

Bundling and Optimal Auctions of Multiple Products

CHRISTOPHER AVERY
Harvard University

and

TERRENCE HENDERSHOTT
University of Rochester

First version received August 1996; final version accepted October 1999 (Eds.)

We study the optimal (*i.e.* revenue maximizing) auction of multiple products. We make three major points. First, we extend the relationship between price discrimination and optimal auctions from the single-product case to the multiple-product case. A monopolist setting prices for multiple products may offer discounts on purchases of bundles of products; similarly, the optimal auction of multiple products facilitates price discrimination by allocating products inefficiently to customers who are willing to purchase both products. Second, we demonstrate that optimal auctions are qualitatively distinct from monopoly sales of multiple products. Because of uncertainty about the values of other consumers, two products are bundled *probabilistically* in an optimal auction for a customer who is willing to buy both of them. A customer may then receive a discount on a lower-valued product without receiving a higher-valued product. Third, we show that in an optimal auction of two products the allocation of one product may vary with the amount of competition for the other product.

1. INTRODUCTION

We study the optimal (*i.e.* revenue maximizing) auction of multiple products. When consumer types/valuations are multidimensional, many important properties do not carry over from single-dimensional auctions. When the consumer's type is unidimensional, the surplus for each given consumer type t is given by a path integral from 0 (or the lowest possible type) to t . Because the form of this integral is fixed, the "virtual utility" for a given consumer does not depend on the distributions of other bidders. When the consumer's type is multidimensional, however, there are many possible paths for determining surplus to a given consumer type.¹ Thus, the virtual utility to a given consumer type will generally depend on the distributions of other consumers.

We demonstrate three important properties of multiproduct auctions. First, we show that an optimal auction favours allocations that bundle products to a consumer who is interested in both products. Second, we show that bundling takes on a different form in an auction than it does in monopoly pricing. Rather than deterministic bundling—shown by McAfee and McMillan (1988) and McAfee, McMillan, and Whinston (1989) to be optimal when a monopolist sells two products at once—the optimal auction of multiple products often requires *probabilistic bundling*. When a customer reveals a high value for

1. In some special cases, such as Armstrong (1996) and Section 4 of Armstrong (2000), the straight-line path from a given type to the lowest possible type is the optimal integral for computing surplus. In these instances, properties from unidimensional auctions do carry over to the multidimensional case.

product B , that may increase her chances of receiving product A . Unlike deterministic bundling, however, her high value for product B may cause her to receive product A even in cases where she does not receive product B . Third, we show that products A and B can emerge endogenously as either substitutes or complements in terms of expected revenue in an optimal auction. Additional competition for one product has an indeterminate effect on the allocation of the other product: an increase in competition for product A can favour or impede allocations of product B in an optimal auction.

Our formulation, which utilizes a binary distribution of values to simplify the multidimensional incentive constraint, is closely related to the work of Armstrong and Rochet (1999) (which extends earlier work by Dana (1993) and Rochet (1995)), and Armstrong (2000). Those papers concentrate on deviations from the first-best efficient allocation of products. In contrast, we focus on the distinction between optimal auctions of multiple products and optimal auctions of single products. Our work is also distinct from earlier papers because we draw an explicit connection between price discrimination in monopoly pricing and bundling in optimal auctions for multiple products. Our analysis extends Bulow and Roberts (1989), which connected price discrimination and optimal auctions for one-dimensional (single-product) problems.

The paper proceeds as follows. Section 2 presents the model. Section 3 extends the marginal revenue approach of Bulow and Roberts (1989) to our two-product model. Section 4 demonstrates properties of optimal auctions in this framework. Section 5 considers two related extensions. Section 6 concludes.

2. THE MODEL

We study the pricing problem faced by a multiproduct monopolist with one unit to sell of each of two products, A and B . We assume that the monopolist has constant marginal costs of producing each product, c^A and c^B .² The monopolist faces both *multiproduct consumers*, who could conceivably purchase either or both products, and *single-product consumers*, who value only one of the two products. All bidders are risk neutral, and all multiproduct consumers have additive valuations for the products. If a multiproduct consumer with values (v^A, v^B) receives product A with probability ρ^A and product B with probability ρ^B , and pays a transfer of T , then that consumer's net utility is $\rho^A v^A + \rho^B v^B - T$. We emphasize the connection between multidimensional incentive constraints and bundling by explicitly excluding synergies and negative externalities between the products.

We allow continuous distributions for the values for the single-product consumers and an arbitrary number of single-product consumers for each product. We define v_i^k to be the realized value for single-product consumer i on product k and assume that the single-product consumers' values are independent of each other and of the values of multiproduct consumers. We also assume that the marginal revenues of single-product consumers are increasing in their values, $MR_i(v) = v - (1 - \Phi_i(v))/\phi_i(v)$ is weakly increasing in v , where $\phi_i(v)$ and $\Phi_i(v)$ are the density and cumulative distribution function for v , respectively.³

The realized values of individual bidders for each product affect the seller's opportunity cost for selling the product to the multiproduct consumer. Specifically, the opportunity cost for selling product k to a multiproduct bidder rather than an individual bidder

2. Alternatively c^A and c^B can be thought of as the monopolist's value for the products.

3. We could drop this assumption and iron out any nonmonotonicities (Bulow and Roberts (1989)) before proceeding. We also refer to single-product consumers as individual bidders.

(or not selling it at all) is the maximum (realized) marginal revenue for a single-product consumer bidding for product k : $\max [\max_i MR_i(v_i^k), c^k]$. With uncertainty about individual bidders' values, the marginal opportunity cost is $MC^k(z) = F_k^{-1}(z)$, where F_k^{-1} is the inverse of the cumulative distribution function for $\max [\max_i MR_i(v_i^k), c^k]$. This implies that $MC^k(z)$ is increasing and that the total opportunity cost for selling product k is $C^k(z) = \int_{c^k}^z MC^k(u) du$. Note that the number of single-product bidders is only relevant in that additional bidders stochastically increase the competition, and hence the "cost," for a product.

