

1 Exponents

1.1 Definitions

Let n be a whole number (*i.e.*, $n = 0$ or 1 or $2 \dots$).¹ Let x be a real number. The expression x^n is shorthand for multiplying x n -times by itself; that is,

$$x^n = \underbrace{x \times \cdots \times x}_{n \text{ times}} .$$

Note the convention that $x^0 = 1$ for all x . The n term in x^n is the *exponent*. Sometimes x^n is described in words as “taking x to the n th power.”

1.2 Addition and multiplication of exponents

One rule of exponents is

Rule 1.

$$x^n \times x^m = x^{n+m} .$$

For example, $2^2 \times 2^3 = 2^{2+3} = 2^5$. (As is readily verified: $2^2 = 4$, $2^3 = 8$, $4 \times 8 = 32$, and $2^5 = 32$.) Rule 1 follows because

$$\begin{aligned} x^n \times x^m &= \underbrace{x \times \cdots \times x}_{n \text{ times}} \times \underbrace{x \times \cdots \times x}_{m \text{ times}} \\ &= \underbrace{x \times \cdots \times x}_{(n+m) \text{ times}} \\ &= x^{n+m} . \end{aligned}$$

A second rule of exponents is

Rule 2.

$$(x^n)^m = x^{n \times m} = x^{nm} .$$

¹Actually, one does *not* need to limit n to the whole numbers; it is certainly valid to let n be any real number. That is, for instance, $x^{1/2}$ is a valid expression (it is the same as \sqrt{x}). Even x^π is a valid expression. When, however, n is not a whole number, the interpretation of x^n is somewhat more involved. It is worth noting, however, that all the rules set forth in this section apply even if n is not a whole number.

Observe the convention that the multiplication of two variables (*e.g.*, n and m) can be expressed as nm . As an example, $(2^2)^3 = 2^6$. (As is readily seen: $2^2 = 4$, $4^3 = 64$, and $2^6 = 64$.) Rule 2 follows because

$$\begin{aligned}
 (x^n)^m &= \underbrace{x^n \times \cdots \times x^n}_{m \text{ times}} \\
 &= \underbrace{\underbrace{x \times \cdots \times x}_{n \text{ times}} \times \cdots \times \underbrace{x \times \cdots \times x}_{n \text{ times}}}_{m \text{ times}} \\
 &= \underbrace{x \times \cdots \times x}_{(n \times m) \text{ times}} \\
 &= x^{nm}.
 \end{aligned}$$

Question 1: Simplify the following.

$$\begin{array}{llll}
 \text{Ex.} & x^3 x^2 = x^5 & \text{(a)} & x^2 x^4 \\
 & \text{(d)} & (x^1)^3 & \text{(e)} & x^2 x & \text{(f)} & x x^4 & \text{(g)} & x^0 x^6 \\
 & \text{(h)} & (x^0)^3 & \text{(i)} & (x^{10})^0
 \end{array}$$

The rules for adding and multiplying exponents obey the usual rules of arithmetic; hence, for instance:

Rule 3. $x^{n+m} = x^n x^m = x^m x^n = x^{m+n}$.

Rule 4. $x^{nm} = (x^n)^m = (x^m)^n = x^{mn}$.

Rule 5. $x^{p(n+m)} = (x^{n+m})^p = (x^n x^m)^p = x^{np} x^{mp} = x^{np+mp}$.

1.3 More rules for exponents

Rule 6. $(xy)^n = x^n y^n$.

Rule 7. $(x^n y^m)^p = x^{np} y^{mp}$.

Rule 8 (Rule of the Common Exponent I). $z^p x^n y^n = z^p (xy)^n$.

Rule 9 (Rule of the Common Exponent II). $z^{p+n} x^n = z^p (zx)^n$.

Rule 10 (Rule of the Common Exponent III). $z^{pn} x^n = (z^p x)^n$.

Question 2: Simplify the following (some expressions may already be in their simplest form).

$$\text{Ex. } x^3y^3 = (xy)^3 \quad (\text{a}) \quad (3x)^2 \quad (\text{b}) \quad 4y^2 \quad (\text{c}) \quad x^2y^3z^2$$

Question 3: For each of the following state whether the expression is *valid* (i.e., always true) or *invalid* (i.e., sometimes or always false).

$$\begin{array}{lll} \text{Ex. } z^{pn}x^n = z^p(xz)^n \Leftarrow \text{invalid} & \text{Ex. } 8x^2 = 2(2x)^2 \Leftarrow \text{valid} & \\ (\text{a}) \quad 32x^2 = 8(2x)^2 & (\text{b}) \quad z^{m+n}x^m y^n = (xz)^m(yz)^n & (\text{c}) \quad (x^n y^{n+1})^p = y(xy)^{np} \\ (\text{d}) \quad x^4y^2 = (x^2y)^2 & (\text{e}) \quad x^4y^2 = x^2(xy)^2 & (\text{f}) \quad (-x)^2x^2 = -x^4 \end{array}$$

1.4 Some exponent no-noes

The following are common *mistakes* to be avoided:

No no 1. $(x + y)^n \neq x^n + y^n$; that is, *exponentiation is not distributive over addition*. For example, $(2 + 3)^2 = 5^2 = 25$, whereas $2^2 + 3^2 = 4 + 9 = 13$.

No no 2. $x^{pn}y^n \neq x^p(xy)^n$. For example, $2^6 \times 3^2 = 2^{3 \times 2}3^2 = 64 \times 9 = 576$, whereas $2^3(2 \times 3)^2 = 8 \times 36 = 288$. (Also recall Rule 10.)

1.5 Negative exponents

The notation x^{-n} means $(1/x)^n$ (obviously $x \neq 0$). Observe that, because

$$x^n x^{-n} = x^n \left(\frac{1}{x}\right)^n = \left(x \times \frac{1}{x}\right)^n = 1^n = 1 (= x^0)$$

(where the third equality follows from Rule 6), we have that x^{-n} is the multiplicative inverse² of x^n . It is also true that

$$x^n \times \frac{1}{x^n} = 1.$$

Because multiplicative inverses are unique, this means

Rule 11. $(1/x)^n = 1/x^n$.

