Second-degree Price Discrimination with a Continuum of Types

Economics 201B

A MONOPOLIST WISHES to practice second-degree price discrimination via quantity discounts when there is a continuum of types.

Assumptions

- There is a continuum of types; specifically, a consumer's type, θ, is an element of the real interval $[\theta_0, \theta_1]$.
- Marginal cost is a constant, c.
- Because marginal cost is a constant, there is no loss of generality in considering a single consumer whose type is drawn from the distribution $F(\cdot) : [\theta_0, \theta_1]$.[1]
- Assume $F(\cdot)$ is differentiable. Let $f(\cdot)$ be the derivative (the density function). Assume $f(\theta) > 0$ for all $\theta \in [\theta_0, \theta_1]$.
- A type-θ consumer's utility is $b(x, \theta) - T$, where x is consumption of the good in question and T is the amount paid to the firm (the price of a package with x units in it).
- A Spence-Mirrlees condition holds:

$$\frac{\partial^2 b(x,\theta)}{\partial\theta\partial x} > 0\,. \tag{SM}$$

- A consumer knows his type. The monopolist knows only that θ is drawn from $[\theta_0, \theta_1]$ according to the distribution $F(\cdot)$. The monopolist also knows the functional form, $b(\cdot, \cdot)$.

Properties

P1. $\partial b/\partial x > 0$ for all $x > 0$. This follows because $\partial b/\partial x$ is the inverse demand curve and inverse demand curves are positive (at least over the relevant region of analysis).

[1] As will be obvious later, we could multiply the expression for profits below by N, the population, without effecting the solution to the maximization problem.

P2. $\partial b/\partial\theta > 0$ for all $x > 0$. Proof: Let $\theta' > \theta$. Observe[2]

$$\begin{aligned} b(x,\theta') - b(x,\theta) &= \int_0^x \frac{\partial b(z,\theta')}{\partial x} dz - \int_0^x \frac{\partial b(z,\theta)}{\partial x} dz \\ &= \int_0^x \left(\int_\theta^{\theta'} \frac{\partial b(z,t)}{\partial\theta\partial x} dt \right) dz \\ &> 0 \,. \end{aligned}$$

The last line follows because $x > 0$, $\theta' > \theta$, and the quantity being integrated (the integrand) is positive by (SM).

Notation

- The package intended for type θ has $x(\theta)$ units in it.
- The charge for an $x(\theta)$-unit package is $T(\theta)$.
- In equilibrium, a type-θ consumer purchases the package intended for him or her. Let $U(\theta)$ denote the equilibrium level he or she enjoys; that is,

$$U(\theta) \equiv b\big(x(\theta), \theta\big) - T(\theta) \,. \tag{1}$$

- A *feasible price discrimination scheme* is a pair of function $x(\cdot)$ and $T(\cdot)$ such that for every type θ, a type-θ consumer prefers an $x(\theta)$-unit package at price $T(\theta)$ to any other sized package and to not purchasing at all.[3] In other words, the scheme is feasible if it is *incentive compatible* and *individually rational* for each type to purchase the package intended for him or her.

Analysis

The monopolist wishes to design a feasible scheme that maximizes her profit, which is $T(\theta) - cx(\theta)$ if the customer proves to be type θ. This profit needs to be multiplied by the number (probability) of type-θ customers, which can be thought of as being $f(\theta)$. Hence, (expected) profit is

$$\Pi = \int_{\theta_0}^{\theta_1} \big(T(\theta) - cx(\theta)\big) f(\theta) d\theta \,. \tag{2}$$

Because the scheme must be feasible, it must satisfy both (IR) and (IC) constraints; that is, respectively, consumers must participate and they must buy the appropriate package. The (IR) constraints are simply

$$U(\theta) \geq 0 \quad \text{for all } \theta \in [\theta_0, \theta_1] \,. \tag{IR}$$

[2] This is the first of many times where we will employ the fundamental theorem of calculus to write the difference of a function in terms of an integral of its derivative.

[3] Because we could set $x(\theta) = 0$ for some types if necessary, we can view all consumers as purchasing (some perhaps "buying" 0) in equilibrium provided that none of them would get negative utility.

The (IC) constraints are tougher. It is clearly not feasible to check directly, for *each* of a continuum of θs, that each θ prefers his intended package to the *continuum* of alternative packages. We will need to find another way to check (IC).

Consider two arbitrary types, θ and θ'. One must be larger than the other and there is no loss of generality in assuming it is θ' (*i.e.,* take $\theta' > \theta$). Incentive compatibility tells us that θ' can't want to purchase the package intended for θ and *vice versa*. This insight yields:

$$\begin{aligned} U(\theta') &\geq b\big(x(\theta),\theta'\big) - T(\theta) \text{ and} \\ U(\theta) &\geq b\big(x(\theta'),\theta\big) - T(\theta')\,. \end{aligned} \tag{RP}$$

These two expressions have been labeled (RP) because this style of argumentation is known as a *revealed-preference* argument. Using expression (1) to substitute out for $T(\theta)$ and $T(\theta')$ in (RP), we have

$$U(\theta') \geq b\big(x(\theta),\theta'\big) - b\big(x(\theta),\theta\big) + U(\theta) = U(\theta) + \int_\theta^{\theta'} \frac{\partial b\big(x(\theta),t\big)}{\partial\theta} dt \quad \text{and}$$
$$U(\theta) \geq b\big(x(\theta'),\theta\big) - b\big(x(\theta'),\theta'\big) + U(\theta') = U(\theta') - \int_\theta^{\theta'} \frac{\partial b\big(x(\theta'),t\big)}{\partial\theta} dt\,.$$

Observe that we can combine the two expressions as

$$\int_\theta^{\theta'} \frac{\partial b\big(x(\theta'),t\big)}{\partial\theta} dt \geq U(\theta') - U(\theta) \geq \int_\theta^{\theta'} \frac{\partial b\big(x(\theta),t\big)}{\partial\theta} dt\,. \tag{3}$$

We will use expression (3) for two purposes in our analysis.

The first purpose will be to establish that $x(\cdot)$ is a non-decreasing function. To this end, ignore the middle term of (3) and observe that (3) implies

$$\int_\theta^{\theta'} \left(\frac{\partial b\big(x(\theta'),t\big)}{\partial\theta} - \frac{\partial b\big(x(\theta),t\big)}{\partial\theta} \right) dt \geq 0\,.$$

Employing the fundamental theorem of calculus, we have

$$\int_\theta^{\theta'} \left(\int_{x(\theta)}^{x(\theta')} \frac{\partial^2 b(z,t)}{\partial\theta\partial x} dz \right) dt \geq 0\,.$$

The integrand is positive by (SM) and $\theta' > \theta$ by assumption. Therefore, for the inequality to hold, $x(\theta') \geq x(\theta)$. Because monotonic functions are differentiable almost everywhere, this implies that $x(\cdot)$ is differentiable almost everywhere. We've proved:

Lemma 1. *A necessary condition for a second-degree pricing scheme to be feasible is that $x(\cdot)$ be non-decreasing.*

We will show, in a moment, that $x(\cdot)$ non-decreasing is also part of a sufficient condition for feasibility.

We now turn to expression (3)'s second purpose, namely to derive properties of $U(\cdot)$. Fix one end point of integration in (3) and let the other converge towards it. Because $x(\cdot)$ is continuous almost everywhere, it follows that $U(\cdot)$ is absolutely continuous with respect to the Lebesgue measure. It is, therefore, almost everywhere differentiable. To calculate its derivative rewrite (1) as

$$\frac{1}{\varepsilon}\int_{\theta}^{\theta+\varepsilon}\frac{\partial b\big(x(\theta+\varepsilon),t\big)}{\partial\theta}dt \geq \frac{U(\theta+\varepsilon)-U(\theta)}{\varepsilon} \geq \frac{1}{\varepsilon}\int_{\theta}^{\theta+\varepsilon}\frac{\partial b\big(x(\theta),t\big)}{\partial\theta}dt\,. \quad (4)$$

This holds for all $\varepsilon > 0$, so must hold in the limit as $\varepsilon \to 0$. The limit of the left-most term of (4) is the derivative of the left-most term of (3):

$$\frac{\partial b\big(x(\theta),\theta\big)}{\partial\theta} + \int_{\theta}^{\theta}\frac{\partial^2 b\big(x(\theta),t\big)}{\partial\theta\partial x}x'(\theta)dt = \frac{\partial b\big(x(\theta),\theta\big)}{\partial\theta}$$

almost everywhere (even though the integral from θ to θ of anything is zero, things can "go wrong" in the limit if $x'(\theta)$ doesn't exist). The limit of the right-most term of (4) is the derivative of the right-most term of (3): $\partial b\big(x(\theta),\theta\big)/\partial\theta$. So, in the limit, (4) becomes:

$$\frac{\partial b\big(x(\theta),\theta\big)}{\partial\theta} \geq \lim_{\varepsilon\to 0}\frac{U(\theta+\varepsilon)-U(\theta)}{\varepsilon} \geq \frac{\partial b\big(x(\theta),\theta\big)}{\partial\theta}\,.$$

