = \left( j(z_t) - j'(z_t) \left( \frac{1}{1 - \gamma} \right) \right)^{-\frac{1}{\gamma}}$$

(19)

Plugging the definition of $J(K,Q)$ into the Hamilton-Jacobi-Bellman in equation (5) returns a one-dimensional ODE for the function $j(z)$. This substitution also yields one-dimensional value-matching and super-contact conditions. Appendix B contains these equations as well as the formal proofs of optimality and details on the derivations. The system of equations does not have (to the best of our knowledge) a closed-form solution, but Appendix C sketches the numerical technique used to solve for the equilibrium.

3 Equilibrium Prices

The solution to the representative agent’s problem of equation (4) characterizes equilibrium prices.\(^\text{12}\) We are interested in the pricing kernel, the price of oil, and the term-structure of oil futures prices.

3.1 Asset Prices and the Pricing Kernel

We assume our model has dynamically complete markets so the pricing kernel is characterized by the representative agent’s marginal utility (see Duffie (2001)). Since investment in new oil reserves is lumpy, there is a singularity in the pricing kernel and asset prices can jump. Since the investment

\(^{12}\)The fixed cost of building new oil reserves makes decentralizing the economy not straightforward. Specifically, it is hard to reconcile the “zero profit” condition of perfect competition with the need to recover fixed investment costs. See Guesnerie (1975) and Romer (1986), for examples.
in new oil reserves is predictable, the jump in asset prices is also predictable. To rule out arbitrage, all asset prices in the economy jump by the same amount (see Karatzas and Shreve (1998)).

Define the risk-free money-market account with price is $B_t$

$$\frac{dB_t}{B_t} = r_t dt + \Lambda_B dI_t$$

(20)

where $r_t$ is the instantaneous risk-free rate in the no-investment region. $\Lambda_B$ is a jump in financial market prices that can occur when the lumpy investment in the oil industry occurs. The pricing kernel process

$$\frac{d\xi_t}{\xi_t} = -\frac{dB_t}{B_t} - \lambda_{K,t} dw_{K,t} - \lambda_{Q,t} dw_{Q,t}$$

(21)

with $B_0 = \xi_0 = 1$. The pricing kernel dynamic is determined from the representative agent’s marginal utility as:

$$\xi_t = e^{-\rho t} J_K(K_t, Q_t)$$

$$r_t = (1 - \eta) \left( \frac{iQ_t}{K_t} \right)^\eta - \sigma_K (\lambda_{K,t} + \rho_K \lambda_{Q,t})$$

(23)

$$\lambda_{K,t} = -\sigma_K \frac{J_{KK}}{J_K}$$

(24)

$$\lambda_{Q,t} = -\sigma_Q \frac{J_{KQ}}{J_K}$$

(25)

$$\Lambda_B = -\frac{\beta_K}{1 - \beta_K}$$

(26)

It is straightforward to show that each of these pricing kernel terms are functions solely of the state variable $z_t = \log (Q_t/K_t)$.

### 3.2 The Oil Price

We price the value of a barrel of oil using the standard utility indifference argument. We define the spot oil price ($S_t$) as the marginal value of an incremental unit in oil capacity. Specifically, $S_t$ solves $J(K_t, Q_t) = J(K_t + S_t \epsilon, Q_t - \epsilon)$ as $\epsilon \to 0$. With a Taylor expansion, this implies $S_t = J_Q/J_K$ or, equivalently, in terms of the state variable $z_t$,

$$S_t = \frac{e^{-z_t} j'(z_t)}{(1 - \gamma) j(z_t) - j'(z_t)}$$

(27)

Our definition of the oil price, $S_t$ is literally the marginal price of adding one unit of oil to reserves. Since one cannot adjust the rate $\tilde{i}$ at which oil reserves are used in our model, this seems the
appropriate definition. In particular, this implies that the price of oil is different from the marginal productivity of oil, because agents cannot alter the demand rate $i$ (see footnote 8).

Since the price is a function of the state variable $z_t$, we can write the spot price process as

$$\frac{dS_t}{S_t} = \mu_S(z_t) + \sigma_V(z_t)dw_{K,t} + \sigma_V(z_t)dw_{Q,t} + \Lambda_SdI_t$$

We will characterize the drift and volatility shortly. For now, consider the behavior of the oil price at the investment boundary $z_1$ (recall the subscript “1” indicates just prior to investment and “2” indicates just after investment). Substituting $S_t = J_Q/J_K$ into equations (11) and (12) lets us characterize the oil price at the point of investment. First, the value of a unit of oil is equal to its marginal cost at the time of investment. That is,

$$S_2 = \beta_X$$  (28)

Second, just prior to investment, equation (12) implies that

$$S_1 = \frac{\beta_X - \beta_Q}{1 - \beta_K}$$  (29)

As with the pricing kernel, the lumpy investment can result in a jump in the price. At the investment boundary $z_t = z_1$, $dS_t/S_t = \Delta_S$ or $S_2 - S_1 = \Lambda_S S_1$. Therefore,

$$\Lambda_S = \frac{\beta_Q - \beta_K \beta_X}{\beta_X - \beta_Q}$$  (30)

Since oil is not a traded financial asset, the jump in the oil price can differ from the price jump in financial assets. For example, if $\beta_Q = \beta_K \beta_X$, then oil price has no jump. In this case, the fixed cost of oil investment at the point of investment is proportional to aggregate wealth ($K_t + S_2 Q_t$). For simplicity of exposition, we focus on this case in the SMM calibration that follows.

### 3.3 Oil Futures Prices

Given the equilibrium processes for oil price and the pricing kernel, we can characterize the behavior of oil futures prices in our model. Define $F(z, t, T)$ as the date-$t$ futures contract that delivers one unit of oil at date $T$ given that the state of the economy is $z$.

$$\frac{dF_t}{F_t} = \mu_{F,t}dt + \sigma_{F,K,t}dw_{K,t} + \sigma_{F,Q,t}dw_{Q,t} + \Lambda_FdI_t$$  (31)

Since the futures contracts are continuously market-to-market, the value of the futures contract is zero.
where $\mu_{F,t}$, $\sigma_{FK,t}$, $\sigma_{FQ,t}$ and $\Lambda_F$ are determined in equilibrium following Cox, Ingersoll Jr., and Ross (1985) as satisfying the partial differential equation

$$\frac{1}{2}\sigma_z^2 F_{zz} + (\mu_z + \sigma_K (\lambda_{K,t} + \rho_{KQ} \lambda_{Q,t}) - \sigma_Q (\lambda_{Q,t} + \rho_{KQ} \lambda_{K,t})) F_z + F_t = 0 \quad (32)$$

with boundary condition

$$F(z,T,T) = S(z). \quad (33)$$

Note that futures prices do not have any jump at the oil-investment point, $F(z_1,t,T) = F(z_2,t,T)$, implying $\Lambda_F = 0$. 15

### 3.4 Convenience yield

The convenience yield is defined as the benefit that accrues to the holder of the physical commodity, but not to the owner of a futures contract. 16 The convenience yield is analogous to a dividend payment on a standard financial futures. However, unlike for financial futures where the dividend is an explicit monetary payment made to the owner of the underlying, for commodities the convenience yield has to be measured implicitly from the difference between the expected (excess) returns on spot and futures prices (or equivalently, from the difference between the risk-neutral and physical spot price drifts). This is analogous to calculating the implicit convenience yield from the “cost-of-carry” and the slope of the futures curve as in Routledge, Seppi, and Spatt (2000).

The following proposition presents the equilibrium cumulative convenience yield in our economy:

**Proposition 1** The implicit cumulative net convenience yield $Y_t$ has the following dynamics 17:

$$dY_t = y_t S_t dt + \Lambda_Y S_t dI_t \quad (34)$$

where

$$y_t = \frac{-i}{S_t} \left( \frac{\alpha \eta (K_t/t Q_t)^{1-\eta} - S_t}{\alpha \eta \omega Q + \eta} \right) - \delta - \sigma_Q (\theta_{Q,t} + \rho_{KQ} \theta_{K,t}) \quad (35)$$

$$\theta_{Q,t} = -\sigma_Q \frac{Q J_{QQ} Q}{J_Q} \quad (36)$$

$$\theta_{K,t} = -\sigma_K \frac{K J_{KQ} Q}{J_Q} \quad (37)$$

$$\Lambda_Y = \Lambda_B - \Lambda_S. \quad (38)$$

---

15 This condition derives directly from the fact that futures prices are martingales under the risk-neutral measure.

16 In many commodity pricing models the net convenience yield is the second factor used to describe futures prices (see Gibson and Schwartz (1990)). In these models, backwardation (downward sloping forward curve) is implied by the convenience yield. The convenience yield (and its relation to spot prices) can also have an important effect on the degree of mean-reversion in commodity prices (see Casassus and Collin-Dufresne (2005)).

