UNOBSERVED INVESTMENT, ENDogenous QUALITY, AND TRADE∗

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ABSTRACT

A seller can make investments that affect a tradable asset’s future returns. The potential buyer of the asset cannot observe the seller’s investment prior to trade, nor does he receive any signal of it, nor can he verify it in any way after trade. Despite this severe moral hazard problem, this paper shows the seller will invest with positive probability in equilibrium. The outcome of the game is sensitive to the distribution of bargaining power between the parties, with a holdup problem existing if the buyer has the bargaining power. A consequence of the holdup problem is welfare-reducing distortions in investment level, but not necessarily a reduction in the expected amount invested vis-à-vis the situation without holdup.

Keywords: unobserved investment, endogenous quality, trade, holdup

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1 INTRODUCTION

Consider the following situation. One actor (e.g., an inventor or the owner/entrepreneur of a small firm) invests in an asset (a new product, the firm, etc.) that she may ultimately wish to sell to another actor (e.g., a large firm). Trade, if it occurs, does so before the value of the asset is realized. Moreover, the return to the asset is greater in the potential buyer’s hands than it is if it remains in the seller’s hands. Critically, the distribution of returns to either actor depends on the seller’s initial investment. Because the buyer cannot observe the seller’s investment, a moral hazard problem exists; in particular, an equilibrium in which the seller invests a fixed amount and trade always occurs cannot exist because sure-to-happen trade destroys the seller’s incentives to invest \textit{ex ante}. On the other hand, if no trade is anticipated, the seller will invest for her own benefit; but then it would be incredible that trade wouldn’t occur if the opportunity to trade later arose. The questions are then to what extent does the seller invest, with what frequency is there exchange, and what mechanism might the parties employ to facilitate exchange?

In some cases, a possible mechanism might be a revenue-sharing contract. However, one can conceive of contexts in which the returns generated by the asset are unverifiable, indeed even unobservable to the seller after trade. For instance, a large firm could use accounting tricks to obscure the returns ultimately generated by the purchased invention or small firm. Or the returns could be private benefits (e.g., the seller is a home builder, the buyer a would-be home owner). Furthermore, the reason the asset is more valuable in the buyer’s hands could be because of subsequent investments made by the buyer and, thus, a revenue-sharing agreement may be sub-optimal because of the effect it has on the buyer’s investments. Hence, it seems worth considering contexts in which revenue-sharing contracts are infeasible; that is, in which the seller’s payment cannot be tied to the realized return from the asset. This is the situation considered in this paper.

Most of this paper considers the case in which the buyer and seller meet only once the seller has invested. Consequently, no contract is in force at the time the seller invests. This, for instance, corresponds to most situations in which a startup undertakes investment with the possible objective of being acquired later (e.g., a small software startup may entertain dreams of being acquired by Google or Microsoft). Only later (Section 6) and briefly is the situation in which the parties can contract prior to investment considered.

When no contract governs the relation between buyer and seller prior to the seller’s investment, the outcome of the game between them depends on the allocation of bargaining power. Here, the two extremes are considered: either the buyer can make a take-it-or-leave-it offer to the seller or the seller can make such an offer to the buyer. When the buyer has all the bargaining power, a holdup problem arises.\footnote{For classic analyses of the holdup problem, see Williamson (1976), Tirole (1986), and Klein (1988).} Because of the underlying moral hazard problem,
the threat of holdup can lead to either more investment in expectation or less investment in expectation *vis-à-vis* the situation without holdup; that is, the situation when the seller has the bargaining power. Even though investment can be greater when holdup is an issue, welfare will prove to be lower.

The reason welfare is lower is that, to avoid being completely held up, the seller must randomize over her investment levels. In contrast, when she has the bargaining power, she can just play the optimal level of investment given the constraints imposed by the moral hazard problem. That the first equilibrium involves mixing by the investing party is reminiscent of the results in Gul (2001) and Lau (2008), which also consider holdup in the context of unobservable investment. There are, however, important differences between this paper and those earlier ones. For instance, in those earlier papers, the investment pays off only if trade occurs; here, in contrast, the investment yields at least some returns even if trade fails to occur. Another difference is that, in those earlier papers, there was no direct benefit to the non-investing party from the investment, whereas here the non-investing party is a direct beneficiary. A further difference is that, in those earlier papers, it was assumed there could still be gains to trade even if the investing party failed to invest. As noted by Gul, the existence of an equilibrium with a positive probability of investment in the Gul or Lau setting is sensitive to this assumption. Here, in contrast, the analysis encompasses both the cases when trade is valuable absent investment and when it is valueless absent investment; the existence of a mixed-strategy equilibrium here does not depend on which case is considered.

Because the buyer is a direct beneficiary of the seller’s investment, which he cannot observe, the situation considered here is also reminiscent of a hidden-action principal-agent problem. In particular, when contracting occurs after the seller invests, there is a connection between the problem considered here and the issues of renegotiation in agency considered by Fudenberg and Tirole (1990) and Ma (1994). Similar to Fudenberg and Tirole’s results, this paper finds that when the principal (buyer) has the bargaining power, the agent (seller) must mix in equilibrium; and similar to Ma’s results, this paper finds that when the agent (seller) has the bargaining power, the agent need not mix in equilibrium. Again, important differences exist between this paper and this earlier literature. Unlike here, in the principal-agent problem, there is an initial contract (*i.e.*, prior to the agent taking action), which affects which contracts will be offered in renegotiation (*i.e.*, after the agent takes action). Also, unlike the principal-agent problem, here there is no verifiable signal correlated with the agent’s action upon which to base a contract.

As the analysis below shows, the seller will have incentives to invest only if there is a positive probability along the equilibrium path that she will end up in possession of the asset; that is, there must be a positive probability of no trade. That the possibility of no trade is needed as an incentive device bears similarity to the solution of double moral hazard problems considered by Demski and Sappington (1991) and subsequent authors. In Demski and Sappington’s paper, it is impossible to contract directly on the seller’s investment, so the threat of leaving her with the asset should she invest too little can provide her incen-
tives. For these incentives to work, however, the buyer must observe the seller’s investment. As noted, here the buyer cannot observe the seller’s investment. Hence, here, the incentive to invest comes from the fact the seller may wind up with the asset, whereas, in this other literature, the seller’s incentive to invest comes from a desire to avoid winding up with the asset. Indeed, in this other literature, the seller never winds up with the asset on the equilibrium path.

Instances in which contracting is possible prior to the seller’s investment are when a principal hires an agent to invest on his behalf. The agent (seller) could be an independent contractor, a research scientist hired by the principal, or an existing unit within a firm. The contract allows the principal to leave the asset (e.g., research project) in the agent’s hands. Possible examples of such a situation are when engineers from Xerox Parc left to found companies (e.g., Metaphor Computing Systems and Adobe) that built on research done at Xerox Parc. Although \textit{ex post} such departures may have raised questions about the wisdom of Xerox’s management, \textit{ex ante} committing to the possibility of such departures could have been necessary to induce the engineers to expend effort in the first place. The analysis in Section 6 shows that optimal incentives in this case consist of fixed compensation with the possibility of the agent’s (seller’s) being able to keep the asset.

The model is introduced in the next section. Section 3 proves that no pure-strategy equilibrium can exist when the buyer has the bargaining power. It is also shown there that there cannot be a pure-strategy equilibrium in which the seller invests a positive amount. Section 4 characterizes the properties that a contract offered post investment must satisfy in equilibrium. As will be seen, some of these properties also define the optimal \textit{ex ante} contract. Section 5 solves the game for the situation in which the buyer makes a take-it-or-leave-it offer after investment and the one in which the seller does. The two situations are briefly contrasted. As noted, Section 6 considers contracting \textit{ex ante}. Section 7 provides additional discussion and conclusions.

