Three-Period Model for Informed Trading and Portfolio Returns

This document extends the model to three trading rounds. All assumptions of the original two-period model remain the same. Assumption 3 is modified to incorporate the extra trading round.

ASSUMPTION 3*: *In all three trading rounds the market makers condition their pricing functions in an asset on order flow in that asset (and not on order flow in the other asset). After the first trading round, time $t = 1^+$, market makers observe the prices of both assets, infer order flow in both assets, and adjust prices before the second trading round. After the second trading round, time $t = 2^+$, market makers observe the prices of both assets, infer order flow in both assets, and adjust prices before the third trading round.*

At dates $t = 1, 2, 3$, the informed trader submits market orders

$$\begin{aligned} x_1(t) &= \beta_+(t)\,\Delta_1\left((t-1)^+\right) + \beta_-(t)\,\Delta_2\left((t-1)^+\right), \\ x_2(t) &= \beta_-(t)\,\Delta_1\left((t-1)^+\right) + \beta_+(t)\,\Delta_2\left((t-1)^+\right), \end{aligned} \tag{1}$$

where $\Delta_k(t)$ reflects the time-t informational advantage of the informed trader over the market maker (MM) in asset k

$$\Delta_k(t) \equiv V_k - P_k(t), \qquad k = 1, 2. \tag{2}$$

The total order flows take the following form

$$\begin{aligned} y_1(t) &= \beta_+(t)\,\Delta_1\left((t-1)^+\right) + \beta_-(t)\,\Delta_2\left((t-1)^+\right) + u_1(t), \\ y_2(t) &= \beta_-(t)\,\Delta_1\left((t-1)^+\right) + \beta_+(t)\,\Delta_2\left((t-1)^+\right) + u_2(t). \end{aligned}$$

Using the above results, we orthogonalize order flows into "portfolios" as $x_\pm(t)$:

$$\begin{aligned} x_\pm(t) &= x_1(t) \pm x_2(t) = b_\pm(t)\,\Delta_\pm\left((t-1)^+\right), \\ y_\pm(t) &= b_\pm(t)\,\Delta_\pm\left((t-1)^+\right) + u_\pm(t), \\ b_\pm(t) &\equiv \beta_+(t) \pm \beta_-(t), \\ \Delta_\pm(t) &\equiv \Delta_1(t) \pm \Delta_2(t), \\ u_\pm(t) &\equiv u_1(t) \pm u_2(t). \end{aligned} \tag{3}$$

Using this transformation helps to simplify calculations by taking better advantage of the symmetry of the problem. Each trading round the MMs observe order flows only in the stocks they make and therefore react with the following price schedules[1]

$$P_k(t) = P_k\left((t-1)^+\right) + \lambda_t y_k(t), \qquad k = 1, 2, \tag{4}$$

which can be rewritten as

$$P_\pm(t) = P_\pm\left((t-1)^+\right) + \lambda_t y_\pm(t), \quad t = 1, 2, 3. \tag{5}$$

where $P_\pm(t) \equiv P_1(t) \pm P_2(t)$. At times $t^+ = t + \varepsilon$, $t = 1, 2$, the MMs observe the prices of the stocks that they do not make, evaluate the order flows that they did not observe at t, and adjust the prices of their stocks to the full information level according to

$$\begin{aligned} P_1\left(t^+\right) &= P_1\left((t-1)^+\right) + \Lambda_+(t)\,y_1(t) + \Lambda_-(t)\,y_2(t), \\ P_2\left(t^+\right) &= P_1\left((t-1)^+\right) + \Lambda_-(t)\,y_1(t) + \Lambda_+(t)\,y_2(t), \end{aligned} \tag{6}$$

[1] By construction, $P_k(0) = 0$.

or, using the orthogonalized portfolios

$$P_{\pm}\left(t^{+}\right) = P_{\pm}\left((t-1)^{+}\right) + \lambda_{\pm}(t)\, y_{\pm}(t), \quad t = 1,2,3, \tag{7}$$

where

$$\lambda_{\pm}(t) \equiv \Lambda_{+}(t) \pm \Lambda_{-}(t). \tag{8}$$

It also immediately follows that

$$\Delta_{\pm}(t) = \Delta_{\pm}\left((t-1)^{+}\right) - \lambda_t y_{\pm}(t), \quad t = 1,2,3.$$