17 The continuous component of the convenience yield $y_t$ is a function only of $z_t$, but as before, we prefer to present this variable under $\{K_t, Q_t\}$ rather than under $z_t$ to deliver better economic intuition from the result.
The convenience yield is determined implicitly from equilibrium prices using the no-arbitrage condition for tradable assets

\[ E^*_t \left[ \frac{dS_t}{S_t} \right] = \frac{dB_t}{B_t} - \frac{dY_t}{S_t} \]  

(39)

where \( E^*_t \) is the expectation under the equivalent martingale measure. The relation between this expectation and the expectation under physical measure is:

\[ E_t \left[ \frac{dS_t}{S_t} \right] = E^*_t \left[ \frac{dS_t}{S_t} \right] - \frac{d\xi_t}{\xi_t} \frac{dS_t}{S_t}. \]  

(40)

Applying Itô’s lemma to \( S_t = J_{Q_t}/J_K \) and using equation (40) we can determine \( dY_t \) from equation (39).

Equation (34) shows two components of the convenience yield. The first is the absolutely continuous component \( y_t \). It depends on the difference between the marginal productivity of oil \((\alpha \eta(K_t/(\hat{i} Q_t))^{1-\eta})\) and its price \( (S_t) \). Because, the rate of oil extraction \( \hat{i} \) is fixed, the instantaneous marginal productivity of oil differs from the price of holding one unit of oil in reserves. When marginal productivity is higher (lower) than the oil price, there is a positive (negative) convenience yield to owning oil.\(^{18}\) Further, this endogenous convenience yield is increasing in the ratio of capital stock to commodity inventories, \( K_t/Q_t \). This implies that the convenience yield is higher near the investment region when the crude oil is relatively scarce. Finally, risk (in the form of both supply and technology shocks) tends to lower the convenience yield.

The convenience yield also has a singular component, which is due to the predictable jumps that occur in prices at the time of oil investment. If oil were a traded asset then \( \Lambda_Y \) would represent pure arbitrage profits that can be locked in by trading oil prices against any other financial asset. Instead, the commodity is not a financial asset, and its ‘price’ is the shadow value to consumers of using it as an input to production.

Equation (35) gives a clear interpretation of the convenience yield in terms of the marginal productivity of a unit of oil in excess of its financial cost \( S_t \), its physical depreciation \( \delta \) and an adjustment for supply shock risk. Richard and Sundaresan (1981) presents the isomorphism between convenience yield and interest rates in a multi-good economy. Comparing equation (35) with that for the short rate \( r_t \) in equation (23) we see a strong resemblance. Effectively, the convenience yield \( y \) can be interpreted as an interest rate in an economy where we switch numeraire and use the commodity instead of the consumption good. In that economy, \( r_t \) would become a ‘convenience yield’ on the consumption good.

\(^{18}\)If the representative agent could flexibly choose \( \hat{i} \) then, in fact, this component would be zero.
4 Model Estimation

To illustrate the quantitative features and advantages of our general equilibrium model, we undertake three empirical exercises. First we estimate our model using Simulated Method of Moments (SMM) estimation. This lets us explore the quantitative properties of our model with a plausible set of parameters. Second, we estimate a two-regime affine approximation to our general equilibrium model where the two regimes capture the key feature of the scarcity/abundance of oil. Third, we use our model to estimate the risk-premium process associated with oil. In particular, we exploit the theoretical property that the risk premium on an oil futures position depends on the relative scarcity of oil.

4.1 Moments and SMM Estimation

In order to see if our model can capture the interesting features of oil prices, we calibrate our model using Simulated Method of Moments (SMM) (Lee and Ingram (1991) and Duffie and Singleton (1993)). We focus on matching the unconditional mean level of the term-structure of futures prices and their unconditional volatility from NYMEX prices. Since we are interested in the role oil plays in the economy we also try to match the average consumption to GDP ratio, the average oil-consumption to GDP ratio, and the average real interest rate. We use quarterly data of crude oil futures prices and aggregate macroeconomic variables of OECD countries from Q4/1990 to Q2/2008. We use two different time periods for the SMM estimation: Panel A (pre-Irak war) from Q4/1990 to Q1/2003 and Panel B (whole sample) from Q4/1990 to Q2/2008. We do this because in recent years (since the Irak war) average crude oil prices have risen dramatically and have become more volatile. The unconditional volatility of long-maturity futures prices has also increased compared to the beginning of the sample period (see figure 2). The data sources and construction are listed in Appendix D.

Our model has twelve parameters that are listed in Table 1. Since SMM is very computationally intensive, we fix some of the parameters based on other studies. The oil share of income, \( \eta \), is set to 0.04 which is consistent with recent RBC studies that include energy as a production factor (see Finn (1995), Finn (2000) and Wei (2003)). The depreciation rate for oil wells, \( \delta \), is set to 0.1. This is consistent with a storage cost of approximately $4 per barrel per year (in line with Ross (1997)). In data, it is hard to separately observe shocks to the oil-stock and the capital stock. We simply assume that the shocks to capital and oil stocks are independent (\( \rho_{KQ} = 0 \)). We fix the volatility of oil stock using the deviation of annual changes of petroleum consumption in our dataset (i.e., \( \sigma_Q = 0.013 \)). Lastly, we set the rate of time preference, \( \rho \) parameter to 0.05.

The cost structure for the production of new oil is central to our model. We estimate both, the variable cost of oil, \( \beta_X \), and the fixed cost component \( \beta_K \). We choose the parameter \( \beta_Q \) so that the total fixed cost is proportional to aggregate wealth in the economy (\( K_t + S_2Q_t \)) which
implies that the oil price is continuous even at the investment boundary. We also estimate the capital productivity, \(\alpha\), the rate of oil extraction, \(\bar{i}\), the volatility of the capital sector, \(\sigma_K\), and the curvature of the representative agent’s utility function, \(\gamma\).

The unconditional moments in our data are sample averages, \(G_T = \frac{1}{T} \sum_{t=1}^{T} g_t\). Denote the analogous moments in our model as \(G_Z(\psi)\) where \(\psi\) are the parameters we are estimating. Given a particular set of parameters, \(\hat{\psi}\), we solve the Hamilton-Jacobi-Bellman equation in (B5) with the numerical technique described in Appendix C. This determines the endogenous dynamics of our state variable, \(z_t\). To approximate the density function of \(z\), \(f(z; \hat{\psi})\) we simulate the model.\(^{19}\) Using this simulated density, we calculate the moments implied by our model as \(G_Z(\hat{\psi}) = E^Z[g(z; \hat{\psi})] \approx \int g(z)f(z)dz\). We chose parameters to solve

\[
\psi^* = \arg\min_{\psi \in \Psi} [G_Z(\psi) - G_T]'W_T[G_Z(\psi) - G_T] 
\]

where \(\Psi\) is the set of feasible parameters (i.e., where the model is well-defined). \(W_T\) is the weighting or distance matrix. We choose \(W_T\) to be the inverse of the diagonal of the unbiased estimate covariance matrix of the sample averages. This weighting matrix ensures that the scale of each moment condition is the same, and gives more weight to less volatile moments.\(^{20}\)

Table 2 and Figure 3 show the mean term structure of futures prices and their volatility for the whole time period (i.e. from Q4/1990 to Q2/2008). The average futures curve over this period is downward sloping. This “backwardation” is a common feature of oil prices. In fact, in the sample period, 63% of the time the 6-months maturity contract is below the 1-month maturity contract. The model generates a similar average degree of backwardation than in the data. The unconditional volatility is also decreasing in the horizon reflecting the high degree of mean reversion in the oil price (this is often called the “Samuelson effect”). Our model also captures this feature of the data reasonably well. Finally, note from Table 2 that our model implies sensible values for consumption to GDP, oil consumption to GDP, and the real interest rate. We match the mean values of these items almost exactly and closely match the standard deviation of these quantities.

Figure 4 shows the historical and model implied mean futures curve for both time periods. As reported above, the model correctly fits the unconditional futures curves. The figure also shows the average curves conditional on whether the spot price is below or above the average spot price. These conditional curves are not explicitly included as moments for the SMM estimation. However, the model is able to generate similar conditional curves for the Q4/90-Q1/03 period (data vs. model). Indeed, the slope and level of the implied conditional curves are closely related to the ones in the data. For the whole time period, the model fails to replicate the conditional curves. This is especially true for the futures curve conditional on high prices where in the data, the curve is

\(^{19}\)We discretize the state space of \(z_t \in [-20, 10]\) in a grid of 30,000 points and then simulate weekly samples of the state variable for \(10^5\) years.

\(^{20}\)Cochrane (2005) discusses the pros and cons of using different weighting matrices for the estimation.
above $60, while in the model it is around $40.\textsuperscript{21} Nevertheless, the model captures the higher unconditional volatility for the long-maturity futures, which is consistent with the observed data. To see this, compare the long-term spreads between the conditional curves for both sample periods.

