Multi-market Value Creation and Competition*

Qiang Fu

National University of Singapore

Ganesh Iyer

University of California, Berkeley

This version: June 2017

*We thank Jinfeng Lu, Amit Pazgal and Shubhranshu Singh for comments. Addresses: Fu: Department of Business Policy, National University of Singapore, 15 Kent Ridge Drive, Singapore, 119245. Iyer: Haas School of Business, University of California, Berkeley, U.S.A. Email: bizfq@nus.edu.sg, giyer@haas.berkeley.

Multi-market Value Creation and Competition

ABSTRACT

We analyze multi-market interactions between firms which must invest limited budgets in value (surplus) creation as well as in competitive rent-seeking activities. Firms are differentiated on a line segment and compete for multiple markets/prizes which differ in the relative effectiveness of each firm's competitive rent-seeking spending. Each firm faces a dual trade-off: First they must choose how much to invest in value creation versus to spend in rent-seeking competition. Second, they must decide on how to allocate resources across the different markets. When the market values are exogenous (and identical across markets) the intensity of competition is highest for the marginal market in the middle, rather than in (advantaged) markets which are close or in (disadvantaged) markets which are closer to the rival. Counter to what one might expect, greater firm differentiation actually intensifies the competition in the middle market. When firms endogenously invest in value creation, they invest more in value creation in closer markets and the investments decline towards the middle. This results in the most intense competition moving away from the middle to a market in each firm's turf.

"...but I ask you would you rather have butter or guns? Shall we import lard or steel? Prodigy's 1997 album The Fat of The Land."

1 Introduction

Many important economic, political, and business contexts involve rivals who must compete by allocating limited resources simultaneously across multiple markets or prizes. Not only must they decide how much to invest in building different markets, but also the extent to which they must compete in each market. Consider the following examples:

- Drug companies spent an estimated $5.2 billion in 2015 in direct to consumer advertising. For new drugs such advertising may induce patients to ask their doctors about their suitability and thus potentially expand the market. The pharmaceutical industry deploys even larger amounts (e.g., $15 billion in 2012) to conduct detailing and promotional activities to doctors across different geographical markets. Detailing activities by medical representatives not only provides information to doctors about the basic drug action, but also involve efforts to persuade doctors to prescribe the firm's drugs over those of rivals.

- In the mobile phone market leading firms like Samsung and HTC invest in promoting the Android platform to convince consumers to adopt the platform over the iPhone. However they also compete for market share in large Asian markets. HTC's advertising campaign is more effective in its home market in Taiwan than in South Korea, and vice versa for Samsung. Advertising campaigns in this product category can work to increase the generic demand for the product category, or it can persuade consumers about the advantages of a firm's product over its rival(s). Samsung and HTC would have to decide on allocating advertising budgets based on the relative preference for their product in each market.

- In presidential elections rival candidates must decide how to allocate their advertising budgets and their campaigning time across electoral markets which are differentiated according to Democratic and Republican preferences. In the typical case the candidates' spending allocation to a market also influences turnout and therefore the size of the voting market.

These examples represent some general aspects of competitive interactions in a variety of contexts: First the players/firms have limited resources (advertising budgets, sales-force size, canvassing time) and they compete in multiple markets. This means that they have to decide on how much of the resource to allocate to each market, resulting in the decisions across markets to be affiliated. Allocating more to one market means less to others. Second, the markets can be differentiated (as in the mobile phone market) with each firm having home markets with relatively higher consumer preference. Should firms deploy more or less resources in markets where they are stronger?

Second, the examples also highlight a basic business strategy trade-off: Firms have to choose between surplus or value creation versus competing for the markets. In other words, the pie that firms will fight for is in itself endogenous. Further, a firm's investment in creating value in a market can be subject to free-riding by the competitor who can deploy competitive resources to win the market. For example, pharmaceutical reps have to decide how much to focus on providing information about the basic drug action versus on persuading doctors that their drug is relatively superior. Efforts to inform doctors about the basic drug action may also end up benefiting competitors in a category. Similarly, direct to consumer advertising for a new class of drugs by one firm can expand the potential market benefiting all the firms in the category. This last feature is also the classic guns vs. butter trade-off described in the conflict literature (Hirshleifer 1988), except that in this paper we examine the trade-off in the context of multi-market interactions between players with limited resources.

We construct a framework to analyze equilibrium multi-market value creation and the competition for that surplus as represented by the above situations. The anatomy of the game is as follows: Two firms (players) with limited resources compete simultaneously in multiple contests for different markets (prizes) which are located evenly on a unit line. Firms are located at the ends of the line and are differentiated and each firm's relative strength in a market depends upon the distance between the firm's location and that of the market: the further a market is from a firm's location the less effective is the firm's competitive rent-seeking spending. Firms simultaneously choose an allocation of their resource endowment among the markets in order to maximize their expected overall payoffs. In each market they simultaneously choose the investment that will determine the size of the market (value or surplus creation) as well as the competitive outlay to win the market from the rival (rent seeking). Firms' outlays in surplus creation are substitutable and they jointly determine the size of the pie. This then allows for the possibility that the investments in surplus creation by one firm is subject to free-riding by the other. In each market firms' competitive spending jointly determine the winner through a Tullock contest success function. How would the equilibrium allocations for the players be in terms of the surplus creation and the rent-seeking allocations across the different markets?

Consider first the case where each market has the same size which is exogenously fixed. This implies that firms are governed only by the competitive rent-seeking incentive across the markets. Should a firm defend closer markets in its home turf, or should it spend resources to win markets which are farther away and harder to win? We find that each firm's equilibrium resource distribution has a non-monotonic inverted U-shape profile: Each firm spends relatively less resources both in closer markets and in markets which are closer to its competitor. The firms' outlays peak at a market in the middle which implies that the competition will be the most intense for the middle market. It is particularly interesting that greater market differentiation actually leads to a more concentrated resource distribution profile with even more intense competition in the middle market.

The main analysis of the paper considers the general case in which market value is endogenously determined. Firms' choose not only the allocation across the markets, but also how to split the spending in each market between investments in market creation and competitive rent-seeking: The former builds the market value or size, while the latter allows a firm to compete for the value that is created. This analysis can then be seen as investigating the guns vs. butter tradeoff in a world of multi-market competition.

With substitutable value creation efforts, the equilibrium market values are polarized: The home turfs of both firms, i.e., the markets closest to the firm locations have the highest investments and markets which are closer to the middle have lower values. The firms' allocations of competitive rent-seeking outlays are different from the pattern in the exogenous market size case. Firms competitive spending no longer peak in the middle, but rather in each firm's turf. The implication is that firms do not compete most intensely in the most valuable markets, rather the intensity of competition is determined by the trade-off between the equilibrium size of the markets and the ease with which the markets are contestable.

Greater market differentiation leads firms to invest relatively more in their home turfs at the expense of the middle markets and to reduce the overall amount of competitive spending. With asymmetric budgets, the firm with the budget advantage balances its equilibrium actions such that it invests more in value creation and also deploys more in competitive spending. This result contrasts with what might be obtained in a model of single market guns and butter competition: In that case, it is possible that the firm with the lower budget may deploy greater competitive spending even as the firm with the budget advantage invests more in value creation. We also find that as the budget asymmetry increases the firm with the advantage invests in more markets closer to the weaker firm.

2 Literature Review

This paper is related to the guns vs. butter literature initiated by Hirshleifer (1988) which explores the resource allocation choice of players between productive output creating investments and rent-seeking actions. This can be seen as the incentives and the trade-offs faced by firms to collaborate in joint production of value and to compete for that value at the same time. Several papers, e.g., Skaperdas (1992) and Hirshleifer (1994), have been developed along this line to investigate how firms' resource position might affect their allocation strategies. (see Garfinkel and Skaperdas (2007) for a review). Our analysis contributes by considering value creation and competition in multi-market interactions and therefore jointly considers dual trade-offs: the allocation of resources between value creation and competition and the allocation of resources to different markets. In marketing, Bass et al. (2005) analyze a form of the guns vs. butter tradeoff in single market over time in model of dynamic competition in generic and brand advertising. Amaldoss et al. (2000) present a different type trade-off in the context of R&D alliances: Firms in an alliance jointly invest to develop products of higher values in order to compete against rival alliances for a market.

Our analysis is also related to the contest literature and can be seen as new form of multi-dimensional resource allocation contest in which firms allocate resources over a set of differentiated markets. This enriches the framework of the Colonel Blotto game of duopoly conflicts in multiple battlefields in which firms allocate their resources among these battlefields to maximize the sum of rents. This game was first proposed by Borel (1921) and analyzed by Borel and Ville (1938) in a special case of three markets.[1] This paper highlights the strategic effects of two important aspects that is missing in the literature: The Colonel Blotto game assumes zero-sum payoff structure. Our analysis obviously considers both the creation of the pie as well as competition for it.[2] To our knowledge, the existing literature on Colonel Blotto games has not considered a game which incorporates the trade-off between market creation and rent-seeking as well as firms' strategic interaction in multiple differentiated markets. Within the standard Colonel Blotto class of games allowing for endogenous market creation as well as allocation across multiple differentiated contests is analytically challenging. The game form developed in the paper contributes by providing a tractable setup to analyze the guns vs. butter decisions in multi-market contests.

The paper is also related to a growing literature that applies contest/tournament models to marketing and I.O. issues, such as R&D and product development, the incentive mechanism for sales personnel, and sale force allocation, etc. Ridlon and Shin (2013) build a repeated contest model to examine whether a firm should favor its weaker employee in an attempt to maximize the total effort output of the employees. There is also a literature on sales contests which focuses on the optimal design and the role of the prize structure to elicit sales agents efforts (see Kalra and Shi (2001) and Lim, Ahearne, and Ham 2009). Finally, Amaldoss and Staelin (2010), and Chen and Lim (2013) study contests between teams/alliances instead of between individual players.