All of Rules 1–10 are valid with negative exponents (provided x , y , or z are not zero).

It is readily seen that

Rule 12. $(x/y)^n = (y/x)^{-n}$.

²The *multiplicative inverse* of a number x is the number y such that $xy = 1$. For instance, $1/2$ is the multiplicative inverse of 2 . Note if y is the multiplicative inverse of x , then x is the multiplicative inverse of y .

Note that

$$(x + y)^{-n} = \frac{1}{(x + y)^n}$$

but *not*

$$\frac{1}{x^n} + \frac{1}{y^n};$$

that is,

No no 3. Observe

$$(x + y)^{-n} \neq \frac{1}{x^n} + \frac{1}{y^n}.$$

For example, $(2+3)^{-2} = 5^{-2} = 1/25$, whereas $1/2^2 + 1/3^2 = 1/4 + 1/9 = 13/36$.

Question 4: Simplify the following (some expressions may already be in their simplest form).

$$\text{Ex. } x^{-3}y^3 = \left(\frac{y}{x}\right)^3 \quad (\text{a}) \quad (x/2)^{-2} \quad (\text{b}) \quad x^2y^{-4} \quad (\text{c}) \quad x^2y^3z^{-2}$$

Question 5: For each of the following state whether the expression is *valid* or *invalid* (assume x , y , and z are all non-zero).

$$\begin{array}{ll} \text{Ex. } z^{-pn}x^n = (xz)^n/z^p \Leftarrow \text{invalid} & \text{Ex. } x^2/8 = (x/2)^2/2 \Leftarrow \text{valid} \\ (\text{a}) \quad 32x^{-2} = 8(2/x)^2 & (\text{b}) \quad z^{m-n}x^m y^n = (xz)^m (y/z)^n \\ (\text{c}) \quad (x^n y^{n-1})^p = y^{-p}(xy)^{np} & (\text{d}) \quad x^4 y^2 = x^6 (y/x)^2 \\ (\text{e}) \quad x^4 y^2 = x^8 (y/x)^2 & (\text{f}) \quad (-x)^2 x^{-2} = -1 \end{array}$$

2 Square Roots

2.1 Definitions

The positive square root of a number $x \geq 0$ is denoted \sqrt{x} ; that is, \sqrt{x} is the positive number such that $\sqrt{x} \times \sqrt{x} = x$. Observe the convention that $\sqrt{x} \geq 0$ (and equal to zero only if $x = 0$). As a convention from now on, assume that, if we write \sqrt{x} , we are limiting attention to $x \geq 0$.

Because the product of two negative numbers is positive (e.g., $-2 \times (-2) = 4$), each number $x \geq 0$ also has a *negative* square root. It is denoted $-\sqrt{x}$. Hence, as an example, the positive square root of 3 is denoted $\sqrt{3}$ and the negative square root of 3 is denoted $-\sqrt{3}$.

2.2 Rules for square roots

By definition,

Rule 13. $(\sqrt{x})^2 = x$ and $\sqrt{x^2} = |x|$. That is, squaring and taking square roots are inverse operations.

Also by definition, the square root of xy is \sqrt{xy} . But observe

$$(\sqrt{x}\sqrt{y})^2 = (\sqrt{x}\sqrt{y})(\sqrt{x}\sqrt{y}) = \underbrace{\sqrt{x}\sqrt{x}}_{=x} \underbrace{\sqrt{y}\sqrt{y}}_{=y} = xy,$$

where the second equality follows because multiplication is associative (*i.e.*, we are free to rearrange the order of the terms). But this says, then, that $\sqrt{x}\sqrt{y}$ is also the square root of xy ; from which we can conclude:

Rule 14. $\sqrt{xy} = \sqrt{x}\sqrt{y}$.

Rule 14 is useful for simplifying expressions. For instance, $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$.

2.3 Some square root no-noes

The following are common *mistakes* to be avoided:

No no 4. $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$; that is, taking square roots is not distributive over addition. For example, $\sqrt{9+16} = \sqrt{25} = 5$, whereas $\sqrt{9} + \sqrt{16} = 3+4 = 7$.

Recall that the sum of the length of two sides of a triangle must exceed the length of the third side. For this reason, this last “no no” is sometimes called the *triangle inequality* (can you see why?) and stated as

Rule 15 (Triangle inequality). $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ and equal only if x or y or both is (are) zero.

No no 5. $\sqrt{xy} \neq x\sqrt{y}$; that is, a multiplicative constant cannot be passed “through” the square roots operator. Recall, too, Rule 14.

Question 6: For each of the following state whether the expression is *valid* or *invalid* (assume x and y are all greater than or equal to zero and w and z are all positive).

Ex. $\sqrt{x^2y} = x\sqrt{y} \Leftarrow$ valid	Ex. $\sqrt{x/8} = \sqrt{x}/8; \Leftarrow$ invalid
(a) $\sqrt{w+z} = \sqrt{w} + \sqrt{z}$	(b) $\sqrt{8x^4} = 2x^2\sqrt{2}$
(c) $\sqrt{x^{2n}} = x^n$	(d) $2 + \sqrt{w} = \sqrt{4+w}$
(e) $\sqrt{x(w+z)} = \sqrt{x}\sqrt{w+z}$	(f) $\sqrt{z^2 - 2z + 1} = z - 1 $

3 Polynomials

3.1 Definitions

A *polynomial* is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the a_m are the *coefficients* and the powers are whole numbers. For example, the expression $4x^3 - 2x^2 + 8x - 12$ is a polynomial. The coefficients are 4, -2 , 8, and -12 . Not every term needs be present; hence, $7x^3 + 4x - 13$ is also polynomial. Equivalently one can consider missing terms to simply have zero coefficients; that is,

$$7x^3 + 4x - 13 = 7x^3 + 0x^2 + 4x - 13.$$

Typically, if a coefficient is 1 or -1 it is not written; that is, it is more typical to write

$$x^4 + 2x^3 - 5x^2 - x + 8$$

than to write

$$1x^4 + 2x^3 - 5x^2 - 1x + 8$$

(although there is nothing wrong with writing the latter).