But this means that

$$U'(\theta) = \frac{\partial b\big(x(\theta),\theta\big)}{\partial\theta}$$

almost everywhere. This in turn implies

$$U(\theta) = U_0 + \int_{\theta_0}^{\theta}\frac{\partial b\big(x(t),t\big)}{\partial\theta}dt\,. \quad (5)$$

From P2, we know $\partial b/\partial\theta \geq 0$ (and strictly grater if $x > 0$). Hence, $U(\theta) \geq U(\theta_0) = U_0$ for all θ. It follows that (IR) is met for all θ if it is met for θ_0; that is, if

$$U_0 \geq 0\,. \quad (\text{IR}')$$

Note, from expression (1), that $b\big(x(\theta_0),\theta_0\big) - T(\theta_0) = U_0$. Clearly, the firm can choose $T(\theta_0)$ so that (IR′) holds.

Our analysis to this point has shown that *necessary* conditions for a pricing scheme to be feasible are (i) that $x(\cdot)$ be non-decreasing (Lemma 1) and (ii) that expression (5) holds for some $U_0 \geq 0$. The next question is whether any scheme that satisfies these two properties is feasible; that is, are these conditions sufficient for a scheme to be feasible? The answer is yes, as shown by the following proposition. Before stating that proposition, note, by virtue of (1) and (5) that we can define $U(\cdot)$ and $T(\cdot)$ in terms of the primitive $b(\cdot,\cdot)$ given an $x(\cdot)$.

Proposition 1. *Any pricing scheme that satisfies* (5) *and in which* $x(\cdot)$ *is a non-decreasing function is feasible.*

Proof: Obviously, IR is no problem—we can always set $T(\theta_0)$ to satisfy it once we've found an IC scheme. So all we need to do is show that any scheme in which $x(\cdot)$ is non-decreasing and (5) holds is IC. This means we need for any pair θ and θ' that

$$U(\theta) \geq b\big(x(\theta'), \theta\big) - b\big(x(\theta'), \theta'\big) + U(\theta') . \tag{6}$$

Assume, first, that $\theta < \theta'$, so that $x(\theta') \geq x(t)$ for all $t \in [\theta, \theta')$. Using expression (5) and the fundamental theorem of calculus, expression (6) can be rewritten as

$$\int_{\theta}^{\theta'} \frac{\partial b\big(x(\theta'), t\big)}{\partial \theta} dt \geq \int_{\theta_0}^{\theta'} \frac{\partial b\big(x(t), t\big)}{\partial \theta} dt - \int_{\theta_0}^{\theta} \frac{\partial b\big(x(t), t\big)}{\partial \theta} dt = \int_{\theta}^{\theta'} \frac{\partial b\big(x(t), t\big)}{\partial \theta} dt ;$$

or, rearranging further, as

$$\int_{\theta}^{\theta'} \left(\frac{\partial b\big(x(\theta'), t\big)}{\partial \theta} - \frac{\partial b\big(x(t), t\big)}{\partial \theta} \right) dt \geq 0 . \tag{7}$$

That is, if (7) holds, then (6) holds. Because (i) $x(\theta') \geq x(t)$ for all $t \in [\theta, \theta')$ and (ii) condition (SM) holds, $\partial b\big(x(\theta'), t\big)/\partial\theta \geq \partial b\big(x(t), t\big)/\partial\theta$. Hence, (7) is true, which means (6) holds.

If $\theta > \theta'$, then observe we can rewrite (6) as

$$\int_{\theta'}^{\theta} \left(\frac{\partial b\big(x(t), t\big)}{\partial \theta} - \frac{\partial b\big(x(\theta), t\big)}{\partial \theta} \right) dt \leq 0 , \tag{8}$$

which holds true because, now, $x(\theta) \geq x(t)$ and (SM). But (8) implies (6).

We've thus shown that any scheme such that $x(\cdot)$ is non-decreasing and (5) holds is IC. ■

Using (1) and (5), we can write

$$T(\theta) = b\big(x(\theta), \theta\big) - U_0 - \int_{\theta_0}^{\theta} \frac{\partial b\big(x(t), t\big)}{\partial \theta} dt . \tag{9}$$

From Proposition 1, there is no loss of generality in restricting attention to tariffs of this form. Plugging this into (2), we have

$$\Pi = \int_{\theta_0}^{\theta_1} \left(b\big(x(\theta), \theta\big) - U_0 - \int_{\theta_0}^{\theta} \frac{\partial b\big(x(t), t\big)}{\partial \theta} dt - cx(\theta) \right) f(\theta) d\theta . \tag{10}$$

The firm's problem can be stated as choosing $x(\cdot)$ to maximize (10) subject to the constraint that $U_0 \geq 0$ (*i.e.,* IR) and subject to $x(\cdot)$ be non-decreasing. The former constraint is clearly binding as the firm's profit is decreasing in U_0. Hence, we know the firm will set $U_0 = 0$. For the moment, we will treat the constraint that $x(\cdot)$ be non-decreasing as a non-binding constraint.

Were it not for the integral within the integral in (10), maximization would be easy—just maximize the expression pointwise. Fortunately, using integration by parts, we can get rid of that middle integral. Recall that $d\bigl(1-F(\theta)\bigr)/d\theta = -f(\theta)$. Hence,

$$\begin{aligned}\int_{\theta_0}^{\theta_1}\left(\int_{\theta_0}^{\theta}\frac{\partial b\bigl(x(t),t\bigr)}{\partial\theta}dt\right)f(\theta)d\theta &= -\bigl(1-F(\theta)\bigr)\int_{\theta_0}^{\theta}\frac{\partial b\bigl(x(t),t\bigr)}{\partial\theta}dt\Bigg|_{\theta_0}^{\theta_1}\\ &\quad+\int_{\theta_0}^{\theta_1}\bigl(1-F(\theta)\bigr)\frac{\partial b\bigl(x(\theta),\theta\bigr)}{\partial\theta}d\theta\\ &= \int_{\theta_0}^{\theta_1}\bigl(1-F(\theta)\bigr)\frac{\partial b\bigl(x(\theta),\theta\bigr)}{\partial\theta}d\theta\,.\end{aligned}$$

(Recall $1-F(\theta_1)=0$—the probability of a type greater than θ_1 is zero—and that the integral from θ_0 to θ_0 is zero.)

Substituting that back into (10), we have

$$\begin{aligned}\Pi &= \int_{\theta_0}^{\theta_1}\left(\Bigl[b\bigl(x(\theta),\theta\bigr)-cx(\theta)\Bigr]f(\theta)-\bigl(1-F(\theta)\bigr)\frac{\partial b\bigl(x(\theta),\theta\bigr)}{\partial\theta}\right)d\theta\\ &= \int_{\theta_0}^{\theta_1}\left(b\bigl(x(\theta),\theta\bigr)-cx(\theta)-\frac{1-F(\theta)}{f(\theta)}\frac{\partial b\bigl(x(\theta),\theta\bigr)}{\partial\theta}\right)f(\theta)d\theta\,. \qquad (11)\end{aligned}$$

Pointwise optimization with respect to $x(\cdot)$ yields the first-order conditions:

$$\frac{\partial b\bigl(x(\theta),\theta\bigr)}{\partial x}-\left(c+\frac{1-F(\theta)}{f(\theta)}\frac{\partial^2 b\bigl(x(\theta),\theta\bigr)}{\partial\theta\partial x}\right)=0\,. \qquad (12)$$

Provided the $x(\cdot)$ that solves these equations is non-decreasing, we're done. At this level of generality, we can't prove that it is; but, as we will see, in most problems it is.

Observe that the first term in (12) is inverse demand for type θ. We know that *efficiency* dictates that inverse demand be equated to marginal cost, c. Here, because we know the cross partial is positive by (SM), inverse demand is being equated to an amount that is greater than c (except for the highest type, θ_1; recall $1-F(\theta_1)=0$). Because demand curves slope down, this means that $x(\theta)<x^*(\theta)$ for all $\theta<\theta_1$. In other words, we have distortion downward except at the top. At the top, as usual, we have efficiency. We earlier saw that $U(\theta)\geq U(\theta_0)$ and strictly greater if $x(\theta)>0$. In other words, while the bottom type earns no information rent, all other types do earn a rent (at least if they purchase a positive amount). Finally, consider the term