[1] Gross and Wagner (1950) generalize the analysis to a finite number of markets. A number of studies apply the model to various contexts, including campaign financing allocations (Lake, 1978), advertising (Friedman, 1958), and military defense (Clark and Konrad, 2007, and Kovenock and Roberson, 2009), etc.

[2] Kvasov (2007) and Roberson and Kvasov (2012) relax the usual assumption that resources are forfeited if they were not used for rent-seeking competitions.

3 The Model

Consider two firms/players indexed by $i = 1, 2$, that are located at the two ends of a unit line with firm 1 located at zero and the other firm at location 1. Each firm is endowed with a fixed competitive resource budget m_i. Without loss of generality, assume that $m_1 \geq m_2 > 0$. Suppose that the line segment has a set of $2n + 1$ markets (or prizes) which are equally spaced and indexed by $k = 1, \ldots, 2n + 1$. Market $k = 1$ is at firm 1's location, while the market $k = 2n+1$ is at firm 2's location. Each of these k's could represent consumer market for a product, or an electoral market, or different R&D projects that firms may invest in.

Firms utilize their endowment by simultaneously choosing the amount b_{ik} to invest in surplus creation in market k as well as the amount x_{ik} with which to compete for the market k. The decisions b_{ik} can for example represent investments in awareness advertising to build primary/product category demand. In the pharmaceutical context these can be direct to consumer advertising which informs the market about the basic drug and its action thereby expanding primary demand, or in the electoral context these can be activities to increase voter turnout. The decisions x_{ik} represent competitive or rent-seeking activities that are directed at winning the market from the rival. These could include comparative advertising or competitive promotional spending to convince consumers to buy from a firm rather than from the rival.

3.1 Exogenous Market Values

We begin with a basic analysis of competition between the firms when each market or prize has a fixed exogenous value $v > 0$. This means that each firm would choose a competitive allocation strategy and firm i's allocation strategy can be represented by a vector $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,2n+1})$, subject to its budget constraints, i.e., $\sum_{k=1}^{2n+1} x_{i,k} \leq m_i$. The effectiveness of a firm's competitive allocation in a market depends on the distance between its own location and the targeted market. For an arbitrary market k given the firms' choice of competitive outlays x_{ik} effective outlays y_{ik} are given by:

$$\begin{aligned} y_{1,k} &= [1-\frac{t(k-1)}{2n+1}]x_{1,k}^r, \text{ and,} \\ y_{2,k} &= [1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}^r. \end{aligned}$$

Thus a firm's competitive outlay is relatively more effective in a market that is closer to it than to its rival. The effectiveness of a firm's spending depends upon the distance between the firm and the market, and $t \in (0,1]$ measures the effectiveness loss caused by distance. In other words, it measures the extent of differentiation between the firms in their ability to compete for the different markets. Therefore in this competitive contest each firm is favored in its own turf because its spending has greater relative effectiveness. Thus markets $\{1,\ldots,n\}$ is the "turf" of firm 1, and markets $\{n+2,\ldots,2n+1\}$ that of firm 2.

In each market k, the winner of the market value v is determined by a standard Tullock contest success function: Each firm i wins with a probability

$$p_{i,k} = \frac{y_{i,k}}{y_{1,k}+y_{2,k}}.$$

Note that ties are fairly broken if both firms place zero outlays in any market. The function $p_{i,k}$ can alternatively be interpreted as a share of market value firm i secures from market k when the rent is divided proportionally. The parameter $r \in (0,1)$ measures the overall productivity or the "discriminatory power" of the competitive (rent-seeking) technology with a higher r implying that the marginal output of a firm's competitive outlay is also higher. With a larger r, the winner is determined more by the superior spending of a firm rather than the noise factors inherent in the contest.

Each firm chooses its allocation strategy to maximize its aggregate payoff from all the markets $\pi_i(\mathbf{x}_i;\mathbf{x}_j) = \sum_{k=1}^{2n+1} \frac{y_{i,k}}{y_{1,k}+y_{2,k}} v$. For a given allocation strategy $\mathbf{x}_j$ by its rival, a firm i will solve a constrained maximization problem given by

$$\begin{aligned} \max_{\mathbf{x}_i} \pi_i(\mathbf{x}_i;\mathbf{x}_j) \\ s.t. \sum_{k=1}^{2n+1} x_{i,k} \leq m_i \\ x_{i,k} \geq 0, \forall k \in \{1,\ldots,2n+1\}. \end{aligned}$$

The firm's allocation problem to maximize payoffs embeds some essential trade-offs: First given that the resource endowment is limited, investing more in any given market necessarily means reducing the allocation for one or more of the other markets. Second, the market differentiation represented by t implies that each firm has to decide how much to invest in its home turf versus attack its rival's turf. We establish the unique pure strategy equilibrium of this multi-market game. Obviously, because unused budgets do not have any outside option value each firm must exhaust its budget and $\sum_{k=1}^{2n+1} x_{i,k} = m_i$.[3] The first step in identifying the equilibrium is the following Lemma:

Lemma 1 *There exists no equilibrium in which a firm places zero outlay in any market, i.e.,* $x_{i,k} > 0$, $\forall k \in \{1, \ldots, 2n+1\}$.

Suppose there were to exist a market where both firms did not allocate any resources, then it will be optimal for one of the firms to shift an infinitesimal amount of resource from elsewhere to this market. Doing so would provide the firm with an incremental payoff of v while having a negligible effect on the payoff of the firm from the alternative market. Similarly, if only one firm were to not allocate any resources in a given market, the other firm would want to reduce its outlay to be negligibly small. Thus in equilibrium both firms compete for each one of the available markets by placing positive resource allocations.

We now proceed to describe the interior equilibrium of the game in which $x_{i,k} > 0$, $\forall k \in \{1, \ldots, 2n+1\}$. Define $\lambda = \frac{m_2}{m_1}$. The following proposition characterizes the equilibrium:

Proposition 1 *There exists a unique pure strategy Nash equilibrium in the game with fixed market values. In each market k, firms' allocate* $x_{1,k} = \frac{m_1 \phi(k)}{\sum_{k=1}^{2n+1} \phi(k)}$ *and* $x_{2,k} = \frac{m_2 \phi(k)}{\sum_{k=1}^{2n+1} \phi(k)}$*, where* $\phi(k)$ *is given by*

$$\phi(k) = \frac{\left\{ r[1 - \frac{t(k-1)}{2n+1}][1 - \frac{t(2n+1-k)}{2n+1}]\lambda^r \right\}}{\left\{ [1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\lambda^r \right\}^2}.$$

Firms spend the same proportion $\phi(k) \big/ \sum_{k=1}^{2n+1} \phi(k)$, of their resource budget in each market. Hence, the ratio between firms' bids remains constant across markets, i.e., $\frac{x_{2,k}}{x_{1,k}} = \lambda = \frac{m_2}{m_1}$. One might think that a firm can have lower incentives to spend in a market that is farther away and in the rival's home turf. But then it is also the case that it is easier for

[3] Allowing for positive outside value does not affect the insights. Greater outside option would lead to less being allocated to rent-seeking competition.

the rival to defend a closer market because its spending in such a market is more efficient. The rival strategically lowers its spending in its home turf. In this model, these opposing incentives cancel out leading firms to choose a constant ratio of resource allocation across markets.[4]

The main point of interest in this proposition is the manner in which the firms split their resource budgets among the $2n+1$ markets. Define $\tilde{\phi}(k) \triangleq \phi(k|\,t) \big/ \sum_{k=1}^{2n+1} \phi(k|\,t)$, which is the portion of resource each firm allocates in equilibrium to a market k. In the following proposition, we explore the properties of the function $\tilde{\phi}(k)$ with respect to k.

Proposition 2 *i.) The function $\tilde{\phi}(k)$ first increases with k and peaks at a cutoff $k^* \geq n+1$. It then decreases if $k^* < 2n+1$. ii.) When firms are symmetric ($m_1 = m_2$), the peak k^* is located at the middle market, i.e., $k^* = n+1$; when firms are asymmetric, i.e., $m_1 > m_2$, the peak k^* is located right to the mid-point, i.e., $k^* > n+1$.*

The resource allocation function $\tilde{\phi}(k)$ is non-monotonic in k. Consider the case of symmetric firms ($m_1 = m_2$). Each firm's allocation increases as it moves towards the center on its home turf peaking at the market $n+1$ in the middle and so the firms compete most intensely in the middle of the market. In markets close to its location a firm strategically withdraws and allocates less because its spending is more effective. Whereas it also allocates less in markets that are farther away and in the competitor's turf precisely because its spending is relatively less effective. This leads to an inverted U-shaped equilibrium resource allocation profile with the maximum allocation by both firms in the middle.

When the firms are asymmetric and $m_1 > m_2$, the spending peak, k^*, shifts to firm 2's turf. The most intense competition takes place in a market in the weaker firm 2's turf and the greater the asymmetry between the firms the closer is the peak to firm 2's location. The general wisdom in the contest literature is that a more balanced playing field leads to more competition. In any given market k, the balance in the playing field depends on a) the effectiveness of the firms' spending which is determined by the distance between the market

[4]The equilibrium specified in Proposition 1 remains the same even in a sequential move game in which one firm first commits to its spending allocation and then the rival firm chooses its allocation strategy upon observing the first mover's strategy. In other words, the possibility of commitment by one of the firms does not change the equilibrium incentives. See the Appendix for details.

k and each firm; and b) firms' budgets. Symmetric firms have an advantage in their own turfs, and therefore the highest spending and competition occurs at the mid point. When they are asymmetric, the most intense competition occurs in a market which is in firm 2's turf, where the closer distance of the market to firm 2 can counteract the disadvantage of its smaller budget.

