Some Notes on Product Differentiation

MBA 299 — Spring 2003

1 Introduction

These notes explore the product-differentiation model in greater depth than the material covered in class.

2 The Model

Our analysis of the product-differentiation model requires a few steps:

1. Discussion of consumers’ preferences.

2. Going from these preferences to market demand in the case of undifferentiated products.

3. Reviewing why undifferentiated products leads to the Bertrand trap.

4. Derivation of firm-specific demands when products are maximally differentiated.

5. Derivation of equilibrium of pricing game given maximally differentiated products.

2.1 Consumer preferences

The differentiation of the good or service in question can be viewed in terms of some scale. Since the scale is arbitrary, we’re free to assign 0 to one extreme and 1 to the other extreme (as you’ll see this particular scaling is very convenient). For example, the product could be wine and the scale could measure the sweetness of the wine. Zero would correspond to the most dry wine and one would correspond to the sweetest wine. A wine with a score—*or location*—of \( \frac{1}{2} \) would have a sweetness midway between the most dry and sweetest. Or, as a second example, consider a one-mile street viewed from a fixed perspective: Zero is the left end of the street and one is the right end. A location of \( \frac{1}{3} \) would be a location one-third of a mile from the left end of the street. Let \( y \) denote the score or location of the product (e.g., \( y = \frac{4}{5} \) would be a wine with a sweetness score of .8 or a store located one-fifth of a mile from the right end of the street).

Consumers have preferences over the location of the product or service in this product space. In particular, let a given consumer’s benefit from consuming
a product at location $y$ be

$$B_L(y) = V - t \times (L - y)^2,$$

where $L$ is this particular consumer’s most-preferred location (e.g., she would most prefer to consume a wine with a sweetness score of $L$); $V$ is her value from getting the product at her most-preferred location (e.g., getting a wine with a sweetness score of $L$); $t$ is a measure of her disutility from getting a product at a location other than her most-preferred location. The squared term, $(L - y)^2$, reflects how far the product she consumes is from her most-preferred product. Observe that her utility is decreasing in that distance; that is, the farther away the product is from what she most likes, the less she values it.

Different consumers have different $L$s (e.g., some people like dry wines more and some like sweet wines more). Assume there are $N$ total people in the market. Assume that their $L$s are distributed uniformly between 0 and 1. This last assumption means, for example, that $\frac{1}{5}$ of the population most prefer a wine drier than $\frac{4}{5}$ or that $\frac{1}{3}$ of the population most prefer a wine drier than $\frac{2}{3}$. In general, one $L$th of the population most prefer a product with a location, $y$, of $L$ or less.

In what follows, we assume that each consumer wants to buy at most one unit of the product and that the product is indivisible (e.g., you buy a bottle of wine or you don’t).

Also assume that regardless of its location, each unit of the good or service costs the firm producing it $c$. In what follows, assume the $V > c + \frac{5}{4}t$.\(^1\) We will also restrict attention to the case where there are only two firms, $X$ and $Z$.\(^2\) Finally, other than stated above, all the assumptions of the Bertrand model continue to hold.

### 2.2 Market demand with undifferentiated products

Suppose the firms, $X$ and $Z$, both locate their products at 0. That is, $y_X = y_Z = 0$. Consider a price of $p$ for the product. Who will buy it? Well, a consumer will buy it only if her consumer surplus from buying it is at least zero. Her consumer surplus is her benefit minus the price:

$$B_L(0) - p = V - t L^2 - p$$

(recall $y = 0$). Observe that consumer surplus is decreasing in $L$; that is, a consumer whose most-preferred location is farther from zero gets less consumer surplus than one whose most-preferred location is closer.

To derive market demand, we need to find the consumer who is indifferent between buying and not buying. By definition, that consumer would earn 0

\(^1\)Clearly, $V > c$, otherwise the market wouldn’t exist. Having $V > c + \frac{5}{4}t$ ensures that, in equilibrium, all consumers want to buy. If $c < V < c + \frac{5}{4}t$, then the equilibrium would have some consumers priced out of the market. Although this would complicate the analysis, it wouldn’t substantially change the results, so we’ll consider only the $V > c + \frac{5}{4}t$ case.

\(^2\)In the lecture notes, the two firms were called 0 and 1, respectively. However, it will be clearer in the analytics if they’re denoted by letters rather than numbers.
consumer surplus if she buys. Hence, the marginal consumer—call her location $L_M$—is the one for which

$$V - tL^2 - p = 0.$$  \hspace{1cm}

Solving for $L$, we find

$$L_M = \sqrt{\frac{V - p}{t}}.$$  \hspace{1cm}

Since consumer surplus is decreasing in $L$ (given that the product is located at 0), a consumer with an $L < L_M$ will strictly prefer to buy and a consumer with an $L > L_M$ will strictly prefer not to buy. What proportion of consumers have an $L < L_M$? Well, since consumer preferences are uniformly distributed, $L_M \times 100\%$ of the population have an $L < L_M$. Given that there are $N$ total people, this means that the number of consumers who want to buy at $p$—and hence the number of units that are demanded at $p$—is $NL_M$ or

$$N\sqrt{\frac{V - p}{t}}.$$  \hspace{1cm}

Since we’ve expressed the units demanded as a function of price, $p$, this last expression is the market demand curve; that is,

$$D(p) = N\sqrt{\frac{V - p}{t}}.$$

2.3 Bertrand competition with undifferentiated products

Since the products are undifferentiated, all the assumptions of the Bertrand model hold. Hence each firm’s firm-specific demand is

$$D_i(p_i, p_j) = \begin{cases} 0, & \text{if } p_i > p_j \\ \frac{N}{2}\sqrt{\frac{V - p_i}{t}}, & \text{if } p_i = p_j \\ N\sqrt{\frac{V - p_i}{t}}, & \text{if } p_i < p_j \end{cases},$$

where $i$ indexes the firm in question and $j$ denotes its rival. Given this demand structure, it’s readily shown that the Nash equilibrium is $p_X = p_Z = c$; that is, both firms price at marginal cost and earn zero economic profit.

2.4 Firm-specific demand with differentiated products

Now suppose that firm $X$ locates at 0 and firm $Z$ locates at 1 (e.g., $X$ produces an extremely dry wine and $Z$ produces an extremely sweet wine). We again need to determine firm-specific demand. There are two conditions that must be satisfied if a given consumer located at $L$ is to buy from a given firm: (i) she must get non-negative consumer surplus if she does; and (ii) she must get at least as much consumer surplus buying from the given firm as from its rival. Let’s conjecture that, in equilibrium, only the second condition is relevant; that
The Model

is, in equilibrium, all consumers buy from one or the other firm. We will, of course, need to verify this conjecture is true at the end, but at the moment we can simply accept it. Under this conjecture, the marginal consumer, $L_M$, is now the consumer who gets equal consumer surplus from the two firms:

$$B_L(0) - p_X = B_L(1) - p_Z$$

or

$$V - tL^2 - p_X = V - t(L - 1)^2 - p_Z.$$  

Observe, first, that as $L$ gets smaller the left-hand side increases (since $L$ is getting closer to 0) and the right-hand side decreases (since $L$ is getting farther from 1). Conversely, if $L$ gets larger, then the left-hand side decreases (since $L$ is getting farther from 0) and the right-hand side increases (since $L$ is getting closer to 1). This means that if $L_M$ is indifferent between $X$ and $Z$, then all consumers with $L < L_M$ strictly prefer to buy from $X$ and all consumers with $L > L_M$ strictly prefer to buy from $Z$. To figure out what $L_M$ is, we need to solve this last equation for $L$. First, expand the quadratic:

$$V - tL^2 - p_X = V - t(L^2 - 2L + 1) - p_Z.$$  

Then cancel like terms:

$$-p_X = 2tL - t - p_Z.$$  

Solving for $L$:

$$L_M = \frac{t - p_X + p_Z}{2t}.$$  

Since all consumers with $L < L_M$ will buy from $X$, $X$ gets $L_M \times 100\%$ proportion of the market; which leaves the remaining $(1 - L_M) \times 100\%$ proportion to firm $Z$. Consequently, $X$ sells $NL_M$ units and $Z$ sells $N(1 - L_M)$ units. Since these numbers of units are functions of the prices, we have, by substituting for $L_M$, the firm-specific demands:

$$D_X(p_X, p_Z) = N \frac{t - p_X + p_Z}{2t}$$

and

$$D_Z(p_Z, p_X) = N \left(1 - \frac{t - p_X + p_Z}{2t}\right) = N \frac{t - p_Z + p_X}{2t}.$$  

Note that both demands slope down in own-firm price.

2.5 Equilibrium

To determine the equilibrium prices, we must first determine each firm’s best responses and, then, look for mutual best responses. Given a belief that its rival
will charge \( p_j \), firm \( i \) (\( i = X \) or \( Z \)) wants to choose the price that will maximize its profit. That is, its best response to \( p_j \) is the price, \( p_i \), that maximizes its profit. Firm \( i \)'s profit is

\[
(p_i - c) \times D_i(p_i, p_j),
\]

since it makes a per-unit profit of \( p_i - c \) and it sells \( D_i(p_i, p_j) \) units. To find the \( p_i \) that maximizes profit, observe that firm \( i \)'s marginal benefit from increasing its price is the increase in price, \( \Delta p \), times the units it sells, approximately \( D_i(p_i, p_j) \). The marginal cost from doing so is that it sells \( -\Delta D \) fewer units, which costs it approximately \( p_i - c \) per unit. Observe that

\[
\Delta D = \frac{\Delta D}{\Delta p} \times \Delta p.
\]

Moreover, that fraction—the change in demand per unit change in price—is just the slope of the firm-specific demand curve; i.e.,

\[
\frac{\Delta D}{\Delta p} = \frac{-N}{2t}.
\]

Putting marginal benefit on the left and marginal cost on the right and setting them equal, as required to maximize profit, yields

\[
D_i(p_i, p_j) \times \Delta p_i = (p_i - c) \times (-\Delta D_i)
\]

\[
N \frac{t - p_i + p_j}{2t} \Delta p_i = (p_i - c) \frac{-\Delta D_i}{\Delta p_i} \Delta p_i.
\]

Cancelling the \( \Delta p_i \) from both sides and substituting for \( \frac{\Delta D_i}{\Delta p_i} \) yields

\[
N \frac{t - p_i + p_j}{2t} = (p_i - c) \frac{N}{2t}.
\]

Cancelling like terms from both sides and solving for \( p_i \) yields

\[
p_i = \frac{t + p_j + c}{2}.
\]

That is, we’ve found that \( X \)'s best response to \( p_Z \) is

\[
p_X = \frac{t + p_Z + c}{2}
\]

and that \( Z \)'s best response to \( p_X \) is

\[
p_Z = \frac{t + p_X + c}{2}.
\]

In equilibrium, both firms must be playing best responses (i.e., we must have mutual best responses). That means that, in equilibrium, both of the last two equations must hold true. Solving this system of two equations in two unknowns yields

\[
p_X = p_Z = c + t.
\]
Hence, the equilibrium price is greater than marginal cost, which means that product differentiation does, indeed, allow firms to avoid the Bertrand trap. Note, too, that since prices are equal, \( L_M = \frac{1}{2} \); hence, each firm’s profit is \( \frac{N^2}{4} \).

We’re not quite done, however. Recall we only *conjectured* that everyone would buy in equilibrium. We need to verify that conjecture now. To do this, it suffices to make sure that the marginal consumer would want to buy. Since, by definition, she’s indifferent between \( X \) and \( Z \), we may as well assume she buys from \( X \). Her consumer surplus if she does so is

\[
B_{\frac{1}{2}}(0) - p_X = V - t \left( \frac{1}{2} \right)^2 - (c + t)
\]

\[
= V - c - \frac{5}{4}t
\]

but by assumption that’s positive—our conjecture is correct.