With no single-product consumers, the monopolist's opportunity costs are fixed at (c^A, c^B) per unit sold and the monopolist's maximization problem can be solved as a linear programme. This is the case studied by Armstrong (2000) and discussed in Section 5.1 of this paper.⁴

With any number of single-product consumers, but only one multiproduct consumer, the monopolist's maximization problem for allocations to the multiproduct consumer takes the form of a nonlinear pricing problem. That nonlinear pricing problem is a special case of the more general multidimensional screening problem studied by Armstrong and Rochet (1999). Although, for expositional ease, we focus on the case with a single multiproduct consumer throughout this paper, our results should carry over to the case of many multiproduct consumers and many single-product consumers, as briefly discussed in Section 5. However, with many multiproduct consumers the set of constraints governing the monopolist's maximization problem expands rapidly, making the analysis quite complex.

In our context, an allocation of z units of product k to the multiproduct consumer with valuations (v^A, v^B) means that the multiproduct consumer receives (one unit of) product k with probability z and does not receive product k with probability $1 - z$. Further, the allocation of product k depends on the realization of the monopolist's opportunity cost for product k (*i.e.* the values of individual-product consumers interested in product k) and the multiproduct consumer receives product k iff its realized opportunity cost is less than $MC^k(z)$. Like Armstrong (2000) and Armstrong and Rochet (1999), we utilize a binary distribution where the multiproduct consumer has only two possible values for each product. The possible values "High" and "Low" are the same for each product, with $v_H^A = v_H^B = H$ and $v_L^A = v_L^B = L$, and $\Delta = H - L$. These simplifications enable us to demonstrate the qualitative differences between unidimensional and multidimensional auctions with minimal technical complications. A consumer with a given pair of values $v^A = i, v^B = j$ is said to be of type ij (or (i, j)), and we denote the probability of type ij by α_{ij} . The marginal probabilities for product k are denoted by α_H^k and α_L^k .

3. MARGINAL REVENUES IN A MULTIPRODUCT MODEL

In this section, we develop methods to extend the marginal revenue approach of Bulow and Roberts from the single-dimensional case to the multidimensional case. We use a direct revelation mechanism design approach to solve for the monopolist's optimal pricing policy. The monopolist solicits reports from the consumers and then determines the production and allocation of those products on the basis of those reports. We use ρ_{ij}^k to denote the probability that product k is allocated to a consumer type ij , and R_{ij} to denote that consumer's expected surplus. Where necessary, we will add a superscript I to

4. Armstrong also assumes that there are multiple multiproduct bidders with values drawn from the same distribution.

$\rho_{ij}^k, \rho_{ij}^{k,l}$, to indicate the case where each of two consumers values only an individual product.

3.1. A multiproduct consumer

The set of incentive constraints for a multiproduct consumer includes the following inequalities.

$$R_{HH} \geq R_{HL} + \rho_{HL}^B \Delta; \quad (1a)$$

$$R_{HH} \geq R_{LH} + \rho_{LH}^A \Delta; \quad (1b)$$

$$R_{HH} \geq R_{LL} + (\rho_{LL}^A + \rho_{LL}^B) \Delta; \quad (1c)$$

$$R_{HL} \geq R_{LL} + \rho_{LL}^A \Delta; \quad (2)$$

$$R_{LH} \geq R_{LL} + \rho_{LL}^B \Delta; \quad (3)$$

$$R_{LL} \geq 0. \quad (4)$$

Equations (1a), (1b), (2), and (3) are the local downward reporting constraints that ensure that the consumer cannot gain by misreporting L for one product when his true value for that product is H . The monopolist's maximization problem based on all constraints (1) through (4) is often referred to as "the relaxed problem."

Equation (1c) ensures that the consumer cannot gain by misreporting both values as L . We replace (1c) with the requirements that $\rho_{LH}^A \geq \rho_{LL}^B$ and $\rho_{HL}^B \geq \rho_{LL}^A$.⁵ Equation (4) is the individual rationality constraint, which causes the monopolist to set $R_{LL} = 0$, but has no other effect on the optimal auction.

We focus on constraints (1a), (1b), (2), and (3) in the remainder of our analysis. We can rewrite constraints (1a) and (1b) as

$$R_{HH} \geq \max \left\{ R_{HL} + \rho_{HL}^B \Delta, R_{LH} + \rho_{LH}^A \Delta \right\}. \quad (1^*)$$

The maximization operator in equation (1*) distinguishes the multiple-product case from the single-product case. With a single product, there is only one way to underreport one's value, but with two products, it is possible to underreport on either product or on both simultaneously. There is no obvious method for identifying which incentive constraints are binding for a buyer who bids on multiple products; the max operator allows either (1a) or (1b) (or both of them) to be binding. Our restriction that the values are drawn from binary distributions minimizes the additional complexity due to the multidimensional incentive constraints because (H, H) is the only pair of values that gives a multiproduct consumer the choice of which value to underreport.

3.2. Single-product consumers

To facilitate comparison, we perform an extended thought experiment to distinguish the multiproduct case from the single-product case. What would happen if we "split the multiproduct consumer in two," replacing the multiproduct bidder with a pair of single-product consumers (consumers "A" and "B"), but maintaining the same joint distribution of

5. Although these are stronger requirements than (1c), they always hold in the optimal solution, as discussed in Footnote 10 below (see also Armstrong and Rochet (1999)).

values as for the multiproduct bidder?⁶ The full set of incentive constraints for consumer A, who values product A (similar incentive constraints hold for consumer B), is

$$\begin{aligned}
 R_H^{A,I} &\geq R_L^{A,I} + \frac{\alpha_{HH}}{\alpha_H^A} \rho_{LH}^{A,I} \Delta + \frac{\alpha_{HL}}{\alpha_H^A} \rho_{LL}^{A,I} \Delta \\
 &= R_L^{A,I} + \frac{\Delta}{\alpha_H^A} (\alpha_{HH} \rho_{LH}^{A,I} + \alpha_{HL} \rho_{LL}^{A,I}), \tag{5}
 \end{aligned}$$

$$R_L^{A,I} \geq 0. \tag{6}$$

It is straightforward to translate the incentive constraints for single-product consumers into marginal revenues (or equivalently into “virtual utilities”). Bulow and Roberts use the term marginal revenue in their analysis of the optimal auction of a single product because the coefficient $MR(v)$ represents the change in revenue *per unit increase* in $\rho(v)$ conditional on the report of v in an incentive-compatible allocation mechanism.