Table 1 reports the SMM estimates for both panels of data. These estimates all seem reasonable and in line with other RBC models. The one notable parameter estimate is the relatively low estimate of risk-aversion close to 0.5.\textsuperscript{22} The estimation results show that the volatility of capital, $\sigma_K$, the marginal cost, $\beta_X$, and the extraction rate of oil, $\bar{i}$, increase when considering the whole time period. The volatility of capital increases from 0.39 (Panel A) to 0.42 (Panel B) to match the higher volatility of crude oil prices in recent times. The marginal cost of oil production increases from $18.1$ (Panel A) to $26.9$ (Panel B). A higher marginal cost yields higher average crude oil prices and is consistent with the recent entry of low-yield wells that produce oil that is costlier to refine.\textsuperscript{23} Finally, a higher demand rate of oil is necessary to match a higher oil-consumption to GDP ratio that arises because oil-consumption is measured in dollars.

Overall, with sensible model parameters, we match the term-structure of futures prices and volatilities reasonably well. The model also produces reasonable quantities for output, oil consumption, and real interest rates. Therefore, we use this parametrization to explore the quantitative implications of the model.

4.1.1 Oil spot price

Figure 5 plots the equilibrium oil spot price against the state variable, $z_t$. Recall that $z_t$, the log of the oil to capital ratio, is a measure of the relative scarcity of oil. Interestingly, the oil price is non-monotonic in the state variable and reaches a maximum that is greater than the marginal extraction cost $\beta_X$ at $z_{S_{\text{max}}}$, which lies between the investment threshold $z_1$ and the replenishment level $z_2$ when there is a fixed cost component ($\beta_K, \beta_Q > 0$). This happens because oil price is driven by both current and anticipated changes in oil reserves. Since oil investment is periodic, as oil is depleted, the stock of oil falls and this is reflected in a lower value of $z_t$. When oil is plentiful ($z_t \geq z_{S_{\text{max}}}$), the decreased stock of oil leads to an increase in the spot price of oil. We define this zone as the \textit{abundance region}, because there is a relative abundance of oil in the economy. The fixed cost involved in adding new oil stocks implies that it is not optimal to make a new investment as soon as the spot price (marginal benefit of oil) reaches the marginal cost of adding new oil.

\textsuperscript{21}This occurs because for the estimated parameters, the model is unable to reach crude oil prices above $50$ (as we will see later, the price has an upper bound at $S_{\text{max}} = 48.04$).

\textsuperscript{22}In the time-additive utility specification we employ here, the risk-aversion parameter is the only curvature in the utility specification and reflects attitudes to both risk and time aggregation. We leave a recursive utility model to future research. See Epstein and Zin (1989), Duffie and Epstein (1992) and Backus, Routledge, and Zin (2005).

\textsuperscript{23}The \textit{World Economic Outlook} from the IMF (2008) reports the annual marginal cost of producing a barrel of oil from 1991 to 2007. This cost is defined as the average of the highest-cost (or bottom quartile) producers, based on a survey of listed oil companies. In recent years, the marginal cost increased steadily from $21.4$ in 2002 to $74.2$ in 2007. Moreover, the average marginal cost for the 1991-2002 period was $18.8$, while for the 1991-2007 period was $28.5$. These figures are in line with our SMM estimates.
\( \beta_x \). Therefore the spot price rises above \( \beta_x \) as oil is depleted. However, closer to the investment threshold, the oil price reflects the likelihood of a near-term lumpy investment in new oil. In this zone, as oil is depleted and \( z_t \) decreases, the probability of hitting the investment increases and the oil price decreases in anticipation of investment in new oil reserves. This defines the scarcity region due to the relative scarcity of oil stocks. Lastly, note that in this parametrization with a fixed cost that is proportional to aggregate wealth, the oil price is unchanged at the point of investment \((S(z_1) = S(z_2))\).

Figure 5 shows that the oil price cannot be written as a one-dimensional Markov process, despite that fact that it is (see equation (27)) a function of \( z_t \), which itself follows a (one-dimensional) Markov process. However, the figure suggests that the price process can be made Markov if, in addition to the price level \( S_t \) we also know a discrete state variable, say \( \varepsilon_t \), a proxy for the relative scarcity/abundance regime (i.e., the fact that \( z_t \) is less or greater than \( z_{S_{\text{max}}} \)). We formalize this observation in the proposition below, which will be useful for our next empirical exercise, where we model the time-series of oil prices as a regime switching model (as we do not observe the state variable \( z_t \)).

**Proposition 2** The oil price in equation (27) is governed by the following two-regime stochastic process

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu_S(S_t, \varepsilon_t)dt + \sigma_{SK}(S_t, \varepsilon_t)dw_{K,t} + \sigma_{SQ}(S_t, \varepsilon_t)dw_{Q,t} + \Lambda_S dt \\
\mu_S(S_t, \varepsilon_t) &= r(S_t, \varepsilon_t) - y(S_t, \varepsilon_t) + \\
&\quad \sigma_{SK}(S_t, \varepsilon_t) \left\{ \lambda_K(S_t, \varepsilon_t) + \rho_{KQ}\lambda_Q(S_t, \varepsilon_t) \right\} + \\
&\quad \sigma_{SQ}(S_t, \varepsilon_t) \left\{ \lambda_Q(S_t, \varepsilon_t) + \rho_{KQ}\lambda_K(S_t, \varepsilon_t) \right\} \\
\sigma_{SK}(S_t, \varepsilon_t) &= \lambda_K(S_t, \varepsilon_t) - \theta_K(S_t, \varepsilon_t) \\
\sigma_{SQ}(S_t, \varepsilon_t) &= \lambda_Q(S_t, \varepsilon_t) - \theta_Q(S_t, \varepsilon_t)
\end{align*}
\]

where

\[
\varepsilon = \begin{cases} 
1 & \text{if } z > z_{S_{\text{max}}} \\
2 & \text{if } z_1 < z \leq z_{S_{\text{max}}}
\end{cases}
\]

and where \( r(S_t, \varepsilon_t) = r_t \), \( \lambda_K(S_t, \varepsilon_t) = \lambda_{K,t} \), \( \lambda_Q(S_t, \varepsilon_t) = \lambda_{Q,t} \), \( \Lambda_S \) is from equation (30) and \( y(S_t, \varepsilon_t) \) is the convenience yield. \( y(S_t, \varepsilon_t) \), \( \theta_K(S_t, \varepsilon_t) \) and \( \theta_Q(S_t, \varepsilon_t) \) are defined in Proposition 1.

**Proof** First, we apply Itô’s lemma to the definition of the spot price \( S_t = J_Q/J_K \). The dynamics of \( S_t \) depends on the third order terms \( J_{KKK}, J_{KKQ}, J_{KQQ} \) and \( J_{QQQ} \). We differentiate the HJB equation in (5) to simplify the resulting SDE for \( S_t \) and obtain equations (42) to (45). To show that the dynamics of \( S_t \) depends only on \( \{S_t, \varepsilon_t\} \), we note that there is a one-to-one mapping between \( \{S_t, \varepsilon_t\} \) and \( z_t \). Using the reduction of states variables presented in subsection 2.2.3, we obtain that all the variables in Proposition 2 are only a function of \( z_t \) and thus of \( \{S_t, \varepsilon_t\} \).
Figure 6 plots the conditional instantaneous return and conditional instantaneous volatility of return as a function of $S_t$. As you would expect from Figure 5, the drift and volatility differ across the scarcity- and abundance-of-oil regions. Since the oil price is bounded by 0 and $S_{max}$, it is mean reverting (negative drift in the top panel of Figure 6). However, the rate of mean reversion is much higher in the scarcity region. A positive shock to the capital stock reduces the oil to capital ratio ($z_t$). However, the effect of this change on the oil price differs across regions and this is reflected in the correlation. This is directly important to the risk premium properties of oil futures prices and we return to this topic below. Lastly, note that the level of the volatility depends on the level of the spot price relative to the maximum spot price, $S_{max}$. At $S_{max}$ the volatility is zero since the price will decrease almost surely.

4.1.2 Oil futures prices

Using the estimated parameters of our model, we can solve for futures prices. Sample futures prices are plotted in Figure 7. As above, we can characterize the futures prices in terms of the current spot price and the scarcity/abundance region. The thick futures curves indicate the abundance region and the thin lines indicate the scarcity. Broadly, the futures prices in our model capture the futures prices in the data (see Figure 2). When the oil is low, oil is plentiful and the state variable is far from the investment trigger. This means that the supply of oil decreases on average, so the expected price in the future is above the current price. In these situations the futures curves are upward-sloping or in contango.

