Robustness of Adaptive Expectations as an Equilibrium Selection Device – Supplementary Notes*

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March 2, 2000

Abstract

These are supplementary notes for Lettau and Van Zandt (2000). They include proofs of certain details in the main paper, and a few extensions. To be in synch, these notes and the main paper should have the same date.

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*Gorkem Celik contributed as research assistant to Section 9. See additional acknowledgements in the main paper.

†The views expressed in this paper are those of the authors and do not necessarily reflect those of the Federal Reserve Bank of New York or the Federal Reserve System.

‡This research supported in part by grants SBR-9223917 and IRI97-11303 from the National Science Foundation, and was written in part while visiting Northwestern University 1998–1999.

Contents

1 An OLG model with constant government deficit

The model in Section 2 of the main paper is a reduced form of a basic OLG model with fiat money whose supply is increased to finance a constant government deficit. In this section, we outline such an OLG model, using the notation of the main paper. This material is not new but is provided for the convenience of the reader.

Trade and consumption of a single perishable good take place in periods $t \in \{0, 1, \dots\}$. For each period t, there is a representative agent, called generation t, who lives only during periods t and $t+1$ (youth and old age, resp.) and whose consumption in these periods is denoted c_{1t} and c_{2t}, resp. All generations have the same consumption set $X \subset \mathbb{R}^2$, the same endowment $(e_1, e_2) \in X$, and the same utility function $U \colon X \to \mathbb{R}$. There is also a storable good called "money" that has no consumption value. The initial stock of money is $m_{-1} > 0$, which is supplied inelastically during period 0 by a generation named -1 that has no other role. In youth, agents in other generations can choose to trade part of their endowment for money. In old-age, agents supply their holding of money inelastically. In addition, in each period, the government issues new money to purchase $\delta > 0$ units of the good, in order to finance a constant real deficit.

We consider only the case in which the government can finance the deficit, and hence money has value. Let p_t be the price of the perishable consumption good in period t in terms of money, and let $\pi_{t+1} := 1p_{t+1}/p_t$ be the inflation factor in period $t+1$. When agents trade in their youth, they form point expectations about the price in the next period. Given a current price p_t and a price expectation $p^e_{t+1} \in (0, \infty)$, generation t chooses planned consumption (c_{1t}, c_{2t}) and money purchases $m_t \geq 0$ in youth that solve

$$
\begin{aligned}
(1) \qquad & \max_{(c_{1t}, c_{2t}) \in X,\ m_t \in \mathbb{R}_+} && U(c_{1t}, c_{2t}) \\
& \text{subj. to:} && c_{1t} \leq e_1 - m_t/p_t \\
& && c_{2t} \leq e_2 + m_t/p^e_{t+1}\ .
\end{aligned}
$$

We assume that this problem has a unique solution, which satisfies the constraints with equality. The solution depends only on the expected inflation factor $\pi^e_{t+1} := p^e_{t+1}/p_t$, and we denote the net supply $e_1 - c_{1t}$ of the good during youth by $S(\pi^e_{t+1})$. The constraint $m_t \geq 0$ implies $S(\pi^e_{t+1}) \geq 0$. Until this constraint is binding, S is equal to the ordinary supply curve of a consumer facing relative prices π^e_{t+1}; once the constraint is binding, S is equal to 0.

Recall the following assumption from the main paper:

Assumption 1

1. *There is* $\pi^a \in (1, \infty)$ *such that* $S(\pi) = 0$ *if and only if* $\pi \geq \pi^a$*;*
2. S *is continuous everywhere and is continuously differentiable on* $(0, \pi^a)$*;*
3. $S'(\pi) < 0$ *for* $\pi \in (0, \pi^a)$ *and* $S'(\pi^a) := \lim_{\pi \uparrow \pi^a} S'(\pi) < 0$*;*
4. $\lim_{\pi \downarrow 0} S(\pi) > \delta$.