Sales to a single-product consumer with value L increase the surplus to single-product consumers with value H . The marginal revenue for a single-product consumer with value L for product A is the weighted difference $MR_L^{A,I} = L - (\alpha_H^A / \alpha_L^A) \Delta$. The marginal revenue for a single-product consumer with value H is simply H , as sales to this consumer do not increase surplus to single-product consumers with value L . With independent private values, the marginal revenue (or virtual utility) for a single-product consumer is a function only of that consumer’s type. If the values of single-product consumers are correlated, then the marginal revenue for a given consumer type will vary with the values of other consumers: $MR_{LH}^{A,I} = L - (\alpha_{HH} / \alpha_{LH}) \Delta$, $MR_{LL}^{A,I} = L - (\alpha_{HL} / \alpha_{LL}) \Delta$. Similar formulas apply for product B.

3.3. Marginal revenues for multiproduct consumers

Multidimensional mechanism design is complicated by the fact that the marginal revenue for a multiproduct consumer depends on the set of binding incentive constraints for that consumer. For example, suppose that the monopolist has assigned $\rho_{HL}^B + \rho_{LL}^A > \rho_{LH}^A + \rho_{LL}^B$. Then (1a), (2), and (3) will be binding, but (1b) will not. At that point, there is no change in consumer surplus with an increase in ρ_{LH}^A , for that only affects constraint (1b), which is not binding. In other words, the monopolist has an incentive to increase ρ_{LH}^A when $\rho_{HL}^B + \rho_{LL}^A > \rho_{LH}^A + \rho_{LL}^B$, for this increases revenues without increasing consumer surplus. We shall say that an allocation is *balanced* if $\rho_{HL}^B + \rho_{LL}^A = \rho_{LH}^A + \rho_{LL}^B$. In this case, constraints (1a) and (1b) are jointly binding.

To simplify the exposition of our results we focus on the case where the allocation is balanced, but our results apply beyond this case.⁷ Lemma 1 provides conditions such that the optimal allocation will be balanced. All proofs are contained in the Appendix.

Lemma 1. *When the values have nonnegative correlation, $r = \alpha_{HH} \alpha_{LL} - \alpha_{HL} \alpha_{LH} \geq 0$, the optimal allocation is balanced.*

6. Note that consumers A and B are meant to be distinct from the single-product consumers whose values underlie the monopolist’s cost function $C^k(z)$.

7. Our results apply directly whenever the solution to the relaxed problem is the solution to the full problem—*i.e.*, when there is not large asymmetry between the two products or there is sufficiently strong positive correlation between the products (see Armstrong and Rochet (1999)). When the optimal allocation is unbalanced, constraints beyond (1) through (4) (*e.g.* the constraint that a type (L, H) consumer should not report type (H, L)) may bind. Although our results qualitatively hold in this case, the technical details are rather complex; we refer the interested reader to Armstrong (2000), Armstrong and Rochet (1999), or our working paper (1998) for more details.

In a balanced allocation, the monopolist pairs each allocation of a product along one path from (H, H) to (L, L) with an allocation of a product along the other path from (H, H) to (L, L) . There are four such ways to pair products, with the probabilities associated with each bundle listed below.

1. Pair ρ_{LH}^A with ρ_{HL}^B , $\alpha_1 = \alpha_{LH} + \alpha_{HL}$.
2. Pair ρ_{LH}^A with ρ_{LL}^A , $\alpha_2 = \alpha_L^A$.
3. Pair ρ_{HL}^B with ρ_{LL}^B , $\alpha_3 = \alpha_B^L$.
4. Pair ρ_{LL}^A with ρ_{LL}^B , $\alpha_4 = 2\alpha_{LL}$.

Throughout the paper, we refer to each of these pairings as “bundles” 1 through 4. A balanced allocation can always be represented as a linear combination of bundles 1 through 4. Note that bundles 2 and 3 correspond to a marginal price L on a single product, while bundle 1 corresponds to bundling in the traditional sense (a price of $H + L$ for purchasing both products). The monopolist’s problem can be represented in terms of allocation weights placed on these bundles, $\bar{\rho} = (\rho_1, \rho_2, \rho_3, \rho_4)$, where $\rho_{LH}^A = \rho_1 + \rho_2$; $\rho_{LL}^A = \rho_2 + \rho_4$; $\rho_{HL}^B = \rho_1 + \rho_3$; $\rho_{LL}^B = \rho_3 + \rho_4$. Total sales to types with value L are then given by $\sum_{x=1}^4 \alpha_x \rho_x$, while the total (opportunity) costs of these sales are given by $\alpha_{LH} C^A(\rho_{LH}^A) + \alpha_{LL} C^A(\rho_{LL}^A) + \alpha_{HL} C^B(\rho_{HL}^B) + \alpha_{LL} C^B(\rho_{LL}^B)$.

In a balanced allocation, the change in expected revenue for increasing the allocation weight on a given bundle is the weighted difference between the revenues from additional sales (net of costs) and additional rents to other consumers. For example, each increase in bundling method 1 produces additional sales to consumers of types (L, H) and (H, L) while increasing the rent for consumers of type (H, H) . The net effect on revenues for the four bundles is

$$\pi_1 = (\alpha_{HL} + \alpha_{LH})L - \alpha_{HH}\Delta; \quad (7)$$

$$\pi_2 = \alpha_L^A L - \alpha_H^A \Delta; \quad (8)$$

$$\pi_3 = \alpha_L^B L - \alpha_H^B \Delta; \quad (9)$$

$$\pi_4 = 2\alpha_{LL} L - (\alpha_{LH} + \alpha_{HL} + \alpha_{HH})\Delta. \quad (10)$$

We define the cost for further allocations of bundle x from $\bar{\rho}$, $C_x(\bar{\rho})$, as the weighted sum of the components of bundle x . In the case of bundle 1, $C_1(\bar{\rho}) = \alpha_{LH} MC^A(\rho_{LH}^A) + \alpha_{HL} MC^B(\rho_{HL}^B) = \alpha_{LH} (\partial C^A(\bar{\rho})) / \partial \rho^1 + \alpha_{HL} (\partial C^B(\bar{\rho})) / \partial \rho_1$. Similar formulas apply to the remaining bundles.

The revenue values π_x do not depend on $\bar{\rho}$ and serve as the analogues to marginal revenues (virtual utilities) for realized values in a single-dimensional auction. Wherever possible, the monopolist increases allocations to a bundle x if $\pi_x - C_x(\bar{\rho}) > 0$, so that bundle x is profitable. The additional complication, however, is that changing ρ_x requires a change of allocations in multiple auctions—one for each component of bundle x .

