Answers to Problem Set #4

Economics 201B

1. As a preliminary, note $b(x,\theta) = \int_0^x p(t,\theta)dt = \theta\bigl(kx - \frac{1}{2}x^2\bigr)$. Of course, $\partial b/\partial x = p(x,\theta)$.

   (a) Expression (12) can be written

   $$0 = p\bigl(x(\theta),\theta\bigr) - \left(c + \frac{1-F(\theta)}{f(\theta)}\frac{\partial p\bigl(x(\theta),\theta\bigr)}{\partial\theta}\right)$$
   $$= \theta\bigl(k - x(\theta)\bigr) - \left(c + \frac{1-F(\theta)}{f(\theta)}\bigl(k - x(\theta)\bigr)\right)$$

   Solving for $x(\cdot)$ yields:

   $$\hat{x}(\theta) = k - \frac{c}{\theta - \frac{1-F(\theta)}{f(\theta)}}\,.$$

   (b) Observe, from the last expression, that $\hat{x}(\cdot)$ is monotonic only if $\theta - \frac{1-F(\theta)}{f(\theta)}$ is non-decreasing in θ (at least for θ such that $\hat{x}(\theta) > 0$). This would be not be feasible if the derivative of the Mills ratio was greater than one.

   (c) Observe this means

   $$F(\theta) = \frac{\theta - \theta_0}{\theta_1 - \theta_0} \text{ (so } 1 - F(\theta) = \frac{\theta_1 - \theta}{\theta_1 - \theta_0}\text{) and } f(\theta) = \frac{1}{\theta_1 - \theta_0}\,.$$

   Hence,

   $$\hat{x}(\theta) = k - \frac{c}{2\theta - \theta_1}$$

   for $\theta \geq \bigl(\theta_1 + c/k\bigr)/2 \equiv \bar{\theta}$; and

   $$T(\theta) = b\bigl(\hat{x}(\theta),\theta\bigr) - \int_{\theta_0}^{\theta}\frac{\partial b\bigl(\hat{x}(t),t\bigr)}{\partial\theta}dt \qquad \text{(from (9))}$$
   $$= \theta\bigl(k\hat{x}(\theta) - \frac{1}{2}\hat{x}(\theta)^2\bigr) - \int_{\tilde{\theta}}^{\theta}\bigl(k\hat{x}(t) - \frac{1}{2}\hat{x}(t)^2\bigr)dt$$
   $$= \frac{k^2(2\tilde{\theta} - \theta_1)\tilde{\theta}(2\theta - \theta_1)^2 + c^2\bigl(\theta_1\tilde{\theta} - 2\theta(2\tilde{\theta} - \theta)\bigr)}{2(\theta_1 - 2\tilde{\theta})(2\theta - \theta_1)^2}\,,$$

   where $\tilde{\theta} = \max\{\theta_0, \bar{\theta}\}$.

2. There are a number of ways to answer this question. This is just one way.

   - Let the seller's profit be $t - cx$ (*i.e.,* $s = -t$ and $w(x, \theta) = -cx$). The assumption of a constant marginal cost, c, because it makes it feasible to work with a single customer without worrying about the overall cost of total output.
   - Let the customer's utility be $u(x, \theta) - t$, where $\theta \in [\theta_L, \theta_H] \equiv \Theta$ is type. Further assume that $u(\cdot, \theta)$ is strictly concave for all θ; that $\partial u(0, \theta)/\partial x > c$ for all $\theta \in (\theta_L, \theta_H]$; and there exists an $x^F(\theta)$ for all $\theta \in \Theta$ such that $\partial u\big(x^F(\theta), \theta\big)/\partial x = c$.
   - Let $\theta \sim F$, where F is differentiable with density f and $f(\theta) > 0$ $\forall \theta \in \Theta$.
   - Assume $u(0, \theta) = 0$ for all $\theta \in \Theta$. Note the reservation utility is, thus, 0.
   - Assume the Spence-Mirrlees condition:

$$\frac{\partial^2 u}{\partial \theta \partial x} > 0\,.$$

   - The above assumptions are sufficient for the standard framework. If you further wished to guarantee that the optimal allocation, $x^*(\cdot)$, could be found by point-wise maximization of

$$\Sigma(x, \theta) \equiv u(x, \theta) - cx - \frac{1 - F(\theta)}{f(\theta)} \frac{\partial u(x, \theta)}{\partial \theta}\,,$$

     then you need to further assume:
     - For all x, $\partial u/\partial \theta \geq 0$;
     - The function $\Sigma(\cdot, \theta)$ is strictly quasi-concave for all $\theta \in \Theta$; and
     - The cross partial $\frac{\partial^2 \Sigma}{\partial \theta \partial x}$ is non-negative for all x.

3. (a) This is without loss of generality because if $f_n = 0$, then it is as if the nth type doesn't exist (*i.e.,* could be ignored).

   (b) This represents the information rent that a type $n + 1$ agent would get from mimicking a type n agent.

   (c) The Spence-Mirrlees (SM) condition in this context is that $\partial \phi/\partial x$ be monotonic in n. The screening condition certainly implies that: $R_n(\cdot)$ strictly increasing means that

$$0 < R_n'(x) = \frac{\partial \phi(x, n)}{\partial x} - \frac{\partial \phi(x, n+1)}{\partial x}\,;$$

   hence, $\partial \phi/\partial x$ is decreasing in n. So SC $\Rightarrow$ SM. Clearly, SM implies that $R_n(\cdot)$ is increasing. Given the assumption that $\phi(0, n) = 0$ for all n, it follows that SM also implies that $R_n(x) > 0$ for all $x > 0$. Hence, the only property of SC not implied by SM is that $R_n(\cdot)$ be convex.

(d) Following the hints:

i. As usual, it is easier to work with utility than payments. Set $u_n = s_n - \phi(x_n, n)$. Observe, then, $s_n = u_n + \phi(x_n, n)$. Using this substitution, revealed preference tells us for any m and n,

$$u_n \geq \overbrace{u_m + \phi(x_m, m)}^{s_m} - \phi(x_m, n) \tag{1}$$
$$u_m \geq \underbrace{u_n + \phi(x_n, n)}_{s_n} - \phi(x_n, m) \tag{2}$$

Assume $n > m$. Rearrange (1) and (2) as

$$\phi(x_n, m) - \phi(x_n, n) \geq u_n - u_m \geq \phi(x_m, m) - \phi(x_m, n)\,. \tag{3}$$

Now, we can proceed in one of two ways. First, using part (c) — specifically, that SC ⇒ SM — we could use the fact that

$$\phi(x, m) - \phi(x, n) = \int_0^x \left(\frac{\partial \phi(z, m)}{\partial z} - \frac{\partial \phi(z, n)}{\partial z} \right) dz$$

and the fact that $\partial\phi(z, m)/\partial z > \partial\phi(z, n)\partial n$ (the Spence-Mirrlees condition) to conclude that the left-most term of (3) can exceed the right-most term of (3) only if $x_n \geq x_m$. Alternatively, observe that

$$\phi(x, m) - \phi(x, n) = \sum_{j=m}^{n-1} R_j(x)\,;$$

hence (3) can be rewritten as

$$\sum_{j=m}^{n-1} R_j(x_n) \geq u_n - u_m \geq \sum_{j=m}^{n-1} R_j(x_m)\,. \tag{4}$$

The screening-condition requires that $R_j(\cdot)$ be strictly increasing for all j; therefore, (4) can hold only if $x_n \geq x_m$.

Note, as a "bonus," we've also established, given that $R_j(x) > 0$ for $x > 0$, that $u_n \geq u_m$. In other words, equilibrium utilities must be non-deceasing in type.

ii. We need to establish that the adjacent IC constraints imply (1) and (2) for all n and m given that x_j is non-decreasing in j. Fix an n and consider any $m < n - 1$ (obviously (1) and (2) hold if $m = n - 1$ because those are just the adjacent IC constraints). Observe

$$\begin{aligned} u_n - u_m = \sum_{j=m}^{n-1} (u_{j+1} - u_j) \geq \sum_{j=m}^{n-1} R_j(x_j) \geq \sum_{j=m}^{n-1} R_j(x_m) \\ = \phi(x_m, m) - \phi(x_m, n)\,, \end{aligned} \tag{5}$$

where the first inequality follows from the adjacent IC constraints and the second inequality follows because x is non-decreasing in type, so $x_m \leq x_j$ if $j \geq m$. Observe that (5) can be rearranged to yield (1). Similarly, we have

$$u_n - u_m = \sum_{j=m}^{n-1}(u_{j+1} - u_j) \leq \sum_{j=m}^{n-1} R_j(x_{j+1}) \leq \sum_{j=m}^{n-1} R_j(x_n) \\ = \phi(x_n, m) - \phi(x_n, n)\,, \tag{6}$$

where the first inequality follows from the adjacent IC constraints and the second follows because x is non-deceasing in type. Expression (6) can be rearranged to yield (2). If $m > n+1$, then just reverse the definitions of m and n in the above. Hence, we've established that a non-decreasing allocation profile and the adjacent IC constraints are sufficient to have *all* the IC constraints hold.

iii. Fix an n. By assumption $x_{n+1} \geq x_n$. By construction, $u_n = \sum_{j=1}^{n-1} R_j(x_j)$ (where we utilize the convention that $\sum_{j=1}^{0}$ is zero). Consider one adjacent IC constraint:

$$u_n \geq u_{n+1} - R_n(x_{n+1})$$

which is equivalent to

$$0 \geq R_n(x_n) - R_n(x_{n+1})\,,$$

which holds because x is non-decreasing in type and $R_n(\cdot)$ is an increasing function. Now consider the other:

$$u_{n+1} \geq u_n + R_n(x_n)$$

which is equivalent to

$$R_n(x_n) \geq R_n(x_n)\,,$$

which holds trivially.

iv. The program (P) given the constraints (IR) and (IC) is obviously what the principal wishes to maximize. We've established that the allocation profile must be non-decreasing, so we're done if we can show that

$$\sum_{n=1}^{N} f_n u_n = \sum_{n=1}^{N}(1 - F_n)R_n(x_n)\,. \tag{7}$$

Simple algebra reveals that (7) follows if $\mathbf{u}$ is such that

$$u_1 = 0 \text{ and } u_n = \sum_{j=1}^{n-1} R_j(x_j) \text{ if } n > 1\,. \tag{8}$$

But clearly, the principal wants to make the **u**'s as small as possible. Clearly, $u_1 = 0$ is the smallest possible value of u_1. And, as just shown, the smallest IC values of the u_n, $n > 1$, are those defined by (8). This completes the proof.