Effect of Market Differentiation and Other Comparative Statics

How does the extent of differentiation of the market affect the equilibrium strategies? We now turn our attention to this question in the n next Proposition:

Proposition 3 *i) For symmetric budgets, the ratio* $\frac{\tilde{\phi}(\{k+1\}|t)}{\tilde{\phi}(\{k\}|t)}$ *strictly increases with* t *for* $k < n+1$*, while it strictly decreases for* $k \geq n+1$*, i.e., rent-seeking resources are increasingly spent at the mid-point. ii) For asymmetric budgets,* $\frac{\partial k^*}{\partial t} \leq 0$*, the spending peak* k^* *shifts towards the left side of the line.*

Note that $\frac{\tilde{\phi}(\{k+1\}|t)}{\tilde{\phi}(\{k\}|t)}$ is the ratio of spending on market $k+1$ as compared to market k and this ratio strictly increases with market differentiation for $k < n+1$, while it strictly decreases for $k \geq n+1$. As t increases, distance of a market from a firm causes a greater attenuation of the firm's spending effectiveness and so each firm gets a greater advantage in markets which are in its own turf. This reduces a firm's spending in more remote markets and therefore provides incentives for the rival to also spend less in protecting closer markets. Thus we get the somewhat unexpected result that greater t leads to increasingly intense competition in marginal markets which are in the middle. This result is interesting precisely because an increase in t is equivalent to the market being more differentiated between the firms. And greater differentiation leads firms to compete more in the middle, rather than in closer markets where they have a natural advantage.

When firms have asymmetric budgets the location of the spending peak k^* depends on several factors, such as the extent of market differentiation, the asymmetry between their budgets (λ), and the contest technology. It is interesting to note that $\frac{\partial k^*}{\partial t} \leq 0$. Increasing differentiation leads firms to concentrate their resources on the markets on the left: Firm 1 "retreats" as t increases and concentrates more spending in closer markets. Therefore,

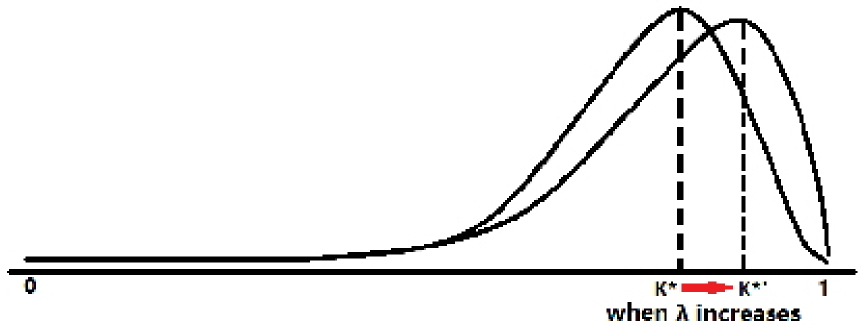

Figure 1: Distribution of Competitive Spendings

the market with the most intense competition moves closer to Firm 1 which has the budget advantage. Greater differentiation reduces Firm 1's spending effectiveness in markets which are farther away offsetting its resource advantage.

We can also show that $\frac{\partial k^*}{\partial \lambda} \leq 0$. In other words as the asymmetry between the firms increases (i.e., $\lambda = \frac{m_2}{m_1}$ decreases) the spending peak k^* is pushed rightward into Firm 2's turf and firms increasingly concentrate their resources to compete in markets which are on the right. Intuitively, the weaker firm (Firm 2) is induced to focus its limited budget on closer markets. This allows Firm 1 to divert its spending from its home turf and move them to markets closer to Firm 2. Thus Firm 2 substitutes for its lack of resources by the effectiveness of its spending in closer markets. Figure 1 illustrates the comparative statics. Finally, increases of r leads k^* to move rightward: As r increases the contest for the markets becomes more discriminatory and this magnifies Firm 2's disadvantage forcing it to focus more on closer markets.

4 Endogenous Value Creation: Guns vs. Butter

We now analyze the full model in which the firms compete by spending in both value creation as well as rent seeking. Firm i's spending each market k is a pair $(b_{i,k}, x_{i,k})$: Recall that $b_{i,k}$ is a productive investment that increases the surplus or value created in market k, while $x_{i,k}$ is the firm's competitive rent-seeking spending that helps it to get a larger share of the surplus in the market. Firms simultaneously commit to their strategy $(\mathbf{b}_i, \mathbf{x}_i)$, where $\mathbf{b}_i$

is the vector $(b_{i,1}, \ldots, b_{i,2n+1})$, with $\sum_{k=1}^{2n+1} b_{i,k} + \sum_{k=1}^{2n+1} x_{i,k} \leq m_i$. The surplus created in market is given by $v_k = v_k(b_{1,k}, b_{2,k})$ and is increasing and strictly concave in each argument. Further, $v_k(0,0) = 0$ and $v_k(b_{1,k}, b_{2,k}) > 0$ if $\max(b_{1,k}, b_{2,k}) > 0$. We can then show:

Lemma 2 *There exists no equilibrium in which both firms do not make positive productive investments in a market k, i.e., $v_k > 0$, $\forall k \in \{1, \ldots, 2n+1\}$.*

Suppose there were to be a market in which neither firm invests. In that case both firms do not have to deploy competitive rent-seeking spending in that market. This means that one of the firms can gain from decreasing its investment in some other market where it faces competition and shifting it to this market. Thus we have that all markets will have positive surplus, and given this we know that the competitive rent-seeking spending must be positive everywhere as well.

To carry the analysis further we assume that $v_k = \sqrt{b_{1,k} + b_{2,k}}$. This functional form represents the contexts of multi-market competition which motivate our analysis in which the investments in value creation are substitutable and hence subject to free riding. For example, in pharmaceutical markets the firms might invest in detailing activities to inform doctors about the basic drug which can create primary demand. Similarly, major cell phone manufacturers like Samsung and HTC have promoted the Android platform to move consumers from the iPhone to Android. The incremental demand for Android phones created by these promotional activities can be competed for by rivals in a market. Similarly, in the early days of the nascent satellite radio market, both the major competitors Sirius and XM invested in advertising which jointly expanded the overall category, in addition to brand-specific advertising (see Bass et al. 2005).

4.1 Symmetric Budgets

This game above characterizes the classic guns vs. butter problem in a multi-market setting which highlights two simultaneous and related trade-offs: How to allocate the limited budget between value (surplus) creation and competition? And how to allocate the budget across the different markets. To explore these trade-offs, consider first the case where firms are ex ante symmetric, with $m_1 = m_2 = m$. For simplicity, we focus on the case of $r = 1$. The following proposition characterizes the unique symmetric equilibrium of this game.

Proposition 4 *There exists a unique symmetric equilibrium. In the equilibrium, Firm 1 makes positive investment $b_{1,k}$ in surplus creation only for market $k \leq n+1$ (Firm 2's strategy is symmetric for market $k \geq n+1$) given by*

$$b_{1,k} = \begin{cases} \left\{ \frac{[1-\frac{t(k-1)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} \right\}^2 & \text{if } k \leq n; \\ \left\{ \frac{[1-t\frac{n}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} \right\}^2 \diagup 2 & \text{if } k = n+1; \\ 0 & \text{if } k > n+1, \end{cases} \tag{1}$$

where $\tilde{\mu}$ is the Kuhn-Tucker multiplier and will be defined in the Appendix.

The equilibrium surplus created in market k is

$$v_k = \begin{cases} \frac{[1-\frac{t(k-1)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}}, & \text{if } k \leq n+1; \\ \frac{[1-\frac{t(2n+1-k)}{2n+1}]}{2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} & \text{if } k \geq n+1 \end{cases}.$$

Finally, the two firms choose the same equilibrium competitive (rent-seeking) spending in each market which are given by:

$$x_{i,k} = \frac{[1-\frac{t(2n+1-k)}{2n+1}][1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2 \tilde{\mu}} v_k.$$

The proposition identifies the value creation and competition trade-off. Firms have the incentive to invest in value creation only in markets which are in their own turf and in the middle $n+1$ market, but they compete for every market. Specifically, Firm 1's investments in creating value decreases with k for k for $k < n+1$, while Firm 2's investments decreases between from market $2n+1$ to $k > n+1$. Thus firms concentrate more on building value in closer markets and the markets 1 and $2n+1$ end up being the one with the highest value, while the middle market $n+1$, generates the least amount of equilibrium surplus. Figure 2 shows the distribution of surplus across the markets in equilibrium.

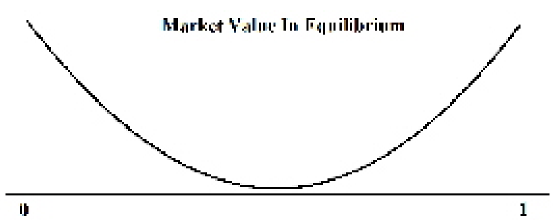

Figure 2a

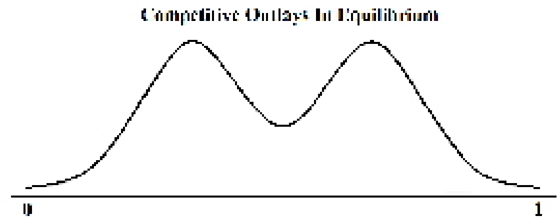

Figure 2b

Figure 2: Distribution of Market Values and Competitive Outlays

Clearly the endogenous market creation profile will affect the competitive rent-seeking spending incentives as the value of each market is no longer the same. Markets which are closer to each firm are larger, providing incentives for firms to invest more in defending them from the rival's competitive spending. However, there is also the countervailing incentive to compete more intensely for markets which are closer to the middle where each firm's relative dis/advantage in spending effectiveness is not too large. The tension between these two forces determines the equilibrium competitive spending profile $x_{i,k}$.