3.2 Arithmetic on polynomials

Polynomials can be added, subtracted, multiplied, and divided. Adding and subtracting polynomials is relatively easy—all one does is add or subtract the coefficients. For example, to add $3x^2 + 6x - 5$ to $2x^3 - 3x + 1$ one writes

$$\begin{array}{r} 2x^3 + 0x^2 - 3x + 1 \\ + 0x^3 + 3x^2 + 6x - 5 \\ \hline 2x^3 + 3x^2 + 3x - 4 \end{array}$$

Rule 16. *The sum of the polynomials $a_nx^n + \cdots + a_0$ and $b_nx^n + \cdots + b_0$, where some of the a_m or b_m or both may be zero, is*

$$(a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0).$$

For example, to subtract $2x^2 + 3x - 5$ from $x^2 + 6x - 1$ one writes

$$\begin{array}{r} x^2 + 6x - 1 \\ - (2x^2 + 3x - 5) \\ \hline -x^2 + 3x + 4 \end{array}$$

Rule 17. *Polynomial $a_nx^n + \cdots + a_0$ minus polynomial $b_nx^n + \cdots + b_0$, where some of the a_m or b_m or both may be zero, is*

$$(a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \cdots + (a_1 - b_1)x + (a_0 - b_0).$$

Polynomials can be *scalar-multiplied*; that is, all coefficients multiplied by the same number (the *scalar*). For example, 3 times $x^2 + 6x - 1$ is $3x^2 + 18x - 3$.

Rule 18 (Scalar multiplication). *The polynomial $a_nx^n + \cdots + a_0$ times b is the polynomial $ba_nx^n + \cdots + ba_0$.*

Polynomials can also be multiplied by other polynomials. In doing so, each term of one polynomial is multiplied by every term of the other polynomial, and all products then summed. For instance, to multiply $x^2 + 2x - 3$ by $2x^2 - x + 2$

one writes

$$\begin{array}{r}
 \\
 \phantom{\frac{2x^2}{2x^2}} \phantom{\frac{4x}{x}} \phantom{\frac{6}{2}} \\
 \frac{2x^2}{2x^2} \frac{4x}{x} \frac{6}{2} \phantom{\text{top line} \times 2} \\
 \frac{2x^2}{2x^2} \frac{4x}{x} \frac{6}{2} \phantom{\text{top line} \times 2} \phantom{\text{top line} \times (-x)} \\
 \frac{2x^2}{2x^2} \frac{4x}{x} \frac{6}{2} \phantom{\text{top line} \times 2} \phantom{\text{top line} \times (-x)} \phantom{\text{top line} \times 2x^2} \\
 \hline
 2x^4 4x^3 6x^2 7x 6 \frac{2x^2}{2x^2} \frac{4x}{x} \frac{6}{2} \phantom{\text{top line} \times 2} \phantom{\text{top line} \times (-x)} \phantom{\text{top line} \times 2x^2} \\
 \hline
 2x^4 3x^3 6x^2 7x 6 \frac{2x^2}{2x^2} \frac{4x}{x} \frac{6}{2} \phantom{\text{top line} \times 2} \phantom{\text{top line} \times (-x)} \phantom{\text{top line} \times 2x^2} \\
 \frac{2x^2}{2x^2} \frac{4x}{x} \frac{6}{2} \phantom{\text{top line} \times 2} \phantom{\text{top line} \times (-x)} \phantom{\text{top line} \times 2x^2} \\
 \frac{2x^2}{2x^2} \frac{4x}{x} \frac{6}{2} \phantom{\text{top line} \times 2} \phantom{\text{top line} \times (-x)} \phantom{\text{top line} \times 2x^2} \phantom{\text{sum of lines 3-5}}
 \end{array}$$

Rule 19. *The product of the polynomials $a_nx^n + \dots + a_0$ and $b_nx^n + \dots + b_0$ is*

$$\begin{aligned}
 b_nx^n (a_nx^n + \dots + a_0) &+ b_{n-1}x^{n-1} (a_nx^n + \dots + a_0) + \\
 &\dots + b_1x (a_nx^n + \dots + a_0) + b_0 (a_nx^n + \dots + a_0) .
 \end{aligned}$$

In some instances, one polynomial divides evenly into another. For example, let's divide $x^3 - 5x^2 + 7x - 2$ by $x^2 - 3x + 1$:

$$\begin{array}{r}
 \\
 \phantom{\frac{x}{x^2}} \phantom{\frac{2}{x}} \\
 \left| \frac{x}{x^2} \frac{2}{x} \phantom{\text{multiply divisor by } x} \\
 \left| \frac{x}{x^2} \frac{2}{x} \phantom{\text{multiply divisor by } x} \phantom{\text{difference}} \\
 \left| \frac{x}{x^2} \frac{2}{x} \phantom{\text{multiply divisor by } x} \phantom{\text{difference}} \phantom{\text{multiply divisor by } -2} \\
 \left| \frac{x}{x^2} \frac{2}{x} \phantom{\text{multiply divisor by } x} \phantom{\text{difference}} \phantom{\text{multiply divisor by } -2} \phantom{\text{difference}} \\
 \left| \frac{x}{x^2} \frac{2}{x} \phantom{\text{multiply divisor by } x} \phantom{\text{difference}} \phantom{\text{multiply divisor by } -2} \phantom{\text{difference}} \\
 \left| \frac{x}{x^2} \frac{2}{x} \phantom{\text{multiply divisor by } x} \phantom{\text{difference}} \phantom{\text{multiply divisor by } -2} \phantom{\text{difference}} \phantom{\text{difference}} \\
 \left| \frac{x}{x^2} \frac{2}{x} \phantom{\text{multiply divisor by } x} \phantom{\text{difference}} \phantom{\text{multiply divisor by } -2} \phantom{\text{difference}} \phantom{\text{difference}}
 \end{array}$$

