Answers to Exam

Economics 201B — First Half

1. (a) Observe, first, that no consumer ever wishes to consume more than 3/2 liters (*i.e.,* 1.5 liters). To see this, observe that, even if the beverage were free, the consumer's utility is declining if $x > a/b$. Specifically,

$$x^* = \arg\max_x ax - \frac{1}{2}bx^2 \Rightarrow a - bx^* = 0 \Rightarrow x^* = \frac{a}{b}\,.$$

Hence, a consumer's willingness-to-pay as a function of liters *on sale*, ℓ, is

$$W(\ell) = \begin{cases} a\ell - \frac{1}{2}b\ell^2\,, \text{ if } \ell \leq \frac{a}{b} \\ \frac{a^2}{2b}\,, \text{ if } \ell > \frac{a}{b} \end{cases}\,. \tag{1}$$

Hence, from (1), it follows that a 2-liter bottle can command at most $a^2/(2b)$ given that $2 > a/b$ (recall $\frac{3}{2}b > a$). So, provided $a^2/(2b) \geq 2c$, the profit-maximizing price for a two-liter bottle is $a^2/(2b)$. If that inequality is not met, then any price for the two-liter bottle that results in no sales is profit-maximizing (*i.e.,* if that inequality is not met, then the firm should not sell two-liter bottles).

(b) From expression (1), the firm can charge, at most, $a - \frac{1}{2}b$ for a one-liter bottle. Its profit from doing so is $a - \frac{1}{2}b - c$. The answer, therefore, depends on whether

$$a - \frac{1}{2}b - c \gtrless \frac{a^2}{2b} - 2c\,. \tag{2}$$

Clearly if $a = b$, the left-hand side exceeds the right-hand side, which means the firm would do better to sell one-liter bottles. It should also be clear that $a = b$ maximizes the difference between the left-hand side and the right-hand side; hence, as $a \uparrow 3b/2$, the right-hand side is increasing faster than the left-hand side. At the limit, $a = 3b/2$, the left-hand side is $b - c$ and the right-hand side is $\frac{9}{8}b - 2c$. The firm could only ever do better selling two-liter bottles if $8c < b$.

2. (a) There are only two unit prices worth considering, v and r. If the latter, then Diane sells to all customers and her expected profit per customer is

$$\phi \times \underbrace{r}_{\text{one sold to Vulcan}} + (1-\phi) \times \underbrace{2r}_{\text{two sold to Romulan}} = 2r - \phi r\,. \tag{3}$$

If, instead, she sets the price at v, she sells to Vulcans only and her expected profit is ϕv. That exceeds (3) if

$$\phi > \frac{2r}{v+r}\,. \tag{4}$$

We can, thus, conclude

$$p = \begin{cases} r\,, & \text{if (4) does not hold} \\ v\,, & \text{if (4) does hold} \end{cases}\,.$$

(b) It should be evident that the two-pack is intended for Romulans and the single crystal for the Vulcans. This is second-degree price discrimination via quantity discounts, so the usual IR and IC constraints hold:

$$\begin{aligned} 2r - T_2 &\geq 0 && \text{(IR-Romulan)} \\ v - T_1 &\geq 0 && \text{(IR-Vulcan)} \\ 2r - T_2 &\geq 2r - 2T_1 && \text{(IC-Romulan)} \\ v - T_1 &\geq v - T_2\,, && \text{(IC-Vulcan)} \end{aligned}$$

where T_n is the price of an n-pack. Note that (IC-Romulan) arises because a Romulan could obtain two crystals by buying two one-packs. Note, too, that (IC-Vulcan) arises because a Vulcan could obtain one crystal by buying a two-pack and tossing away one crystal (space, the final dumping ground). Inspection (or linear programming) quickly reveals that $T_1 = v$ and $T_2 = 2r$ satisfies all constraints. Because this solution gives Diane all the surplus, it must be profit maximizing. Her expected profit per customer is $\phi v + (1-\phi)2r$, which is clearly greater than her profit under linear pricing.

3. The key is to set up the payments so that we achieve budget balance and so that they put each citizen on the same margin on which the social planner operates. Budget balancing can be achieved by using the payment scheme:

$$p_i(\boldsymbol{\theta}) = \rho_i(\theta_i) - \frac{1}{I-1}\sum_{j\neq i}\rho_j(\theta_j)\,.$$

Were the λ_i's all the same, then we know that we set

$$\rho_i(\theta_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left\{\sum_{j\neq i} v_j[x^*(\theta_i,\boldsymbol{\theta}_{-i}),\theta_j]\right\} + h_i\,,$$

where h_i is an arbitrary constant. We do this because it will put citizen i on the same margin as the social planner (*i.e.,* his problem is equivalent to hers). For that reason, when the λ_i's vary, we have reason to expect that the following will work:

$$\rho_i(\theta_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}} \left\{ \frac{\lambda_j}{\lambda_i} \sum_{j \neq i} v_j[x^*(\theta_i, \boldsymbol{\theta}_{-i}), \theta_j] \right\} + h_i \,. \tag{5}$$

[One could receive full credit for the problem stopping here. The rest of the answer is for educational purposes.] To verify that (5) works, observe that each i announces a type $\hat{\theta}_i$ to maximize:

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}} \left\{ v_i[x^*(\hat{\theta}_i, \boldsymbol{\theta}_{-i}), \theta_i] + \mathbb{E}_{\boldsymbol{\theta}_{-i}} \left\{ \frac{\lambda_j}{\lambda_i} \sum_{j \neq i} v_j[x^*(\hat{\theta}_i, \boldsymbol{\theta}_{-i}), \theta_j] \right\} + h_i \right\}$$
$$= \mathbb{E}_{\boldsymbol{\theta}_{-i}} \left\{ v_i[x^*(\hat{\theta}_i, \boldsymbol{\theta}_{-i}), \theta_i] + \frac{\lambda_j}{\lambda_i} \sum_{j \neq i} v_j[x^*(\hat{\theta}_i, \boldsymbol{\theta}_{-i}), \theta_j] \right\} + h_i \,,$$

which is equivalent to maximizing

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}} \left\{ \sum_{j=1}^{I} \lambda_j v_j[x^*(\hat{\theta}_i, \boldsymbol{\theta}_{-i}), \theta_j] \right\} . \tag{6}$$

In other words, if $\hat{\theta}_i = \theta_i$ maximizes (6), then we've shown this mechanism works. By definition,

$$\sum_{j=1}^{I} \lambda_j v_j[x^*(\hat{\theta}_i, \boldsymbol{\theta}_{-i}), \theta_j] \leq \sum_{j=1}^{I} \lambda_j v_j[x^*(\theta_i, \boldsymbol{\theta}_{-i}), \theta_j] \,,$$

for all $\hat{\theta}_i$ and θ_i. So it follows that

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}} \left\{ \sum_{j=1}^{I} \lambda_j v_j[x^*(\hat{\theta}_i, \boldsymbol{\theta}_{-i}), \theta_j] \right\} \leq \mathbb{E}_{\boldsymbol{\theta}_{-i}} \left\{ \sum_{j=1}^{I} \lambda_j v_j[x^*(\theta_i, \boldsymbol{\theta}_{-i}), \theta_j] \right\} .$$

Hence, truth-telling maximizes (6) and, therefore, is a best response. The mechanism, thus, works.

4. (a) Given a sharing rule ϕ, the agent will choose a to maximize

$$\phi G q a - d(a) \tag{7}$$

if he chooses to work for the principal (accepts the contract). He will do so if the maximized value of (7) exceeds 0. By assumption, were $\phi = 1$, then the maximized value of (7) would strictly exceed zero, so there exist values of ϕ such that the IR constraint is met. The function (7) is concave in a so the first-order condition is sufficient for defining the agent's best response to ϕ:

$$a^*(\phi) = \begin{cases} 0\,, \text{ if } \phi Gq - d'(a) < 0 \; \forall a \\ d'^{-1}(\phi Gq)\,, \text{ otherwise} \end{cases} .$$

The optimal ϕ is, thus, the solution to

$$\max_{\phi}\,(1-\phi)Gqa^*(\phi) \tag{8}$$

subject to

$$\phi Gqa^*(\phi) - d\left(a^*(\phi)\right) \geq 0\,.$$

Because the principal gets nothing if the agent doesn't work for her and because, as noted, the IR constraint can be met, there will be some $\phi \in (0,1)$ that solves this problem and yields the principal a positive expected profit.

(b) In the first best, the principal can employ a forcing contract; that is, she maximizes $qaG - w$ subject to $w - d(a) \geq 0$. The IR constraint binds, so we may substitute it into the maximand to find that the first-best problem is

$$\max_a qaG - d(a)\,.$$

The solution to this, which exists by assumption, is the first-best action, a^F. Note, critically, that because there is no ϕ on the first term, this $a^F > a^*(\phi)$ for all $\phi < 1$. Put differently, if the principal wants first-best action, she must set $\phi = 1$. She wouldn't do this, of course, if she could use just a pure-sharing contract (she gets nothing, whereas we saw in (a) she had a positive expected profit for some $\phi < 1$). But consider a contract of the form $w_s = R_s - T$, where $T = qa^F G - d(a^F)$ (*i.e.,* a contract in which $\phi = 1$). It is readily verified that this contract satisfies the IR constraint. It also leads the agent to choose $a = a^F$. And the principal captures all the expected surplus as she always gets T, which has been set equal to the maximized expected surplus. In other words, this contract achieves the first best.

(c) Part (a) is like simple monopoly pricing. The parameter ϕ acts like the price and, in part (a), is being asked to do double duty: It both provides incentives (*i.e.,* determines allocation) and it is used to divide surplus. Not surprisingly, the first best does not attain. In part (b), we have a second instrument, T, which acts like the entry fee in a two-part tariff. Now we can use ϕ solely for incentive (*i.e.,* allocative) purposes, so we can achieve a first-best action. The second instrument is, then, used to capture all the surplus for the principal (who is like a monopolist).

(d) The non-standard assumption is the agent's risk *neutrality*. If he were risk *averse*, then he would demand further compensation for the risk he must bear under the contract $w_s = R_s - T$. In other words, if the agent is induced to choose $a = a^F$, then he receives more than first-best compensation in expectation; which means the first best is not achieved. Alternatively, the principal distorts the action asked of him, which also means the first best is not achieved.