We can note from the Proposition that the competitive spending (rent-dissipation) rate for market k is $[1-\frac{t(2n+1-k)}{2n+1}][1-\frac{t(k-1)}{2n+1}]/\tilde{\mu}\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2$. As in the exogenous market value case, it strictly increases with k for $k<n+1$, and strictly decreases for $k \geq n+1$. But the equilibrium investment in value creation has an opposite profile leading to the highest amount of surplus v_k created at the market coincident with the firms, while the middle market has the lowest surplus size. Thus the incentive to compete most intensely for the middle market is offset by the fact that in equilibrium it will have the smallest size. Hence, the distribution of competitive rent-seeking bids are no longer single-peaked as in the case of fixed market values. Figure 2b. shows the distribution of competitive rent-seeking expenditures.

To examine the distribution of the competitive spending further, note that by symmetry, we will have in equilibrium that $x_{i,k}=x_{i,(2n+1)-(k-1)}$, and so without loss of generality, we focus on the left side of the line for Firm 1, i.e., $k \leq n+1$. Recall $x_{i,k}=\frac{[1-\frac{t(2n+1-k)}{2n+1}][1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2\tilde{\mu}}$

and $v_k = \frac{[1-\frac{t(k-1)}{2n+1}]}{2\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\}\tilde{\mu}}$, which gives

$$x_{i,k} = \frac{[1 - \frac{t(2n+1-k)}{2n+1}][1 - \frac{t(k-1)}{2n+1}]^2}{2\tilde{\mu}^2 \left\{[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}]\right\}^3},$$

Because $[1 - \frac{t(k-1)}{2n+1}] + [1 - \frac{t(2n+1-k)}{2n+1}] = (2 - \frac{t \cdot 2n}{2n+1}) > 0$, to evaluate $x_{i,k}$ with respect to k, it is sufficient to examine the numerator. We have

$$\begin{aligned} & \frac{d[1 - \frac{t(2n+1-k)}{2n+1}][1 - \frac{t(k-1)}{2n+1}]^2}{dk} \\ = \; & \frac{t[(2n+1) - t(k-1)]}{(2n+1)^2} [t(4n+3-3k) - (2n+1)] . \end{aligned}$$

The sign depends on the size of $[t(4n+3-3k) - (2n+1)]$.

Corollary 1 *(a) Within each half of the line, the distribution of firms' competitive rent-seeking expenditures is in general non-monotonic. The competitive spending $x_{i,k}$s reach their peak symmetrically in two markets $\tilde{k}_1$ and $\tilde{k}_2$, with $\tilde{k}_1 = 2(n+1) - \tilde{k}_2 < n+1$ and correspondingly $\tilde{k}_2 = 2(n+1) - \tilde{k}_1 > n+1$.*

(b) The locations of the peak competitive spending move toward the midpoint as t increases and the firms becomes more differentiated .

The distribution of competitive spending is no longer monotonic within each firm's turf. As already described above the middle market no longer faces the most intense competition. Rather in an interesting contrast to the case of exogenous market surplus values, we get that each firm chooses the highest the competitive spending at two symmetrically located markets on either side of the middle market $k = n+1$. Thus when firms endogenously create the market, the most intense competition shifts to each firm turf, and this reflects the balance between fighting in markets where there is more surplus and where competitive spending is more effective in winning the market from the rival.

The particular pattern depends on the size of t. Consider, for instance, the case of $t = 1$. The expression $[t(4n+3-3k) - (2n+1)]$ reduces to $2(n+1) - 3k$. The competitive spending $x_{i,k}$ increases and then decreases after reaching an interior peak. In contrast, suppose that t is sufficiently small, i.e., $t \leq \frac{2n+1}{4n}$, the most intense competitions simply occur at each firm's home court because in this case, $[t(4n+3-3k)-(2n+1)]_{k=1} \leq 0$. The second part of the corollary shows an interesting effect of firm differentiation on the competitive spending. Indeed the markets with the most intense competition move closer to the middle even as differentiation increases.

We now examine how market differentiation affects the extent to which firms invest in value creation versus spending on competitive rent-seeking. This analysis is tractable for the three-market case, with $n = 1$, for which we get:

Proposition 5 *Consider a three-market case with $n = 1$. When t increases, firms invest less in the middle market and invest more on their home markets, i.e., $\frac{dv_2}{dt} < 0$ and $\frac{dv_1}{dt}, \frac{dv_3}{dt} > 0$. They deploy less in competitive rent-seeking activities in all markets, i.e., $\frac{dx_k}{dt} < 0, \forall k \in \{1, 2, 3\}$.*

The degree of market differentiation affects both i) firms' division of resources between rent-seeking activities and value-creating investments, and ii) their resource allocations across market. With greater market differentiation both firms invest less in the middle market, and more in their own turfs. A more differentiated market leads to lower effectiveness of a firm's competitive spending in the remote market, and strengthens its advantage at its own turf. This increases the return to the firm's investment at its own turf, as it is more able to protect it from possible predation. Hence, firms shift investments in value creation back to closer markets in their own turfs. At the same time, the greater advantage at home market turns off the rival firm's ability to win those markets, which leads each firm to reduce its competitive expenditure accordingly. Further, because firms invest less in the middle market, the reduced value also elicits less competitive spending in the middle.

4.2 Asymmetric Budgets

Finally, consider the general case that allows firms to be endowed with asymmetric resource budgets with $m_1 > m_2$. Despite the asymmetry, Lemma 2 continues to hold: There exists no equilibrium in which a market ends up with zero surplus. As a result, both firms will invest in competitive rent-seeking activities in all markets, i.e., $x_{i,k} > 0, \forall i \in \{1, 2\}, k \in \{1, \ldots, 2n+1\}$. While a closed-form solution to the equilibrium cannot be obtained due to the nonlinearity of production function and the budget asymmetry, it is still possible to characterize some important properties of the equilibrium.

Proposition 6 *When $m_1 > m_2$ we have that in equilibrium:*

a. *Firm 1, invests more in value creation and also deploys higher competitive rent-seeking spending.*

b. *The equilibrium surplus across markets, v_k, is distributed as a U-shaped curve, strictly decreasing first and then strictly increasing.*

c. *A firm makes more investments in value creation in markets closer to its own location.*

d. *Firm 1 makes investment in value creation in markets 1 to $\bar{k}$, with $\bar{k} \geq n+1$, while only Firm 2 invests in markets $\bar{k}+1$ to $2n+1$. Firm 2 may also invest on $\bar{k}$ (in knife-edge case), but for markets $\{1, \ldots, 2n+1\} \setminus \{\bar{k}\}$ only one firm invests.*

In the equilibrium, the ratio between firms' competitive rent-seeking expenditures remains constant across all markets (i.e., $\frac{x_{1,k}}{x_{2,k}} = \frac{\mu_2}{\mu_1}$), where μ_1 and μ_2 are the Kuhn-Tucker multipliers for firms 1's and 2's constrained maximization problem. This leads to the result in part (a) of the proposition that the firm with the budget advantage spends more in the rent-seeking competition. But at the same time it also invests more in value creation. This contrasts with an important finding in the literature on single-market guns and butter competition that the firm with the budget advantage invests more in value creation (butter) but it is the weaker firm with the smaller budget that invests more in guns (see Skaperdas 1992). Because the joint surplus is subject competition, this leads to the firm with the smaller budget ending up with a larger expected payoffs. In contrast, part (a) of proposition shows that this result does not carry over to guns and butter competition in multi-market settings. With multiple markets and with market differentiation, firms have to trade-off where to invest as well as how much to invest in value creation and in rent seeking competition. This allows for greater productive investments in closer markets by Firm 1 where Firm 2's competitive spending is less efficient.

As shown in the proposition, in general, in each market only one firm makes productive investments and so the equilibrium value creation profiles of the two firms are mutually exclusive. This can be seen as a strategic attempt by each firm to reduce free-riding of their value creation investments. This is different from the symmetric case, where we see (minimal) overlap: i.e., both firms overlap in a single market in the middle ($n + 1$). The overlap is thus an artifact of symmetry.

In the asymmetric case, Firm 1 invests on strictly more number of markets than Firm 2. The distribution of value creation is similar to that in the symmetric case. It is intuitive that each firm is more willing to invest in markets closer to its own position. As a result, the market values are distributed as a U-shape curve: Firm 1's investments strictly decrease toward the other end of the line until it stops investing; in contrast, firm 2's investments pick up in markets closer to the right end of the line. This observation is qualitatively similar to that in the symmetric case. In the symmetric case, equilibrium productive investment is minimized at the middle market, i.e., $n + 1$. In the asymmetric case, as expected it is minimized at a market to the right of the middle point, because of the asymmetry. In fact if the asymmetry is sufficiently large, Firm 2 may stop investing on value creation, and focus its resource only on competitive rent-seeking activities.

We can also establish some comparative statics pertaining to how the extent of asymmetry in the firms' budgets affect their strategies: Specifically, greater asymmetry in the budget between the firms (i.e., $\frac{m_1}{m_2}$) leads to increases in the equilibrium $\bar{k}$: i.e., Firm 1 makes productive investments in more markets while Firm 2 invests in fewer markets. Consistent with this result greater budget asymmetry also leads to increases the ratio of firms' competitive expenditure ($\frac{x_{1,k}}{x_{2,k}}$). Firm 1 therefore deploys relatively higher spending in rent-seeking. Thus the overall message is that in the presence of market differentiation and multi-market interactions greater budget advantage leads a firm to balance its actions such that it not only invests relatively more in value creation, but also competes with greater resources.

5 Conclusion

Our analysis brings together two basic trade-offs that are common to many important economic and business contexts: First, firms must invest resources in value creation which they can profitably extract. But in competitive markets they also have to compete for the market with rivals. This leads to the guns-and-butter-like trade-off of allocating limited resources to creating the market versus competing for it? Second, firms competing in differentiated markets must also decide how to allocate resources between different markets where they have more or less competitive advantage. In this paper we develop and analyze a model of multi-market competition which captures these trade-offs and their effects on firm strategies.