Observe that such division is like long division of multi-digit numbers. The first step is to determine the coefficient and power that, when multiplied by the divisor (*i.e.*, $x^2 - 3x + 1$), will match the coefficient and power of the highest-power term of the polynomial being divided. In this case, if $x^2 - 3x + 1$ is multiplied by x , the resulting highest-power term, x^3 , matches the highest-power term of the polynomial being divided. An x is therefore placed above the bar. The resulting product, $x^3 - 3x^2 + x + 0$, is then subtracted from the polynomial being divided. This leaves a remainder, $-2x^2 + 6x - 2$. What times the divisor will match the first term (*i.e.*, $-2x^2$) of this remainder? The answer is -2 , so a -2 is added to the term above the bar. The resulting product, $-2x^2 + 6x - 2$, is then subtracted from this remainder. Because the resulting difference is zero, we're done. That is, we've established that

$$(x - 2) \times (x^2 - 3x + 1) = x^3 - 5x^2 + 7x - 2 .$$

Question 7: Carry out the following operations:

- (a) $6x^3 + 3x + 5$ plus $-x^2 - x + 6$.
- (b) $-x^4 + x^3 + 2x^2 + 6$ plus $2x^4 - 2x^2$.
- (c) $x^4 + x^3 + x^2 + x + 1$ minus $2x^3 + 2x^2 + x$.
- (d) $3x^3 + 2x^2 + x + 1$ minus $x^3 - 5x^2 - 3x - 2$.
- (e) $x + 3$ times $x^2 + 2x - 1$.
- (f) $2x^2 - 3$ times $x^2 - 2x + 1$.
- (g) Divide $12x^2 - 3$ by $2x - 1$.
- (h) Divide $2x^3 + 6x^2 + 5x + 2$ by $x + 2$.

3.3 Solving polynomial equations

A polynomial equation is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0,$$

where some of the coefficients may be zero. For example,

$$4x^3 + x - 3 = 10x^2 - 3x + 2 \tag{1}$$

is a polynomial equation.

To solve a polynomial equation, the first step is to put the equation in *standard form*. Standard form is where all the x terms are on one side of the equals sign and a 0 is on the other side. Note this is a straightforward transformation:

$$(a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \cdots + (a_1 - b_1)x + (a_0 - b_0) = 0.$$

So, for instance, putting expression (1) into standard form yields:

$$4x^3 - 10x^2 + 4x - 5 = 0.$$

A solution to a polynomial equation in standard form is known as a *root* of that equation. For example, the roots of the equation $x^3 + x^2 - x - 1 = 0$ are 1 and -1 .

It is important to recognize that not every polynomial equation in standard form has a solution.³ For example, the expression $x^2 + 1 = 0$ has no solution. Also it is important to recognize that some polynomial equations will have multiple solutions—as we saw was the case for $x^3 + x^2 - x - 1 = 0$, which has two solutions, 1 and -1 .

3.3.1 Factoring

Solving polynomial equations in standard form is intimately related to *factoring*. A polynomial is *factored* if it is written as the product of terms of the form $(Ax + B)$. For instance, the polynomial $x^2 - 2x + 1$ can be factored as $(x - 1)(x - 1)$; that is,

$$x^2 - 2x + 1 = (x - 1)(x - 1).$$

Some other examples:

- $x^2 - 1 = (x - 1)(x + 1)$.
- $6x^2 + 17x + 5 = (3x + 1)(2x + 5)$.
- $x^3 + x^2 - x - 1 = (x - 1)(x + 1)(x + 1)$.
- $2x^3 + 7x^2 - 2x - 7 = (x - 1)(x + 1)(2x + 7)$.

³To be technical a real-number solution.

The relation between factoring and solving polynomial equations follows because, recall, anything times zero is zero. Hence, if a polynomial can be factored, then, by setting each of the factored terms (*i.e.*, terms of the form $Ax + B$) equal to zero and solving for x , we get a solution to the polynomial equation in standard form. For example, $x = -1/3$ is a solution to

$$6x^2 + 17x + 5 = 0$$

because

$$6x^2 + 17x + 5 = (3x + 1)(2x + 5),$$

and $x = -1/3$ is a solution to $3x + 1 = 0$. To check:

$$\begin{aligned} \left(3 \times \frac{-1}{3} + 1\right) \left(2 \times \frac{-1}{3} + 5\right) &= (-1 + 1) \left(\frac{-2}{3} + 5\right) \\ &= 0 \left(\frac{-2}{3} + 5\right) \\ &= 0. \end{aligned}$$

where the last equality follows because zero times anything is zero.

Rule 20. *If the polynomial in a polynomial equation in standard form can be factored as $(A_1x + B_1)(A_2x + B_2) \dots (A_nx + B_n)$, then the solutions of the polynomial equation are $x = -B_1/A_1$, $x = -B_2/A_2$, \dots , $x = -B_n/A_n$.*

Note sometimes one is given a polynomial equation already factored. For instance,

$$(2x - 4)(x + 5)(3x - 9) = 0.$$

In such cases, solving is easy, just set each $Ax + B$ term equal to zero and solve (*e.g.*, in this case, $2x - 4 = 0$ yields $x = 2$ as a solution; $x + 5 = 0$ yields $x = -5$ as a solution; and $3x - 9 = 0$ yields $x = 3$ as a solution).

Question 8: Solve the following equations by factoring. Find all solutions.

- (a) $2x^3 + 7x^2 - 2x - 7 = 0$.
- (b) $x^2 - 6x + 9 = 0$.
- (c) $x^2 + 6x + 9 = 0$.
- (d) $2x^2 - 3x + 1 = 0$

Question 9: What are the solutions to the following equations.