2 Model

There are two risk-neutral parties, a seller and a buyer. The seller owns an asset (an invention, small firm, etc.) in which she can invest. Let $I \in [0, \infty)$ denote her investment. After investing, an opportunity arises in which the seller can sell the asset to a buyer. To motivate trade, assume the buyer, if he acquires the asset, can take a subsequent action, $b \in B \subseteq [0, \bar{b}]$, that affects the asset’s return (to him at least). Assume $0 \in B$ and $B \setminus \{0\} \neq \emptyset$. Finally, the asset yields a return, $r$, to its then owner, where the return depends on investments made in it. Note the realization of $r$ occurs after the point at which exchange can occur. The payoffs to the buyer and seller—\textit{ignoring} transfers—are, respectively,

$$U_B = \begin{cases} 0, & \text{if no exchange} \\ r - b, & \text{if exchange} \end{cases} \quad \text{and} \quad U_S = \begin{cases} r - I, & \text{if no exchange} \\ -I, & \text{if exchange} \end{cases},$$

where “exchange” means ownership of the asset was passed to the buyer. Not surprisingly, the buyer takes no action if there isn’t exchange.
Given investment $I$ and action $b$, the return $r$ has an expected value $R(I, b)$. The following assumption is maintained throughout the analysis:

**Assumption 1.** The expected return function, $R : \mathbb{R}_+ \times B \to \mathbb{R}$, has the following properties:

(i) For all $b \in B$, $R(\cdot, b) : \mathbb{R}_+ \to \mathbb{R}$ is a twice continuously differentiable, strictly increasing, and strictly concave function;

(ii) For any $b \in B$, there exists an $\hat{I}(b) < \infty$ such that $\partial R(I, b)/\partial I < 1$ if $I > \hat{I}(b)$;

(iii) $\partial R(0, 0)/\partial I > 1$; and

(iv) For any $I \in \mathbb{R}_+$, $\arg\max_{b \in B} R(I, b) - b$ exists and, if $I > 0$, it is a subset of $B \setminus \{0\}$.

Define $V(I) = \max_{b \in B} R(I, b) - b$. Finally, assume:

(v) There exists an $I^* > 0$ such that $I^* = \arg\max_I V(I) - I$.

It follows immediately from Assumption 1(iv) that $V(I) > R(I, 0)$ for all $I > 0$; that is, the asset is always more valuable in the buyer’s hands than the seller’s given investment. The asset may or may not be more valuable in the buyer’s hands absent investment. Note that $I^*$ is the first-best (welfare-maximizing) level of investment.

The expected return to the seller if she retains ownership is $R(I, 0)$. As this quantity plays an important role in what follows, define the function $\rho : \mathbb{R}_+ \to \mathbb{R}$ as $\rho(I) = R(I, 0)$. By Assumption 1(i), $\rho(\cdot)$ is a strictly increasing function. It hence has an inverse: Let $\iota(\cdot)$ denote that inverse; that is, $\iota(\rho(I)) \equiv I$ and $\rho(\iota(R)) \equiv R$. Assumption 1(i) entails that $\iota(\cdot)$ is twice continuously differentiable, strictly increasing, and strictly convex.

Assumption 1(i)–(iii) ensure that the program

$$\max_I \rho(I) - I \tag{1}$$

has a unique, interior solution. Call it $\hat{I}$. Observe $\hat{I}$ is what the seller would invest were later trade infeasible (i.e., under autarky).

Program (1) is equivalent to the program

$$\max_R R - \iota(R) \tag{2}$$

In particular, Assumption 1(i)–(iii) ensure that program (2) has a unique, interior solution. Call it $\hat{R}$. Of course, $\hat{R} = \rho(\hat{I})$ and $\hat{I} = \iota(\hat{R})$. By Assumption 1(i), the function being maximized in (2) is concave. An implication of that is

$$1 \geq \iota'(R) \tag{3}$$

for all $R \in [0, \hat{R}]$ (with strict inequality for $R < \hat{R}$).
3 Preliminary Analysis

Consider first the situation in which no contract exists between buyer and seller at the time the seller invests. Buyer and seller meet after the seller’s investment and means of exchange established. Although there are many possible bargaining games that could be considered, attention is limited here to take-it-or-leave-it (TIOLI) bargaining. That is, either the buyer or the seller has the ability to make a TIOLI offer to the other, where an offer consists of a contract or mechanism for the parties to play.

The purpose of this section is to motivate the following analysis by showing that no pure-strategy equilibrium can exist.

**Proposition 1.** If buyer has the ability to make a take-it-or-leave-it offer to the seller, then no pure-strategy equilibrium exists. The same is true if the seller can make a take-it-or-leave-it offer unless welfare given exchange and no investment exceeds maximum possible welfare given no exchange (i.e., unless \( V(0) \geq \hat{R} - \iota(\hat{R}) \)).

**Proof:** Suppose, first, it is the buyer who can make a TIOLI offer to the seller. The proof is by contradiction. Let \( I > 0 \) be the seller’s investment given her pure strategy. Equivalently, let \( R = \rho(I) \) be the expected return absent trade. Because her investment is sunk, the seller will accept an offer from the buyer if and only if her expected payment is at least \( R \). The buyer, having the bargaining power, will not offer more than necessary; hence, his best response is to offer \( R \). Suppose the seller deviated to \( I = 0 \). Her expected profit would go from \( R - \iota(R) \) to \( R \). Hence, there is no pure-strategy equilibrium in which the seller invests \( I > 0 \). Suppose she invests \( I = 0 \). The buyer’s best response is \( \rho(0) \); that is, the seller’s profit is \( \rho(0) - \iota(\rho(0)) \). But \( \hat{R} > \rho(0) \) maximizes \( R - \iota(R) \). The seller would again deviate.

Now suppose it is the seller who makes a TIOLI offer to the buyer. The proof is again by contradiction. If the buyer expects \( I > 0 \), he is willing to pay \( V(I) \). This is, thus, the price the seller will charge. But again the seller has no incentive to actually invest. If the buyer expects \( I = 0 \), he is willing to pay \( V(0) \). It is readily seen that a pure-strategy equilibrium exists in which the seller invests nothing and the buyer is willing to pay at most \( V(0) \) if and only if \( V(0) \geq \hat{R} - \iota(\hat{R}) \).

An immediate corollary is

**Corollary 1.** The first-best outcome is not attainable as an equilibrium if the terms of trade are set after the seller invests.

**Proof:** The first best requires the seller play \( I^* > 0 \) as a pure strategy. No such pure-strategy equilibrium exists.
4 MECHANISM DESIGN

After the seller has sunk her investment, a means of (possibly) arranging trade is a mechanism.

4.1 CHARACTERIZATION

In light of the revelation principle, attention can be restricted to direct-revelation mechanisms. Because the seller is the only actor with private information, a mechanism must induce her to reveal her investment (equivalently, the expected value of \( r \) should trade not occur). Let \( R \) denote the seller’s expected return if she retains ownership. Consistent with the mechanism-design literature, call \( R \) the seller’s type. Because the seller’s type could be any \( R \) in the range of \( \rho(\cdot) \), the set of possible values for \( R \) is \( [\rho(0), \infty) \). As will be seen later, however, there is no loss in restricting attention to \( R = [R, \bar{R}] \subseteq [\rho(0), \bar{R}] \). A mechanism is, then, a pair \( \langle x(\cdot), t(\cdot) \rangle \), where \( x: \mathcal{R} \to [0, 1] \) is the probability ownership of the asset is transferred to the buyer; and \( t: \mathcal{R} \to \mathbb{R} \) is the payment to the seller.