The "efficient" prices (7) are quoted to the informed trader in the next trading round. The price impacts λ_t and $\lambda_{\pm}(t)$ are given by

$$\lambda_t = \frac{E\left[V_1 y_1(t)\right]}{E\left[(y_1(t))^2\right]}, \quad t = 1,2,3,$$

and subsequently

$$\lambda_{\pm}(t) = \frac{E\left[V_{\pm} y_{\pm}(t)\right]}{E\left[(y_{\pm}(t))^2\right]}, \quad t = 1,2.$$

where $V_{\pm} \equiv V_1 \pm V_2$. The last result directly follows from the fact that the information sets of the portfolios are orthogonal to each other because both the fundamentals $\{V_{+}, V_{-}\}$ and the liquidity order flows $\{u_{+}, u_{-}\}$ are orthogonal. Therefore, the orthogonalized portfolios are traded independently. At date $t = T = 4$, the fundamentals are revealed and final payoffs are realized. Finally we define the vectors $\mathbf{x}(t) = (x_{+}(t), x_{-}(t))'$, $\mathbf{y}(t) = (y_{+}(t), y_{-}(t))'$, $\mathbf{\Delta}(t) = (\Delta_{+}(t), \Delta_{-}(t))'$, $\mathbf{P}(t) = (P_{+}(t), P_{-}(t))'$, $\mathbf{V} = (V_{+}, V_{-})'$, $\mathbf{u}(t) = (u_{+}(t), u_{-}(t))'$, and matrices

$$\mathbf{b}(t) = \begin{pmatrix} b_{+}(t) & 0 \\ 0 & b_{-}(t) \end{pmatrix}, \tag{9}$$
$$\lambda(t) = \begin{pmatrix} \lambda_{+}(t) & 0 \\ 0 & \lambda_{-}(t) \end{pmatrix}.$$

Using the orthogonalization prices and strategies take the following form in vector notation

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{b}(t)\,\mathbf{\Delta}\left((t-1)^{+}\right), \\ \mathbf{P}(t) &= \mathbf{P}\left((t-1)^{+}\right) + \lambda_t \mathbf{y}(t), \\ \mathbf{P}\left(t^{+}\right) &= \mathbf{P}\left((t-1)^{+}\right) + \lambda(t)\,\mathbf{y}(t), \ \mathbf{P}(0) = (0,0)'. \end{aligned} \tag{10}$$

The model is solved by finding a vector of trading strategies, $\mathbf{x}^{*}(t)$, which maximizes the informed trader's expected profits over all three trading rounds

$$\mathbf{x}^{*}(t) = \underset{\{x_{+}(t), x_{-}(t)\}}{\arg\max} \underbrace{\mathrm{E}\left[\sum_{t=1}^{3} \mathbf{x}(t)'\,\mathbf{\Delta}(t)\,|\widetilde{\mathbf{V}} = \mathbf{V}\right]}_{\text{informed trader's expected profits}}. \tag{11}$$

The following theorem summarizes the solution of the model.

THEOREM 3: *There exists a linear solution to the informed trader's profit maximization problem* (11) *characterized by the following parameters.*
At time $t = 1$ *the informed trader's strategy,* $\mathbf{x}(1) = \mathbf{b}(1)\mathbf{V}$, *is characterized by* $\{b_+(1), b_-(1)\}$ *which are the solutions to the system of nonlinear equations:*

$$b_\pm(1) = \frac{1 - 2\Gamma^\pm \lambda_\pm(1)}{2\lambda_1 - \Gamma^\pm \lambda_\pm^2(1)}, \tag{12}$$
$$\Gamma^\pm \equiv \frac{1}{4\lambda_3}\left(1 - \lambda_\pm(2) b_\pm(2)\right)^2 + b_\pm(2)\left(1 - \lambda_2 b_\pm(2)\right).$$

The market maker pricing function is $\mathbf{P}(1) = \lambda_1(\mathbf{b}(1)\mathbf{V} + \mathbf{u}(1)) = \lambda_1 \mathbf{y}(1)$, *with:*

$$\lambda_1 = \frac{b_+(1) + b_-(1)}{b_+^2(1) + b_-^2(1) + \rho\left(b_+^2(1) - b_-^2(1)\right) + 2}. \tag{13}$$

At time $t = 1^+$ *the asset prices are updated using both order flows* $\mathbf{P}(1^+) = \lambda(\mathbf{1})\mathbf{y}(1)$ *with*

$$\lambda_\pm(1) = \frac{b_\pm(1)(1 \pm \rho)}{b_\pm^2(1)(1 \pm \rho) + 1}. \tag{14}$$