- (a) $(x - 3)(x - 2)(x - 1) = 0$.
- (b) $(2x - 6)(5x + 15)(2x - 1) = 0$.
- (c) $(x/3 - 1)(2x + 2)(x - 7) = 0$.
- (d) $(6 - x)(2x - 1)(x - 3) = 0$.

3.3.2 Two special cases

If the polynomial in question has no a_0 term, then $x = 0$ must be a solution to the standard-form polynomial equation. For instance,

$$6x^5 + 3x^3 + 5x^2 - 8x = 0$$

has, as a solution, $x = 0$. To see, this, observe that

$$6x^5 + 3x^3 + 5x^2 - 8x = x(6x^4 + 3x^2 + 5x - 8)$$

and recall that zero times anything is zero.

Rule 21. A polynomial equation of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x = 0$ has a solution of $x = 0$.

A second special case is the following:

Rule 22. Consider a polynomial equation of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$. If the sum of the coefficients (i.e., $a_n + a_{n-1} + \cdots + a_1 + a_0$) equals zero, then $x = 1$ is a solution of the equation.

To see why Rule 22 is true, recall that one to any power is one. Hence, the value of any polynomial evaluated at $x = 1$ is equal to the sum of its coefficients. Therefore, if those coefficients sum to zero, it will be the case that $x = 1$ is a solution. For instance, consider $7x^7 - 14x^3 + 7 = 0$. The sum of the coefficients $7 + (-14) + 7 = 0$. So $x = 1$ is a solution because

$$\begin{aligned} 7 \times 1^7 - 14 \times 1^3 + 7 &= 7 \times 1 - 14 \times 1 + 7 \\ &= 0. \end{aligned}$$

Note that these two special rules only tell us about a solution to the equations in question. These equations may have other solutions as well. For instance, consider

$$2x^2 - x - 1 = 0.$$

By Rule 22, $x = 1$ is a solution. But if we factor the polynomial we find

$$2x^2 - x - 1 = (x - 1)(2x + 1);$$

so $x = -1/2$ is *also* a solution.

Question 10: Find at least one solution to each of the following equations.

- (a) $5x^4 + 2x^2 = 0$.
- (b) $-3x^5 + x^4 + x^2 + 1 = 0$.
- (c) $x^4 + 4x^3 + 4x^2 + x = 0$.
- (d) $3x^3 - 2x^2 - 1 = 0$.
- (e) $x^3 - 4x^2 + 3x = 0$. (Can you find two solutions to this equation?)

3.3.3 From solutions to factors

Observe that if $x = s$ is a solution to a polynomial equation in standard form, then $(x - s)$ will be a factor of the polynomial. This proves useful because if we know a solution, then we have a tool to help factor the polynomial. To illustrate, consider $3x^2 - x - 2 = 0$. From Rule 22 it must be that $x = 1$ is a solution to this equation. But this also means that $(x - 1)$ is a factor of the polynomial. Suppose we want to find another factor? That is, we seek $(Ax + B)$ such that

$$(x - 1)(Ax + B) = 3x^2 - x - 2.$$

We can find $(Ax + B)$ by dividing $3x^2 - x - 2$ by $(x - 1)$:

$$\begin{array}{r}
 \overline{3x 2} \\
 x-1 \overline{) 3x^2 - x - 2} \\
 \underline{3x^2 3x} \\
 2x - 2 \\
 \underline{2x 2} \\
 0
 \end{array}
 \quad \begin{array}{l}
 \text{multiply divisor by } 3x \\
 \text{difference} \\
 \text{multiply divisor by } 2 \\
 \text{difference}
 \end{array}$$

Question 11: Factor the following polynomials.

Ex. $2x^2 - x - 1$. (*Answer: By making this a standard-form polynomial equation, $2x^2 - x - 1 = 0$, we see that $x = 1$ is a solution by Rule 22. Hence $x - 1$ is a factor. Dividing $2x^2 - x - 1$ by $x - 1$ yields $2x + 1$ as the other factor. So the answer is $(x - 1)(2x + 1)$.)*)

- (a) $x^2 - x$.
- (b) $x^2 - 5x + 4$.
- (c) $2x^2 + 8x - 10$.
- (d) $x^3 - x^2 - x + 1$.

3.3.4 Equations of the form $ax^2 + bx + c = 0$

If a polynomial equation of the form $ax^2 + bx + c = 0$ has a solution, then the solution or solutions are given by the *quadratic formula*:

Rule 23 (Quadratic formula). *If*

$$ax^2 + bx + c = 0$$

has a solution, then its solution or solutions are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (2)$$

Some quick observations:

1. Because one cannot take the square root of a negative number, we see that $ax^2 + bx + c = 0$ can only have solutions if $b^2 \geq 4ac$.
2. The equation $ax^2 + bx + c = 0$ will have *one* solution only if $b^2 = 4ac$. If $b^2 > 4ac$, then it will have two solutions.

Examples:

- $x^2 - 2x - 2 = 0$. So solutions are

$$\frac{-(-2) + \sqrt{(-2)^2 - 4 \times 1 \times (-2)}}{2 \times 1} \text{ and } \frac{-(-2) - \sqrt{(-2)^2 - 4 \times 1 \times (-2)}}{2 \times 1},$$

which simplify to

$$\frac{2 + \sqrt{12}}{2} = 1 + \sqrt{3} \text{ and } \frac{2 - \sqrt{12}}{2} = 1 - \sqrt{3},$$

respectively.

- $x^2 + 2x + 1 = 0$. Observe $b^2 = 4ac$ (i.e., $2^2 = 4 \times 1 \times 1$), so there is only one solution: $x = \frac{-b}{2a} = -2/2 = -1$.
- $x^2 + 5x + 4 = 0$. So solutions are

$$\frac{-5 \pm \sqrt{25 - 16}}{2} = \frac{-5 \pm 3}{2} = -1 \text{ or } -4.$$

Question 12: Solve the following equations:

- (a) $4x^2 + 5x + 1 = 0$.
- (b) $2x^2 + 4x + 2 = 0$.
- (c) $x^2 - 4x + 4 = 0$.
- (d) $x^2 - 4x - 4 = 0$.