It is also possible to decompose π_x into marginal revenues for each component of x , but this decomposition varies with $\bar{\rho}$. The marginal revenues for the individual components of (say) bundle 1 depend on how the rent to type (H, H) associated with bundle 1, $\alpha_{HH}\Delta$, is apportioned between the types (L, H) and (H, L) . In general, these marginal revenues will be uniquely determined in association with the optimal allocation. Examples 2b and 3 demonstrate the relationship between these marginal revenues and $\bar{\rho}$.⁸ We can

8. A combination of such constraints serves to limit and/or determine the individual marginal revenues MR_{LH}^A , MR_{LL}^A , MR_{HL}^B , MR_{LL}^B given ρ . Given $\bar{\rho}$, the marginal revenues satisfy the conditions $MR_{ij}^k = MC^k(\rho_{ij}^k)$ for each $\rho_{ij}^k \in (0, 1)$. Those equalities combine with the more general constraint that the weighted sum of marginal revenues for each component of bundle y must equal π_y (e.g. $\alpha_{LH} MR_{LH}^A + \alpha_{HL} MR_{HL}^B = \pi_1$) to limit the marginal revenues for ρ_{ij}^k such that $\rho_{ij}^k = 0$ or 1.

provide bounds for these marginal revenue values based on extreme cases where the entire increase in rent produced by allocations of a bundle is assigned to one of its two component types.

Lemma 2. *The marginal revenues for different consumer types fall within the following ranges:*

$$\begin{aligned}
 MR_{LH}^A &\in \left[L - \frac{\alpha_{HH}}{\alpha_{LH}} \Delta, L \right]; \\
 MR_{HL}^B &\in \left[L - \frac{\alpha_{HH}}{\alpha_{HL}} \Delta, L \right]; \\
 MR_{LL}^A &\in \left[L - \frac{\alpha_H^A}{\alpha_{LL}} \Delta, L - \frac{\alpha_{HL}}{\alpha_{LL}} \Delta \right]; \\
 MR_{LL}^B &\in \left[L - \frac{\alpha_H^B}{\alpha_{LL}} \Delta, L - \frac{\alpha_{LH}}{\alpha_{LL}} \Delta \right].
 \end{aligned}$$

These bounds enable us to draw direct comparisons between the monopolist’s treatment of multiproduct and single-product consumers.

4. PROPERTIES OF OPTIMAL AUCTIONS

The monopolist faces a constrained maximization problem

$$\text{(RMP)} \quad \max_{\rho} L(\bar{\rho}) = \sum_{x=1}^4 \pi_x \cdot \rho_x - \sum_{k=A,B} C^k(\bar{\rho}),$$

subject to: 1. $\rho_x \geq 0 \forall x$; 2. $\rho_1 + \rho_2 \leq 1$; $\rho_1 + \rho_3 \leq 1$; $\rho_2 + \rho_4 \leq 1$; $\rho_3 + \rho_4 \leq 1$; 3. $\rho_4 \leq \rho_1$. Constraint groups 1 and 2 ensure that each allocation probability satisfies $0 \leq \rho_{ij}^k \leq 1$.⁹

If $MC^k(z)$ varies with z , the maximization problem becomes a nonlinear programme and the first-order conditions for an optimal allocation are then

$$\frac{\partial L(\bar{\rho})}{\partial \rho_x} = \pi_x - C_x(\bar{\rho}) = 0,$$

for each bundle x such that ρ_x can be increased or reduced without violating one of the resource constraints in 1 through 3. The first-order conditions require further that $\pi_x - C_x(\bar{\rho}) \geq 0$ if ρ_x can be reduced but not increased without violating a resource constraint and similarly that $\pi_x - C_x(\bar{\rho}) \leq 0$ if ρ_x can be increased but not reduced without violating a resource constraint. Because the revenues are linear in ρ_x and the marginal opportunity costs are increasing in ρ_x , any solution to the first-order conditions is a global optimum.

The first-order conditions enable us to characterize properties of the optimal auction (see Armstrong and Rochet (1999) for a more complete characterization of a related problem or our working paper (1998) for an algorithmic solution). For example, if there is strong positive correlation, $r \geq \alpha_{LH} \alpha_{HL} / \alpha_{LL}$, then the optimal auction consists of separate

9. Constraint 3 is binding in cases where $MR_{LH}^A < MR_{LL}^A$ and $MR_{HL}^B < MR_{LL}^B$, which are analogous to the case of a nonmonotonic marginal revenue curve in a single-product auction. In the case of strong positive correlation, the requirement that $\rho_1 \geq \rho_4$ causes the monopolist to adjust these probabilities in tandem, a procedure that is analogous to ironing the marginal revenue curve in a single-product problem.

auctions for the two products: $\rho_{LH}^A = \rho_{LL}^A$, $\rho_{HL}^B = \rho_{LL}^B$. An allocation with $\rho_{LH}^A = \rho_{LL}^A$, $\rho_{HL}^B = \rho_{LL}^B$ can be represented as $\rho_1 = \rho_4 = 0$, $\rho_2 = \rho_{LH}^A$, $\rho_3 = \rho_{HL}^B$. This allocation is optimal iff $\pi_1 \leq C_1(\bar{p})$ when $\pi_2 = C_2(\bar{p})$ and $\pi_3 = C_3(\bar{p})$. These three conditions occur together exactly when $r \geq \alpha_{LH}\alpha_{HL}/\alpha_{LL}$.¹⁰ Beyond determining whether or not any bundling occurs in auctions, our approach allows us to demonstrate that bundling in multiproduct auctions is not deterministic, though it is generally deterministic in multiproduct monopoly pricing (McAfee and McMillan (1988)).

Proposition 1. *If the optimal auction allocates product k to a multiproduct bidder of type ij when the opportunity cost for that allocation is mr_{ij}^k , then that multiproduct bidder receives product k whenever the opportunity cost of that allocation is less than mr_{ij}^k .*

Proposition 1 indicates that a multiproduct bidder of a given type receives a product whenever the opportunity cost for that product is below a particular value regardless of the outcome on the other product. The cutoff value is precisely the marginal revenue for allocating that product to the multiproduct bidder. Setting $\rho_{ij}^k = \tau$ means that the monopolist sells product k to the multiproduct consumer if the opportunity cost for product k is $C^k(\tau)$ or less, *i.e.* the monopolist sells product k to the multiproduct consumer with probability τ .