When the market size is fixed then the firms' allocation strategies are only governed by the competitive rent seeking incentives and each firm's equilibrium resource allocation strategies follow an inverted-U profile. A firm spends less in closer markets where its spending is relatively more effective because it is easier to defend these markets from competition. At the same time it also spends relatively less in far away markets precisely because its spending is relatively less effective and it is harder to win these markets from competition. Thus the most intense competition is for a market in the middle. Further, and counter to intuition, the competition for the marginal market in the middle becomes more intense even as the firms become more differentiated.

The main analysis considers the dual trade-off in which firms decide how much to invest in creating value in each market as well as how much to spend in competing for the markets. In equilibrium a firm invests more in closer markets and the investment profile declines monotonically. For symmetric firms this leads to the most intense competition to move away from the middle market to one in each firm's turf. We also find that with asymmetric budgets the firm with the advantage invests more in both value creation as well as in competitive rent-seeking spending. As the budget asymmetry increases the firm with the advantage invests in more markets closer to the weaker firm. Greater market differentiation leads to more value creation by firms in their home turfs and a reduction in the overall amount of competitive spending.

An aspect of the problem that we have not explored is the role of potential uncertainty of firms about their rivals. For instance, a firm might be uncertain about the size of its rival's resource budget and the nature of this uncertainty should have a bearing on the extent to which firms invest in value creation versus rent seeking. The analysis of multi-market guns and butter contests under incomplete information is a challenging problem which may be investigated in future work.

References

[1] Amaldoss, W., Meyer, R.J., Raju, J., and Rapoport, A. 2000. Collaborating to Compete. *Marketing Science* 19(2), 105–126.

[2] Bass, F.M., Krishnamoorthy, A., Prasad, A., and Sethi, S.P. 2005. Generic and Brand Advertising Strategies in a Dynamic Duopoly. *Marketing Science* 24(4), 556–568.

[3] Borel, E. 1921. La théorie du jeu les équations intégrales à noyau symétrique. Comptes Rendus de l'Académie 173, 1304–1308; English translation by Savage, L. 1953. The theory of play and integral equations with skew symmetric kernels. *Econometrica* 21, 97–100.

[4] Borel, E., Ville, J. 1938. *Application de la théorie des probabilitiés aux jeux de hasard.* Paris: Gauthier-Villars; reprinted in Borel E., Chéron, A. (Eds), *Théorie mathematique du bridge à la portée de tous.* Paris: Editions Jacques Gabay.

[5] Chen, H., and Lim, N. 2013. Should Managers Use Team-Based Contests?. *Management Science* 59, 2823-2836.

[6] Clark, D.J., and Konrad, K.A. 2007. Contests with Multi-tasking. *Scandinavian Journal of Economics* 109(2), 303-319.

[7] Friedman, L. 1958. Game-theory models in the allocation of advertising expenditures. *Operations Research* 6(5), 699-709.

[8] Garfinkel, M. R., and Skaperdas S. 2007. Economics of Conflict: An Overview, in Sandler T., and Hartley K. (Eds), *Handbook of Defense Economics.* Elsevier.

[9] Gross O.A., and Wagner, R.A. 1950. A Continuous Colonel Blotto Game. *Technical Report RM-408.* RAND Corporation, Santa Monica, CA.

[10] Hirshleifer, J. 1988. The Analytics of Continuing Conflict. *Synthese* 76 (2), 1988, 201-233.

[11] Hirshleifer J. 1994. The Dark Side of the Force, *Economic Inquiry* 32, 1-10.

[12] Kalra, A., and Shi, M. 2001. Designing Optimal Sales Contests: A Theoretical Perspective. *Marketing Science* 20(2), 170-193.

[13] Kovenock, D., and Roberson, B. 2012. Coalitional Colonel Blotto Games with Application to the Economics of Alliances. *Journal of Public Economic Theory* 14(4), 653-676.

[14] Kvasov, D. 2007. Contests with Limited Resources. *Journal of Economic Theory* 136(1), 738-748.

[15] Kvasov, D., and Roberson, B. 2012. The Non-constant-sum Colonel Blotto Game. *Economic Theory* 51(2), 397-433.

[16] Lake, M. 1978. *A New Campaign Resource Allocation Model.* Reprited in Brams, S.J., Schotter, A., and Schwodiauer, G. (Eds.), *Applied Game Theory.* Wurzburg, West Germany: Physica-Verlag.

[17] Lim, N., Ahearne, M., and Ham, S. 2009. Designing Sales Contests: Does the Prize Structure Matter?. *Journal of Marketing Research* 46, 356-371.

[18] Ridlon, R., and Shin, J. 2013. Favoring the Winner or Loser in Repeated Contests. *Marketing Science* 32(5), 768-785.

[19] Skaperdas, S. 1992. Cooperation, Conflict, and Power in the Absence of Property Rights. *The American Economic Review* 82(4), 720-739.

Appendix

Proof of Lemma 1

The proof is by contradiction. Suppose there exists a market $k' \in \{1, \ldots, 2n+1\}$ such that $x_{i,k'} = x_{j,k'} = 0$. Then let firm i deviate by finding an infinitesimal ε, such that it places an outlay of ε in market k', but reduces its bid in some other market k'' by ε. In this case, it will gain v at market k' with probability one, but its probability of winning v in market k'' decreases negligibly. By continuity, the firm must get strictly better with such a deviation, which establishes the contradiction.

Suppose now that $x_{j,k'} > 0$ and $x_{i,k'} = 0$. Now firm j can always gain by reducing $x_{j,k'}$ to an infinitesimally small ε and reallocating to other markets. This means that firm i has the incentive to deviate from $x_{i,k'} = 0$, which establishes a contradiction.

Proof of Proposition 1

Evaluating π_i with respect to an arbitrary $x_{i,k}$ yields

$$\begin{aligned}
\frac{\partial \pi_1}{\partial x_{1,k}} &= \frac{\left\{ r[1 - \frac{(k-1)}{2n+1}] \frac{k}{2n+1} x_{1,k}^{r-1} x_{2,k}^{r-1} \right\} x_{2k}}{\left\{ [1 - \frac{(k-1)}{2n+1}] x_{1,k}^r + \frac{k}{2n+1} x_{2,k}^r \right\}^2} v; \\
\frac{\partial \pi_2}{\partial x_{2,k}} &= \frac{\left\{ r[1 - \frac{(k-1)}{2n+1}] \frac{k}{2n+1} x_{1,k}^{r-1} x_{2,k}^{r-1} \right\} x_{1k}}{\left\{ [1 - \frac{(k-1)}{2n+1}] x_{1,k}^r + \frac{k}{2n+1} x_{2,k}^r \right\}^2} v.
\end{aligned}$$

An interior optimum must satisfy

$$\begin{aligned}
\frac{\partial \pi_1}{\partial x_{1,k}} &= \frac{\partial \pi_{1k'}}{\partial x_{1,k'}} = \mu_1; \\
\frac{\partial \pi}{\partial x_{2,k}} &= \frac{\partial \pi_2}{\partial x_{2,k'}} = \mu_2,
\end{aligned}$$

$\forall k, k' \in \{1, \ldots, 2n+1\}, k \neq k'$. Hence, we must have

$$\frac{x_{2,k}}{x_{1,k}} = \frac{x_{2,k'}}{x_{1,k'}} = \frac{\mu_1}{\mu_2}, \forall k, k' \in \{1, \ldots, 2n+1\}, k \neq k'.$$

Define $\lambda \equiv \frac{\mu_1}{\mu_2}$. We then have $x_{2,k} = \lambda x_{1,k}$ for all k. We can then rewrite $\frac{\partial \pi_i}{\partial x_{i,k}}$ as

$$\frac{\partial \pi_i}{\partial x_{i,k}} = \frac{\left\{ r[1 - \frac{(k-1)}{2n+1}] \frac{k}{2n+1} \lambda^r \right\}}{\left\{ [1 - \frac{(k-1)}{2n+1}] + \frac{k}{2n+1} \lambda^r \right\}^2} \frac{v}{x_{i,k}},$$

Define $\phi(k) \equiv \frac{\left\{r[1-\frac{(k-1)}{2n+1}]\frac{k}{2n+1}\lambda^r\right\}}{\left\{[1-\frac{(k-1)}{2n+1}]+\frac{k}{2n+1}\lambda^r\right\}^2}$. We then have $x_{i,k} = \frac{\phi(k)}{\phi(k')}x_{i,k'}$ and we must recursively obtain $x_{i,k} = \frac{\phi(k)}{\phi(1)}x_{i,1}$. Further, the resource (budget) constraints can be rewritten as

$$\begin{aligned}
\sum_{k=1}^{2n+1} x_{1,k} &= [\frac{\sum_{k=1}^{2n+1}\phi(k)}{\phi(1)}]x_{1,1} = m_1; \\
\sum_{k=1}^{2n+1} x_{2,k} &= \lambda\sum_{k=1}^{2n+1} x_{1,k} = \lambda[\frac{\sum_{k=1}^{2n+1}\phi(k)}{\phi(1)}]x_{1,1} = m_2.
\end{aligned}$$

As a result, we have $\lambda = \frac{m_2}{m_1}$ and $x_{1,1} = \frac{m_1\phi(1)}{\sum_{k=1}^{2n+1}\phi(k)}$, and therefore $x_{1,k} = \frac{m_1\phi(k)}{\sum_{k=1}^{2n+1}\phi(k)}$, and $x_{2k} = \frac{m_2\phi(k)}{\sum_{k=1}^{2n+1}\phi(k)}$. This proves the Proposition.