3.3.5 An extended example

Consider

$$x^3 + 3x^2 + 3x + 1 = 8x, \quad (3)$$

which we seek to solve.

First step is to put it into standard form:

$$x^3 + 3x^2 - 5x + 1 = 0. \quad (4)$$

Next, observe that the coefficients sum to 0. So, using Rule 22, we know that $x = 1$ is one solution. We therefore know that $(x - 1)$ is a factor. Let's proceed to find the other factors and, thus, solutions:

$$\begin{array}{r}
 \quad \quad \quad x^2 \quad + \quad 4x \quad - \quad 1 \\
 x-1 \left| \begin{array}{r} x^3 \quad + \quad 3x^2 \quad - \quad 5x \quad + \quad 1 \\ x^3 \quad - \quad x^2 \end{array} \right. \quad \text{multiply divisor by } x^2 \\
 \hline
 \quad \quad \quad 4x^2 \quad - \quad 5x \quad + \quad 1 \quad \text{difference} \\
 \quad \quad \quad \underline{4x^2 \quad - \quad 4x} \quad \quad \quad \text{multiply divisor by } 4x \\
 \quad \quad \quad \quad - \quad x \quad + \quad 1 \quad \text{difference} \\
 \quad \quad \quad \quad - \quad \underline{x \quad + \quad 1} \quad \text{multiply divisor by } -1 \\
 \quad \quad \quad \quad \quad \quad 0 \quad \text{difference}
 \end{array}$$

That is, we have

$$(x - 1)(x^2 + 4x - 1) = x^3 + 3x^2 - 5x + 1.$$

Because zero times anything is zero, the other solutions to expression (4) must make

$$x^2 + 4x - 1 = 0.$$

We can solve this last equation using the quadratic formula:

$$x = \frac{-4 \pm \sqrt{16 + 4}}{2} = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}.$$

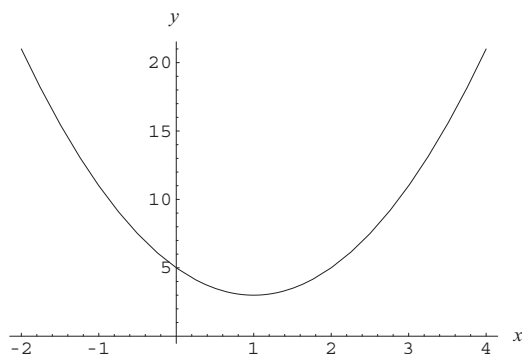


Figure 1: A parabola, $y = 2(x - 1)^2 + 3$.

Hence the solutions to equation (4)—and, thus, equation (3)—are $x = 1$, $x = -2 + \sqrt{5}$ and $x = -2 - \sqrt{5}$.

Question 13: Find *all* solutions to the following equations.

- (a) $x^3 - 3x^2 + 3x - 1 = 0$.
- (b) $x^3 - 5x^2 + 8x - 4 = 0$.
- (c) $x^3 + 3x^2 - 4 = 0$.

3.4 Parabolas

3.4.1 Definitions

A *parabola* is a figure derived by plotting $y = A(x - B)^2 + C$, where A , B , and C are constants and x and y are variables (see, *e.g.*, Figure 1). If $A > 0$, then the parabola is like a smile (*i.e.*, looks something like this \smile). If $A < 0$, then the parabola is like a frown (*i.e.*, looks something like this \frown).

3.4.2 Vertices

Because any number squared is greater than or equal to zero and equal to zero only if the number being squared is zero (*i.e.*, $x^2 \geq 0$ and $x^2 = 0$ only if $x = 0$), It follows that the $(x - B)^2$ part of the parabola is smallest when $x = B$. Hence, when $A > 0$, this means the bottom of the parabola must be reached at $x = B$. If $A < 0$, then things are turned on their heads and, hence, the top of the parabola must be reached at $x = B$.

Rule 24. Consider a parabola of the form $y = A(x - B)^2 + C$. If $A > 0$, then the bottom of the parabola is reached at $x = B$. If $A < 0$, then the top of the parabola is reached at $x = B$.

When $x = B$, it follows that $y = C$. So the coordinates of the bottom (if $A > 0$) or top (if $A < 0$) of a parabola are (B, C) . The point (B, C) is known as the *vertex* of the parabola.

3.4.3 Quadratics and parabolas

If we expand the formula for a parabola, we get

$$A(x - B)^2 + C = A(x^2 - 2Bx + B^2) + C = Ax^2 - 2ABx + (AB^2 + C). \quad (5)$$

Observe that this just a quadratic expression. In other words, any expression of the form $ax^2 + bx + c$ is the formula for some parabola. From expression (5), we see that whether the parabola is a “smile” or a “frown” depends on whether $a > 0$ (smile) or $a < 0$ (frown).

Rule 25. *A quadratic expression of the form $y = ax^2 + bx + c$ is the formula for a parabola. It is a “smiling” parabola if $a > 0$. It is a “frowning” parabola if $a < 0$.*

If we are given a quadratic expression $y = ax^2 + bx + c$, how do we solve for the parabola’s vertex? The answer is by matching the terms in $y = ax^2 + bx + c$ with those in expression (5):

- a matches to A ;
- b matches to $-2AB$; and
- c matches to $AB^2 + C$.

Substituting:

- $A = a$; hence,
- $B = \frac{b}{-2A} = \frac{b}{-2a} = -\frac{b}{2a}$; hence,
- $C = c - AB^2 = c - a \left(-\frac{b}{2a}\right)^2 = c - a \left(\frac{b^2}{4a^2}\right) = c - \frac{b^2}{4a}$.

To summarize:

Rule 26. *A quadratic expression of the form $y = ax^2 + bx + c$ describes a parabola with vertex $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.*

Question 14: Find the vertices of the following parabolas:

- $y = 2x^2 - 3x + 2$.
- $y = -x^2 + x + 1$.
- $y = 25x^2 - 5x + 1$.