Proposition 1 is quite general, as the proof applies directly for any discrete distribution of values for the multiproduct consumer.

Corollary 1. *The optimal allocation for the two-value case has bundling discounts: $\rho_{LH}^{A,I} \leq \rho_{LH}^A$ and $\rho_{HL}^{B,I} \leq \rho_{HL}^B$, and marginal price increases: $\rho_{LL}^{k,I} \geq \rho_{LL}^k$.*

Corollary 1 follows immediately from Proposition 1 in tandem with Lemma 2, which indicated that $MR_{LH}^A \geq MR_{LH}^{A,I}$, $MR_{HL}^B \leq MR_{HL}^{B,I}$, $MR_{LL}^A \leq MR_{LL}^{A,I}$, and $MR_{LL}^B \leq MR_{LL}^{B,I}$.¹¹ Thus, the monopolist sells the lower-valued product more frequently to a multiproduct consumer than to single-product customers with the same joint distribution of values when the realized values are (H, L) or (L, H) . Similarly the monopolist sells both products more frequently to single-product consumers than to a multiproduct consumer when the realized values are (L, L) . These results are most familiar in monopoly pricing for multiple products with constant marginal costs (McAfee, McMillan, and Whinston (1989)). A sale of both products to a multiproduct customer with values (H, L) (when type (L, L) would receive nothing) induces a bundling discount because it corresponds to a price of H for a single product and $H + L$ for both products simultaneously.

Our initial discussion of constraints (1a) and (1b) indicated that bundling of products to multiproduct consumers of types (L, H) and (H, L) achieves two sales at the cost of one increment (rather than two) to rent. In contrast, bundling of products to a multiproduct consumer of type (L, L) achieves two sales at the cost of three increments (rather than two) to rent. Thus, the monopolist favours a multiproduct bidder relative to two single-product bidders in the allocation of the lower valued of two products, while the monopolist favours the two single-product bidders in the allocation of two equally (low) valued products.¹² In monopoly pricing, bundling always causes the monopolist to allocate a high-value product together with a low-value product to the same customer who would

10. Case A of Proposition 1 in Armstrong and Rochet (1999) and part (i) of Lemma 2 in Armstrong (2000) derive this same independence condition using a different method.

11. Proposition 1 of Armstrong and Rochet (1999) implies this same result.

12. This is the intuition behind McAfee, McMillan, and Whinston's (1989) results.

not receive that low-value product on its own. But Proposition 1 implies that bundling will take on a more complex form in an optimal auction.

Corollary 2. *An increase in the realized value for individual bidder i does not affect the marginal revenues for the multiproduct consumer. Therefore, a change in the realized values of individual bidders for one product does not affect the allocation of the other product.*

The marginal revenues for the multiproduct bidder are a function of the allocation probabilities \bar{p} , but these probabilities are aggregate probabilities, calculated before the individual bidders reveal their types. Therefore, the marginal revenues for the multiproduct bidder vary only with the distributions for the values of individual-product bidders, not with the realizations of the values from those distributions.

Corollary 2 indicates that bundling in optimal auctions will be *probabilistic*: the allocation of product A depends only on the realized values of product A . Thus, if the monopolist wishes to increase the allocation of both products to a multiproduct bidder with values (L, H) , he will not be able to guarantee that he will offer the products in a package to that bidder. In some instances, he will sell product A to the multiproduct bidder while allocating product B to a single-product bidder who covets it. A bidder of type ij could only be assured of receiving both products together if MR_{ij}^A varied with the allocation of product B —which would violate Proposition 1.

Probabilistic bundling pairs an allocation of product A for realized costs of the form (“Low,” “High”) with an allocation of product B for the separate realized costs of the form (“High,” “Low”). The monopolist achieves the effect of bundling the products even though it is possible that only one of the two is actually sold to the multiproduct bidder. Example 1 demonstrates the value of probabilistic bundling when the costs of the monopolist are literally either “High” or “Low.” When costs are drawn from continuous distributions, as in Examples 2b and 3, the cutoff values for probabilistic bundling are determined within the monopolist’s maximization problem. Proposition 1 extends to any distribution of values, implying that bundling in optimal auctions generally takes a probabilistic form.

Example 1. The multiproduct customer takes values from the distribution $H = 5$, $L = 2$, and $\alpha_{HH} = \alpha_{HL} = \alpha_{LH} = \alpha_{LL} = 1/4$ and throughout the examples we assume the monopolist has zero marginal costs. We choose these values to emphasize the value of bundling. The monopolist would never sell a low-valued product to two single-product customers drawing values from this distribution, but it is profitable to bundle the products for a multiproduct consumer with type (L, H) or (H, L) because $\pi_1 = 1/4$. The optimal allocation does not use any of the other bundles because $\pi_2 = \pi_3 = -1/2$, $\pi_4 = -5/4$. There is one individual customer competing for each product and their values are drawn independently from the binary distribution with values of 6 and 0 equally likely for each product. Then the marginal revenue of allocating products to the individual bidders is 6 for a value of 6 and negative for a value of 0.

The monopolist implements the optimal auction by allocating a product to a multiproduct bidder with values (H, H) , (L, H) , or (H, L) whenever the single-product bidder for that product has a value of 0.¹³ The allocation probabilities for an optimal auction

13. Because the monopolist will always allocate the relevant product to a single-product bidder with a value of 6 and will never allocate the relevant product to a single-product bidder with a value of 0, any optimal trading mechanism begins by offering the products separately to the individual bidders for a price of 6 each.

are then $\rho_{Hj}^A = \rho_{LH}^A = 1/2$, $\rho_{iH}^B = \rho_{HL}^B = 1/2$, $\rho_{LL}^A = \rho_{LL}^B = 0$. The surplus to a multiproduct bidder with values (2, 5) or (5, 2) is zero because $\rho_{LL}^A = \rho_{LL}^B = 0$. That fixes the prices for allocations to these bidders to their true values and in turn gives the prices for a multiproduct bidder with values (5, 5). We summarize the allocations and prices for the multiproduct bidder in Table 1.¹⁴

TABLE 1
Selling prices for the multiproduct bidder in Example 1

Values	(6, 6)	(6, 0)	(0, 6)	(0, 0)
(5, 5)	—	$p^B = 3.5$	$p^A = 3.5$	$p^{AB} = 7$
(5, 2)	—	$p^B = 2$	$p^A = 5$	$p^{AB} = 7$
(2, 5)	—	$p^B = 5$	$p^A = 2$	$p^{AB} = 7$
(2, 2)	—	—	—	—

With this allocation and pricing policy, the monopolist receives expected revenues of 42/16 from sales to the multiproduct bidder. The monopolist bundles products to the multiproduct bidder across the set of realized values for the single-product bidder. Product A for a report of (2, 5) is bundled with product B for a report of (5, 2); the price of 5 for product A for the set of values [(0, 6), (5, 2)] links with the price of 2 for product B for the set of values [(6, 0), (5, 2)] to create a total price for the two products of 7. It is common for the multiproduct bidder to receive the low-valued product for a report of (2, 5) or (5, 2) without receiving the high-valued product.