One key substantive component of this assumption is that $\pi^a > 1$, which means that when the prices of youth and old-age consumption are equal, the consumer prefers to trade consumption today for consumption tomorrow. This will lead to the potential use of money as a medium for such trades, and is often called the "classical" case. This assumption holds,

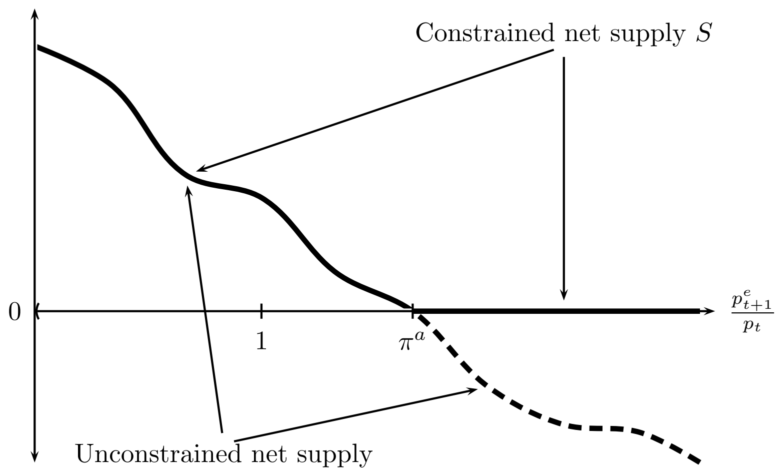

FIGURE 1. An illustration of Assumption 1. The wavy line, solid and dashed, is the unconstrained net supply of a two-period household in an OLG model. The solid line, is S, the actual supply curve given that households can save by buying money but cannot borrow.

for example, if U is monotone and symmetric and $e_1 > e_2$. The other key substantive component is that S is strictly decreasing until the constraint $m_t \geq 0$ binds. This holds if c_1 and c_2 are gross substitutes. Assumption 1 is illustrated in Figure 1.

Remark 1 Recall Example 2.1 in the main paper, in which $S(\pi) = a - b\pi$ for $\pi \in (0, \pi^a]$, where $a > b > 0$ and $\pi^a = a/b > 1$. S has these properties when U is Cobb-Douglas and e_1 is sufficiently larger than e_2. For example, if the domain of U is $\mathbb{R}^2_+$ and if $U(c_1, c_2) = c_1 c_2$, then $S(\pi) = (e_1 - e_2\pi)/2$ and $a > b \Leftrightarrow e_1 > e_2$.

2 Proof of Proposition 2.1

The following is Proposition 2.1 in the main paper. Here we provide a complete proof.

Proposition 1 *A price path* $\{p_t\}_{t=0}^{\infty} \in \mathbb{R}^{\infty}_{++}$ *is an equilibrium if and only if* $p_0(S(\pi_1^e) - \delta) = m_{-1}$ *and, for* $t \geq 1$,

$$\pi_t = \frac{S(\pi_t^e)}{S(\pi_{t+1}^e) - \delta} . \tag{2}$$

PROOF: Recall the period-t market clearing condition stated in Section 2 of the main paper:

$$p_t S(\pi_{t+1}^e) = m_{t-1} + p_t \delta . \tag{3}$$

Suppose $\{p_t\}_{t=0}^{\infty}$ is an equilibrium. Then equation (3) for period 0 is $p_0(S(\pi_1^e) - \delta) = m_{-1}$. Let $t \geq 1$. Equation (3) must hold for periods $t-1$ and t:

$$p_{t-1} S(\pi_t^e) = m_{t-2} + p_{t-1}\delta \tag{4}$$

$$p_t S(\pi_{t+1}^e) = m_{t-1} + p_t \delta . \tag{5}$$

Substitute $m_{t-1} = m_{t-2} + p_{t-1}\delta$ into equation (4) and then subtract this from equation (5):

$$p_t S(\pi^e_{t+1}) - p_{t-1} S(\pi^e_t) \;=\; p_t \delta \;.$$

Divide through by p_{t-1}, substitute $\pi_t = p_t/p_{t-1}$, and then rearrange to obtain equation (2).

