Answers to Problem Set #3

Economics 201B

1. To be done in section.

2. (a) In the full-information case, the principal maximizes $x - s$ over x and s subject to the constraint that $s - \theta x^2 \geq 0$ (*i.e.,* subject to the constraint that the agent participate). Clearly, the constraint binds — the principal wishes to make s as small as possible — so the problem is equivalent to the unconstrained problem

$$\max_x x - \theta x^2 .$$

The solution to which is $x^F(\theta) = \frac{1}{2\theta}$. Hence, $s^F(\theta) = \frac{1}{4\theta}$.

(b) Observe that the *higher* θ, the worse the type. That is, in terms of the notation in "Hidden-Information Agency," $C_E(x) = x^2$ and $C_I(x) = 2x^2$. The solution can be found from expressions (8) and (9) of "Hidden-Information Agency" (see page 8): $x^*(1) = x^F(1) = 1/2$ and $x^*(2)$ solves

$$\max_x x - 2x^2 - \frac{1-\frac{1}{2}}{\frac{1}{2}} \underbrace{\left(2x^2 - x^2\right)}_{R(x)} .$$

So $x^*(2) = 1/6$. Hence, $s^*(2) = 2x^*(2)^2 = 1/18$ and $s^*(1) = x^*(1)^2 + R[x^*(2)] = 1/4 + 1/36 = 10/36$.

(c) Note that now the payment from principal to agent can be contingent on the realization of σ as well as the announcement of θ. Let $S(\sigma, \theta)$ denote such a payment. Observe the (IR) and (IC) constraints become:

$$\frac{3}{4}S(1,1) + \frac{1}{4}S(2,1) - x(1)^2 \geq 0 \quad \text{(IR-1)}$$

$$\frac{3}{4}S(2,2) + \frac{1}{4}S(1,2) - 2x(2)^2 \geq 0 \quad \text{(IR-2)}$$

$$\frac{3}{4}S(1,1) + \frac{1}{4}S(2,1) - x(1)^2 \geq \frac{3}{4}S(1,2) + \frac{1}{4}S(2,2) - x(2)^2 \quad \text{(IC-1)}$$

$$\frac{3}{4}S(2,2) + \frac{1}{4}S(1,2) - 2x(2)^2 \geq \frac{3}{4}S(2,1) + \frac{1}{4}S(2,2) - 2x(1)^2 \quad \text{(IC-2)}$$

If we thought of *all* constraints as binding, then we have a system of four linear equations in four unknowns. Generically, such systems have solutions. Moreover, if this one does with $x(\theta) = x^F(\theta)$, then the first best is achieved. To see if this system has a solution, write it in matrix form

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} & -\frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} S(1,1) \\ S(2,1) \\ S(1,2) \\ S(2,2) \end{pmatrix} = \begin{pmatrix} x^F(1)^2 \\ 2x^F(2)^2 \\ x^F(1)^2 - x^F(2)^2 \\ 2x^F(2)^2 - 2x^F(1)^2 \end{pmatrix}.$$

Recall that a system has a solution if the matrix of coefficients (the first matrix) has a non-zero determinant. It is readily seen that the determinant of that matrix is $-\frac{1}{4} \neq 0$. So the optimal mechanism can be found by solving the above set of equations for the $S(\cdot,\cdot)$ values. Observe that this achieves the first best, the same answer as part (a). This suggests that as long as we have risk-neutral agents and no constraints on transfers (positive or negative), then a little bit of information is all that's necessary to achieve the first best (you should be able to see that the determinant is non-zero provided $\Pr\{\sigma = \theta\} \neq 1/2$).

3. (a) Because the types have the "opposite" order from "Hidden-Information Agency" it pays to consider "new" types $\tau \equiv -\theta$. Note that $\tau \in [-2,-1]$ with a uniform distribution. We want to see if the problem fits the standard framework. Clearly, the utilities are additively separable and the social welfare function, $x + \tau x^2$, is globally concave for all τ. To check Spence-Mirrlees, observe that $\partial u/\partial x = 2\tau x$, which is increasing in type as required by Spence-Mirrlees. Observe that virtual surplus (see page 29 of "Hidden-Information Agency") is

$$\begin{aligned}\Sigma(x,\tau) &= x + \tau x^2 - \frac{-1-\tau}{1} \underbrace{x^2}_{\partial u/\partial \tau} \\ &= x + (1+2\tau)x^2 .\end{aligned}$$

Observe, too, that this problem satisfies the assumptions of Proposition 4 of "Hidden-Information Agency" (page 30). Hence, the solution is $x^*(\tau) = -\frac{1}{2(1+2\tau)}$; or, in terms of θ, $x^*(\theta) = \frac{1}{2(2\theta-1)}$. Recall that

$$\begin{aligned}s^*(\tau) &= v_L - \underbrace{\tau\left(\frac{-1}{2(1+2\tau)}\right)^2}_{u(x(\tau),\tau)} + \int_{-2}^{\tau} \underbrace{\left(\frac{-1}{2(1+2t)}\right)^2}_{\partial u/\partial \tau} dt \\ &= 0 - \frac{\tau}{4(1+2\tau)^2} - \frac{1}{8(2\tau+1)} - \frac{1}{24}.\end{aligned}$$

(b) The values of $x^*(1)$ and $x^*(2)$ are the same as in Part 2. The value of $s^*(2)$ is also the same (no rent to the worst type). Finally, $s^*(1) =$

$\frac{1}{3} > \frac{10}{36}$. That $s^*(1)$ is greater is not surprising given the existence of intermediate types who make it harder to induce truthtelling (*e.g.*, it's more tempting for a type 1 to announce type 1.5 than to announce type 2). For this reason, one might have hypothesized that $x^*(2)$ would have been lower than in Part 2 to economize on the information rent; however, given the distributional assumption, this proved not to be the case.