Let \( U(R) = (1 - x(R))R + t(R) \) denote the seller’s utility if she truthfully announces her type (note, at this point, her investment is sunk). In equilibrium, the seller must tell the truth. Hence, if the seller’s type is truly \( R \), she cannot do better announcing an \( R' \neq R \); that is,

\[
U(R) \geq (1 - x(R'))R + t(R') = U(R') + (1 - x(R'))(R - R'),
\]

where the equality follows from (4). Similarly, if her type is \( R' \), she must prefer to announce that than announce \( R \):

\[
U(R') \geq U(R) + (1 - x(R))(R' - R). \tag{6}
\]

Expressions (5) and (6) can be combined to obtain:

\[
(1 - x(R))(R - R') \geq U(R) - U(R') \geq (1 - x(R'))(R - R') \tag{7}
\]

By considering \( R > R' \), the following lemma is immediate from (7):

**Lemma 1.** If the mechanism induces truth-telling (is incentive compatible), then the probability of trade, \( x(\cdot) \), is non-increasing in the seller’s type (level of investment).

The properties of \( U(\cdot) \) are critical to the analysis.

**Lemma 2.** If the mechanism induces truth-telling, then \( U(\cdot) \) is a convex function.

**Proof:** Pick \( R \) and \( R' \) in \( \mathcal{R} \). Pick a \( \lambda \in (0, 1) \). Define

\[
R_\lambda = \lambda R + (1 - \lambda)R'.
\]
Truth-telling implies
\[
\lambda U(R) \geq \lambda U(R_\lambda) + \lambda (1 - x(R_\lambda))(R - R_\lambda) \quad \text{and} \quad (8)
\]
\[
(1 - \lambda)U'(R') \geq (1 - \lambda)U(R_\lambda) + (1 - \lambda)(1 - x(R_\lambda))(R' - R_\lambda). \quad (9)
\]
Adding (8) and (9) yields:
\[
\lambda U(R) + (1 - \lambda)U'(R') \geq U(R_\lambda) + (1 - \lambda) \left( \lambda R + (1 - \lambda)R' - R_\lambda \right). \tag{10}
\]

The result follows.

The preceding two lemmas lead to

**Proposition 2.** Necessary conditions for a mechanism to induce truth-telling are (i) that \(x(\cdot)\) be non-increasing and (ii) that
\[
U(R) = \bar{U} - \int_R^R 1 - x(z) dz,
\]
where \(\bar{U}\) is a constant (\(\bar{R}\), recall, is sup \(R\)).

**Proof:** Part (i) follows from Lemma 1. Convex functions are absolutely continuous (see, e.g., van Tiel, 1984, p. 5). Every absolutely continuous function is the integral of its derivative (see, e.g., Yeh, 2006, Theorem 13.17, p. 283). By dividing (7) by \(R - R'\) and taking the limit as that difference goes to zero, it follows that \(U'(R) = 1 - x(R)\) almost everywhere. Expression (10) follows.

As is typically true in mechanism-design problems, these conditions are also sufficient:

**Proposition 3.** Any mechanism in which (i) \(x(\cdot)\) is non-increasing and (ii) expression (10) holds induces truth-telling.

**Proof:** Suppose the seller’s type is \(R\) and consider any \(R' < R\). We wish to verify (5):
\[
U(R) - U(R') = \int_{R'}^R (1 - x(z) dz \geq \int_{R'}^R (1 - x(R')) dz = (1 - x(R'))(R - R'),
\]
where the first equality follows from (10) and the inequality follows because \(x(\cdot)\) is non-increasing. Expression (5) follows. The case \(R' > R\) is proved similarly and, so, omitted for the sake of brevity.

4.2 Consequences

The results of the previous subsection establish the following results.

Anticipating the mechanism to be played, the seller will be willing to invest \(\iota(R)\) if and only if it maximizes \(U(R) - \iota(R)\). It follows:
Proposition 4. If \( I > 0 \) is a level of investment chosen by the seller with positive probability in equilibrium, then the subsequent probability of trade given that investment is \( 1 - \iota'(\rho(I)) \).

Proof: Let \( R = \rho(I) \). Since, by supposition, the seller chooses \( R \) with positive probability in the equilibrium, \( R \in \mathcal{R} \). Hence, \( R \) must satisfy the first-order condition
\[
0 = U'(R) - \iota'(R) = 1 - x(R) - \iota'(R).
\]
(11)
The result follows.

Immediate corollaries are

Corollary 2. If trade always occurs in equilibrium, then the seller invests nothing.

Corollary 3. There is no equilibrium in which the seller invests more than her autarky level of investment, \( \hat{I} \).

A consequence of this analysis is that there is no loss of generality with respect to the design of a mechanism in assuming \( \mathcal{R} = [\rho(0), \hat{R}] \). This does not mean that the seller necessarily plays all \( R \) in \( [\rho(0), \hat{R}] \) with positive probability, rather that the mechanism can accommodate all such \( R \).

4.3 Participation Constraints

Lemma 3. On the equilibrium path, the seller’s expected utility must be at least \( \hat{R} - \iota(\hat{R}) \).

Proof: A course of action available to the seller is to invest \( \iota(\hat{R}) \) and decline to trade. This represents an expected utility of \( \hat{R} - \iota(\hat{R}) \). Hence, any strategy played in equilibrium by the seller other than this must yield at least as great an expected utility.

Like the seller, the buyer could also decline to play the mechanism. Normalize his utility in this case to 0. Let \( F : \mathcal{R} \rightarrow [0, 1] \) be the seller’s strategy, then the following is required:

Lemma 4. On the equilibrium path, it must be that
\[
\int_{\mathcal{R}} (x(R)V(\iota(R)) - t(R)) dF(R) = \int_{\mathcal{R}} (x(R)V(\iota(R)) + (1 - x(R))R - U(R)) dF(R) \geq 0.
\]
(12)
5  EQUILIBRIUM

5.1  THE BUYER HAS ALL THE BARGAINING POWER

Here, consider the situation in which the buyer makes a Tioli offer of a mechanism to the seller. The seller will, thus, be held to her reservation utility. That is, expression (11) must hold for any $R > \rho(0)$ that the seller chooses with positive probability. In light of Lemma 3, it follows that

$$U(R) - \iota(R) = \bar{U} - \iota(\bar{R}),$$

where the equality follows (10) and Proposition 4. Given it is the buyer who makes the Tioli offer, (13) is binding.

In what follows, assume the seller’s strategy, $F : (R, \bar{R}]$ is differentiable. As is standard, let $f(R)$ denote this derivative. Assume $f(R) > 0$ for all $R \in (R, \bar{R}]$. These assumptions will be shown consistent with an equilibrium below. Because the seller never invests more than $\bar{R}$, there is no loss of generality in assuming $\bar{R} \leq \bar{\bar{R}}$.

The buyer seeks to choose $x(\cdot)$ and $\bar{U}$ to maximize

$$\int_{R}^{\bar{R}} \left( x(R)V(\iota(R)) + (1 - x(R))R - \bar{U} + \int_{R}^{\bar{R}} (1 - x(z))dz \right) dF(R).$$

Using integration by parts, this expression becomes:

$$- \bar{U} + \left( x(R)V(\iota(R)) + (1 - x(R))R + \int_{R}^{\bar{R}} (1 - x(z))dz \right) F(R)$$

$$+ \int_{R}^{\bar{R}} \left( x(R)V(\iota(R)) + (1 - x(R))R + \frac{F(R) - F(R)}{f(R)}(1 - x(R)) \right) f(R)dR$$

$$= -\bar{U} + \left( x(R)V(\iota(R)) + (1 - x(R))R \right) F(R)$$

$$+ \int_{R}^{\bar{R}} \left( x(R)V(\iota(R)) + (1 - x(R))R + \frac{F(R) - F(R)}{f(R)}(1 - x(R)) \right) f(R)dR.$$

From Proposition 4, for the seller to play an $R > R \geq \rho(0)$, with positive probability, the buyer’s choice of $x(R)$ must be $1 - \iota'(R)$. Differentiating (15) pointwise for $R > R$, consistency with both Proposition 4 and optimization by the buyer are met if

$$\frac{F(R)}{f(R)} = V(\iota(R)) - R,$$

because, then, the buyer is indifferent as to his choice of $x(\cdot)$ and might as well choose $x(\cdot) = 1 - \iota'(\cdot)$. 
Using (16), we can rewrite the buyer’s expected utility, expression (15), as

\[
x(R)V(\iota(R)) + (1 - x(R))R F(R) - \bar{U} + \int_{\bar{U}}^{R} (F(R) + Rf(R))dR = \hat{R} - \bar{U} + \left( V(\iota(R)) - R \right)F(R)x(R),
\]

where the equality follows by “undoing” the product rule of differentiation. Recalling that the seller’s participation constraint will bind, it follows from (13) and this last expression that the buyer’s expected utility equals \[
\left( V(\iota(R)) - R \right)F(R)x(R) + \hat{R} - \iota(\hat{R}) - \left( \hat{R} - \iota(\hat{R}) \right).
\]

Given that \(\hat{R}\) uniquely maximizes \(R - \iota(R)\), it follows that the buyer’s expected utility cannot exceed \[
\left( V(\iota(R)) - R \right)F(R)x(R) + \hat{R} - \iota(\hat{R}) - \left( \hat{R} - \iota(\hat{R}) \right).
\]

We are now in position to establish:

**Proposition 5.** There exists a subgame-perfect equilibrium of the game in which the buyer makes the seller a take-it-or-leave-it offer in which the seller plays a mixed strategy whereby she chooses \(R \in [\rho(0), \hat{R}]\) according to the distribution function

\[
F(R) = \exp \left( - \int_{\rho(0)}^{R} \frac{1}{V(\iota(z)) - z} dz \right) \tag{17}
\]

and the buyer offers the mechanism \(\langle x(\cdot), t(\cdot) \rangle\) such that

\[
x(R) = \begin{cases} 
1, & \text{if } R = \rho(0) \\
1 - \iota'(R), & \text{if } R > \rho(0)
\end{cases}
\]

and

\[
t(R) = \hat{R} - \iota(\hat{R}) + \iota(R) - (1 - x(R))R. \tag{18}
\]

**Proof:** Expression (17) is the solution to the differential equation (16). By Assumption 1(iv), \(V(\iota(z)) - z > 0\) for \(z > \rho(0)\). So if \(\hat{R} > \rho(0)\), then

\[
\int_{\rho(0)}^{\hat{R}} \frac{1}{V(\iota(z)) - z} dz < \infty;
\]

hence, \(F(\hat{R}) > 0\). This implies \(x(\hat{R}) = 1\). But \(x(\hat{R}) = 1\) is inconsistent with Proposition 4 if \(\hat{R} > \rho(0)\). It must therefore be that \(\hat{R} = \rho(0)\). Expression (18) follows from Proposition 2 because \(1 - x(R) = \iota'(R)\). The remainder of
As an example, suppose that $R(I, b) = \alpha(1 + \sqrt{I})(1 + \sqrt{b})$ and $\mathcal{B} = \mathbb{R}_+$, where $\alpha \in (0, 2)$. Straightforward calculations reveal:

$$I^* = \frac{\alpha^2}{(2 - \alpha)^2}; \hat{I} = \frac{\alpha^2}{4}; \nu(R) = \frac{(R - \alpha)^2}{\alpha^2}; \text{ and } V(\nu(R)) - R = \frac{R^2}{4}.$$  

The Proposition 5 equilibrium will thus be characterized by

$$F(R) = \exp\left(-\frac{4}{R}\right) \exp\left(\frac{8}{\alpha(2 + \alpha)}\right); x(R) = 1 - \frac{2(R - \alpha)}{\alpha^2};$$

and $t(R) = 1 + \frac{1}{4}\alpha(4 + \alpha) - R + \frac{1}{\alpha^2}(3R^2 - 4\alpha R)$.

Expected welfare is

$$\frac{1}{4}\alpha \left(4 + \alpha + \alpha \exp\left(-\frac{4}{2 + \alpha}\right)\right).$$

This exceeds welfare given autarky by

$$\frac{\alpha^2}{4} \exp\left(-\frac{4}{2 + \alpha}\right).$$

First-best welfare is $\frac{2\alpha}{2 - \alpha}$. For instance, if $\alpha = 1$, first-best welfare is 2, equilibrium welfare is approximately 1.3159, and welfare under autarky is 1.25. The equilibrium probability of exchange does not have a convenient closed-form solution. If $\alpha = 1$, that probability is approximately 0.6017.

Is the equilibrium in Proposition 5 unique? Within a broad class of possible strategies, the answer is yes.

**Definition 1.** A mixed strategy for the seller, $F : [\underline{R}, \overline{R}] \subseteq [\rho(0), \hat{R}] \to [0, 1]$, is piecewise absolutely continuous if, for a finite sequence $R_1 < \cdots < R_N$, $F(\cdot)$ is absolutely continuous on all segments $(R_n, R_{n+1})$, $n = 0, \ldots, N$, where $R_0 \equiv \underline{R}$ and $R_{N+1} \equiv \overline{R}$ and discontinuous at each $R_n$, $n = 1, \ldots, N$.

Because a constant function is absolutely continuous, observe that mixed strategies in which the seller mixes over a discrete set of investment levels are included in this definition. The strategy in Proposition 5 is also an element of the set of strategies defined in Definition 1 (for it, $N = 1$ and $R_0 = R_1 = \rho(0)$). Moreover, it is the only strategy within this set that can be played in equilibrium:

**Proposition 6.** For the game in which the buyer makes the seller a take-it-or-leave-it offer and the seller is limited to piecewise absolutely continuous strategies, the equilibrium in Proposition 5 is unique.

The proof can be found in the appendix.

---

2 If $F(\cdot)$ were not discontinuous at an $R_n$, then there is nothing special about that $R_n$ insofar as $F(\cdot)$ will be absolutely continuous on $(R_{n-1}, R_{n+1})$. 
5.2 The Seller Has All the Bargaining Power

Now, consider the situation in which the seller makes a tioli offer of a mechanism to the buyer. The buyer can “hold” the seller to a particular mechanism by “threatening” to believe the offer of any other mechanism signals the seller has chosen \( I = 0 \) with certainty. As with all threats, the threat must be credible and, if credible, effective (i.e., the seller cannot do better to deviate even if it induces the buyer to believe \( I = 0 \)).