When the spot price of oil is high (near $S_{max}$), the futures curve is downward sloping since the expected change in spot price is negative. More interestingly for intermediate prices, the futures curve can be hump-shaped. These hump-shaped curves highlight that the spot price is not sufficient to characterize the futures curve. The futures curve are steeper in the scarcity region. This is implication of the high likelihood of a near-term investment in new oil in this zone. For example, in Figure 7 consider the curve with a current spot price $S_t = 30$ in the scarcity region (thin line). In the near term, there is an expected drop in spot price from the likely new investment. This is followed by an average price increase as the new oil is depleted. In contrast, a current spot price $S_t = 30$, but in the abundance region (thick line), reflects an anticipated increase in the spot price as oil is depleted. Possible new investment in oil influences the longer horizon prices.

For intuition, we have discussed futures shape in terms of expected future prices, but this is not the full story in this model where risk-premia play a central role (and futures prices are risk-adjusted expected future spot prices). In fact, it is straightforward to interpret the dynamic properties of the futures curve slope in terms of convenience yield. In general, the futures curve are steeper when the spot price is in the scarcity region, because the convenience yield is higher in this region (see figure 8).

---

24 This is the traditional arbitrage pricing approach, e.g., Casassus and Collin-Dufresne (2005).
4.1.3 Oil futures volatility

The volatility of the futures contract are shown in Figure 9. To compare the futures volatility for different oil spot prices we show the relative volatility which we define as $\sigma_F(S_t, \varepsilon_t; T - t)/\sigma_S(S_t, \varepsilon_t)$. This ratio corresponds to the inverse of the optimal hedge ratio, which is the number of futures contracts in a portfolio that minimizes the risk exposure of one unit of oil. Since the futures price with zero maturity is the spot price, this ratio is 1 when $t = T$.

As above, we can characterize the volatilities in terms of the current spot price and the scarcity/abundance region. The thick futures curves indicate the abundance region and the thin lines indicate the scarcity region. In general, the volatilities are much lower for higher maturities. This is a consequence of mean reversion in spot prices (i.e. the Samuelson Effect). The figure highlights that the hedge-ratio depends on both the level of the spot price and the scarcity/abundance region. In contrast, many commodity price models impose log-linear relation between spot and futures prices that implies a hedge ratio that is independent of price (e.g., Schwartz (1997)). Second, the curves are non-monotonic in the maturity horizon. For high prices, the expected investment in oil (rise in supply) is reflected in the futures contract and also in the volatility. For short maturities and very high prices the relative volatility has an abrupt behavior because the volatility of the spot price is very low (recall that $\sigma_S(S_{\text{max}}, \varepsilon, t) = 0$). Eventually, the relative volatility could be negative at longer horizons. This reflects the expected transition of the state-variable across the two regimes where the spot price volatilities change sign.

4.1.4 Backwardation and Contango

The interesting features of the spot and futures prices all stem from the property that the dynamic properties of the oil price are different in the scarcity and abundance regions. This is only relevant empirically if the (endogenous) state variable has a distribution that accesses these regions frequently enough to matter. The fact that our model matches the unconditional average pattern of futures prices (see Table 1) is encouraging. However, given our parametrization we can directly calculate the relative frequency. Figure 10 plots the unconditional distribution for the state variable, $z_t$. The mapping from $z_t$ to the spot price is overlaid to highlight that both, the scarcity and the abundance regions, are visited frequently. The economy is in the scarcity region 14% of the time and in the abundance region 86% of the time. The lower panel of Figure 10 show the related distribution for the spot price. The spot price is frequently near its maximum. This implies that, consistent with the data, futures prices in our model are often downward sloping in our calibration.

4.2 Regime-Switching Estimation

While our structural model captures many interesting features of the data, it is challenging to compute. In comparison, commodity pricing models in practice are often based on an exogenous
specification of a spot price and a convenience yield (e.g., a slope of a futures curve). These two-factor models are flexible enough to capture many salient features of the data while remaining tractable. Our second empirical exercise links our general equilibrium model to the more tractable reduced-form contingent claims approach. As seen in proposition 2 and figure 6, the stochastic process for the equilibrium spot price differs across the scarcity and abundance regions. However, within those regions a linear approximation to the drift and volatility functions works quite well. Hence we can estimate a two-factor model where the second factor is an indicator for the scarcity/abundance regions.

The linear approximation of our structural model is

\[
dS_t = \mu_S(S_t, \varepsilon_t)S_t dt + \sigma_S(S_t, \varepsilon_t)S_t dw_{S,t}
\]

where

\[
\mu_S(S, \varepsilon) = \alpha + \kappa \varepsilon (\log[S_{\text{max}}] - \log[S])
\]

(48)

\[
\sigma_S(S, \varepsilon) = \sigma \varepsilon \sqrt{\log[S_{\text{max}}] - \log[S]}
\]

(49)

and \( \varepsilon_t \) is a two-state Markov chain with transition (Poisson) probabilities

\[
P_t = \begin{bmatrix}
1 - \lambda_1 dt & \lambda_1 dt \\
\lambda_2 dt & 1 - \lambda_2 dt
\end{bmatrix}
\]

(50)

To estimate the model we use quasi-maximum likelihood technique of Hamilton (1989) to estimate a two-regime model.\(^{25}\) A by-product of the estimation technique are the smoothed inferences for each regime.\(^{26}\) The data here consists of weekly Brent crude oil prices between Apr-1983 and Apr-2005.\(^{27}\)

The parameter estimates and standard errors of our model are given in Table 3. Although we do not impose this directly on the estimation, regime one corresponds to the abundance region and regime two corresponds to the scarcity region. Most parameters are significant implying that there are clearly two regimes in the data. The parameters also vary across regimes implying that these regimes are significantly different. The parameter \( \alpha = -0.184 \) is negative and significant implying that the process for the price has an upper bound at \( S_{\text{max}} \) which is estimated at $87.27 per barrel (\( \log[S_{\text{max}}] = 4.469 \)). Also consistent with the general equilibrium model, the mean reversion in the abundance regime is stronger than in the scarcity regime. Lastly, the abundance regime is characterized to be less volatile than the scarcity regime (\(|\sigma_1| < |\sigma_2|\)).

\(^{25}\)Given we are working with weekly data, we do a quasi-maximum likelihood estimation by considering only the first two moments of the distribution.

\(^{26}\)We follow Kim (1994) algorithm, which is a backward iterative process that starts from the smoothed probability of the last observation.

\(^{27}\)We deflate the price by US Consumer Price Index. The average price is 16.29 dollars per barrel in 1983 prices (or 31.53 dollars per barrel in 2005 prices). The annualized standard deviation of weekly returns is 38\%. The skewness in crude oil prices for this period is 1.09 and the excess kurtosis is 0.42.
The resulting parameter estimates on the drift and volatility are in line with the model. This is useful since the stochastic process is, up to the two-regimes, affine, one can apply standard derivative pricing techniques. In addition the smoothed inference about whether the state of the economy is in the scarcity or abundance of oil regions are also consistent with our general equilibrium model. Figure 11 shows the crude oil price and the inferred probability of being in the scarcity state (regime 2). The economy stays, on average one year in the abundance regime. This is also the most frequent regime with the economy here approximately 79.5% of the time in it (i.e., \( \lambda_2 / (\lambda_1 + \lambda_2) = 0.795 \)). The economy typically stays in the second regime, the scarcity regime, several months. This mirrors closely the calibration of our general equilibrium model in the prior section. We use these inferred probabilities to investigate the risk premium properties of the model next.

### 4.3 Commodity Risk Premium

Interestingly, in our model the risk-premium on oil is state-dependent. Figure 12 plots the risk premium (in our calibrated model) as a function of the oil-to-capital state variable. The risk premium is positive in the abundance region and negative in the scarcity region. Recall, from Figure 6 (lower panel) that the sign of the volatility of the spot price differs across the scarcity/abundance regions. This implies that the correlation between shocks to the consumption good and the spot price of oil also changes sign across the regions. In the scarcity region, a long position in a near-term oil future hedges negative shocks to the consumption good. A negative shock to the stock of the consumption good in the scarcity region implies an increased oil price (and an increased price in near-term futures prices) since the relative stocks of commodity decreases. In the abundance region, the opposite is true and a long-position in futures earns a positive risk premium.

Can we detect this time variation in oil risk premium? To investigate we use our empirical proxy for the scarcity region (from the two-regime estimation above) and see if the risk premium, relative to the simple benchmark of U.S. equities, is sensitive to the regime. We consider the following specification:

\[
\begin{align*}
    r_{j,t+1}^e &= a_t + b_t r_{M,t+1}^e + \epsilon_{t+1} \\
    a_t &= a_0 + a_1 \hat{p}_t \\
    b_t &= b_0 + b_1 \hat{p}_t
\end{align*}
\]

The return on a long position in oil is \( r_{j,t+1}^e \) and is computed two ways. We use the return on a spot investment in oil (analogous to a buy and hold strategy). As an alternative, we use a fully collateralized long futures position in oil.\(^{28}\) We proxy aggregate risk with U.S. equities (CRSP value weighted index). As a proxy for the scarcity/abundance region, \( \hat{p}_t \), we use our inferred (smoothed)

---

\(^{28}\)This is the more common approach for investors to take positions in energy markets. See Gorton and Rouwenhorst (2006) and Erb and Harvey (2006).
probability of being in the scarcity state estimated using the regime switching model in the previous section (see Figure 11).