At time $t = 2$ *the informed trader's strategy,* $\mathbf{x}(2) = \mathbf{b}(2)\boldsymbol{\Delta}(1^+)$, *is characterized by* $\{b_+(2), b_-(2)\}$ which are the solutions to the system of nonlinear equations:

$$b_\pm(2) = \frac{2\lambda_3 - \lambda_\pm(2)}{4\lambda_2\lambda_3 - \lambda_\pm^2(2)}. \tag{15}$$

The market maker pricing function is $\mathbf{P}(2) = \mathbf{P}(1^+) + \lambda_2(\mathbf{x}(2) + \mathbf{u}(2))$, *with:*

$$\lambda_2 = \frac{b_+(1) E\left[V_+\Delta_+(1)\right] + b_-(1) E\left[V_-\Delta_-(1)\right]}{b_+^2(1) E\left[\Delta_+^2(1)\right] + b_-^2(1) E\left[\Delta_-^2(1)\right] + 4}, \tag{16}$$

where

$$E\left[V_\pm\Delta_\pm(1)\right] = 2(1 \pm \rho)\left(1 - \lambda_\pm(1) b_\pm(1)\right), \tag{17}$$
$$E\left[\Delta_\pm^2(1)\right] = 2(1 \pm \rho)\left(1 - \lambda_\pm(1) b_\pm(1)\right)^2 + \frac{\lambda_\pm^2(1)}{2}.$$

At time $t = 2^+$ *the asset prices are updated using both order flows* $\mathbf{P}(2^+) = \mathbf{P}(1^+) + \lambda(\mathbf{2})\mathbf{y}(2)$ *with*

$$\lambda_\pm(2) = \frac{b_\pm(2)(1 \pm \rho)\left(1 - \lambda_\pm(1) b_\pm(1)\right)}{b_\pm^2(2)\left((1 \pm \rho)\left(1 - \lambda_\pm(1) b_\pm(1)\right)^2 + \lambda_\pm^2(1)\right) + 1}. \tag{18}$$

At time $t = 3$ *the informed trader's strategy is* $\mathbf{x}(3) = \mathbf{b}(3)\boldsymbol{\Delta}(2^+)$ *and the market maker pricing function is* $\mathbf{P}(3) = \mathbf{P}(2^+) + \lambda_3(\mathbf{x}(3) + \mathbf{u}(3))$, *with*

$$x_\pm(3) = \frac{1}{2\lambda_3}\Delta_\pm\left(2^+\right), \tag{19}$$

and

$$\lambda_3 = \frac{1}{2}\sqrt{2E\left[V_1\Delta_1(2)\right] - E\left[\Delta_1^2(2)\right]}, \tag{20}$$

where

$$E\left[V_1\Delta_1\left(2\right)\right] = E\left[V_+\Delta_+\left(1\right)\right]\left(1-\lambda_+\left(2\right)b_+\left(2\right)\right) + E\left[V_-\Delta_-\left(1\right)\right]\left(1-\lambda_-\left(2\right)b_-\left(2\right)\right), \tag{21}$$

$$\begin{aligned} E\left[\Delta_1^2\left(2\right)\right] &= 2(1+\rho)\left(1-\lambda_+\left(1\right)b_+\left(1\right)\right)^2\left(1-\lambda_+\left(2\right)b_+\left(2\right)\right)^2 \\ &+ \frac{\lambda_+^2\left(1\right)}{2}\left(1-\lambda_+\left(1\right)b_+\left(1\right)\right)^2 + \frac{\lambda_+^2\left(2\right)}{2} \\ &+ 2(1-\rho)\left(1-\lambda_-\left(1\right)b_-\left(1\right)\right)^2\left(1-\lambda_-\left(2\right)b_-\left(2\right)\right)^2 \\ &+ \frac{\lambda_-^2\left(1\right)}{2}\left(1-\lambda_-\left(1\right)b_-\left(1\right)\right)^2 + \frac{\lambda_-^2\left(2\right)}{2}. \end{aligned}$$

Proof: The proof follows the logic of Theorem 1 in the paper with modifications due to the presence of the extra trading round. Using the same approach is can be shown that the conjectured solution is optimal.

To illustrate the properties of the informed trader's within-asset and cross-asset trading intensities in periods 1 and 2 we graph the model's solution similar to Panel A of Figure 2 in the main manuscript. The covariance and VAR results for the three period model have the same signs as those in the two-period model.

Plot of $\beta_{\pm}(1)$ and $\beta_{\pm}(2)$: