Answers to Problem Set #6

Economics 201B

1. (a) $a^*(\boldsymbol{\tau})$ solves the first-order condition[1]

$$\sum_{n=1}^{N} \lambda_n(\tau_n - a) = 0\,.$$

Hence,

$$a^*(\boldsymbol{\tau}) = \frac{\sum_{n=1}^{N} \lambda_n \tau_n}{\sum_{n=1}^{N} \lambda_n}\,.$$

It follows from the Groves-Clarke Theorem (Proposition 12 of the Notes) that

$$\begin{aligned} P_n(\boldsymbol{\tau}) &= \sum_{j \neq n} \lambda_j \big(\tau_j a^*(\boldsymbol{\tau}) - \frac{a^*(\boldsymbol{\tau})^2}{2}\big) + h_n(\boldsymbol{\tau}_{-n}) \\ &= \sum_{j \neq n} \lambda_j \left(\tau_j \frac{\sum_{k=1}^{N} \lambda_k \tau_k}{\sum_{k=1}^{N} \lambda_k} - \frac{1}{2}\big(\frac{\sum_{k=1}^{N} \lambda_k \tau_k}{\sum_{k=1}^{N} \lambda_k}\big)^2\right) + h_n(\boldsymbol{\tau}_{-n})\,. \end{aligned}$$

If the mechanism is balanced, then $\sum_{n=1}^{N} P_n(\boldsymbol{\tau}) \equiv 0$. This means we can differentiate it with respect to any τ and get a derivative that is identically equal to zero:

$$\begin{aligned} 0 \equiv \frac{\partial \sum_{n=1}^{N} P_n(\boldsymbol{\tau})}{\partial \tau_1} &= \frac{\partial}{\partial \tau_1}\left((N-1)\sum_{n=1}^{N} \lambda_n \big(\tau_n a^*(\boldsymbol{\tau}) - \frac{a^*(\boldsymbol{\tau})^2}{2}\big) + \sum_{n=1}^{N} h_n(\boldsymbol{\tau}_{-n})\right) \\ &= (N-1)\lambda_1 a^*(\boldsymbol{\tau}) + \sum_{n=2}^{N} \frac{\partial h_n(\boldsymbol{\tau}_{-n})}{\partial \tau_1} \end{aligned}$$

(note the use of the envelope theorem in going from the first to second line). Because this last expression is also an identity, we can differentiate it with respect to τ_2 (or any other τ_n, $n \geq 2$) and get a derivative that is also identically equal to zero:

$$\begin{aligned} 0 \equiv \frac{\partial^2 \sum_{n=1}^{N} P_n(\boldsymbol{\tau})}{\partial \tau_1 \partial \tau_2} &= (N-1)\lambda_1 \frac{\partial a^*(\boldsymbol{\tau})}{\partial \tau_2} + \sum_{n=3}^{N} \frac{\partial^2 h_n(\boldsymbol{\tau}_{-n})}{\partial \tau_1 \partial \tau_2} \\ &= (N-1)\frac{\lambda_1 \lambda_2}{\sum_{n=1}^{N} \lambda_n} + \sum_{n=3}^{N} \frac{\partial^2 h_n(\boldsymbol{\tau}_{-n})}{\partial \tau_1 \partial \tau_2}\,. \end{aligned}$$

[1] Although not stated, each $\lambda_n \geq 0$.

Observe that this last differential equation can be solved; which means that one *can* construct a balanced dominant-strategy mechanism for this problem.

(b) $a^*(\boldsymbol{\tau})$ solves the first-order condition

$$\sum_{n=1}^{N}\left(\frac{\tau_n}{a}-1\right)=0\,.$$

Hence,

$$a^*(\boldsymbol{\tau})=\frac{\sum_{n=1}^{N}\tau_n}{N}\,.$$

It follows from the Groves-Clarke Theorem (Proposition 12 of the Notes) that

$$\begin{aligned}P_n(\boldsymbol{\tau})&=\sum_{j\neq n}\Big(\tau_j\ln\big(a^*(\boldsymbol{\tau})\big)-a^*(\boldsymbol{\tau})\Big)+h_n(\boldsymbol{\tau}_{-n})\\&=\sum_{j\neq n}\left(\tau_j\ln\left(\frac{\sum_{k=1}^{N}\tau_k}{N}\right)-\frac{\sum_{k=1}^{N}\tau_k}{N}\right)+h_n(\boldsymbol{\tau}_{-n})\,.\end{aligned}$$

If the mechanism is balanced, then $\sum_{n=1}^{N}P_n(\boldsymbol{\tau})\equiv 0$. This means we can differentiate it with respect to any τ and get a derivative that is identically equal to zero:

$$\begin{aligned}0\equiv\frac{\partial\sum_{n=1}^{N}P_n(\boldsymbol{\tau})}{\partial\tau_1}&=\frac{\partial}{\partial\tau_1}\left((N-1)\sum_{n=1}^{N}\Big(\tau_n\ln\big(a^*(\boldsymbol{\tau})\big)-a^*(\boldsymbol{\tau})\Big)+\sum_{n=1}^{N}h_n(\boldsymbol{\tau}_{-n})\right)\\&=(N-1)\ln\big(a^*(\boldsymbol{\tau})\big)+\sum_{n=2}^{N}\frac{\partial h_n(\boldsymbol{\tau}_{-n})}{\partial\tau_1}\end{aligned}$$

Unlike part (a), if we try the trick of continually differentiating this expression, we won't end up with a solvable differential equation. To be solvable, at the very least, we would need

$$0\equiv(N-1)\frac{\partial^{N-2}\ln\big(a^*(\boldsymbol{\tau})\big)}{\partial\tau_2\cdots\partial\tau_{N-1}}+\frac{\partial^{N-1}h_N(\boldsymbol{\tau}_{-N})}{\partial\tau_1\cdots\partial\tau_{N-1}}\,,$$

but that is clearly impossible (the left term will depend on τ_N while the right term won't). So no balanced mechanism exists.

(c) Clearly, the social planner would never set a at any values other than 0 or 1. She will choose the former if $\sum_{n=1}^{N}\tau_n/N<1$ and the latter if $\sum_{n=1}^{N}\tau_n/N>1$. Hence, by Groves-Clarke,

$$P_n(\boldsymbol{\tau})=\sum_{j\neq n}\big(\tau_j a^*(\boldsymbol{\tau})-a^*(\boldsymbol{\tau})\big)+h_n(\boldsymbol{\tau}_{-n})\,.$$

Because $a^*(\cdot)$ is not differentiable, we can't use the "derivative test" outlined above. Observe, however, that

$$\sum_{n=1}^{N} P_n(\boldsymbol{\tau}) = \begin{cases} \sum_{n=1}^{N} h_n(\boldsymbol{\tau}_{-n}) \text{ if } a^*(\boldsymbol{\tau}) = 0 \\ (N-1)\sum_{n=1}^{N}(\tau_n - 1) + \sum_{n=1}^{N} h_n(\boldsymbol{\tau}_{-n}) \text{ if } a^*(\boldsymbol{\tau}) = 1 \end{cases} .$$

Consequently, budget balancing would require

$$\sum_{n=1}^{N} \partial P_n(\boldsymbol{\tau})/\partial \tau_j = \begin{cases} \sum_{n\neq j} \partial h_n(\boldsymbol{\tau}_{-n})/\partial \tau_j = 0 \text{ if } a^*(\boldsymbol{\tau}) = 0 \\ N-1+\sum_{n\neq j} \partial h_n(\boldsymbol{\tau}_{-n})/\partial \tau_j = 0 \text{ if } a^*(\boldsymbol{\tau}) = 1 \end{cases} .$$

Because knowing whether $a = 0$ or 1 depends on knowing *all* the τs, while each $h_n(\cdot)$ can depend on only $N-1$ τs, it won't be feasible to solve this problem generally. That is, a balanced mechanism cannot be designed.

2. We solved for $a^*(\cdot)$ above. Recall that

$$P_n(\boldsymbol{\tau}) = \rho_n(\tau_n) - \frac{1}{N-1}\sum_{j\neq n} \rho_j(\tau_j) .$$

Hence, below, we give only the ρ functions.

(a) $\rho_n(\tau_n) = \mathbb{E}_{\boldsymbol{\tau}_{-n}} \left\{ \sum_{j\neq n} \lambda_j \left(\tau_j \frac{\sum_{k=1}^{N}\lambda_k\tau_k}{\sum_{k=1}^{N}\lambda_k} - \frac{1}{2}\left(\frac{\sum_{k=1}^{N}\lambda_k\tau_k}{\sum_{k=1}^{N}\lambda_k}\right)^2 \right) \right\} + h_n$.

(b) $\rho_n(\tau_n) = \mathbb{E}_{\boldsymbol{\tau}_{-n}} \left\{ \sum_{j\neq n} \left(\tau_j \ln\left(\frac{\sum_{k=1}^{N}\tau_k}{N}\right) - \frac{\sum_{k=1}^{N}\tau_k}{N} \right) \right\} + h_n$.

(c) $\rho_n(\tau_n) = \mathbb{E}_{\boldsymbol{\tau}_{-n}} \left\{ \sum_{j\neq n} \left(\tau_j a^*(\boldsymbol{\tau}) - a^*(\boldsymbol{\tau}) \right) \right\} + h_n$.

3. (a) Revealed preference tells us

$$\delta_i(\nu_i,\boldsymbol{\nu}_{-i})\nu_i - p_i(\nu_i,\boldsymbol{\nu}_{-i}) \geq \delta_i(\nu_i',\boldsymbol{\nu}_{-i})\nu_i - p_i(\nu_i',\boldsymbol{\nu}_{-i}) \text{ and} \quad (1)$$
$$\delta_i(\nu_i',\boldsymbol{\nu}_{-i})\nu_i' - p_i(\nu_i',\boldsymbol{\nu}_{-i}) \geq \delta_i(\nu_i,\boldsymbol{\nu}_{-i})\nu_i' - p_i(\nu_i,\boldsymbol{\nu}_{-i}) . \quad (2)$$

Suppose $\delta_i(\nu_i,\boldsymbol{\nu}_{-i}) = \delta_i(\nu_i',\boldsymbol{\nu}_{-i})$, then (1) requires $p_i(\nu_i,\boldsymbol{\nu}_{-i}) \leq p_i(\nu_i',\boldsymbol{\nu}_{-i})$, but (2) requires $p_i(\nu_i,\boldsymbol{\nu}_{-i}) \geq p_i(\nu_i',\boldsymbol{\nu}_{-i})$, which means they must be equal.

(b) This follows immediately from (a).

(c) We want $\delta_i(\nu_i,\boldsymbol{\nu}_{-i}) = 1$ if $\nu_i > v^{**}$ and we want $\delta_i(\nu_i,\boldsymbol{\nu}_{-i}) = 0$ if $\nu_i < v^{**}$. Consider $\nu_i = v^{**}+\varepsilon$ and $\nu_i = v^{**}-\varepsilon$, $\varepsilon > 0$. Then

$$v^{**} + \varepsilon - P_1 \geq -P_0 \text{ and} \quad (3)$$
$$v^{**} - \varepsilon - P_1 \leq -P_0 . \quad (4)$$

Because (3) and (4) hold for arbitrary ε, it must be that $P_1 - P_0 = v^{**}$.

(d) [Note: contrary to what was asked, it is not generally possible to have a balanced mechanism.[2]] Fix $P_0(\boldsymbol{\nu}_{-i}) \equiv T_0$, a constant. Let $P_1(\boldsymbol{\nu}_{-i}) = \max_{j \neq i} \nu_j + T_0$. Set

$$\delta_i(\nu_i, \boldsymbol{\nu}_{-i}) = \begin{cases} 1\,, & \text{if } \nu_i > \max_{j \neq i} \nu_j \\ 0\,, & \text{if } \nu_i < \max_{j \neq i} \nu_j \end{cases}$$

(recall ties can be ignored). To see that truth-telling is a dominant strategy, we only need to consider what happens if a lie changes δ from what it would be if an agent told the truth (recall part (a)):

$$\nu_i - \max_{j \neq i} \nu_j - T_0 \geq -T_0 \text{ (lie results in not getting good)} \tag{5}$$

$$-T_0 \geq \nu_i - \max_{j \neq i} \nu_j - T_0 \text{ (lie results in getting good).} \tag{6}$$

Expression (5) holds because if truth-telling gets you the good, then $\nu_i > \max_{j \neq i} \nu_j$. Expression (6) holds because if truth-telling means you don't get the good, then $\nu_i < \max_{j \neq i} \nu_j$.

[2] It would be possible to make it balanced if one of the agents was certain to have a value of 0; he could be the "seller" and collect all the money paid in.