Converse: Suppose that $p_0(S(\pi^e_1) - \delta) = m_{-1}$ and equation (2) holds for $t \geq 1$. Then equation (3) holds by assumption for period 1; we show by induction that it holds for period $t \geq 2$, given that it holds for period $t-1$. From equation (3) for period $t-1$, $p_{t-1}(S(\pi^e_t) - \delta) = m_{t-2}$, and from $m_{t-1} = m_{t-2} + p_{t-1}\delta$, we have $p_{t-1} S(\pi^e_t) = m_{t-1}$. Combining this with equation (2) for period t, $S(\pi^e_t) = \pi_t S(\pi^e_{t+1}) - \pi_t \delta$, yields $p_{t-1}\pi_t S(\pi^e_{t+1}) - p_{t-1}\pi_t\delta = m_{t-1}$, or $p_t S(\pi^e_{t+1}) = m_{t-1} + p_t\delta$. This is equation (3) for period t. □

3 Steady states in the affine example

Sections 2 of the main paper identifies low and high steady states $\pi^L(\delta)$ and $\pi^H(\delta)$. In this section we derive π^L and π^H for the affine example; see Figure 2.2 in the main paper for the corresponding graph.

Proposition 2 *Consider the affine example, in which $S(\pi) = a - b\pi$. Let $\hat{\delta} > 0$. For $\delta \in (0, \hat{\delta})$,*

$$\left\{\pi^L(\delta), \pi^H(\delta)\right\} \;=\; \left(a + b - \delta \pm \sqrt{(a+b-\delta)^2 - 4ab}\right)/2b \;.$$

Furthermore, for $\delta \in (0, \hat{\delta})$, $\pi^L(\delta)$ and $\pi^H(\delta)$ are the only steady states. If $\delta = \hat{\delta}$, there is a single steady state. If $\delta > \hat{\delta}$, there are no steady states.

Proof: The condition $\pi = W(\pi, \pi)$ for a steady state is

$$\begin{aligned} \pi &= \frac{a - b\pi}{a - b\pi - \delta} \\ 0 &= b\pi^2 - (a + b - \delta)\pi + a \;. \end{aligned}$$

The solutions to this quadratic equation are

$$\pi \;=\; \frac{a + b - \delta \pm \sqrt{(a+b-\delta)^2 - 4ab}}{2b} \;.$$

Let $h(\delta)$ be the term in the square-root; $h(\delta) := (a+b-\delta)^2 - 4ab$. There is a single solution if $h(\delta) = 0$; there are two (resp., no) real solutions if $h(\delta) > 0$ (resp., $h(\delta) < 0$). To conclude the proof, we have to show that $h(\hat{\delta})$ is the cutoff above which h is negative and below which h is positive. Since $h'(\delta) = -2(a + b - \delta) < 0$ (given that $\delta < S(0) = a$), it suffices to note that

$$h(\hat{\delta}) \;=\; \left(a + b - a - b + 2\sqrt{ab}\right)^2 - 4ab \;=\; 0 \;.$$

□

4 Rational expectations law of motion

Section 4 of the main paper sketches the derivation of a REE law of motion. Here we provide a full derivation, and find its functional form for the affine case. See Figures 4.1 in the main paper for the corresponding graph.

First we state precisely the sense in which equation (4.1) characterizes the REE:

Proposition 3 $\{\pi_t\}_{t=1}^{\infty}$ *is a REE inflation path if and only if* $S(\pi_1) > \delta$ *and, for* $t \geq 1$*,*

$$S(\pi_{t+1}) = S(\pi_t)/\pi_t + \delta \tag{6}$$

PROOF: Suppose $\{\pi_t\}_{t=1}^{\infty}$ is the inflation path associated with a REE price path $\{p_t\}_{t=0}^{\infty}$. Proposition 2.1, along with $\pi_t^e = \pi_t$, imply that $p_0(S(\pi_1) - \delta) = m_{-1}$ and hence $S(\pi_1) > \delta$, and that (6) holds for $t \geq 1$.