Let \( \Pi \) denote the seller’s expected utility in equilibrium. It follows that
\[
\Pi = U(R) - \iota(R) \text{ for all } R \text{ that the seller plays with positive probability in equilibrium.}
\]
Substituting into the buyer’s participation condition, expression (12), and invoking Proposition 4, it must be that
\[
\int_{R} \left( (1 - \iota'(R))V(\iota(R)) + \iota'(R)R - \iota(R) \right) dF(R) \geq \Pi. 
\] (19)

Observe that the expression in the large parentheses in (19) is expected social surplus given the constraint that the trade probabilities must satisfy \( x(R) = 1 - \iota'(R) \) for any \( R > \rho(0) \) played with positive probability by the seller. Because the expression in large parentheses is continuous in \( R \) on \( [\rho(0), \hat{R}] \), a compact space, it has a maximum, \( M \). Define
\[
S = \begin{cases} 
\mathcal{M}, & \text{if } \mathcal{M} > V(0) \\
V(0), & \text{if } \mathcal{M} \leq V(0) 
\end{cases}.
\] (20)

Let \( \mathcal{R}_M \) be the set of maximizers of the expression in large parentheses in (19). Define
\[
R^s = \begin{cases} 
R \in \mathcal{R}_M, & \text{if } \mathcal{M} > V(0) \\
\rho(0), & \text{if } \mathcal{M} \leq V(0) 
\end{cases}.
\]
The objective is to show there exists an equilibrium in which the seller plays \( R^s \) with certainty and offers a direct-revelation mechanism in which
\[
x(R^s) = \begin{cases} 
1 - \iota'(R^s), & \text{if } R^s > \rho(0) \\
1, & \text{if } R^s = \rho(0) 
\end{cases}.
\] (21)

To that end, consider a mechanism in which \( x(R) = x(R^s) \) for all \( R \in [\rho(0), \hat{R}] \) and
\[
(\hat{R} - R^s)(1 - x(R^s)) + x(R^s)V(\iota(R^s)) + (1 - x(R^s))R^s \\
\geq \hat{U} \geq (\hat{R} - R^s)(1 - x(R^s)) + \iota(R^s) + \max \{V(0), \hat{R} - \iota(\hat{R})\}. 
\] (22)

Lemma 5. The interval defined by (22) is non-empty (i.e., a \( \hat{U} \) satisfying that expression exists).

Proof: If \( R^s = \rho(0) \), the result follows because, then,
\[
V(0) \geq \max_{R} \left( (1 - \iota'(R))V(\iota(R)) + \iota'(R)R - \iota(R) \right) \geq \hat{R} - \iota(\hat{R}),
\]
where the last inequality follows because \( \iota'(\hat{R}) = 1 \). Suppose \( R^s > \rho(0) \). The interval is non-empty because, rearranging terms,

\[
(1 - \iota'(R^s))V(\iota(R^s)) + \iota'(R^s)R^s - \iota(R^s) = S \geq \max \{ V(0), \hat{R} - \iota(\hat{R}) \}.
\]

\[ \blacksquare \]

**Proposition 7.** For any \( \hat{U} \) satisfying (22), there exists a perfect Bayesian equilibrium in which the seller invests \((R^s)\) with certainty and offers the buyer, on a take-it-or-leave-it basis, a mechanism with that \( \hat{U} \) and a fixed probability of trade \( x(R^s) \), where \( x(R^s) \) is defined by (21). The buyer accepts this mechanism, but believes the offer of any other mechanism to indicate the seller has chosen not to invest (i.e., \( I = 0 \)) and plays accordingly.

**Proof:** It is rational for the buyer to accept this mechanism if he believes the seller has chosen \( R^s \): From (14), the buyer’s expected utility is

\[
x(R^s)V(\iota(R^s)) + (1 - x(R^s))R^s - \hat{U} + (\hat{R} - R^s)(1 - x(R^s)) \geq 0,
\]

where the inequality follows from (22). If the seller will offer that mechanism, she does best to play \( R^s \): Her expected utility from playing \( R \) is

\[
\hat{U} - \int_R^\hat{R} (1 - x(z))dz - \iota(\hat{R}).
\]

Differentiating with respect to \( R \) yields

\[
1 - x(R^s) - \iota'(R^s).
\]

Because \( \iota(\cdot) \) is convex, this expression is positive for \( R < R^s \), equal to zero for \( R = R^s \), and negative for \( R > R^s \); that is, it is optimal for the seller to play \( R^s \) if she will offer the mechanism. Finally, given the buyer’s beliefs, if the seller deviates to another mechanism, she will be unable to sell the asset for more than \( V(0) \). Consequently, her payoff will be at most \( \max \{ V(0) - \iota(R), \hat{R} - \iota(\hat{R}) \} \), depending on whether trade occurs or not. Hence, at best, her payoff is \( \max \{ V(0), \hat{R} - \iota(\hat{R}) \} \). This is less than her expected payoff from offering the mechanism,

\[
\hat{U} - (\hat{R} - R^s)(1 - x(R^s)) - \iota(R^s),
\]

by (22).

Recall the example following Proposition 5. It is readily verified for that example that

\[
V(0) = \hat{R} - \iota(\hat{R}) = \frac{\alpha(4 + \alpha)}{4}.
\]

Note that \( \rho(0) = \alpha \) in this example. Calculations reveal that \( \iota'(\alpha) = 0 \). Hence, \( S = M \). The derivative of the expression in large parentheses in (19) is

\[
V'(\iota(R))\iota'(R)(1 - \iota'(R)) - \left( V(\iota(R)) - R\right)\iota''(R).
\]

(24)
Because $\iota'(\hat{R}) = 1$ and $\iota(\cdot)$ is convex, (24) is negative evaluated at $R = \hat{R}$. In this example, $V'(I) \to \infty$ as $I \to 0$; but taking limits as $R \to \rho(0) = \alpha$ reveals that (24) converges to $(\alpha + 1)/2 > 0$. It follows that $\rho(0) < R^s < \hat{R}$.

Calculations show

$$R^s = \frac{1}{6} \left( -4 + 2\alpha + \alpha^2 + \sqrt{16 + 32\alpha + 20\alpha^2 + 4\alpha^3 + \alpha^4} \right).$$

For instance, if $\alpha = 1$, then equilibrium investment is $\frac{61}{18} - \frac{7}{18} \sqrt{73} \approx 0.0662$; the equilibrium probability of trade is $\frac{1}{12} (19 - \sqrt{73}) \approx .8713$; and welfare is $\frac{1}{108} (-287 + 53\sqrt{73}) \approx 1.5355$. For $\alpha = 1$,

$$\bar{U} \in \left[ \frac{1}{72} (191 - 11\sqrt{73}) , \frac{1}{216} (-271 + 73\sqrt{73}) \right] \approx [1.3474, 1.6329].$$

The equilibrium of Proposition 7 can be given a non-mechanism-design interpretation along the lines of the solution to the credence-good problem proposed in Fong (2005).3 Suppose the seller simply makes a toli offer to the buyer at price of $V(\iota(R^s))$. If the buyer believes the seller has invested $\iota(R^s)$, then the buyer is indifferent between accepting and rejecting and, so, is willing to mix. Suppose he plays the strategy of accepting with probability $\pi(R^s)$, where $\pi(R^s)$ is defined by (21). It is readily seen the seller’s expected payoff is $S$, so she has no incentive to deviate by investing a different amount if she intends to offer the asset at price $V(\iota(R^s))$. If the buyer believes a sale price less than $V(\iota(R^s))$ means the seller invested nothing and a price greater than $V(\iota(R^s))$ means the seller has invested no more than $\iota(R^s)$, then the seller cannot profit via a deviation in which she sets a price other than $V(\iota(R^s))$. This argument establishes:

**Proposition 8.** There exists a perfect Bayesian equilibrium in which the seller invests $\iota(R^s)$ with certainty and offers the buyer, on a take-it-or-leave-it basis, the asset for a price of $V(\iota(R^s))$. They buyer plays the mixed strategy by which he accepts with probability $\pi(R^s)$ given by (21) and rejects with probability $1 - \pi(R^s)$. The buyer believes an offer at any price less than $V(\iota(R^s))$ means the seller invested nothing and he believes an offer at any price greater than $V(\iota(R^s))$ means the seller has invested no more than $\iota(R^s)$.