The results are consistent with the model in that the risk-premium is sensitive to the scarcity and abundance regimes. Columns 1 and 3 in table 4 document the well known fact that risk premium in oil markets are, if they exist, not easy to detect. Unconditionally, an investment in oil does not bear systematic risk. The intercept from the unconditional regression is also not significant. Including the inferred probability of being in the scarcity regime, reported in columns 2 and 4 of Table 4, has a significant impact. Including the regime improves the fit of the model (the $R^2$ increase from 1% to 5% and from 1% to 4%, respectively). Consistent with our model, the risk premium appears lower and negative in scarcity regime. The coefficient estimate of $b_1$ is negative and significant. In contrast, the risk premium in the abundance region is not different from zero.

We tried several alternative specifications for the returns and proxy for the scarcity/abundance regime. The results are broadly consistent. The risk-premium is negative and significant in the (proxy to) the scarcity region. These results may help understand why detecting risk premia in commodity trading strategies has thus far produced ambiguous results (e.g., Gorton and Rouwenhorst (2006) and Erb and Harvey (2006)). However, we leave a joint estimation of the risk-premium properties and all the parameters of our model for future research.29.

5 Conclusion

We developed a two-good general equilibrium model where one of the goods, oil, is not directly consumed but is essential for production of the consumption good. The key assumptions in the model are that oil investment is irreversible and occurs at a cost that includes a fixed component. These assumptions imply that oil investment is periodic and lumpy. As a result, the implied oil price dynamics differ when the economy is near the point of new oil investment, where the crude oil is relatively scarce, than far from the investment threshold, when there is plenty of oil. This implies a rich behavior for the term structure of futures oil price. We estimate key parameters of our model using a Simulated Method of Moments estimation to match features of the economy (oil-to-GDP ratio, interest rates) and properties of oil prices. In particular, we match the term structure of oil futures average prices and price volatility. Since much of the predictive power of the model comes from the differing spot-price behavior across the scarcity/abundance regions, we consider a simpler “reduced-form” model. We estimate a two-regime switching model where the spot price within the regime is affine. We find that not only does this empirical specification fit the data reasonably well, it is also a sensible approximation of the general equilibrium model. This characterization, then, captures the key properties of the general equilibrium model and maintains much of the tractability of standard commodity derivative pricing. Lastly, we use our general equilibrium setting to explore

29 In particular, the use of the smoothed probabilities from our linear regime shifting model are estimated from the whole data set and are subject to a look-ahead bias (see Lettau and Ludvigson (2001).)
the risk-premium characteristics of a long investment in oil futures. Immediately, the two-regime 
property for oil prices translates to a risk premium that is state dependent. In fact, in our model 
the risk premium is positive in one state and negative in the other. Empirically, controlling for the 
scarcity/abundance regime helps to identify the risk premium in an oil position.
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Appendix

A Sufficient Conditions for Existence of a Solution

We note that our problem is slightly different than in traditional models without fixed costs such as Dumas (1992) or Kogan (2001). Indeed, unlike in these models the no-transaction cost problem does not provide for a natural upper bound. Indeed, in our case, if we set \( \beta_K = \beta_Q = \beta_X = 0 \) the value function becomes infinite, since it is then optimal to build an infinite number of oil wells (at no cost). Thus unlike in these papers, it is natural to expect that sufficient conditions on the parameters for existence of the solution should depend on the marginal cost of building an oil well (as well as other parameters). Indeed, intuitively, if the marginal costs of an additional oil well is too low relative to the marginal productivity of oil in the \( K \)-technology one would expect the number of oil wells built (and thus the value function) to be unbounded. To establish reasonable conditions on the parameters we consider the case where there are only variable costs (\( \beta_K = \beta_Q = 0 \) and \( \beta_X > 0 \)), but where the investment decision is perfectly reversible. Let us denote \( J_u(t,K,Q) \) the value function of the perfectly reversible investment/consumption problem. Clearly, the solution to that problem will be an upper bound to the value function of (4).

When the investment decision is perfectly reversible then it becomes optimal to adjust the stock of oil wells continuously so as to keep \( J_u(t,K,Q) = 0 \). This suggests that one can reduce the dimensionality of the problem, and consider as the unique state variable \( W_t = K_t + \beta_X Q_t \) the ‘total wealth’ of the representative agent (at every point in time the agent can freely transform \( Q \) oil wells into \( \beta_X Q \) units of consumption good and vice-versa). Indeed, the dynamics of \( W \) are:

\[
dW_t = (\alpha K_t^{1-\gamma} (\bar{w} t)^\gamma - C_t - \beta_X (\bar{w} t + \delta) Q_t)dt + \sigma_k K_t dw_{K,t} + \beta_X \sigma_Q Q_t dw_{Q,t}
\]

(A1)

Since along each path, the agent can freely choose to adjust the ratio of oil to capital stock \( z_t = \frac{Q_t}{K_t} \) the Cobb-Douglas structure suggests that it will be optimal to maintain a constant ratio, \( Z_t = Z^* \). We may rewrite the dynamics of \( W_t \) as

\[
dW_t = \left( \mu_w^u(Z^*) - c_t^u \right) dt + \sigma_w^u(Z^*) dw_{w,t}
\]

(A2)

where \( w_{w,t} \) is a standard Brownian motion and we define

\[
C_t = c_t^u W_t,
\]

(A3)

\[
\mu_w^u(Z) = \frac{\alpha \bar{w} t^\gamma - \bar{w} t + \delta) \beta_X Z}{1 + \beta_X Z}
\]

(A4)

and

\[
\sigma_w^u(Z) = \sqrt{\frac{\sigma_k^2 + 2 \rho_k \sigma_k \sigma_Q \beta_X Z + (\beta_X Z)^2 \sigma_Q^2}{1 + \beta_X Z}}.
\]

(A5)

The proposition below verifies that if the function

\[
f(Z) = \frac{\rho}{1 - \gamma} - \mu_w^u(Z) + \gamma \frac{\sigma_w^u(Z)^2}{2}
\]

(A6)

admits a global minimum at \( Z^* \) such that

\[
a^u := \frac{1}{\gamma} \left\{ \rho - (1 - \gamma) \left( \mu_w^u(Z^*) - \gamma \frac{\sigma_w^u(Z^*)^2}{2} \right) \right\} > 0
\]

(A7)

then the optimal strategy is indeed to consume a constant fraction of total wealth \( c_t^u = a^u \) and to invest continuously so as to keep \( Q_t/K_t = Z^* \).

Proposition A1 Assume that there are no fixed costs (\( \beta_K = \beta_Q = 0 \)), and that investment is costly (\( \beta_X > 0 \)), but fully reversible. If the function \( f(Z) \) defined in (A6) admits a global minimum \( Z^* \) such that condition (A7) holds then the optimal value function is given by

\[
J_u(t,K,Q) = e^{-\rho t} (a^w)^{1-\gamma} (K + \beta_X Q)^{1-\gamma}
\]

(A8)
The optimal consumption policy is
\[ C^*_t = a^u(K^*_t + \beta_x Q^*_t) \tag{A9} \]
and the investment policy is characterized by:
\[ \frac{Q^*_t}{K^*_t} = Z^*. \tag{A10} \]

**Proof** Applying Itô’s lemma to the candidate value function we have:
\[ \frac{dJ_u(t, K_t, Q_t)}{J_u(t, K_t, Q_t)} + U(t, C_t)dt = (1 - \gamma) \{ h(c_t) - f(Z_t) \} dt + (1 - \gamma)\sigma^u(W(t, C_t)dw, t) \tag{A11} \]
where we have set \( C_t = c_t(K_t + \beta_x Q_t), \sigma^u(W(t, C_t)dw, t) \) are defined in equations (A5) and (A6), respectively, and we have defined:
\[ h(c) = (a^u)^\gamma(c^{1-\gamma}) - \frac{1}{1-\gamma} - c. \]
Note that the function \( h(c) \) is concave and admits a global maximum \( c^*_t = a^u \) with \( h(a^u) = \frac{a^u}{1-\gamma} \). Suppose the function \( f(Z) \) is strictly convex and admits a global minimum at \( Z^* \). Then, if we pick the constant \( a^u \) such that \( h(a^u) = f(Z^*) \), we have for any \( c, Z \):
\[ h(c) - f(Z) \leq h(c^*_t) - f(Z^*) = 0 \]
Thus integrating equation (A11) we obtain:
\[ J_u(T, K_T, Q_T) + \int_0^T U(t, C_t)dt \leq J_u(0, K_0, Q_0) + \int_0^T (1 - \gamma)J_u(t, K_t, Q_t)\sigma^u(W(t, C_t)dw, t) \tag{A12} \]
Taking expectation and using the fact that the stochastic integral is a positive local martingale we obtain:
\[ E \left[ J_u(T, K_T, Q_T) + \int_0^T U(t, C_t)dt \right] \leq J_u(0, K_0, Q_0) \tag{A13} \]
Further we note that for when we choose the controls \( c_t = a^u \) and \( Z_t = Z^* \) then we obtain equality in equation (A12) and further have:
\[ \frac{dJ_u}{J_u} = -a^u dt + (1 - \gamma)\sigma^u(W(t, C_t)dw, t) \tag{A14} \]
which implies that the local martingale is a martingale and thus (A13) obtains with equality. Further we have
\[ \lim_{T \to \infty} E[J_u(T, K_T, Q_T)] = \lim_{T \to \infty} J_u(0, K_0, Q_0)e^{-a^uT} = 0 \]
under the assumption (A7). Letting \( T \to \infty \) in (A13) shows that our candidate value function indeed is the optimal value function and confirms that the chosen controls are optimal. \( \square \)

