Problem Set #4

Economics 201B

1. Consider the following second-degree pricing problem. A monopolist faces a continuum of consumers. Let θ denote type. Assume $\theta \in [\theta_0, \theta_1] \subset \mathbb{R}$. Assume that
$$\theta_0 > \frac{c}{k}\,,$$
where c is the constant marginal cost of production and k is a parameter of inverse demand. Specifically, assume a type-θ's inverse demand is
$$p(x, \theta) = \theta \times (k - x)\,.$$
Assume θ is distributed according to the differentiable distribution function $F(\cdot)$. Let $f(\cdot)$ denote the associated density function (*i.e.,* $f(\theta) = F'(\theta)$). Assume $f(\theta) > 0$ for all $\theta \in [\theta_0, \theta_1]$.

   (a) Let $\hat{x}(\theta)$ denote the solution, under the assumptions of this problem, to equation (12) from the "Second-degree Price Discrimination with a Continuum of Types" handout. What is $\hat{x}(\theta)$?

   (b) What difficulty would emerge if, under the assumptions of this problem,
$$\frac{d}{d\theta}\left(\frac{1 - F(\theta)}{f(\theta)}\right)$$
were greater than 1 for some values of θ?

   (c) Let $F(\cdot)$ be the uniform distribution. What is the optimal second-degree price discrimination scheme?

2. Set up a price-discrimination scheme via quantity discounts as a mechanism-design problem satisfying the assumptions of the standard framework, including the assumptions of Proposition 9 in "Lecture Notes for Economics."

3. Consider the following variant of the standard framework in which there are a *discrete* number of types, N. Let $n \in \{1, \ldots, N\}$ denote a type. Let the probability of the agent being type n be denoted by f_n. Assume $f_n > 0$ for all n.

   (a) Why is this last assumption without loss of generality?

   Define $F_n = \sum_{m=1}^{n} f_m$. The principal's utility is $b(x, n) - s$ and the agent's is $s - \phi(x, n)$, where $x \in \mathbb{R}_+$ is the allocation. Let $V(x, n) = b(x, n) - \phi(x, n)$. The agent's reservation utility is 0. Assume $V(\cdot, n)$ is strictly quasi-concave for all n. Assume $\phi(0, n) = 0$ for all n. Define $R_n(x) \equiv \phi(x, n) - \phi(x, n + 1)$.

(b) What is $R_n(\cdot)$?

Finally, assume:

> **Screening Condition (SC):** For all $n \in \{1, 2, \ldots, N-1\}$, $R_n(\cdot)$ is strictly increasing and convex. Moreover, $R_n(x) > 0$ for all $x > 0$.

(c) How does the Screening Condition relate to the Spence-Mirrlees condition?

(d) Prove the following proposition: *Under the assumptions given, an allocation profile* $(x_1, \ldots, x_N)$ *is implementable if and only if* x_n *is non-decreasing in type* n*. Moreover, the optimal mechanism for the principal solves*

$$\max_{\{x_1,\ldots,x_N\}} \sum_{n=1}^{N} f_n \left(V(x_n, n) - \frac{1-F_n}{f_n} R_n(x_n) \right)$$

subject to the constraint that $x_1 \leq \cdots \leq x_N$*.*

[Hint: It is advisable to proceed as follows:

i. First, establish the necessity of a non-decreasing allocation profile using a revealed-preference argument.

ii. Next, define the *adjacent incentive compatibility constraints* as, for all $n \leq N-1$,

$$u_n \geq u_{n+1} - R_n(x_{n+1})$$
$$u_{n+1} \geq u_n + R_n(x_n)\,,$$

where u_n is type n's equilibrium utility level (*i.e.,* $u_n = s_n - \phi(x_n, n)$). Show that if the adjacent IC constraints hold and the allocation profile is non-decreasing, then *all* the IC constraints hold.

iii. Establish that a non-decreasing allocation profile is sufficient (*i.e.,* implementable) by considering the contract in which $u_1 = 0$ and $u_n = u_{n-1} + R_{n-1}(x_{n-1})$ for $n > 1$. Recall, given the preceding step, that you need only check adjacent IC constraints.

iv. Finally, show the principal's problem is

$$\max_{\{\mathbf{u},\mathbf{x}\}} \sum_{n=1}^{N} f_n \left(V(x_n, n) - u_n \right) \tag{P}$$

(where $\mathbf{u} = (u_1, \ldots, u_N)$ and $\mathbf{x} = (x_1, \ldots, x_N)$) subject to

$$u_n \geq 0 \;\forall n \tag{IR}$$
$$u_n \geq u_j + \phi(x_j, j) - \phi(x_j, n) \;\forall n, j \tag{IC}$$

and, then, show that's equivalent to what you are to show.]