### 5.3 Comparison of Equilibria

Any equilibrium in which the seller invests a positive amount means the probability of trade given the seller has invested $\iota(R)$ must be $1 - \iota'(R)$. Given that constraint, expected social welfare is given by the integral in (19). Because the equilibrium of interest when the seller has all the bargaining power puts all weight on the $R$ that maximizes the integrand, whereas the equilibrium

3In contrast to the analysis here, in Fong, the quality of the good is determined randomly and exogenously. The seller learns the realization of this random process prior to sale, whereas the buyer does not, so a similar asymmetry of information exists.
when the buyer has the bargaining power requires the seller’s strategy be a non-degenerate distribution, it immediately follows that

**Proposition 9.** *Maximum expected equilibrium welfare when the seller can make a take-it-or-leave-it offer to the buyer is greater than expected equilibrium welfare when the buyer can make a take-it-or-leave-it offer to the seller; that is, expected welfare in the equilibrium given in Proposition 7 exceeds expected welfare in the equilibrium given in Proposition 5.*

Intuitively, both the situation in which the seller can make the TIOLI offer and the one in which the buyer can suffer from a moral-hazard problem. Hence, in neither situation is the first best possible (see Corollary 1). The situation is further exacerbated when the buyer can make a TIOLI offer because of the holdup problem. If the buyer knew precisely how much the seller had invested, he would capture all gains to trade, thereby destroying, from an *ex ante* perspective, any incentive for the seller to invest. For investment to happen in equilibrium, the buyer must be uncertain as to how much the seller has invested; that is, the seller must play a mixed strategy. Because the seller mixes over investment levels other than the one that is second-best optimal given the moral-hazard problem (*i.e.*, given the need to make the trade probability equal \(1 - \iota'(R)\)), the need to overcome the holdup problem will further reduce welfare.

Given this discussion, one might expect that the seller’s investment would be lower, on average, when the buyer has the bargaining power than when the seller does. This, however, need not be the case. Recall the previously used example with \(\alpha = 1\). Calculations reveal that expected investment is approximately \(0.0687\), which exceeds the value of \(\iota(R^*)\), approximately \(0.0662\), derived above. These calculations do not, though, reveal a universal truth: Using the same example, but with \(\alpha = 3/2\), \(\iota(R^*) \approx 0.1551\), whereas expected investment when the buyer has the bargaining power is approximately \(0.1270\).

The analysis indicates, therefore, that the problem with holdup is not necessarily that it reduces investment relative to a no-holdup benchmark so much as it distorts investment *vis-à-vis* that benchmark.\(^4\) The desire to evade holdup can induce the seller to over-invest as well as to underinvest.

### 6 Pre-Trade Contracting

In contrast to the situation considered so far, suppose the parties could enter into a contractual relation prior to the seller’s investment. This does not eliminate the moral-hazard problem, so trade must again be stochastic if the seller is to have incentives to invest. In particular, Propositions 2–4 continue to hold. Given that the bargaining the parties do to establish the contract occurs under full information, it is reasonable to presume that bargaining results in a contract that maximizes expected welfare; that is, that maximizes the integral in (19). From the analysis in Section 5.2, it follows that the parties will utilize the

\(^4\)Williamson (1976) made a related observation in noting that a consequence of holdup could be a distortion in the kinds of investment made.
mechanism in Proposition 7, except, because they can make *ex ante* transfers, the constraints on $\hat{U}$ don’t apply. Formally:

**Proposition 10.** Suppose buyer and seller can enter into a contract prior to the seller’s investing. Let $u_B$ and $u_S$ be, respectively, their reservation utility levels (i.e., what they would get were no contract signed and they pursued their next best alternatives). If $u_B + u_S \leq S$, where $S$ is second-best social surplus (i.e., as defined by (20)), then the parties will agree to a contract in which the probability of trade is $x(R^s)$, where $x(R^s)$ is defined by (21), and the seller is paid $T$, where

$$x(R^s)V(\iota(R^s)) - u_B \geq T \geq u_S - R^s(1 - x(R^s)) + \iota(R^s).$$  \hspace{1cm} (25)

**Proof:** Simple algebra confirms the interval defined by (25) is non-empty given the assumption that $u_B + u_S \leq S$. If the seller accepts this contract, her utility, as a function of $R$, is

$$T - R(1 - x(R^s)) - \iota(R).$$  \hspace{1cm} (26)

Differentiating (26) with respect to $R$ yields

$$1 - x(R^s) - \iota'(R) = \iota'(R^s) - \iota'(R),$$

which, given the convexity of $\iota(\cdot)$, is positive for $R < R^s$, zero for $R = R^s$, and negative for $R > R^s$. Hence, she will choose $R = R^s$ in equilibrium. Substituting $R^s$ for $R$ in (26) reveals, given (25), that the seller’s expected equilibrium utility is not less than her reservation utility. Given the seller’s equilibrium play, the buyer’s expected utility is $x(R^s)V(\iota(R^s)) - T$. From (25) it is readily seen his expected utility is not less than his reservation utility. \qed

7 Discussion and Conclusions

This paper has shown that it is possible to induce a seller to invest, with positive probability, in a tradable asset even when the potential buyer cannot observe the seller’s investment. The critical assumption is that the seller expects some return from the asset if trade fails to occur. For this reason, it is possible to overcome, partially, the moral hazard problem that exists given that the seller is otherwise investing on behalf of the buyer. The seller has the maximum incentive to invest when trade is certain *not* to occur. The provision of this maximum incentive is not, in general, optimal because, given investment by the seller, there are gains to trade: the tradable asset is more valuable in the buyer’s hands than the seller’s. A tradeoff thus exists between the provision of incentives *ex ante* and achieving efficiency *ex post*. This paper has shown how this tradeoff can be managed to achieve a second-best outcome.

The focus of the paper has been on the situation in which the relation between seller and buyer is established *after* the seller invests. Consequently, the outcome is sensitive to how buyer and seller bargain. In particular, when the buyer has the bargaining power, a holdup problem exists. As shown, holdup
induces additional distortions with respect to the seller’s investment strategy. Although the consequence of holdup need not be that the seller invests less in expectation, the distortions nevertheless ensure that welfare is reduced vis-à-vis bargaining without a holdup threat (i.e., bargaining in which the seller has the bargaining power).

There remain questions to be addressed. One stems from the fact that the solutions proposed above rely on there being a positive probability of an inefficient allocation ex post; that is, unless there is a positive probability of the asset remaining in the seller’s hands, which is ex post inefficient, there is no means of inducing the seller to invest. One question, then, is why, when the mechanism has left the seller in possession of the asset, don’t the parties renegotiate to an efficient allocation?

This is a question that applies to much of the literature on mechanism design and trade under asymmetric information. \(^5\) In settings in which the uninformed player (and only the uninformed player) can make repeated offers and the uninformed player’s value of trade is independent of investment (as in Gul, 2001, or Lau, 2008), a Coase-Conjecture-like result can be shown to hold, with trade occurring with probability one on the equilibrium path. Moreover, in Gul and Lau, the investing party retains incentives to invest. In contrast, for the problem considered here, if the bargaining game were to yield trade with probability one, then the seller’s investment will never be directly beneficial to her and she can, thus, have no incentive to invest.