We note that in the case where \( \eta = 0 \), then Oil has no impact on the optimal decisions of the agent and the value function \( J_u \) is the typical solution one obtains in a standard Merton (1973) or Cox, Ingersoll Jr., and Ross (1985) economy. In that case, the condition on the coefficient \( a^u \) becomes:
\[ a_0 = \frac{1}{\gamma} \left\{ \rho - (1 - \gamma)(\alpha - \gamma \frac{\sigma_Q^2}{2}) \right\} > 0. \tag{A15} \]

A lower bound to the value function is easily derived by choosing to never invest in oil wells (i.e., setting \( dI_t = 0 \) \( \forall t \)) and by choosing an arbitrary feasible consumption policy \( C^*_t = \alpha K^1_{1-\eta}(\bar{Q}) \eta \). Indeed, in that case we have:
\[ \frac{dK_t}{K_t} = \sigma_K dw, K_t \tag{A16} \]
It follows that if the following condition holds:
\[ a^l := \rho + (1 - \gamma) \left\{ (1 - \eta) \frac{\sigma_K^2}{2} + \eta \left( i + \delta + \gamma \frac{\sigma_Q^2}{2} \right) \right\} + (1 - \eta)^2 \left\{ \frac{\sigma_K^2}{2} - \rho_{KQ} \frac{\sigma_K \sigma_Q}{2} + \frac{\sigma_Q^2}{2} \right\} > 0 \tag{A17} \]
then, we have
\[ J_i(0, K_0, Q_0) := E \left[ \int_0^\infty e^{-\delta t} \frac{(C_i^t)^{1-\gamma}}{1-\gamma} dt \right] = \frac{1}{\alpha^i} \left( C_i^0 \right)^{1-\gamma} \]

(A18)

We collect the previous results and a few standard properties of the value function in the following proposition.

**Proposition A2** If \( a^i, a^u > 0 \), the value function of problem (4) has the following properties.

1. \( J_i(t, K, Q) \leq J(t, K, Q) \leq J_u(t, K, Q) \).
2. \( J(t, K, Q) \) is increasing in \( K, Q \).
3. \( J(t, K, Q) \) is concave homogeneous of degree \((1-\gamma)\) in \( Q \) and \( K \).

For the estimation we select parameters such that conditions (A7) and (A17) are satisfied, i.e., that \( a^i, a^u > 0 \).

## B Homogeneity and Optimality

Subsection 2.2.3 presents how to use the homogeneity to reduce the number of states variables. In this section we offer more details needed to characterize the solution and present a proposition that defines the optimal consumption and investment strategy.

We start by recalling that \( j(z) \) satisfies
\[ J(K, Q) = \frac{K^{1-\gamma}}{1-\gamma} j(z) \]  

(B1)

where \( z \) is the log of the oil wells to numeraire-good ratio \( z = \log \left( \frac{Q}{K} \right) \). The dynamic process for \( z_t \) is in equation (15) where
\[
\begin{align*}
\mu_{zt} &= \left( -(\bar{i} + \delta) - \frac{1}{2} \sigma_q^2 \right) - \left( \alpha(\bar{e}^z)^\gamma - c^*_z - \frac{1}{2} \sigma_k^2 \right), \\
\sigma_z &= \sqrt{\sigma_k^2 - 2 \sigma_{KQ} \sigma_k \sigma_Q + \sigma_Q^2}, \\
\Lambda_z &= z_2 - z_1,
\end{align*}
\]

(B2, B3, B4)

and the consumption rate, \( c^*_z = \frac{C^*_z}{K} \), is a function of \( z \).

Plugging consumption in equation (19) and the definition of \( J(K, Q) \) into the Hamilton-Jacobi-Bellman in equation (5) we obtain one-dimensional ODE for the function \( j \).

\[
\begin{align*}
\theta_0 j(z) + \theta_1 j'(z) + \theta_2 j''(z) + \gamma \left( j(z) - \frac{j'(z)}{1-\gamma} \right)^{1-\frac{1}{\gamma}} + \alpha(\bar{e}^z)^\gamma ((1-\gamma)j(z) - j'(z)) &= 0
\end{align*}
\]

(B5)

where
\[
\begin{align*}
\theta_0 &= -\rho - \gamma(1-\gamma)\frac{\sigma_k^2}{2}, \quad \theta_1 = -(\bar{i} + \delta) + \gamma \sigma_k (\sigma_k \sigma_Q - \rho_{KQ} \sigma_Q) - \frac{\sigma_k^2}{2}, \quad \theta_2 = \frac{\sigma_k^2}{2}
\end{align*}
\]

(B6)

To determine the investment policy, \( \{z_1, z_2\} \), the value-matching condition of equation (8) becomes:
\[ (1 + e^{\bar{z}_1} \beta_X)^{1-\gamma} j(z_1) - (1 - \beta_K + e^{\bar{z}_1} (\beta_X - \beta_Q))^{1-\gamma} j(z_2) = 0 \]

(B7)

Lastly, using the homogeneity there are only two super-contact conditions to determine that capture equations (9), (10), and (11).\(^30\) They are
\[
\begin{align*}
(1 - \gamma) e^{\bar{z}_1} (\beta_X - \beta_Q) j(z_1) - (1 - \beta_K + e^{\bar{z}_1} (\beta_X - \beta_Q)) j'(z_1) &= 0 \quad \text{(B8)} \\
(1 - \gamma) e^{\bar{z}_1} \beta_X j(z_2) - (1 + e^{\bar{z}_1} \beta_X) j'(z_2) &= 0 \quad \text{(B9)}
\end{align*}
\]

\(^30\)In a similar way, if \( \beta_K = \beta_Q = 0 \) the two super-contact conditions presented in footnote (11) become the same condition \((1 + (1 - \gamma) e^{\bar{z}_1} \beta_X) j'(z_1) - (1 + e^{\bar{z}_1} \beta_X) j''(z_1) = 0 \).
The following proposition summarizes the above discussion and offers a verification argument. Let us define the functions:

\[ a(z) := j(z) - \frac{j'(z)}{1 - \gamma} \quad \text{(B10)} \]

\[ F(x, y) := \left( \frac{1 - \beta_K + e^x (\beta_X - \beta_Q)}{1 + \beta_X e^y} \right)^{1-\gamma} \frac{j(y)}{1 - \gamma} - \frac{j(x)}{1 - \gamma} \quad \text{(B11)} \]

**Proposition B1** Suppose that we can find two constants \( z_1, z_2 \) \( (0 \leq z_1 < z_2) \) and a function \( j(\cdot) \) defined on \([z_1, \infty)\), which solve the ODE given in equation (B5) with boundary conditions (B7), (B8), and (B9), such that the following holds:

\[
\begin{align*}
0 &< a(z)^{-1/\gamma} < M_1 \\
0 &< \frac{a(z)}{j(z)} < M_2 \\
F(x, y) &\leq 0, \quad \forall y \geq x \geq z_1 \\
0 &= F(z_1, z_2) \geq F(z_1, y), \quad \forall y \geq z_1
\end{align*}
\]

where \( M_1, M_2 \) are constants.

Then the value function is given by

\[ J(t, K, Q) = e^{-\rho t} K^{1-\gamma} / (1-\gamma) j(z) \quad \text{(B16)} \]

where \( z = \log \frac{Q}{K} \). Further the optimal consumption policy is to set

\[ c(z_t) = a(z_t)^{-1/\gamma}. \]

The optimal investment policy consists of a sequence of stopping times and investment amounts, \( \{(T_i, X_{T_i})\}_{i=0,2,...} \) given by \( T_0 = 0 \) and:

- If \( z_0 \leq z_1 \) then invest (to move \( z_0 \) to \( z_2 \)):

\[ X_0^* = Q_0 \frac{e^{-z_0} (1 - \beta_K) - e^{-z_2} - \beta_Q}{e^{-z_2} + \beta_X} \quad \text{(B17)} \]

Then start with new initial values for the stock of consumption good \( K_0 - \beta(X_0^*, K_0, Q_0) \) and stock of oil wells \( Q_0 + X_0^* \).