Converse: Suppose $\{\pi_t\}_{t=1}^{\infty}$ is a sequence of inflation factors such that $S(\pi_1) > \delta$ and (6) holds for all $t \geq 1$. Let $p_0 = m_{-1}/(S(\pi_1) - \delta)$. Observe that $p_0 > 0$ since $S(\pi_1) > \delta$. Define recursively $p_{t+1} = \pi_t p_t$, so that $\{p_t\}_{t=0}^{\infty}$ is a price path with associated inflation path $\{\pi_t\}_{t=1}^{\infty}$. Letting $\pi_t^e = \pi_t$ for $t \geq 1$, we see that $p_0 = m_{-1}(S(\pi_1^e) - \delta)$ and that (6) holds for $t \geq 1$. Hence, by Proposition 2.1, $\{p_t\}_{t=0}^{\infty}$ is a REE. □

Define $S(0) := \lim_{\pi \downarrow 0} S(\pi)$ and, for $\delta \in [0, S(0))$, let $\pi^{\min}(\delta)$ be the unique π such that $S(\pi)/\pi + \delta = S(0)$; the solution exists and is unique because $S(\pi)/\pi$ is a continuous and strictly decreasing function with range $[0, \infty)$ on the domain $(0, \pi^a]$. For $\pi \in (\pi^{\min}(\delta), \pi^a]$, $S(\pi)/\pi + \delta < S(\pi^{\min}(\delta))/\pi^{\min}(\delta) = S(0)$ since S is decreasing, and so $S^{-1}(S(\pi)/\pi + \delta)$ is well-defined for $\pi \in (\pi^{\min}(\delta), \pi^a)$. We can thus rewrite equation (6) as $\pi_t \in (\pi^{\min}(\delta), \pi^a)$ and $\pi_{t+1} = \Pi(\pi_t)$. Since S and S^{-1} are differentiable, so is Π. This defines the notation for and proves the following corollary:

Corollary 1 $\{\pi_t\}_{t=2}^{\infty}$ *is a REE inflation path if and only if (a)* $S(\pi_1) > \delta$ *and (b) for all* $t \geq 1$, $\pi_t \in (\pi^{\min}(\delta), \pi^a)$ *and* $\pi_{t+1} = \Pi(\pi_t)$*.*

Example 1 Consider the affine example, in which $S(\pi) = a - b\pi$. Then $\Pi(\pi) = (a + b - \delta - a/\pi)/b$, as follows: $S(0) = a$ and the equation $S(\pi)/\pi + \delta = S(0)$ is

$$\begin{aligned} (a - b\pi)/\pi + \delta &= a \\ a - b\pi + \delta\pi &= a\pi \\ a/(a + b - \delta) &= \pi \ . \end{aligned}$$

Since $S^{-1}(z) = (a - z)/b$,

$$\begin{aligned} \Pi(\pi) &= S^{-1}(S(\pi)/\pi + \delta) \\ &= (a - (a - b\pi)/\pi - \delta)/b \\ &= (a + b - \delta - a/\pi)/b \ . \end{aligned}$$

5 Miscellaneous inequalities

The proofs of Propositions 5.1 and 5.2 in Appendix A of the main paper require characterizing the magnitude of the roots of a quadratic equation. This section derives certain simple inequalities used in those proofs. Section 5.1 proves a basic lemma, and then Section 5.2 (resp., 5.3) uses this to prove a claim in the proof of Proposition 5.1 (resp., Proposition 5.2.

5.1 Characterization of roots of a quadratic equation

Lemma 1 *Consider a quadratic equation of the form*

$$x^2 + 2bx + c \;=\; 0\ .$$

The largest norm of the roots is characterized as follows:

	Largest norm of the roots < 1	> 1	$= 1$
$\lvert b\rvert > 1$		*always*	
$\lvert b\rvert \le 1,\ b^2 - c < 0$	$c < 1$	$c > 1$	$c = 1$
$\lvert b\rvert \le 1,\ b^2 - c \ge 0$	$2\lvert b\rvert - c < 1$	$2\lvert b\rvert - c > 1$	$2\lvert b\rvert - c = 1$

PROOF: The roots are $-b \pm \sqrt{b^2 - c}$. One of the roots has norm at least $|b|$; hence $|b| > 1$ implies that the largest norm is greater than 1. Suppose instead $|b| \le 1$.

Suppose $b^2 - c < 0$, so that the roots are imaginary, with real part $-b$ and imaginary part $\pm\sqrt{c - b^2}$. Then the norm of each root is $\sqrt{b^2 + (c - b^2)} = \sqrt{c}$.