$$\frac{1-F(\theta)}{f(\theta)}\frac{\partial b\bigl(x(\theta),\theta\bigr)}{\partial\theta}$$

in expression (11). The partial derivative $\partial b/\partial\theta$ is the marginal contribution to *all* types greater than θ's equilibrium consumer surplus (information rent) from selling $x(\theta)$ units to type θ. How costly such a contribution is to the monopolist depends on how "many" consumers there are of types greater than θ (*i.e.,* $1-F(\theta)$) relative to how "many" type θ there are (*i.e.,* $f(\theta)$). The word "many" is in quotes because we are, of course, working with probabilities and, moreover, because we are being somewhat loose in calling $f(\theta)$ the probability of drawing a type θ. The cross-partial derivative in (12) is the increment to the marginal contribution to the information rents of all types greater than θ. Appropriately weighted (*i.e.,* by $\big(1-F(\theta)\big)/f(\theta)$), it is, from the monopolist's perspective, an additional component to the marginal cost of increasing $x(\theta)$.

To summarize: Let $\hat{x}(\theta)$ be the solution to expression (12). If $\hat{x}(\cdot)$ is a non-decreasing function, then it is the profit-maximizing package-size function. The optimal tariff for an $\hat{x}(\theta)$-unit package is

$$T(\theta) = b\big(\hat{x}(\theta),\theta\big) - \int_{\theta_0}^{\theta} \frac{\partial b\big(\hat{x}(t),t\big)}{\partial \theta} dt \,.$$

Example

Suppose that $\theta_0 > c \geq 0$, that $F(\cdot)$ is the uniform distribution on $[\theta_0, \theta_1]$, and that the inverse demand of type θ is $\theta - x$. Recall that inverse demand is $\partial b/\partial x$. Hence, integrating partial demand with respect to quantity from 0 to x units yields benefit:

$$b(x,\theta) = \int_0^x (\theta - z)dz = \theta x - \frac{1}{2}x^2 \,.$$

Observe that (SM) is met.

Given the uniformity assumption,

$$F(\theta) = \frac{\theta - \theta_0}{\theta_1 - \theta_0} \text{ and } 1 - F(\theta) = \frac{\theta_1 - \theta}{\theta_1 - \theta_0} \,.$$

In addition,

$$f(\theta) = \frac{1}{\theta_1 - \theta_0} \,.$$

Expression (12) becomes

$$\theta - x(\theta) - \Big(c + (\theta_1 - \theta) \times 1\Big) = 0 \,.$$

Solving,

$$\hat{x}(\theta) = \begin{cases} 2\theta - \theta_1 - c \,, \text{ if } \theta > (\theta_1 + c)/2 \\ 0 \,, \text{ if } \theta \leq (\theta_1 + c)/2 \end{cases} \,.$$

$\hat{x}(\cdot)$ is clearly non-decreasing. Define

$$\bar{\theta} = \begin{cases} \theta_0 \,, \text{ if } \theta_0 > (\theta_1 + c)/2 \\ (\theta_1 + c)/2 \,, \text{ if } \theta_0 \leq (\theta_1 + c)/2 \end{cases} \,.$$

Hence, $T(\theta) = 0$ for $\theta < \bar{\theta}$. For $\theta \geq \bar{\theta}$, we have

$$\begin{aligned}
T(\theta) &= \theta\hat{x}(\theta) - \frac{1}{2}\hat{x}(\theta)^2 - \int_{\theta_0}^{\theta} \hat{x}(t)dt \\
&= \theta(2\theta - \theta_1 - c) - \frac{1}{2}(2\theta - \theta_1 - c)^2 - \int_{\bar{\theta}}^{\theta} (2t - \theta_1 - c)dt \\
&= \theta(2\theta - \theta_1 - c) - \frac{1}{2}(2\theta - \theta_1 - c)^2 - \left(\theta^2 - (\theta_1 + c)\theta + \bar{\theta}^2\right) \\
&= \begin{cases} 2(\theta_1 + c)\theta - \theta^2 - \theta_0^2 - (\theta_1 + c)^2/2\,, \text{ if } \bar{\theta} = \theta_0 \\ 4\bar{\theta}\theta - \theta^2 - 3\bar{\theta}^2\,, \text{ if } \bar{\theta} > \theta_0 \end{cases},
\end{aligned}$$

where the last line follows by making the substitution $2\bar{\theta} = \theta_1 + c$.