Proof of Proposition 2

Apparently, $\tilde{\phi}(k|\,t)$ continues to be single-peaked, as the sign is determined by the term

$$\left\{\begin{array}{c}
\left[(2n+1)^2 - t(2n+1)(n-1) - t^2 n\right] \\
-(2n+1)\lambda^r\left[(2n+1) - t(3n+1) + t^2 n\right] \\
-t(1+\lambda^r)\left[(2n+1) - tn\right]k
\end{array}\right\}.$$

Define $\hat{k} = \frac{\left\{\begin{array}{c}\left[(2n+1)^2 - t(2n+1)(n-1) - t^2 n\right] \\ -(2n+1)\lambda^r\left[(2n+1) - t(3n+1) + t^2 n\right]\end{array}\right\}}{t(1+\lambda^r)[(2n+1)-tn]}$. If k is treated as a continuous varaible, the function $\tilde{\phi}(k|\,t)$ is then maximized at $k = \hat{k}$. Hence, $\tilde{\phi}(k|\,t)$ reaches its peak at

$$k^* = \left\{\begin{array}{cc}
0 & \text{if } \hat{k} \le 0 \\
2n+1 & \text{if } \hat{k} \ge 2n+1 \\
\arg\max\limits_{(int(\hat{k}),int(\hat{k})+1)} \tilde{\phi}(k). & \text{if otherwise}
\end{array}\right. .$$

Obviously, $\hat{k}$ decreases with λ. It has a value of precisely $\frac{\left\{\begin{array}{c}\left[(2n+1)^2 - t(2n+1)(n-1) - t^2 n\right] \\ -(2n+1)\left[(2n+1) - t(3n+1) + t^2 n\right]\end{array}\right\}}{2t[(2n+1)-tn]} = \frac{\left\{\begin{array}{c}\left[-(2n+1)(n-1) - tn\right] \\ -(2n+1)\left[-(3n+1) + tn\right]\end{array}\right\}}{2[(2n+1)-tn]} = \frac{4n^2+6n+2-2tn^2-2tn}{2[(2n+1)-tn]} = \frac{\left[2n^2+3n+1-tn^2-tn\right]}{[(2n+1)-tn]} = n+1$. This implies that in the asymmetric case, the peak appears to the right of $n+1$.

Because $\lambda < 1$, λ^r decreases with r. As a result, $\hat{k}$ increases with r.

Proof of Proposition 3

Evaluating $\hat{k}$ with respect to t leads to

$$\frac{\partial \hat{k}}{\partial t} = \frac{1}{\{t(1+\lambda^r)\left[(2n+1)-tn\right]\}^2} \times \left\{ \begin{array}{c} \left[\begin{array}{c} -(2n+1)(n-1)-2tn \\ +(2n+1)(3n+1)\lambda^r - 2tn(2n+1) \end{array} \right] \{t(1+\lambda^r)\left[(2n+1)-tn\right]\} \\ -\left\{ \begin{array}{c} \left[(2n+1)^2 - t(2n+1)(n-1) - t^2 n\right] \\ -(2n+1)\lambda^r\left[(2n+1)-t(3n+1)+t^2 n\right] \end{array} \right\} (1+\lambda^r)\left[(2n+1)-tn\right] \\ +\left\{ \begin{array}{c} \left[(2n+1)^2 - t(2n+1)(n-1) - t^2 n\right] \\ -(2n+1)\lambda^r\left[(2n+1)-t(3n+1)+t^2 n\right] \end{array} \right\} tn(1+\lambda^r) \end{array} \right\}.$$

Let $\varphi(t)$ denote the numerator. We have $\varphi(t)$ rewritten as

$$\begin{aligned} \varphi(t) &= \left\{ \begin{array}{c} \left[\begin{array}{c} -t(2n+1)(n-1)-2t^2 n \\ +t(2n+1)(3n+1)\lambda^r - 2t^2 n(2n+1)\lambda^r \end{array} \right] \{(1+\lambda^r)\left[(2n+1)-tn\right]\} \\ -\left[\begin{array}{c} -t(2n+1)(n-1)-t^2 n \\ +t(2n+1)(3n+1)\lambda^r - t^2 n(2n+1)(2n+1)\lambda^r \end{array} \right] (1+\lambda^r)\left[(2n+1)-tn\right] \\ -\left[(2n+1)^2-(2n+1)^2\lambda^r\right](1+\lambda^r)\left[(2n+1)-tn\right] \\ +\left\{ \begin{array}{c} \left[(2n+1)^2 - t(2n+1)(n-1) - t^2 n\right] \\ -(2n+1)\lambda^r\left[(2n+1)-t(3n+1)+t^2 n\right] \end{array} \right\} tn(1+\lambda^r) \end{array} \right\} \\ &= \left\{ \begin{array}{c} -\left[t^2 n + t^2 n(2n+1)\lambda^r\right]\{(1+\lambda^r)\left[(2n+1)-tn\right]\} \\ -\left[(2n+1)^2-(2n+1)^2\lambda^r\right](1+\lambda^r)\left[(2n+1)-tn\right] \\ +\left\{ \begin{array}{c} \left[(2n+1)^2 - t(2n+1)(n-1) - t^2 n\right] \\ -(2n+1)\lambda^r\left[(2n+1)-t(3n+1)+t^2 n\right] \end{array} \right\} tn(1+\lambda^r) \end{array} \right\} \\ &= \left\{ \begin{array}{c} \left[(2n+1)^2\lambda^r-(2n+1)^2-t^2 n-t^2 n(2n+1)\lambda^r\right]\{(1+\lambda^r)\left[(2n+1)-tn\right]\} \\ +\left\{ \begin{array}{c} \left[(2n+1)^2 - t(2n+1)(n-1) - t^2 n\right] \\ -(2n+1)\lambda^r\left[(2n+1)-t(3n+1)+t^2 n\right] \end{array} \right\} tn(1+\lambda^r) \end{array} \right\} \end{aligned}$$

Further,

$$
\begin{aligned}
& (1+\lambda^r)\left\{\begin{array}{c}
\left[(2n+1)^2\lambda^r-(2n+1)^2-t^2n-t^2n(2n+1)\lambda^r\right](2n+1) \\
+\left\{\begin{array}{c}
\left\{\begin{array}{c}
\left[(2n+1)^2-t(2n+1)(n-1)-t^2n\right] \\
-(2n+1)\lambda^r\left[(2n+1)-t(3n+1)+t^2n\right]
\end{array}\right\} \\
-\left[(2n+1)^2\lambda^r-(2n+1)^2-t^2n-t^2n(2n+1)\lambda^r\right]
\end{array}\right\} tn
\end{array}\right\} \\
= \; & (1+\lambda^r)\left\{\begin{array}{c}
\left[(2n+1)^2\lambda^r-(2n+1)^2-t^2n-t^2n(2n+1)\lambda^r\right](2n+1) \\
+\left\{\begin{array}{c}
\left[2(2n+1)^2-t(2n+1)(n-1)\right] \\
-(2n+1)\lambda^r\left[2(2n+1)-t(3n+1)\right]
\end{array}\right\} tn
\end{array}\right\} \\
= \; & (1+\lambda^r)\left\{\begin{array}{c}
\left[(2n+1)^2\lambda^r-(2n+1)^2\right](2n+1)-\left[t^2n+t^2n(2n+1)\lambda^r\right](2n+1) \\
+2[2n+1)^2-(2n+1)^2\lambda^r]tn \\
+t(2n+1)[(3n+1)\lambda^r-(n-1)]tn
\end{array}\right\} \\
= \; & (1+\lambda^r)\left\{\begin{array}{c}
\left[(2n+1)^2\lambda^r-(2n+1)^2\right](2n+1)-t^2n\left[1+(2n+1)\lambda^r\right](2n+1) \\
+2tn[2n+1)^2-(2n+1)^2\lambda^r] \\
+t^2n(2n+1)[(3n+1)\lambda^r-(n-1)]
\end{array}\right\} \\
= \; & (1+\lambda^r)\left\{\begin{array}{c}
(2n+1)^2\lambda^r-(2n+1)^2][(2n+1)-2tn] \\
+t^2n(2n+1)\left\{\begin{array}{c}
[(3n+1)\lambda^r-(n-1)] \\
-[(2n+1)\lambda^r+1]
\end{array}\right\}
\end{array}\right\} \\
= \; & (1+\lambda^r)\left\{\begin{array}{c}
(2n+1)^2(\lambda^r-1)[(2n+1)-2tn] \\
+t^2n^2(2n+1)(\lambda^r-1)
\end{array}\right\}.
\end{aligned}
$$

It is negative because $\lambda < 1$.

Proof of Lemma 2

Suppose otherwise. Then neither firm would exert competitive rent-seeking effort in a market with no surplus. Hence, a firm can strictly increase its payoff by decreasing its investment from other markets where its rival also invests, but increase its investment in this market where it is the sole claimant of the market surplus.

Proof of Proposition 4

We first demonstrate that in any symmetric equilibrium, firms exert the same amount of rent-seeking effort on every market, i.e., $x_{1,k} = x_{2,k}, \forall k \in \{1, \ldots, 2n+1\}$.

Symmetric equilibrium, with $(b_{1,k}, x_{1,k}) = (b_{2,2n+1-(k-1)}, x_{2,2n+1-(k-1)})$, leads to $\mu_1 = \mu_2$. Hence, $[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}/\left\{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}\right\}^2 = [1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]x_{1,k}/\left\{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}\right\}^2$, which gives $x_{1,k} = x_{2,k}$.