The revenue-maximizing deterministic bundling scheme offers both products in a bundle to the multiproduct consumer for a price of 7 if neither individual bidder purchases, and offers product A (and similarly for product B) to the multiproduct consumer for a price of 5 if an individual bidder purchases a product for 6. This alternative policy produces expected revenues of 41/16.

4.1. Competition and bundling

When there are relatively few bidders in an auction, *e.g.* in a procurement setting where a select group of suppliers are bidding to produce or develop parts of a more complex product, the effects of an additional bidder for one product can have an unexpected impact on the allocation of the other product. Increasing competition for product A due to another single-product bidder is equivalent to increasing the cost distribution for product A such that the resulting distribution first-order stochastically dominates the original cost distribution. As the following result shows, an increase in competition for product A reduces the profitability of all bundles including product A and thus the allocations of product A to the multiproduct consumer.

Proposition 2. *Suppose that the distribution of opportunity costs for sales of product B to the multiproduct bidder is fixed at $F^B(y)$ and consider two separate distributions $F^A(y)$ and $G^A(y)$ for the opportunity costs for product A. Assume that distribution G^A first-order stochastically dominates distribution F^A and that neither distribution has any atoms. Denote the optimal allocation probabilities for these distributions as $\rho^{k,F}$ and $\rho^{k,G}$. Then $\rho_{ij}^{A,F} \geq \rho_{ij}^{A,G}$ for each i, j .*

14. These prices, the p^k s., satisfy *ex post* individual rationality. Any pricing scheme charging the multiproduct bidder 3.5 on average for any product(s) allocated satisfies *ex ante* individual rationality.

In contrast to this result, an increase in competition for product A has an indeterminate effect on the allocation of product B . The marginal revenues for the multiproduct consumer are determined by the interplay of binding constraints and the cost functions. When competition for one product changes (*i.e.* the associated cost function $C^k(z)$ changes), the multiproduct consumer's marginal revenue for the other product changes as well; whether it rises or falls depends on the set of initially binding constraints.

In effect, allocations of products A and B to the multiproduct consumer can be substitutes or complements from the perspective of the monopolist, where this relationship varies with the combination of bundles 1 through 4 that are utilized in the optimal allocation. If only bundling method 1 is utilized for the multiproduct bidder with distribution F (as in Example 1), then ρ_{LH}^A and ρ_{HL}^B are complements since they are always paired together. Then, an increase in competition on product A will reduce the level of bundling and thus reduce ρ_{HL}^B without affecting ρ_{LL}^B . On the other hand, if bundling methods 1 and 2 are both utilized for distribution F , but bundling method 3 is not utilized, then ρ_{LL}^A and ρ_{HL}^B are substitutes, with each competing to be paired with ρ_{LH}^A . In this case, an increase in competition for product A reduces ρ_{LL}^A and may lead to a pure increase in ρ_{HL}^B .

If bundling method 3 is utilized for the multiproduct bidder with distribution F , then any change in ρ_{HL}^B must be offset by an opposing change in ρ_{LL}^B to maintain the first-order conditions for allocation of bundle 3, which has the same cost for both F and G . Then, a change from distribution F to distribution G must have an ambiguous effect on allocations of product B : an increase in one of the terms [ρ_{HL}^B, ρ_{LL}^B] must occur in tandem with a corresponding reduction in the other term. Example 2 illustrates this effect.

Example 2. Suppose that the multiproduct bidder's values are independent and equally likely to be $H = 4$ and $L = 3$ on each product: $\alpha_{HH} = \alpha_{HL} = \alpha_{LH} = \alpha_{LL} = 1/4$. Then $\pi_1 = 5/4, \pi_2 = \pi_3 = 1, \pi_4 = 3/4$. Suppose further that distribution F corresponds to $MC^A(z) = MC^B(z) = 4z$ (*i.e.* the cumulative distributions for opportunity costs are given by $F^A(y) = F^B(y) = y/4$), which would occur with one single-product bidder for each product and each drawing independent values from a $U(2, 4)$ distribution.¹⁵ The monopolist's maximization problem leads to a linear system of first-order conditions with the interior solution $\rho_{LH}^A = \rho_{HL}^B = 5/8, \rho_{LL}^A = \rho_{LL}^B = 3/8$.

Suppose that $G^A(y) = y/8$ so that the single-product bidder for product A draws an independent uniform value on $(4, 8)$. Then the monopolist's maximization problem produces an interior solution with $\rho_{LH}^A = 1/3, \rho_{HL}^B = 7/12, \rho_{LL}^A = 1/6, \rho_{LL}^B = 5/12$. That is, the change from F to G causes ρ_{HL}^B to fall (because bundle 1 became less profitable with the change from F to G) and ρ_{LL}^B to increase (because a reduction in ρ_{HL}^B causes bundle 3 to be more profitable for any given level of ρ_{LL}^B).

Table 2 indicates the marginal revenues and allocation probabilities for the optimal auction in each of these cases. The marginal revenues for individual allocations can be calculated from the marginal costs at the optimal allocation: $MR_{ij}^k = mr_{ij}^k = MC^k(\rho_{ij}^k)$. Specifically, for distribution $F, MR_{LH}^A = MR_{HL}^B = 5/2, MR_{LL}^A = MR_{LL}^B = 3/2$. With the introduction of asymmetry in distribution G , the marginal revenues become asymmetric as well: $MR_{LH}^A = 8/3, MR_{HL}^B = 7/3, MR_{LL}^A = 5/12, MR_{LL}^B = 7/12$.