- If \( z_0 > z_1 \) then set \( X_0^* = 0 \) and define the sequence of \( F \)-stopping times:

\[ T_i = \inf \{ t > T_{i-1} : z_t = z_1 \} \quad i = 1, 2, \ldots \quad \text{(B18)} \]

and corresponding \( \mathcal{F}_{T_i} \)-measurable investments in oil wells:

\[ X_{T_i}^* = Q_r \frac{e^{-z_1} (1 - \beta_K) - e^{-z_2} - \beta_Q}{e^{-z_2} + \beta_X}. \quad \text{(B19)} \]

**Proof** We define our candidate value function as \( J(K, Q, t) = e^{-\rho t} K^{1-\gamma} / (1-\gamma) j(z) \), where \( z = \log(Q/K) \) as before and where we define \( j(z) \) as in the proposition for \( z \geq z_1 \) and where we set

\[ j(z) = \left( \frac{1 - \beta_K + e^x (\beta_X - \beta_Q)}{1 + e^{\beta_X e^y}} \right)^{1-\gamma} j(z_2), \quad \forall z < z_1. \]

Applying the generalized Itô’s lemma to our candidate value function for some arbitrary controls we find:

\[ dJ(t, K_t, Q_t) + U(t, C_t)dt = e^{-\rho t} K_t^{1-\gamma} \left\{ \theta_0(z_t) j(z_t) + \theta_1(z_t) j'(z_t) + \theta_2 j''(z_t) + \theta_3 j''(z_t) \right\} dt \]

\[ + a(z_t) \sigma_K dw_{K,t} + \{ j(z_t) - a(z_t) \} \sigma_Q dw_{Q,t} + F(z_t, z_t) \]
where for simplicity we have defined $\hat{\theta}_0(z) = \theta_0 + (1 - \gamma)\alpha(\hat{z}e^z)^\gamma$ and $\hat{\theta}_1(z) = \theta_1 - \alpha(\hat{z}e^z)^\gamma$ and $C_t = c_t K_t$.

Now the definition of the function $j(z)$ implies that

$$\hat{\theta}_0(z)j(z) + \hat{\theta}_1(z)j'(z) + \theta_2j''(z) + \sup_c \left[ \frac{(c)^{1-\gamma}}{1 - \gamma} - a(z) c \right] \leq 0 \quad \forall z \geq z_1$$

Further, $F(x, y) \leq 0 \forall x \leq y$ with equality only if $x = z_1$ and $y = z_2$. Thus we have that for arbitrary controls

$$J(T, K_t, Q_t) + \int_0^T U(t, C_t) dt \leq J(0, K_0, Q_0) + \int_0^T e^{-\mu t} K_1^{1-\gamma} a(z_t) \sigma_K dw_{K,t} + \int_0^T e^{-\mu t} K_1^{1-\gamma} \{j(z_t) - a(z_t)\} \sigma_Q dw_{Q,t}.$$ (B21)

Taking expectation (using the fact that the stochastic integral is a positive local martingale hence a supermartingale) we obtain that for arbitrary controls

$$E \left[ J(T, K_T, Q_T) + \int_0^T U(t, C_t) dt \right] \leq J(0, K_0, Q_0)$$ (B22)

For the controls proposed in the proposition equation (B21) holds with equality. Further we have for these particular controls:

$$\frac{dJ(t, K_t, Q_t)}{J(t, K_t, Q_t)} = -a(z_t)^{-1/\gamma} \frac{a(z_t)}{j(z_t)} \frac{\sigma_K}{\sigma_J} \left( \frac{a(z_t)}{j(z_t)} \right) \frac{dJ, K_t, Q_t}{w_{J,t}}.$$ (B23)

This implies that (using the assumptions that $\frac{a(z)}{j(z)} \in (0, M_1)$ and $a(z_t)^{-1/\gamma} \in (0, M_2)$) the stochastic integral in (B21) is a martingale and that

$$\lim_{T \to \infty} E[J(T, K_T, Q_T)] = \lim_{T \to \infty} J(0, K_0, Q_0) E \left[ e^{-\int_0^T a(z_t)^{-1/\gamma} \frac{a(z_t)}{j(z_t)} \sigma_K}{\sigma_J} \left( \frac{a(z_t)}{j(z_t)} \right) dt \right] = 0.$$ (B24)

where we have defined a new measure $\tilde{P} \sim P$ by the Radon-Nikodym derivative

$$\frac{d\tilde{P}}{dP} = e^{-\int_0^T \frac{1}{\gamma} \frac{a(z_t)}{j(z_t)} (\sigma_J(z_t))^2 dt + \int_0^T \frac{1}{\gamma} \frac{a(z_t)}{j(z_t)} (\sigma_K(z_t))^2 dt}.$$ (B25)

\[\square\]

C Numerical Techniques

In this appendix we delineate the numerical algorithm used to solve the Hamilton-Jacobi-Bellman equation in (5) with boundary conditions represented by equations (8) to (11).

The first step is to use the homogeneity of the solution to reduce the state space (see Subsection 2.2.3). After this is done, the solution of the problem is represented by a nonlinear second order ODE in the state variable $z_t = \log(Q_t/K_t)$. The boundary conditions are also expressed in terms of $z_t$. Now, we need to determine the value function $j(z)$ in equation (13). The nonlinear HJB equation for $j(z)$ depends on (i) the optimal control $c_t^*$, and on (ii) the optimal investment strategy $\{z_1, z_2\}$ determined by the boundary conditions. Unfortunately, the optimal control itself depends on the value function $j(z)$. This implies that $j(z)$, $c_t^*$, $z_1$ and $z_2$ need to be simultaneously determined.

We use an iterative method to solve for $j(z)$. The main idea is to build a conditionally linear ODE for $j(z)$ so it is possible to apply a finite-difference scheme. The selection of the initial guess is extremely important for the convergence of the iteration. We assume that $j^0(z) = 1$ which corresponds to the solution when the oil is not relevant for the production technology ($\eta = 0$). In this case we also know that it is never optimal to invest $z_1^0 \to \infty$.

For every iteration $m$ (for $m = 0 \ldots \infty$) we do the following steps:
• Determine the optimal consumption \( c^m \) as a function of \( j^m(z) \) using equation (19).

• We recognize that the ODE for \( j^{m+1}(z) \) determines the value function when it is optimal not to invest in new stocks of commodity. We name this function as \( j^{m+1}_{\text{noinv}}(z) \). We calculate the coefficients of the ODE for \( j^{m+1}_{\text{noinv}}(z) \). It is important to notice that this ODE is linear conditional on \( c^m \).

• Determine the optimal commodity/capital ratio \( z_2^{m+1} \) using the super contact condition in equation (B9). Conditional that it is optimal to invest in new commodity stocks, the returning point is always \( z_2^{m+1} \) independent of what was the value of \( z_2 \) before investment was made. Using this argument we define the extended value matching condition as

\[
j^{m+1}_{\text{inv}}(z) = j^m(z_2^{m+1}) \left( \frac{1 - \beta_K + e^t(\beta_X - \beta_Q)}{1 + e^t(z_2^{m+1})}\right)^{1-\gamma}.
\]

This equation represents the value function when the representative agent is forced to invest.

• Use a finite-difference scheme to solve for the value function \( j^{m+1}_{\text{inv}}(z) \). The finite difference discretization defines a tridiagonal matrix that needs to be inverted to determine the value of \( j^{m+1}_{\text{noinv}}(z) \). Instead of doing this, we eliminate the upper diagonal of this matrix. At this point the value of \( j^{m+1}_{\text{noinv}}(z) \) depends only on the value of \( j^{m+1}_{\text{noinv}}(z - \Delta z) \). We choose a \( z_{\text{min}} \) negative enough to ensure that at that level it is optimal to invest, and then we solve the value function for higher \( z \). At every point we choose the maximum of the value from investing \( j^{m+1}_{\text{inv}}(z) \) and the value of no investing which comes from the finite-difference scheme. This maximum determines the value of \( j^{m+1}(z) \). The optimal trigger \( z_2^{m+1} \) is endogenously determined when the representative agent is indifferent between investing and postponing the investment. The algorithm described above is a more efficient way than solving independently for \( j^{m+1}_{\text{inv}}(z) \) and \( j^{m+1}_{\text{noinv}}(z) \) and then choosing \( j^{m+1}(z) = \max(j^{m+1}_{\text{inv}}(z), j^{m+1}_{\text{noinv}}(z)) \).

• Check for the convergence condition. If it not satisfied we start a new iteration with the updated value of \( j^{m+1}(z) \).