3.4.4 Completing the square

Our analysis of going from $y = ax^2 + bx + c$ to the vertex of the parabola also shows that any quadratic expression of the form $y = ax^2 + bx + c$ can be rewritten in the form $y = A(x - B)^2 + C$. Using the above substitutions, we see that

$$ax^2 + bx + c = a \left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}. \quad (6)$$

Expression (6) is sometimes called *completing the square*.

Completing the square can be useful in solving quadratic equations; indeed, it is how the quadratic formula was derived. Consider, the equation

$$ax^2 + bx + c = 0.$$

If we complete the square (*i.e.*, utilize expression (6)), we can rewrite this as

$$a \left(x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a} = 0.$$

Rearranging terms, we get

$$a \left(x + \frac{b}{2a} \right)^2 = \frac{b^2}{4a} - c.$$

Rearranging further, we get

$$\begin{aligned} \left(x + \frac{b}{2a} \right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\ &= \frac{b^2}{4a^2} - \frac{4ac}{4a^2} \\ &= \frac{b^2 - 4ac}{4a^2}. \end{aligned}$$

Observe that if $b^2 - 4ac < 0$, then there is *no* solution—the square of anything (*e.g.*, the left-hand side) can't be negative. So let's proceed for the case in which $b^2 - 4ac \geq 0$: The next step is clearly to take square roots of both sides, recognizing that there are both positive and negative roots (*e.g.*, $z^2 = 4$ has both the solution $z = 2$ and $z = -2$).

$$\begin{aligned} x + \frac{b}{2a} &= \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \\ &= \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

Rearranging we, thus, have

$$\begin{aligned} x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{aligned} \tag{7}$$

By comparing expression (7) with expression (2) from page 11, we see that we have derived the quadratic formula.

Question 15: For each of the following rewrite the quadratic by completing the square. Also, if that quadratic were set equal to zero, determine the solutions (if any) to that equation.

- (a) $x^2 - 9x - 36$.
- (b) $2x^2 + x - 1$.
- (c) $2x^2 - 4x + 5$.

4 Parallel and Perpendicular Lines

4.1 Equations for lines

A *line* has the formula $y = mx + b$, where m and b are constants and x and y are variables. This way of writing a line is also called *slope-intercept* form because m is the *slope* of the line and b is the *intercept* (the point on the y axis at which the line intersects the y axis). If $m > 0$, the line slopes up. If $m < 0$, the line slopes down.

Consider an expression of the form

$$Ax + By = C,$$

where A , $B \neq 0$, and C are constants. This, too, is an expression for a line, which we can see by rearranging:

$$\begin{aligned} By &= -Ax + C \\ y &= -\frac{A}{B}x + \frac{C}{B}. \end{aligned}$$

Hence,

Rule 27. *An expression of the form $Ax + By = C$ is equivalent to the line $y = mx + b$, where $m = -A/B$ and $b = C/B$.*

Question 16: Reexpress the following in slope-intercept form:

- (a) $3x + 2y = 5$.
- (b) $-8x + 4y = 16$.
- (c) $25x - 5y = 100$.

4.2 The line through two points

The line through two distinct points (x_1, y_1) and (x_2, y_2) is the line satisfying the equations:

$$\begin{aligned} y_1 &= mx_1 + b \text{ and} \\ y_2 &= mx_2 + b. \end{aligned}$$

Notice that if we subtract the first equation from the second, we get:

$$y_2 - y_1 = mx_2 - mx_1 = m(x_2 - x_1),$$

from which it follows that

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Substituting that back into the second equation, we get

$$\begin{aligned}y_2 &= \frac{y_2 - y_1}{x_2 - x_1}x_2 + b \\y_2 - \frac{y_2 - y_1}{x_2 - x_1}x_2 &= b \\ \frac{y_2(x_2 - x_1)}{x_2 - x_1} - \frac{x_2(y_2 - y_1)}{x_2 - x_1} &= b \\ \frac{x_2y_1 - x_1y_2}{x_2 - x_1} &= b.\end{aligned}$$

To conclude:

Rule 28. The line $y = mx + b$ through the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ and } b = \frac{x_2y_1 - x_1y_2}{x_2 - x_1}.$$

Question 17: Find the equation for the line through the following points:

- (a) $(2, 3)$ and $(3, 5)$.
- (b) $(2, 4)$ and $(4, 8)$.
- (c) $(1, -1)$ and $(2, 3)$.

4.3 Parallel lines

If $y = mx + b$ is a line and (x_1, y_1) is a point not on that line, we can find a line parallel to $y = mx + b$ that goes through (x_1, y_1) . Call this parallel line $y = Mx + B$. A parallel line has the same slope, so $M = m$. But, then, B is readily found by subtraction:

$$B = y_1 - mx_1.$$

Question 18: For each line and point, find another line parallel to the first that goes through the given point.

- (a) $y = 3x - 2$ and $(1, 5)$.
- (b) $y = -4x + 10$ and $(2, 0)$.
- (c) $3x + 2y = 10$ and $(4, 4)$ (*hint:* turn the line into slope-intercept form first).

4.4 Perpendicular lines

If $y = mx + b$ is a line and (x_1, y_1) is a point, we can find a line perpendicular (*i.e.*, at right angles) to $y = mx + b$ that goes through (x_1, y_1) . Call this perpendicular line $y = Mx + B$. The product of the slopes of two perpendicular lines equals -1 . Hence, $M = -1/m$. But, then, B is readily found by subtraction:

$$\begin{aligned}B &= y_1 - Mx_1 \\ &= y_1 + x_1/m.\end{aligned}$$

Question 19: For each line and point, find another line perpendicular to the first that goes through the given point.

- (a) $y = 3x - 2$ and $(1, 5)$.
- (b) $y = -4x + 10$ and $(2, 2)$.
- (c) $2x + 5y = 10$ and $(4, 4)$ (*hint:* turn the line into slope-intercept form first).