15. A consumer drawing values from $U(a/2, a)$ has $MR(v) = 2v - a$, which is uniform on $(0, a)$. In this case, each single-product consumer can be viewed as drawing a marginal revenue from $U(0,4)$. We assume that the monopolist has no production costs.

TABLE 2
Solution for opportunity-cost cutoffs and probabilities in Example 2

(v^A, v^B)	MR_{v^A, v^B}^A	MR_{v^A, v^B}^B	ρ_{v^A, v^B}^A	ρ_{v^A, v^B}^B	MR_{v^A, v^B}^A	MR_{v^A, v^B}^B	ρ_{v^A, v^B}^A	ρ_{v^A, v^B}^B
(4, 4)	4	4	1	1	4	4	1/2	1
(3, 4)	2.5	4	5/8	1	8/3	4	1/3	1
(4, 3)	4	2.5	1	5/8	4	7/3	1/2	7/12
(3, 3)	1.5	1.5	3/8	3/8	4/3	5/3	1/6	5/12
		Distribution F				Distribution G		

5. EXTENSIONS TO MANY MULTIPRODUCT CONSUMERS

We discuss two separate cases with many multiproduct consumers: (1) without single-product consumers, (2) with single-product consumers.

5.1. *No single-product consumers*

There is a close connection between our results and the results in Armstrong (2000), which focuses on results with more than one multiproduct consumer and no single-product consumers. In Armstrong’s model, a “bundling auction” produces an unbalanced allocation in general, which can only be optimal if $r < 0$.¹⁶

When $r \geq 0$, Armstrong’s optimal allocations are balanced and can be expressed as linear combinations of bundles 2 and 3. The requirement that all products are sold to a multiproduct consumer restricts the set of balanced allocations. To maintain a balanced allocation with all products sold, a unit increase in bundle 1 must be offset by a corresponding change of $(\alpha_{LH} - \alpha_{HL})\alpha_B^L$ units of bundle 3 and reduction of α_{LH}/α_{LL} units of bundle 4.

Thus, when $\pi_1 + ((\alpha_{LH} - \alpha_{HL})/\alpha_L^B)\pi_3 - (\alpha_{LH}/\alpha_{LL})\pi_4 \geq 0$, the monopolist prefers to put as much weight as possible on bundle 1 while maintaining a balanced allocation (this corresponds to a mixed auction). Otherwise, the monopolist prefers to put as much weight as possible on bundle 4 (this corresponds to an independent auction). The condition $\pi_1 + ((\alpha_{LH} - \alpha_{HL})/\alpha_L^B)\pi_3 - (\alpha_{LH}/\alpha_{LL})\pi_4 \leq 0$ can be transformed to produce Armstrong’s condition (“strong positive correlation”) for an independent auction. Interestingly, Armstrong finds the same distributional condition for an independent auction as we find in Section 4, though these conditions arise from quite different calculations.

The optimality conditions for our model generally involve one-for-one comparisons between π_x and C_x for each of bundles $x = 1, 2, 3, 4$. In contrast, the single comparison between $\pi_1 + ((\alpha_{LH} - \alpha_{HL})/\alpha_L^B)\pi_3 - (\alpha_{LH}/\alpha_{LL})\pi_4$ and 0 is paramount in Armstrong’s model. This distinction arises because Armstrong’s maximization problem is a linear programme. With his distributional assumption to ensure that both products are sold, that linear programme produces boundary solutions with efficient allocations; the question is which of the possible boundary solutions is optimal. The addition of single-product consumers in our framework causes the monopolist’s problem to become a nonlinear programme that may produce an interior solution in $\bar{\rho}$. In contrast to Armstrong’s linear programme, the monopolist can (generally) adjust the allocation of a given bundle without any need to make a corresponding adjustment to the allocation of other bundles. Thus,

16. If the allocation is unbalanced with (say) $\rho_{LH}^A + \rho_{LL}^B > \rho_{HL}^B + \rho_{LL}^A$, then $MR_{LH}^A = L - \Delta(\alpha_{HH}/\alpha_{LH})$, its minimum possible value, and $MR_{LL}^A = L - \Delta(\alpha_{LH}/\alpha_{LL})$, its maximum possible value. It would be optimal to reallocate product A from type (L, H) to type (L, L) unless $MR_{LH}^A > MR_{LL}^A$. Here, $MR_{LH}^A > MR_{LL}^A$ simplifies to $r < 0$, which is equivalent to Armstrong’s condition for a bundling auction to be optimal.

the set of (up to four) first-order conditions $\pi_x - C_x(\bar{p}) = 0$ characterizes the optimal allocation in our model.

5.2. *Multiproduct consumers and single-product consumers*

With more than one multiproduct consumer, the existence of single-product consumers causes the monopolist’s maximization problem to take the form of one nonlinear programming problem per multiproduct consumer. These programming problems are intertwined in two ways. First, the resource constraints (constraint group 2 in (RMP)) become residual resource constraints: a multiproduct consumer can only be allocated products that are not allocated to other multiproduct consumers. Second, the marginal opportunity costs for allocations to a multiproduct consumer will now depend on allocations to other multiproduct consumers. They also make it quite difficult to identify a solution to the first-order conditions for each problem.

The nature of the first-order conditions remains unchanged after the addition of more multiproduct consumers. Thus, the properties of multiproduct auctions that we have demonstrated in earlier results carry over directly when there is more than one multiproduct consumer. Example 3 illustrates probabilistic bundling in a two-bidder example.

Example 3. Suppose that there are two symmetric multiproduct bidders who draw independent values from the distribution used in Example 1: $H = 5, L = 2, \alpha_{HH} = \alpha_{HL} = \alpha_{LH} = \alpha_{LL} = 1/4$. Suppose further that there is one single-product consumer for each product. Consumers *A* and *B* draw independent values from $U(3, 6)$ and $U(9/2, 9)$ distributions, respectively, corresponding to $MC^A(z) = 6z, MC^B(z) = 9z$.