Suppose instead $b^2 - c < 0$, so that the roots are real. The largest absolute value of the roots is $|b| + \sqrt{b^2 - c}$. Thus, we check

$$\begin{aligned} |b| + \sqrt{b^2 - c} &\gtreqless 1 \\ \sqrt{b^2 - c} &\gtreqless 1 - |b| \\ b^2 - c &\gtreqless 1 - 2\,|b| + b^2 \\ 2\,|b| - c &\gtreqless 1\ . \end{aligned}$$

□

5.2 A lemma in the proof of Proposition 5.1

The following lemma is stated as an unproved claim in the proof of Proposition 5.1 in the main paper.

Lemma 2 *Consider the quadratic equation*

$$x^2 - (A + B)x + A \;=\; 0\ ,$$

with $0 < A$ *and* $0 < B < 1$*. Both roots lie inside the unit circle if* $A < 1$*, and at least one lies outside the unit circle if* $A > 1$*.*

PROOF: We rewrite this equation as

$$x^2 + 2bx + c \;=\; 0\ ,$$

where $b = -(A+B)/2$ and $c = A$, so that we can refer to Lemma 1. Note that $2\,|b| - c = B$. Consider two cases:

1. Suppose $A < 1$. Then $|b| < 1$, $c < 1$, and $2\,|b| - c < 1$; hence, both roots lie inside the unit circle.
2. Suppose $A > 1$. If $|b| > 1$, then at least one root lies outside the unit circle. Suppose instead that $|b| \le 1$. Then $b^2 \le 1$ and hence $b^c - c < 0$. Since also $c > 1$, at least one root lies outside the unit circle.

□

5.3 A lemma in the proof of Proposition 5.2

Lemma 3 *Let $A, B, C \in \mathbb{R}$, and suppose $B < C$. Then*

$$\left|\frac{A+B}{A+C}\right| > 1 \qquad \text{if and only if} \qquad -(A+B) > A+C \ .$$

PROOF: Suppose $-(A+B) > A+C$.

1. If $A+C \geq 0$, then $A+B < 0$ and $(A+B)/(A+C) < -1$.
2. If $A+C > 0$, then $A+B < A+C < 0$, and $(A+B)/(A+C) > 1$.

Suppose $-(A+B) \leq A+C$.

1. If $A+B \leq 0$, then $0 \leq -(A+B) \leq A+C$ and $-1 \leq (A+B)/(A+C) \leq 0$.
2. If $A+B > 0$, then $0 < A+B \leq A+C$ and $0 < (A+B)/(A+C) \leq 1$.

□

6 Tatônnement stability

We consider the tatônnement stability of equilibria for the constant-gain adaptive expectations rule using current information, which is studied in Section 5.3 in the main paper.

Fix π_t^e and p_{t-1}. When the current price is p_t, the excess demand for the good is

$$Z(p_t) \ := \ m_{t-1}/p_t + \delta - S(\psi(p_t/p_{t-1}, \pi_t^e)) \ .$$

The condition for tatônnement stability is $Z'(p_t) < 0$. This derivative is

$$Z'(p_t) \ = \ -\frac{m_{t-1}}{p_t^2} - \frac{S'(\cdot)\psi_{\pi^i}(\cdot,\cdot)}{p_{t-1}} \ .$$

Let $z(p_t) := (p_t^2/p_{t-1})Z'(p_t)$, so that $z(p_t) = -(m_{t-1}/p_{t-1}) - \pi_t^2 S'(\cdot)\psi_{\pi^i}(\cdot,\cdot)$. From the equilibrium conditions $m_{t-1} = m_{t-2} + p_{t-1}\delta$ and $m_{t-2} = p_{t-1}(S(\pi_t^e) - \delta)$, we have $m_{t-1} = p_{t-1}S(\pi_t^e)$. Hence,

$$z(p_t) \ = \ -S(\pi_t^e) - \pi_t^2 S'(\pi_{t+1}^e)\psi_{\pi^i}(\pi_t, \pi_t^e) \ .$$

Near the steady state π^H with $\delta \approx 0$, we have $\pi_t^e \approx \pi^a$ and $S(\pi_t^e) \approx 0$. Then for $\pi_t \approx \pi^H$,

$$z(p_t) \ \approx \ -(\pi^a)^2 S'(\pi^a)\psi_{\pi^i}(\pi^a, \pi^a) \ > \ 0 \ ,$$

and hence π_t is not tatônnement stable.

However, a very low equilibrium inflation factor (close to zero)—which is not consistent with the steady state π^H—might be tatônnement stable. We illustrate this with the affine example $S(\pi) = a - b\pi$ and with $\psi(\pi^i, \pi^e) = \pi^i$. Then

$$z(p_t) \ = \ a + b\pi_t^e + b\pi_t^2 \ .$$

As shown in Section 3 of these notes, the high steady state is

$$\pi^H(\delta) \;=\; \frac{a+b-\delta+\sqrt{(a+b-\delta)^2-4ab}}{2b} \;.$$

The equilibrium condition $f(\pi_t, \pi_t^e) = 0$ for this example is

$$\begin{aligned} \pi_t(a - b\pi_t - \delta) - a + b\pi_t^e &= 0 \\ b\pi_t^2 - (a-\delta)\pi_t + a - b\pi_t^e &= 0 \;. \end{aligned}$$

The smaller solution is

$$F^1(\pi_t^e) \;=\; \frac{a-\delta-\sqrt{(a-\delta)^2-4b(a-b\pi_t^e)}}{2b} \;.$$

Thus, we have to evaluate $z(p_t)$ at $\pi_t^e = \pi^H(\delta)$ and $\pi_t = F^1(\pi^H(\delta))$. This was done numerically using Mathematica for a random sample of size 100,000, with parameter a normalized to 1 (this represents just a choice of units), b drawn uniformly from $(0,1)$, and δ drawn uniformly from $(0,\hat{\delta})$, where $\hat{\delta}$ is as defined in Section 3 of these notes. (See Appendix A for the printout.) In all cases, the sign was negative, and hence the lower equilibrium inflation rate (i.e., the lower equilibrium price) was tatônnement stable.

Consider now the low steady state π^L and $\delta \approx 0$. At the equilibrium consistent with the steady state,

$$z(p_t) \;\approx\; -S(1) - S'(1)\psi_{\pi^i}(1,1) \;.$$

This is negative—hence the equilibrium is tatônnement stable—if

$$-\frac{S(1)}{S'(1)} \;>\; \psi_{\pi^i}(1,1) \;. \tag{7}$$

If this inequality is reversed, the equilibrium is tatônnement unstable. Compare this with the condition for π^L to be (dynamically) stable, as stated in Proposition 5.2 in the main paper:

$$-\frac{S(1)}{S'(1)} \;<\; \frac{2\psi_{\pi^i}(1,1)}{2-\psi_{\pi^i}(1,1)} \;.$$

Since $(2-\psi_{\pi^i}(1,1))/2 < 1$, if π^L is stable, it is also tatônnement stable. However, if

$$\frac{2\psi_{\pi^i}(1,1)}{2-\psi_{\pi^i}(1,1)} \;<\; -\frac{S(1)}{S'(1)} \;<\; \psi_{\pi^i}(1,1) \;,$$

then π^L is dynamically *unstable* but tatônnement *stable.*

7 A lemma about the OLS rules

Section 7 of the main paper studies the OLS learning rules that are also the subject of Marcet and Sargent (1989b). A fact used there is that if $\pi_t \to \hat{\pi}$, then $\alpha_t \to 1 - \hat{\pi}^2$. This is essentially Lemma 1 in Marcet and Sargent (1989a). Here we restate and prove this result with our notation.

Lemma 4 *For the rules $OLS_{p_{t-1}}$ and OLS_{p_t}, if $\pi_t \to \hat{\pi} \geq 1$, then $\alpha_t \to 1 - \hat{\pi}^{-2}$.*

Proof: For OLS_{p_t}, $\alpha_t = p_{t-1}^2 / \sum_{s=0}^{t} p_{s-1}^2$. Since α_t for $\text{OLS}_{p_{t-1}}$ is equal to α_{t-1} for OLS_{p_t}, it suffices to prove the result for OLS_{p_t}.