5 Rational Expressions

5.1 Solving proportions

A *proportion* or *ratio* expresses the relation between numerical measures (*e.g.*, lengths) as a fraction. For instance, one might say a daughter is $4/5$ ths as tall as her mother. The fraction $4/5$ gives the daughter's height as a proportion of her mother's; alternatively the ratio of the daughter's height, h , to the mother's height, H , is such that

$$\frac{h}{H} = \frac{4}{5}.$$

Sometimes we wish to retrieve the actual numerical measures using our knowledge of the proportion and the other numerical measure. For instance, if we knew the daughter was 4 feet tall, then we can solve this last equation to determine the mother's height:

$$\begin{aligned} h &= \frac{4}{5}H \text{ (multiply both sides by } H) \\ \frac{5}{4}h &= H \text{ (multiply both sides by } 5/4) \\ \frac{5}{4} \times 4 \text{ feet} &= H \text{ (substitute in the daughter's height)} \\ 5 \text{ feet} &= H. \end{aligned}$$

That is, the mother is 5 feet tall.

More generally,

Rule 29. *If ℓ and L are numerical measures and it is known they stand in proportion p to each other, then from any two, the third can be solved for as follows:*

$$p = \frac{\ell}{L}; \ell = pL; \text{ and } L = \frac{\ell}{p}.$$

Question 20: Solve the following.

- (a) Buffy ran the 100 meters in $9/10$ ths the time it took Biff. If Biff ran the race in 12 seconds, how long did Buffy take?
- (b) Mount Everest is approximately twice as tall as the tallest mountain in California, Mt. Whitney. If Mt. Everest is 29,000 feet tall, approximately how tall is Mt. Whitney?

5.2 Solving rational expressions

Consider an equation of the form

$$\frac{x-2}{x+5} = 6.$$

We can solve it by multiplying both sides by $x+5$,

$$\begin{aligned}(x+5)\frac{x-5}{x+5} &= 6(x+5) \\ x-5 &= 6x+30,\end{aligned}$$

then rearranging,

$$-5-30 = 5x,$$

and then solving $5x = -35$, which yields $x = -7$.

More generally, consider the following:

$$\frac{Ax+B}{Cx+D} = \frac{Ex+F}{Gx+H},$$

where we seek to solve for x and the parameters $A-H$ are constants. To solve, multiply both sides by the two denominators (*i.e.*, $Cx+D$ and $Gx+H$):

$$(Cx+D)(Gx+H)\frac{Ax+B}{Cx+D} = (Cx+D)(Gx+H)\frac{Ex+F}{Gx+H}$$

which, simplifying, becomes

$$\begin{aligned}(Gx+H)(Ax+B) &= (Cx+D)(Ex+F) \\ AGx^2 + (BG+AH)x + BH &= CEx^2 + (CF+DE)x + DF;\end{aligned}$$

hence,

$$(AG-CE)x^2 + ((BG+AH) - (CF+DE))x + BH - DF = 0.$$

The last expression is a quadratic in standard form and can be solved using the quadratic formula (*i.e.*, formula (2) on page 11).

Question 21: Solve the following equations for x .

- $\frac{x-10}{x} = 2.$
- $\frac{4}{1+x} = 4.$
- $\frac{A+Bx}{x} = C$, where $A-C$ are constants and $B \neq C$. (*Bonus:* what would be the problem if $B = C$?)
- $\frac{3+x}{2x+1} = 1.$
- $\frac{20-x}{x+5} = \frac{x-2}{x-3}$

6 Checking Your Work

One of the nice features of algebra is that it is often easy to check whether you've made a mistake. Two methods are (1) plug your answer back into the original problem and (2) check the result using arbitrary values.

To illustrate the “plug the solution back in” method. Suppose you are asked to solve $x^2 - 2x + 1 = 0$ and you work out a solution of $x = 2$. Is this correct? Well, you can check by plugging 2 back into the original equation and see if it satisfies the equality:

$$\begin{aligned}2^2 - 2 \times 2 + 1 &= 4 - 4 + 1 \\ &\neq 0;\end{aligned}$$

hence, $x = 2$ does *not* satisfy the equation and so *cannot* be the correct solution.

To illustrate the “arbitrary values” method. Suppose you are asked to solve

$$\frac{x - K}{x} = 2 \tag{8}$$

($K \neq 0$) and you get a solution of $x = 2K$. Is this correct? You could plug $2K$ in and try to verify the equation, but an alternative is to try arbitrary values for K and see what happens. Suppose $K = 1$, then $x = 2$. The original expression, expression (8), would then be

$$\frac{2 - 1}{2} = \frac{1}{2};$$

hence, we see that $x = 2K$ *cannot* be the correct solution. That is, logically, if we write $x = 2K$, what we're saying is that for *all* possible values of K , the solution to equation (8) is $x = 2K$. We can falsify this claim—show that we've made a mistake—simply by finding some value for K for which it doesn't work.

Question 22: For each of the following problems a proposed solution has been offered. Check it using one of the above methods. If the solution is correct, note that it is correct. If it is incorrect, find the correct solution.

- (a) $3x^2 - 7x + 1 = 0$. Proposed solution $x = 2$.
- (b) $\frac{2x-K}{x+1} = 3$. Proposed solution $x = K$.
- (c) $6x^2 - 14x + 4 = 0$. Proposed solution $x = 2$.
- (d) $\frac{Jx+K}{x+1} = 1$, $J \neq 1$. Proposed solution $x = \frac{2-K}{J-1}$ (*hint*: try arbitrary values for J and K).
- (e) $\frac{x+1}{\sqrt{x}} = K$, $K \geq 2$. Proposed solutions

$$x = \left(\frac{K + \sqrt{K^2 - 4}}{2} \right)^2 \text{ and } x = \left(\frac{K - \sqrt{K^2 - 4}}{2} \right)^2.$$