Once \( j(z) \) has converged it is straightforward to calculate spot commodity prices from equation (27). For the futures prices we use an implicit finite-difference technique. This is simpler than the solution for \( j(z) \) since the coefficients of the PDE and boundary conditions and boundaries \( \{z_1, z_2\} \) are known at the beginning of the scheme.

### D Data

For the SMM estimation we use quarterly time series from Q4/1990 to Q2/2008. We build the series of crude oil futures prices and interest rates, private consumption, GDP and petroleum consumption from OECD countries. Crude oil futures prices are obtained from the New York Mercantile Exchange (NYMEX). We use contracts with maturities of 1, 3, 6, 9, 12, 18, 24, 30 and 36 months. If a specific contract is missing, we select the one with the nearest maturity. For the quarterly figures we use the average prices within that period. To get the (annualized) volatility of futures returns, we sample quarterly observations of a GARCH(1,1) estimated separately for each (log) futures series using weekly prices. The volatilities time series are necessary to use the weighting matrix described above. Consumption and output data is from www.oecd.org. The aggregate data is available from Q4/1995 for all OECD members (30 countries). For the initial years we build a proxy for the series with the G7 countries data available from the same site. We assume that the GDP ratio of the G7 countries and all OECD members was constant from Q4/1990 to Q1/1995. Petroleum consumption data for OECD countries is from the U.S. Energy Information Administration site (www.eia.doe.gov). Finally, the interest rate data is from Federal Reserve FRED site [research.stlouisfed.org/fred2](https://research.stlouisfed.org/fred2). To build the real interest rate time series we also use the CPI series, which are obtained from the same site.
Panel B: Q4/1990 to Q2/2008

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
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<tr>
<td>Productivity of capital $K$ (*)</td>
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<td>0.138</td>
</tr>
<tr>
<td>Importance of oil</td>
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<tr>
<td>Demand rate for oil (*)</td>
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<td>Volatility of capital (*)</td>
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<tr>
<td>Volatility of oil stocks</td>
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<td>Correlation of capital and oil shocks</td>
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<tr>
<td>Depreciation of oil (*)</td>
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<td>0.10</td>
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</table>

| Irreversible investment                       |                              |                              |
| Fixed cost ($K$ component) (*)                | $\beta_K$ 0.007              | 0.009                        |
| Fixed cost ($Q$ component) (***)              | $\beta_Q$ 0.132              | 0.242                        |
| Marginal cost of oil (*)                      | $\beta_X$ 18.1               | 26.9                         |

| Agents preferences                            |                              |                              |
| Patience                                      | $\rho$ 0.05                  | 0.05                         |
| Risk aversion (*)                             | $\gamma$ 0.50                | 0.51                         |

Table 1: The table reports the SMM estimates for the sample periods Q4/1990 to Q1/2003 (Panel A) and Q4/1990 to Q2/2008 (Panel B). Only the parameters marked with an asterisk (*) are estimated. The $Q$-component of the fixed cost, denoted with (**), was set to $\beta_Q = \beta_K \beta_X$. 
<table>
<thead>
<tr>
<th>Moment conditions</th>
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<th>Model</th>
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<td>Sample</td>
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<td>0.016</td>
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<td>0.002</td>
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<td>Panel B: Q4/1990 to Q2/2008</td>
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<td>Futures prices - 01</td>
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<td>0.009</td>
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<td>0.008</td>
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<tr>
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<td>0.011</td>
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<td>0.002</td>
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Table 2: The table presents the historical and the model implied moments using the corresponding parameter estimates from Table 1.
Table 3: The table reports the quasi-maximum likelihood (QML) estimates for the regime-switching model for weekly deflated Brent crude oil prices between Jan/1982 and Aug/2003.
<table>
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<th></th>
<th>Oil Spot Price Returns</th>
<th>Oil Futures Returns</th>
</tr>
</thead>
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<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
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<tr>
<td>Constant</td>
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<tr>
<td></td>
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<td>(0.11)</td>
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<td></td>
<td>$a_1$</td>
<td>-0.88</td>
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<tr>
<td></td>
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<td>(0.45)</td>
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<tr>
<td></td>
<td>$b_0$</td>
<td>-0.20</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.47)</td>
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<tr>
<td></td>
<td>$b_1$</td>
<td>-1.23</td>
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<tr>
<td></td>
<td></td>
<td>(2.95)</td>
</tr>
<tr>
<td>$\hat{p}_t$</td>
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<tr>
<td>$r_{M,t+1}^e$</td>
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</tr>
<tr>
<td>$\hat{p}<em>t r</em>{M,t+1}^e$</td>
<td>-1.23</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2.95)</td>
</tr>
<tr>
<td>$R^2$</td>
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</tr>
<tr>
<td>Num obs</td>
<td>240</td>
<td>240</td>
</tr>
</tbody>
</table>

Table 4: The table presents parameter estimates for the regressions: $r_{j,t+1}^e = a_t + b_t r_{M,t+1}^e + \epsilon_t$ where $a_t = a_0 + a_1 \hat{p}_t$, $b_t = b_0 + b_1 \hat{p}_t$. $\hat{p}_t$ is the conditioning variable and is the smoothed inferred probability of being in the scarcity regime. $r_{j,t+1}^e$ is the (log) monthly real excess return of crude oil spot prices (columns (1) and (2)) or the (log) monthly real excess return of the collateralized futures strategy (columns (3) and (4)). $r_{M,t+1}^e$ is the real monthly excess return of the value-weighted CRSP index. The scaling variables $\hat{p}_t$ is the smoothed inferred probability of being in the scarcity regime (from the estimation in Table 3). The $t$-statistics are in parentheses.
Figure 1: The spot price of oil (per barrel, deflated by GDP deflator to 2001 dollars), the GDP quarterly growth rate, NBER recession periods.
Figure 2: Historical crude oil futures curves for both sample periods. The upper plot shows the end-of-month futures curves from Oct/1990 to Mar/2003. The lower plot shows the end-of-month futures curves from Oct/1990 to Jun/2008.
Figure 3: Historical and model implied moments for quarterly crude oil data from Q4/1990 to Q2/2008. The model implied moments are obtained using the parameter estimates from Panel B in Table 1.
Figure 4: Historical and model implied moments of futures curves for both sample periods. Each set of curves has three different moments: the unconditional mean, and two conditional means depending whether the spot price is below or above the average spot price.
Figure 5: Oil price $S_t$ as a function of the logarithm of the oil wells-capital ratio, $z_t$. The vertical dashed-line is at $z_{S\text{max}}$ and separates two regions. The thin line shows the oil price in the scarcity region ($z_1 < z_t \leq z_{S\text{max}}$) and the thick line is the oil price in the abundance region ($z_t \geq z_{S\text{max}}$). We use the parameters from Panel B in Table 1. In particular, the fixed cost components of the investment are $\beta_K = 0.009$ and $\beta_Q = 0.242$, and the marginal cost of oil is $\beta_X = $26.9. The equilibrium critical ratios are $z_1 = -8.91$, $z_{S\text{max}} = -8.269$ and $z_2 = -6.89$. 
Figure 6: Expected return and instantaneous volatility of returns in oil price $S_t$. The horizontal dashed-line separates the two regimes. The thin lines below the dashed-lines show the variables under the scarcity regime and the thick lines under the abundance regime. We use the parameters from Panel B in Table 1. In particular, the fixed cost components of the investment are $\beta_K = 0.009$ and $\beta_Q = 0.242$, and the marginal cost of oil is $\beta_X = $26.9. The endogenous upper bound for the price is $S_{\text{max}} = $48.04.
Figure 7: Futures curves for contracts on oil for different spot prices. The thick curves are for spot prices in the abundance region and the thin lines when the spot price is in the scarcity region. We use the parameters from Panel B in Table 1 and the endogenous upper bound for the price is $S_{max} = 48.04$. 
Figure 8: Convenience yield as a function of the state variable $z_t$ when $i_t = \bar{i}$. The thick line is the convenience yield when the economy is in the abundance region and the thin for the economy in the scarcity region. We use the parameters from Panel B in Table 1 and the endogenous upper bound for the price is $S_{\text{max}} = 48.04$. 
Figure 9: Relative volatility of futures contracts on oil to spot price volatility for different spot prices and maturities. The thick curves are for spot prices in the abundance region and the thin lines when the spot price is in the scarcity region. We use the parameters from Panel B in Table 1 and the endogenous upper bound for the price is $S_{max} = \$48.04$. 


Figure 10: Simulated probability density function for the state variable $z_t$ and the commodity price $S_t$ using the parameters from Panel B in Table 1.
Figure 11: Historical Brent crude oil prices between Apr/1983 and Apr/2005 deflated by the US Consumer Price Index (thick line) and inferred probability of being in the scarcity state (thin line).
Figure 12: Implied commodity risk premium as a function of the state variable $z_t$. The thick line is the risk premium when the economy is in the abundance region and the thin for the economy in the scarcity region. We use the parameters from Panel B in Table 1.