The monopolist only uses bundle 1 in the optimal allocation because $\pi_2, \pi_3, \pi_4 < 0$. The optimal auction sets $C_1 = 1/4$ for each multiproduct bidder. Then the first-order conditions for the multiproduct bidders produce a linear system of equations with solution $\rho_1 = 1/40$. The optimal auction sets $\rho_{LH}^A = \rho_{HL}^B = 1/40, \rho_{LL}^A = \rho_{LL}^B = 0$ for each multiproduct consumer, corresponding to $MR_{LH}^A = 2/5, MR_{HL}^B = 3/5$.

This solution features probabilistic bundling. A consumer of type (L, H) receives product *A* against a rival consumer of type (L, L) and a single-product consumer with marginal revenue $2/5$ or less for product *A* (*i.e.* value 3.2 or less).¹⁷ But a consumer of type (L, H) can only receive product *B* if the single-product consumer for that product has marginal revenue less than 5 (*i.e.* a value less than 7). Thus, it remains common for a consumer of type (L, H) to receive product *A* without receiving product *B*, while a consumer of type (L, L) never receives product *A*.

6. CONCLUSION

This paper uses a parsimonious model to gain insights into the nature of optimal auctions with multiple products. Previous work found conditions under which the multidimensional problem resembles the single-dimensional problem (Armstrong (1996, 2000)). By contrast, we demonstrate three fundamental properties for optimal revelation mechanisms. First, we provide intuition for why monopoly pricing and optimal auctions favour allocations that bundle products to a consumer who is interested in both products. The motivation for bundling is that it facilitates price discrimination. Second, we show that

17. That same consumer receives product *A* half the time against another consumer of type (L, H) in the same situation.

bundling takes on a different form in an auction than it does in direct monopoly pricing: the optimal auction of multiple products may include *probabilistic bundling*. Third, we show that additional competition for one product has an indeterminate effect on the allocation of the other product: an increase in competition for product *A* can favour or impede allocations of product *B* in an optimal auction.

We view these results as the first step toward a complete understanding of the properties of multiproduct auctions. We hope that this paper will be valuable in guiding attempts to prove more general properties of multidimensional optimal revelation mechanisms that distinguish them from their unidimensional counterparts.

Acknowledgements. We are grateful for comments and advice from Mark Armstrong, Jeremy Bulow, Paul Klemperer, Eric Maskin, Preston McAfee, Bob Wilson, and Richard Zeckhauser.

APPENDIX

Proof of Lemma 1. Because Lemma 1 is a special case of Proposition 1 in Armstrong and Rochet (1999) and corresponds to Lemma 2, parts (i) and (iii) of Armstrong (2000), we omit the proof. A proof specific to the context of this paper is available in our 1998 working paper. ||

Proof of Lemma 2. We demonstrate the result for MR_{LH}^A ; an identical argument applies for MR_{HL}^B . Because ρ_{LH}^A only affects incentives through constraint (1b), then the extreme values for MR_{LH}^A must correspond to the cases where (1b) has maximum and minimum effect. If (1b) is solely binding, then the effect on net revenue per additional sale to type *LH* is given by the lower value $L - \Delta(\alpha_{HH}/\alpha_{LH})$. If (1b) is relaxed entirely, then each additional sale to type *LH* simply raises net revenue by L . The derivation for MR_{LH}^A and MR_{LL}^B is similar. ||

Proof of Proposition 1. Suppose that the optimal auction allocates product k to the multiproduct bidder of type ij when the opportunity cost of that allocation is mr_{ij}^k but that the optimal auction does not always allocate product k to that multiproduct bidder when the opportunity cost of that allocation is $s < mr_{ij}^k$. Then the monopolist could achieve the same allocation probabilities for the multiproduct bidder by increasing the probability of an allocation for the opportunity cost s and reducing the probability of an allocation for the opportunity cost mr_{ij}^k by a corresponding amount. The revenue and consumer surplus for the multiproduct bidder are unchanged by this shift of probability, but there is a strict increase in the profits gained from the individual bidders. This is a contradiction. ||

Proof of Proposition 2. Allocations for ρ_{Hj}^A are unaffected by the incentive constraints, so we concentrate on ρ_{ij}^A . Assume that $\rho_{LH}^{A,G} > \rho_{LH}^{A,F}$ (the argument is similar for $\rho_{LL}^{A,G}$) and assume further that none of the resource constraints is binding. The following proof also applies with slight adjustments to the case with some binding resource constraints. At the optimal solution for F , the change to distribution G causes bundles 1, 2, 4 to become less profitable. To maintain the first-order conditions with $\rho_{LH}^{A,G} > \rho_{LH}^{A,F}$, it must be that $\rho_{HL}^{B,G} < \rho_{HL}^{B,F}$ (to maintain FOC for bundle 1) and $\rho_{LL}^{A,G} < \rho_{LL}^{A,F}$ (to maintain FOC for bundle 2). But then (focusing on bundle 3) $\rho_{HL}^{B,G} + \rho_{HL}^{B,G} < \rho_{HL}^{B,F} + \rho_{HL}^{B,F}$ even though the costs are the same for both distributions. The first-order conditions for F and G cannot both hold simultaneously with these values. ||

REFERENCES

- ARMSTRONG, M. (1996), "Multiproduct Nonlinear Pricing", *Econometrica*, **64**, 51–75.
 ARMSTRONG, M. (2000), "Optimal Multi-Object Auctions", *Review of Economic Studies*, **67**, 455–481.
 ARMSTRONG, M. and ROCHET, J.-C. (1999), "Multi-Dimensional Screening: A User's Guide", *European Economic Review*, **43**, 959–979.
 AVERY, C. and HENDERSHOTT, T. (1998), "Bundling and Optimal Auctions of Multiple Products" (Working paper, Harvard University).
 BULOW, J. and ROBERTS, D. J. (1989), "The Simple Economics of Optimal Auctions", *Journal of Political Economy*, **97**, 1060–1090.
 DANA, J. (1993), "The Organization and Scope of Agents: Regulating Multiproduct Industries", *Journal of Economic Theory*, **59**, 228–310.

- McAFEE, P. R. and McMILLAN, J. (1988), "Multidimensional Incentive Compatibility and Mechanism Design", *Journal of Economic Theory*, **46**, 335–354.
- McAFEE, P. R., McMILLAN, J. and WHINSTON, D. (1989), "Multiproduct Monopoly, Component Bundling, and Correlation of Values", *Quarterly Journal of Economics*, **93**, 371–383.
- ROCHET, J.-C. (1995), "Ironing, Sweeping and Multidimensional Screening" (Mimeo).