Define $\tilde{\mu} = \mu_1 = \mu_2$ and $x_k = x_{1,k} = x_{2,k}$. The Kuhn-Tucker conditions can be rewritten as

$$\begin{aligned}
\frac{\partial \pi_1}{\partial x_{1,k}} &= \frac{\partial \pi_2}{\partial x_{2,k}} = \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2 x_k}\sqrt{b_{1,k}+b_{2,k}} = \tilde{\mu}; \\
\frac{\partial \pi_1}{\partial b_{1,k}} &= \frac{[1-\frac{t(k-1)}{2n+1}]}{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \le \tilde{\mu}; \\
\frac{\partial \pi_1}{\partial b_{2,k}} &= \frac{[1-\frac{t(2n+1-k)}{2n+1}]}{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \le \tilde{\mu}.
\end{aligned}$$

It is impossible to have $\frac{\partial \pi_1}{\partial b_{1,k}} = \frac{\partial \pi_1}{\partial b_{2,k}} = \tilde{\mu}$ for $k \neq n+1$. Hence, except for the mid-point, no market has both firms make positive productive investment. In any symmetric equilibrium, $v_k = v_{2n+1-(k-1)}$. We must have $\frac{\partial \pi_1}{\partial b_{1,k}} > \frac{\partial \pi_1}{\partial b_{1,2n+1-(k-1)}}$ and $\frac{\partial \pi_1}{\partial b_{2,k}} < \frac{\partial \pi_1}{\partial b_{1,2n+1-(k-1)}}$. That is, for $k < (>)\ n+1$, only firm 1(2) invests.

We then consider the mid-point, i.e., $k = n+1$. Firms must both invest positively here, because $\frac{\partial \pi_1}{\partial b_{1,k}} = \frac{\partial \pi_1}{\partial b_{2,k}} = \frac{1}{2}\cdot\frac{1}{2v_k}$.

Hence, we rewrite the Kuhn-Tucker conditions as

$$\begin{aligned}
x_k &= \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2 \tilde{\mu}} v_k, \\
\text{and } v_k &= \frac{1}{2}\frac{[1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}}, k \le n+1.
\end{aligned}$$

Recall $\sum_{k=1}^{2n+1} v_k^2 + 2\sum_{k=1}^{2n+1} x_k = 2m$. LHS can further be written as

$$\begin{aligned}
& \sum_{k=1}^{2n+1} v_k^2 + 2\sum_{k=1}^{2n+1} x_k \\
= {} & 2\sum_{k=1}^{n} v_k^2 + 4\sum_{k=1}^{n} x_k + v_{n+1}^2 + 2x_{n+1}.
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{k=1}^n v_k^2 &= \sum_{k=1}^n \frac{1}{4}\frac{[1-\frac{t(k-1)}{2n+1}]^2}{\tilde{\mu}^2\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2},\\
\sum_{k=1}^n x_k &= \sum_{k=1}^n \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2\tilde{\mu}}v_k\\
&= \sum_{k=1}^n \frac{1}{2}\frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^2\tilde{\mu}}\frac{[1-\frac{t(k-1)}{2n+1}]}{\tilde{\mu}\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}}\\
&= \sum_{k=1}^n \frac{1}{2}\frac{[1-\frac{t(k-1)}{2n+1}]^2[1-\frac{t(2n+1-k)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}^3\tilde{\mu}^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&2\sum_{k=1}^n v_k^2 + 4\sum_{k=1}^n x_k\\
=\;& 2\sum_{k=1}^n \frac{[2n+1-t(k-1)]^2}{4\tilde{\mu}^2[(2n+1)-tn]^2}\left\{\frac{1}{4}+2\frac{[2n+1-t(2n+1-k)]}{2[(2n+1)-tn]}\right\}.
\end{aligned}$$

Further, $v_{n+1}^2 = \frac{1}{16\tilde{\mu}^2}$ and $2x_{n+1} = 2(\frac{1}{4\tilde{\mu}}\frac{1}{4\tilde{\mu}}) = \frac{1}{8\tilde{\mu}^2}$. We then have

$$\begin{aligned}
&2\sum_{k=1}^n v_k^2 + 4\sum_{k=1}^n x_k + v_{n+1}^2 + 2x_{n+1}.\\
=\;& \frac{1}{\tilde{\mu}^2}\left\{\sum_{k=1}^n \frac{[2n+1-t(k-1)]^2}{[(2n+1)-tn]^2}\left\{\frac{1}{2}+2\frac{[2n+1-t(2n+1-k)]}{[(2n+1)-tn]}\right\}+\frac{3}{16}\right\}.
\end{aligned}$$

Hence,

$$\tilde{\mu} = \sqrt{\frac{\left\{\sum_{k=1}^n \frac{[2n+1-t(k-1)]^2}{[(2n+1)-tn]^2}\left\{\frac{1}{4}+\frac{[2n+1-t(2n+1-k)]}{[(2n+1)-tn]}\right\}+\frac{3}{32}\right\}}{m}}$$

Proof of Proposition 5

Consider the special case of $n = 1$. We have

$$\begin{aligned}
\tilde{\mu} &= \sqrt{\frac{\left\{\sum_{k=1}^n \frac{[2n+1-t(k-1)]^2}{[(2n+1)-tn]^2}\left\{\frac{1}{4}+\frac{[2n+1-t(2n+1-k)]}{[(2n+1)-tn]}\right\}+\frac{3}{32}\right\}}{m}}\\
&= \sqrt{\frac{\frac{9}{(3-t)^2}\left(\frac{1}{4}+\frac{9}{3-t}\right)+\frac{3}{32}}{m}}.
\end{aligned}$$

We can then calculate v_k by $v_k = \frac{1}{2}\frac{[1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}}$, $k \leq n+1$. We have

$$\begin{aligned}
v_1 &= \frac{1}{2}\frac{[1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} \\
&= \frac{1}{2}\frac{1}{(1+\frac{3-2t}{3})\tilde{\mu}} \\
&= \frac{1}{4}\frac{1}{\frac{3-t}{3}\sqrt{\frac{\frac{9(39-t)}{4(3-t)^3}+\frac{3}{32}}{m}}} \\
&= \frac{3}{4}\frac{1}{\sqrt{\frac{\frac{9(39-t)}{4}+\frac{3(3-t)}{32}}{m}}},
\end{aligned}$$

which is increasing in t. So is v_3.

For v_2, we have

$$\begin{aligned}
v_1 &= \frac{1}{2}\frac{[1-\frac{t(k-1)}{2n+1}]}{\left\{[1-\frac{t(k-1)}{2n+1}]+[1-\frac{t(2n+1-k)}{2n+1}]\right\}\tilde{\mu}} \\
&= \frac{1}{2}\frac{1-\frac{t}{3}}{\left[\left(1-\frac{t}{3}\right)+\left(1-\frac{t}{3}\right)\right]\tilde{\mu}} \\
&= \frac{1}{4\tilde{\mu}},
\end{aligned}$$

which is decreasing in t, because $\tilde{\mu}$ increases with it.

Proof of Proposition 6

Note that the Kuhn-Tucker conditions laid out above continue to hold in the asymmetric case:

$$\begin{aligned}
\frac{\partial \pi_1}{\partial x_{1,k}} &= \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{\left\{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}\right\}^2}\sqrt{b_{1,k}+b_{2,k}} = \mu_1; \\
\frac{\partial \pi_1}{\partial b_{1,k}} &= \frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \leq \mu_1; \\
\frac{\partial \pi_2}{\partial x_{2,k}} &= \frac{[1-\frac{t(k-1)}{2n+1}][1-\frac{t(2n+1-k)}{2n+1}]x_{1,k}}{\left\{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}\right\}^2}\sqrt{b_{1,k}+b_{2,k}} = \mu_2; \\
\frac{\partial \pi_1}{\partial b_{2,k}} &= \frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}\frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \leq \mu_2.
\end{aligned}$$

By the argument laid out above, the ratio between the firms' competitive rent-seeking outlays is constant across all markets, $\frac{x_{1,k}}{x_{2,k}} = \frac{\mu_2}{\mu_1}$. As a result, in each market k, firm 1 wins with a probability

$$p_{1,k} = \frac{[1-\frac{t(k-1)}{2n+1}]\mu_2}{[1-\frac{t(k-1)}{2n+1}]\mu_2 + [1-\frac{t(2n+1-k)}{2n+1}]\mu_1},$$

and firm 2 wins with the complementary probability. Obviously, as k increases, i.e., on a market further away from the left end, $p_{1,k}$ strictly decreases and $p_{2,k}$ strictly increases.

We now lay out the following arguments successively, which build the proof for the proposition.

Claim 1 There exists at most one market in which both firms make positive productive investments.

Assume otherwise that there are markets $k, k' \in \{1, \ldots, 2n+1\}$, such that $b_{i,k}, b_{i,k'} > 0$, $\forall i \in \{1,2\}$. This implies

$$\begin{aligned}
&\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k} + [1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} \frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \\
= \; &\frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'} + [1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}} \frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}} = \mu_1; \\
&\frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k} + [1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} \frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \\
= \; &\frac{[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'} + [1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}} \frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}} = \mu_2;
\end{aligned}$$

This further leads to

$$\begin{aligned}
&\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k} + [1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} \frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \\
&+ \frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k} + [1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} \frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \\
= \; &\frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'} + [1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}} \frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}} \\
&+ \frac{[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'} + [1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}} \frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}},
\end{aligned}$$

which gives $b_{1,k} + b_{2,k} = b_{1,k'} + b_{2,k'}$. However,

$$\begin{aligned}
&\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k} + [1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} \frac{1}{2\sqrt{b_{1,k}+b_{2,k}}} \\
= \; &\frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'} + [1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}} \frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}}
\end{aligned}$$

implies $b_{1,k}+b_{2,k} \neq b_{1,k'}+b_{2,k'}$, because $\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} \neq \frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}$ by the fact $\frac{x_{1,k}}{x_{2,k}} = \frac{x'_{1,k}}{x'_{2,k}} = \frac{\mu_2}{\mu_1}$. Contradiction.