On the other hand, it may be possible for the parties to commit to their take-it-or-leave-it offers; for instance, by developing reputations not to continue negotiations. For example, some divorce lawyers—known as “bombers”—have developed reputations for sticking to their take-it-or-leave-it offers. A company that sought to provide its engineers and scientists incentives along the lines of Proposition 10 would necessarily have to develop a reputation to let the engineers and scientists walk away with positive probability.

It could also be that there is a relatively narrow window in which exchange can occur. If each round of negotiation takes non-negligible time, then the seller could end up with the asset because the parties simply run out of time to bargain. Unlike bargaining under symmetric information, in which bargaining is typically reached in a single round, with asymmetric information there can be multiple rounds of bargaining on the equilibrium path (see, e.g., Spier, 1992). Modeling such bargaining games is beyond the scope of the present paper, but it seems reasonable to predict that such models will again find a tradeoff between ex ante incentives and ex post efficiency.

In the model considered here, the motive for trade is that the buyer is able to take a further action that raises the return from the asset. Other rationales for trade could also exist: For instance, the returns generated by the asset are idiosyncratic to the owner and the buyer’s distribution of returns given any investment level dominate the seller’s distribution according to a stochastic

\(^5\)Among the earlier cited papers, for instance, this question applies to both Fudenberg and Tirole (1990) and Demski and Sappington (1991).
order such as first-order stochastic dominance. In fact, this motivation is simply an alternative interpretation of the analysis above: Let the buyer’s “action,” $b$, be 0 or 1, with the latter simply indicating possession.\footnote{Under this alternative interpretation, the buyer’s expected utility would be $R(I, 1)$ rather than $R(I, 1) - 1$; it is readily seen that change has no effect on the analysis above.}

Another motivation for trade could be different risk tolerances. For instance, the seller, as an individual entrepreneur, could be risk averse, whereas the large company that might buy her out could be risk neutral. Assuming a risk-averse seller complicates the analysis because the units of the parties’ payoffs are no longer the same (the buyer’s remains money, but the seller’s is now utils). In addition, the seller will now care about the riskiness of the returns as well as their expected value, which means she could be making investment decisions on two margins: risk and return (e.g., if returns were distributed normally, she would be concerned with both mean and variance). An analysis of the problem with differing attitudes toward risk remains a topic for future research.

A final extension to mention is intermediate information structures. Above, information was wholly asymmetric. Were it wholly symmetric, it is readily seen an equilibrium exists in which the seller invests at her autarky level (i.e., $i(R)$) if buyer has the bargaining power and she invests at the first-best level (i.e., $I^*$) if she has the bargaining power. Intermediate structures are also conceivable. For instance, the buyer could observe an imperfect signal of the seller’s investment.\footnote{An imperfect signal is meaningful only when the seller mixes over investment levels in equilibrium. When she plays a pure strategy with respect to investment, the signal is irrelevant because the buyer can perfectly forecast the seller’s investment because he knows the equilibrium of the game.} Another alternative is that one or both parties observe a signal predictive of the asset’s future returns.

**APPENDIX A: PROOF OF PROPOSITION 6**

**Proof:** Let $\mathcal{D} = \{R_1, \ldots, R_N\}$. Because $F(\cdot)$ is a distribution, it is right continuous, so $\lim_{R \downarrow R_n} F(R) = F(R_n)$ for all $R_n \in \mathcal{D}$. Define $F(R_n^+) = \lim_{R \uparrow R_n} F(R)$. Note $F(R_0^-) = 0$. Because, as a distribution, $F(\cdot)$ is non-decreasing, each point of discontinuity (i.e., $R_n \in \mathcal{D}$) is a jump up in $F$. Because it is a distribution, it is right continuous, so $\lim_{R \downarrow R_n} F(R) = F(R_n)$. Define $F(R_n^-) = \lim_{R \uparrow R_n} F(R)$. Note $F(R_0^-) = 0$ and $F(R_n) - F(R_n^-) > 0$ for all $R_n \in \mathcal{D}$. This last point implies that the seller plays each $R_n \in \mathcal{D}$ with positive probability and, thus, by Proposition 4, $x(R_n) = 1 - i'(R_n)$.

Making use of Fubini’s Theorem, the buyer’s expected utility, expression...
Appendix

(14), can be rewritten as
\[
-\bar{U} + \int_{R_0}^R \left( x(R)V(\iota(R)) + (1 - x(R))R \right) dF(R)
\]
\[
+ \sum_{n=0}^{N} \left( (F(R_n) - F(R_n^-)) \int_{R_n}^R (1 - x(z)) dz \right)
\]
\[
+ (F(R_{n+1}^-) - F(R_n)) \int_{R_{n+1}}^R (1 - x(z)) dz
\]
\[
+ \int_{R_n}^{R_{n+1}} \left( F(R) - F(R_n) \right) (1 - x(R)) dR
\].

Canceling like terms, this expression becomes
\[
-\bar{U} + \int_{R_0}^R \left( x(R)V(\iota(R)) + (1 - x(R))R \right) dF(R)
\]
\[
+ \sum_{n=0}^{N} \int_{R_n}^{R_{n+1}} F(R)(1 - x(R)) dR.
\]

Because \( F(\cdot) \) is absolutely continuous on each segment, it is differentiable almost everywhere on each segment and, moreover, it is the integral of its derivative. Let \( f(\cdot) \) denote its derivative. The buyer’s expected utility can thus be rewritten as
\[
-\bar{U} + \sum_{n=0}^{N} \left( (F(R_n) - F(R_n^-)) \left( x(R_n)V(\iota(R_n)) + (1 - x(R_n))R_n \right) \right)
\]
\[
+ \int_{R_n}^{R_{n+1}} \left( x(R)V(\iota(R)) + (1 - x(R))R \right) f(R) dR + \int_{R_n}^{R_{n+1}} F(R)(1 - x(R)) dR
\].

Some consequences of this last expression are:

1. If \( F(R_0) > 0 \), then \( x(R_0) = 1 \) given the mechanism the buyer offers must maximize his expected utility. If \( x(R_0) = 1 \), then \( R_0 = \rho(0) \) by Proposition 4.

2. Suppose \( \iota(R') \) and \( \iota(R'') \) are two possible investment levels in equilibrium, \( R' < R'' \). Then \( F(R') < F(R'') \). To prove this note it must hold if \( R' < R_n \leq R'' \) for some \( R_n \); hence, suppose \( R_n \leq R' < R'' < R_{n+1} \) for some \( n = 0, \ldots, N \). If \( F(R') = F(R'') \), then \( f(\cdot) \) equals zero almost everywhere. Consequently, the only way that \( x(R), R \in (R', R''), \) enters into the buyer’s expected utility expression is as the integral
\[
\int_{R'}^{R''} F(R)(1 - x(R)) dR = F(R') \int_{R'}^{R''} (1 - x(R)) dR.
\]
Given the buyer offers a mechanism that maximizes his expected utility given the relevant constraints, he must set \( x(R) = x(R''') \) for \( R \in (R', R'') \) given that \( x(\cdot) \) must be non-increasing. Because \( \nu(R''') \) is in the set of investments over which the seller mixes, Proposition 4 implies \( x(R''') = 1 - \nu'(R''') \). But then

\[
U'(R) - \nu'(R) = 1 - x'(R''') - \nu'(R) = \nu'(R''') - \nu'(R) > 0
\]

for all \( R \in (R', R'') \), where the inequality follows because \( \nu(\cdot) \) is convex. Hence,

\[
(U(R) - \nu(R)) - (U(R') - \nu(R')) = \int_{R'}^{R} (U'(R) - \nu'(R)) dR > 0
\]

for any \( R \in (R', R'') \), which implies the seller would never play \( R' \), a contradiction. Observe, \textit{inter alia}, this proof can be extended to show there is no equilibrium in which the seller mixes over a finite number of investment levels (let \( R' = R_n \) and \( R'' = R_{n+1} \), but work with \( F(R_{n+1}) \) to show \( F(R_n) \neq F(R_{n+1}) \)).