Claim 2 In the equilibrium, market rents are distributed as a U-shaped curve: It strictly decreases with k until a point $\bar{k}$, and then strictly increases.

Consider three arbitrary adjacent markets, $k-1, k$ and $k+1$. Assume that $b_{1,k-1}+b_{2,k-1}, b_{1,k+1}+b_{2,k+1} \leq b_{1,k}+b_{2,k}$. Recall the Kuhn-Tucker conditions. Because $p_{1,k}$ strictly decreases with k, and $p_{2,k}$ strictly increases with k, we must have the following: If firm 1 makes productive investment on $k-1$, then it cannot invest on k; if firm 2 makes productive investment on $k+1$, then it cannot invest on k.

Suppose $b_{1,k-1} > 0$, $b_{2,k+1} > 0$, then $b_{1,k}+b_{2,k} = 0$. Contradiction.

Suppose $b_{1,k-1} = 0$, and $b_{2,k+1} > 0$. Then only firm 1 has productive investment on k. This implies $b_{2,k-1} > 0$. By Kuhn-Tucker condition, it also implies $b_{2,k-1} > b_{1,k}$, which leads to contradiction. By the same logic, we conclude that it is impossible to have $b_{2,k+1} = 0$, and $b_{1,k-1} > 0$.

Suppose $b_{1,k-1} = b_{2,k+1} = 0$. This implies $b_{2,k-1}, b_{1,k+1} > 0$. Because firm 1 invests positively on $k+1$ but not $k-1$, we must have $b_{1,k+1} < b_{2,k-1}$. Because firm 2 invests positively on $k-1$ but not $k+1$, we must have $b_{1,k+1} > b_{2,k-1}$. Contradiction.

Claim 3 Suppose that there exists a k_0, with $b_{1,k_0}, b_{2,k_0} > 0$, then we must have (1) for all $k < k_0$, $b_{1,k} > 0$ and $b_{2,k} = 0$, and (2) for all $k > k_0$, $b_{2,k} > 0$ and $b_{1,k} = 0$.

Suppose otherwise that there is a $k < k_0$, with $b_{2,k} > 0$. Then we must have $b_{1,k} = 0$ by Claim 1. Because firm 2 has positive productive investment at both k and k_0, we must have

$$\begin{aligned}
&\frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} \frac{1}{2\sqrt{b_{2,k}}} \\
= &\frac{[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}}{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}+[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}} \frac{1}{2\sqrt{b_{1,k_0}+b_{2,k_0}}} = \mu_2,
\end{aligned}$$

which implies $b_{2,k} < b_{1,k_0}+b_{2,k_0}$, because $\frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} < \frac{[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}}{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}+[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}}$. However, because firm 1 invests zero on k, we have

$$\begin{aligned}
&\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} \frac{1}{2\sqrt{b_{2,k}}} \\
< &\frac{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}}{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}+[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}} \frac{1}{2\sqrt{b_{1,k_0}+b_{2,k_0}}} = \mu_1,
\end{aligned}$$

which implies $b_{2,k} > b_{1,k_0}+b_{2,k_0}$, because $\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} > \frac{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}}{[1-\frac{t(k_0-1)}{2n+1}]x_{1,k_0}+[1-\frac{t(2n+1-k_0)}{2n+1}]x_{2,k_0}}$. Contradiction.

The same argument applies to the second part of the claim.

Claim 4 Suppose there does not exist a k_0 as described in Claim 3. Consider two arbitrary markets $k, k' \in \{1, \ldots, 2n+1\}$, $k < k'$. If $b_{1,k} = 0$, then $b_{1,k'} = 0$; similarly, if $b_{2,k'} = 0$, then $b_{2,k} = 0$.

Suppose $b_{1,k} = 0$, then $b_{1,k'} > 0$. Then we must have $b_{2,k} > 0$. The Khun-Tucker condition requires

$$\begin{aligned} & \frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} \frac{1}{2\sqrt{b_{2,k}}} \\ < \quad & \frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}} \frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}} = \mu_1. \end{aligned}$$

This implies $b_{2,k} > b_{1,k'}+b_{2,k'}$ because $\frac{[1-\frac{t(k-1)}{2n+1}]x_{1,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} > \frac{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}$. However, $b_{2,k} > 0$ implies

$$\begin{aligned} \mu_2 \quad = \quad & \frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} \frac{1}{2\sqrt{b_{2,k}}} \\ \geq \quad & \frac{[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}} \frac{1}{2\sqrt{b_{1,k'}+b_{2,k'}}}, \end{aligned}$$

which implies $b_{2,k} < b_{1,k'}+b_{2,k'}$ because $\frac{[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}}{[1-\frac{t(k-1)}{2n+1}]x_{1,k}+[1-\frac{t(2n+1-k)}{2n+1}]x_{2,k}} < \frac{[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}{[1-\frac{t(k'-1)}{2n+1}]x_{1,k'}+[1-\frac{t(2n+1-k')}{2n+1}]x_{2,k'}}$. Contradiction.

The same argument applies to the second part of the claim.

Claim 5 Suppose there exist two markets $k, k' \in \{1, \ldots, 2n+1\}$, $k < k'$, with $b_{1,k}, b_{1,k'} > 0$, then $b_{1,k} > b_{1,k'}$.

Because $p_{1,k}$ strictly decreases, by Kuhn Tucker condition, we must have $b_{1,k}+b_{2,k} > b_{1,k'}+b_{2,k'}$. By Claims 1, 3 and 4, we have $b_{1,k} > b_{1,k'} + b_{2,k'}$.

Claim 6 Suppose there exist two markets $k, k' \in \{1, \ldots, 2n+1\}$, $k < k'$, with $b_{2,k}, b_{2,k'} > 0$, then $b_{2,k} < b_{2,k'}$.

This is implied by the proof of Claim 5.

Define $k_1 = \max(k|\, b_{1,k} > 0, b_{2,k} = 0)$ and $k_2 = \max(k|\, b_{1,k} = 0, b_{2,k} > 0)$. If k_0 exists, by definition, $k_0 = k_1 + 1 = k_2 - 1$; if k_0 does not exist, $k_2 = k_1 + 1$.

Claim 7 $\mu_1 < \mu_2$.

Suppose $\mu_1 \geq \mu_2$. This implies that firm 1's spending on rent seeking is no more than firm 2's on every market. As a result, $\frac{\partial \pi_1}{\partial b_{1,k}} \leq \frac{\partial \pi_1}{\partial b_{1,2n+1-(k-1)}}$. This implies that $k_1 \leq 2n+1$, i.e., firm 1 makes productive investments on a smaller number of markets than firm 2. At the same time, $v_{1,k} \leq v_{2,2n+1-(k-1)}$ must hold to make sure that firm 1 invests for $k \leq k_1$.

These facts imply that firm 1 spend less than firm 2 on both rent seeking and productive investment, which contradict the fact that firm 1 has a bigger budget, since a firm in this game has no reason to leave resource unused.

Claim 8 $k_1 \geq 2n+1$

Given $\mu_1 < \mu_2$, the claim is self-evident by the same argument that proves Claim 7.

Assume that k_0 does not exist. We have for $k \leq k_1$,

$$\begin{aligned}
\frac{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}2b_{1,k} &= x_{1,k}=\frac{\mu_2}{\mu_1}x_{2,k};\\
\frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}^2}{4\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2} &= b_{1,k}\mu_1.
\end{aligned}$$

Hence, for $k \leq k_1$,

$$\begin{aligned}
b_{1,k} &= \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}^2}{4\mu_1\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2},\\
x_{1,k} &= \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}^2\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}{2\mu_1\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3},\\
x_{2,k} &= \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}^2\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}{2\mu_1\mu_2\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3}
\end{aligned}$$

For $k \geq k_2$,

$$\begin{aligned}
\frac{[1-\frac{t(k-1)}{2n+1}]\mu_2}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}2b_{2,k} &= x_{2,k};\\
\frac{\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{4\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2} &= b_{2,k}\mu_2.
\end{aligned}$$

Hence, for $k \geq k_2$,

$$
\begin{aligned}
b_{2,k} &= \frac{\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{4\mu_2\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}, \\
x_{2,k} &= \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{2\mu_2\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3}, \\
x_{1,k} &= \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{2\mu_1\mu_2\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3}
\end{aligned}
$$

The equilibrium is determined by the following equations:

$$
\begin{aligned}
& \textstyle\sum_{k=1}^{k_1} b_{1,k} + \sum_{k=1}^{k_1} x_{1,k} + \sum_{k=k_2}^{2n+1} x_{1,k} \\
= \;& \frac{1}{4\mu_1}\textstyle\sum_{k=1}^{k_1} \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}^2}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2} \\
& +\frac{1}{2\mu_1}\textstyle\sum_{k=1}^{k_1} \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}^2\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3} \\
& +\frac{1}{2\mu_1\mu_2}\textstyle\sum_{k=k_2}^{2n+1} \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3} \\
= \;& m_1; \\
& \textstyle\sum_{k=k_2}^{2n+1} b_{2,k} + \sum_{k=1}^{k_1} x_{2,k} + \sum_{k=k_2}^{2n+1} x_{2,k} \\
= \;& \frac{1}{4\mu_2}\textstyle\sum_{k=k_2}^{2n+1} \frac{\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2} \\
& +\frac{1}{2\mu_2}\textstyle\sum_{k=k_2}^{2n+1} \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^2}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3} \\
& +\frac{1}{2\mu_1\mu_2}\textstyle\sum_{k=1}^{k_1} \frac{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2\right\}^2\left\{[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}}{\left\{[1-\frac{t(k-1)}{2n+1}]\mu_2+[1-\frac{t(2n+1-k)}{2n+1}]\mu_1\right\}^3} \\
= \;& m_2.
\end{aligned}
$$