3. Suppose \( \nu(R') \) and \( \nu(R'') \) are two possible investment levels in equilibrium, \( R' < R'' \). Then \( F(R') < F(R) < F(R'') \) for all \( R \in (R', R'') \). From the previous result if

\[
R' < R_n < R_n < R''
\]

then the claim is true for all \( R \in [R_n, R_n) \). Hence, there is again no loss in restricting attention to the case in which \( R_n < R' < R_n < R_{n+1} \) for some \( n = 0, \ldots, N \). Suppose, first, that \( F(R') = F(R) \). Let \( S = \sup \{ R | F(R) = F(R') \} \). By assumption \( S > R' \). Because \( S < R'' < R_{n+1} \), \( F(\cdot) \) is continuous at \( S \), hence \( F(S) = F(R') \). Because \( F(\cdot) \) increases at \( S \), \( S \) must be a possible play of the seller. But then we have a contradiction of the previous step. Hence, \( F(R') < F(R) \) for all such \( R \). Suppose \( F(R'') = F(R) \). Let \( I = \inf \{ R | F(R) = F(R'') \} \). By assumption \( I < R'' \) and it was just established that \( I > R' \). It follows that \( F(\cdot) \) is continuous at \( I \), hence \( F(I) = F(R'') \). Because \( F(\cdot) \) increases at \( I \), \( I \) must be a possible play of the seller. But then we have a contradiction of the previous step (\textit{i.e.}, step 2).

4. Because the previous two steps establish that \( F(\cdot) \) is everywhere increasing, it must be that \( f(R) > 0 \) for almost every \( R \). Hence, almost every \( R \) is possible, which means \( x(R) = 1 - \nu'(R) \) almost everywhere. In fact, because \( x(\cdot) \) must be non-increasing and \( \nu'(\cdot) \) is continuous, it must be that \( x(R) = 1 - \nu'(R) \) everywhere on \( (R', R) \).

5. It can now be shown that \( F(R_n) > F(R_{n-1}) \) is impossible in equilibrium for \( R_n > R_{n-1} \). Suppose not and pick an \( \epsilon \) such that \( R_{n-1} < R_n - \epsilon < R_n \). As a
slight abuse of notation, define $R_{\varepsilon} = R_n - \varepsilon$. Suppose the buyer deviated from offering a mechanism with $x(R) = 1 - \iota'(R)$ to offering the following

$$
\tilde{x}(R) = \begin{cases} 
  x(R), & \text{if } R \notin [R_{\varepsilon}, R_n] \\
  x(R_{\varepsilon}), & \text{if } R \in [R_{\varepsilon}, R_n]
\end{cases}.
$$

Note $\tilde{x}(\cdot)$ is non-increasing, as required. The change in the buyer’s expected utility would be

$$
(F(R_n) - F(R_n^{-}))(x(R_{\varepsilon}) - x(R_n)) \left( V(\iota(R_n)) - R \right) + \int_{R_n}^{R_{\varepsilon}} (x(R_{\varepsilon}) - x(R)) \left( V(\iota(R)) - R - \frac{F(R)}{f(R)} \right) f(R)dR. \quad (27)
$$

Because the buyer cannot wish to deviate in this way, (27) must be negative for all such $\varepsilon$. Because the asset is always more valuable in the buyer’s hands than the seller’s, the first line of (27) is positive given $x(R_{\varepsilon}) > x(R_n)$. Hence, the integral must be negative for all $\varepsilon$. This implies

$$
V(\iota(R)) - R - \frac{F(R)}{f(R)} < 0 \quad (28)
$$

almost everywhere in some neighborhood $(R_n - \delta, R_n)$. Define $R_{\delta} = R_n - \delta$. Consider, instead, a deviation in which the buyer offers

$$
\hat{x}(R) = \begin{cases} 
  x(R), & \text{if } R \notin [R_{\delta}, R_n] \\
  x(R_n), & \text{if } R \in [R_{\delta}, R_n]
\end{cases}.
$$

Note $\hat{x}(\cdot)$ is non-increasing, as required. The change in the buyer’s expected utility would be

$$
\int_{R_{\delta}}^{R_n} (x(R_n) - x(R)) \left( V(\iota(R)) - R - \frac{F(R)}{f(R)} \right) f(R)dR.
$$

But this is positive by (28), which means the buyer would wish to deviate. Reductio ad absurdum, there is no equilibrium in which $F(R_n) > F(R_n^{-})$ for $R_n > \bar{R}$.

6. It has been established that the buyer’s equilibrium expected utility can be written as (15). Observe that expression can be rewritten as

$$
\bar{R} - \bar{U} + \left( x(R) V(\iota(R)) + (1 - x(R)) \bar{R} \right) F(R)
$$

$$+ \int_{\bar{R}}^{R} x(R) \left( V(\iota(R)) - \bar{R} \right) f(R) - F(R)) dR. \quad (29)
$$
7. In keeping with step 4, let \( x(R) = 1 - \iota'(R) \) for \( R \in (\bar{R}, \tilde{R}) \). Given the buyer must offer \( x(\cdot) \) in equilibrium, it follows that, for any \( R' \in (\bar{R}, \tilde{R}) \), the buyer cannot prefer to offer \( x(R) - \varepsilon, \varepsilon > 0 \), for all \( R \in (R', \bar{R}) \). Hence, from (29),

\[
\int_{R'}^{\bar{R}} \left( V(\iota(R)) - R \right) f(R) - F(R) \, dR \geq 0 \quad (30)
\]

for all \( R' \in (\bar{R}, \tilde{R}) \). Suppose (30) were a strict inequality on a set of \( R > R' \) of positive measure for some \( R' \). Then the buyer would do better to deviate to

\[
\tilde{x}(R) = \begin{cases} 
  x(R), & \text{if } R < R' \\
  x(R'), & \text{if } R > R' 
\end{cases}
\]

This last claim can be established by integrating by parts the change in the buyer’s expected utility from this deviation:

\[
(x(R') - x(R)) \int_{R}^{\bar{R}} \left( V(\iota(z)) - z \right) f(z) - F(z) \, dz \bigg|_{R'}^{\bar{R}} + \int_{R'}^{\bar{R}} \left( \int_{R}^{\bar{R}} \left( V(\iota(z)) - z \right) f(z) - F(z) \, dz \right) \left( -x'(R) \right) \, dR. \quad (31)
\]

The first line of (31) is zero, term A is positive by the assumption (30) is positive for a positive measure of \( R > R' \), and term B is positive because \(-x'(R) = \iota''(R)\) and \( \iota(\cdot) \) is convex. Hence, (31) is positive, which means the buyer would wish to deviate. \textit{Reductio ad absurdum}, it must be that (30) is zero for almost every \( R' \). Because integrals are continuous, (30) is zero for all \( R' \). Hence, the function of \( R' \) defined by the integral in (30) is constant, so its derivative,

\[
\left( V(\iota(R')) - R' \right) f(R) + F(R') = 0, \quad (32)
\]

almost everywhere.

Expression (32) is the same differential equation as (16), so it follows that the seller’s strategy must satisfy (16). It was shown in the text preceding Proposition 5 that if the seller’s strategy satisfied that differential equation, then \( \bar{R} = \hat{R} \) in equilibrium. Because \( F(\hat{R}) = 1 \), the differential equation has a unique solution, given by (17). \( \blacksquare \)
References