Inverting α_t and replacing p_{t-1}/p_{s-1} by $\prod_{j=s}^{t-1} \pi_j$ yields, for t and t' such that $0 < t' < t$,

$$\alpha_t^{-1} \;=\; \sum_{s=0}^{t} \prod_{j=s}^{t-1} \pi_j^{-2} \;=\; \left(\sum_{s=0}^{t'} \prod_{j=s}^{t'-1} \pi_j^{-2} \right) \prod_{j=t'}^{t-1} \pi_j^{-2} + \sum_{s=t'+1}^{t} \prod_{j=s}^{t-1} \pi_j^{-2}$$

$$= \alpha_{t'}^{-1} \prod_{j=t'}^{t-1} \pi_j^{-2} + \sum_{s=t'+1}^{t} \prod_{j=s}^{t-1} \pi_j^{-2} \; .$$

Let $\pi > 1$ and suppose that $\pi_t \geq \pi$ (resp., $\pi_t \leq \pi$) for $t > t'$. Then

$$\alpha_t^{-1} \overset{(\geq)}{\leq} \alpha_{t'}^{-1} \pi^{-2(t-t')} + \sum_{s=t'+1}^{t} \pi^{-2(t-s)}$$

for $t > t'$. Since the limit of the r.h.s. as $t \to \infty$ is $(1 - \pi^{-2})^{-1}$, $\liminf_{t\to\infty} \alpha_t \geq 1 - \pi^{-2}$ (resp., $\limsup_{t\to\infty} \alpha_t \leq 1 - \pi^{-2}$). If $\pi_t \to \hat{\pi}$, then the first (resp., second) inequality must hold for any π such that $1 < \pi < \hat{\pi}$ (resp., $\pi > \hat{\pi}$). Hence, $\alpha_t \to 1 - \hat{\pi}^{-2}$. □

8 The simplest constant-gain expectations rules

Section 5 of the main paper characterizes stability for constant-gain expectations rules. The simplest example of such rules is $\psi(\pi_t^i, \pi_t^e) = \pi_t^i$. Here we state and derive the results for this example.

Example 2 Suppose $\pi_{t+1}^e = \pi_{t-1}$ for $t \geq 2$. Then equation (5.1) in the main paper is

$$\pi_{t+1}^e \;=\; \frac{S(\pi_{t-1}^e)}{S(\pi_t^e) - \delta} \; .$$

π^H is unstable for $\delta \approx 0$, and π^L is stable for $\delta \approx 0$ if $-S(1)/S'(1) > 1$. In the affine example, this condition is $a > 2b$.

Example 3 Suppose π_1^e is an initial condition and $\pi_{t+1}^e = \pi_t$ for $t \geq 1$. Then the equilibrium condition $\pi_t = W(\pi_{t+1}^e, \pi_t^e)$ can be written

$$\pi_t(S(\pi_t) - \delta) \;=\; S(\pi_t^e) \; . \tag{8}$$

To apply the implicit function theorem at a steady state $\hat{\pi}$, we must have $S(\hat{\pi}) - \delta + \hat{\pi}S'(\hat{\pi}) \neq 0$ (otherwise, the system is also unstable). Then, for π_t^e in a neighborhood of $\hat{\pi}$, the equilibrium selection $\pi_t = F(\pi_t^e)$ satisfies

$$F'(\pi_t^e) \;=\; \frac{S'(\pi_t^e)}{S(\pi_t) - \delta + \pi_t S'(\pi_t)} \; .$$

Since $\psi(\pi_t, \pi_t^e) = \pi_t$, our reduced-form difference equation is $\pi_{t+1}^e = F(\pi_t^e)$. At the steady state π^H and for $\delta \approx 0$,

$$F'(\pi^H) \;\approx\; \frac{S'(\pi^a)}{\pi^a S'(\pi_a)} \;=\; 1/\pi^a \;<\; 1 \; .$$

Hence, π^H is stable for $\delta \approx 0$. At the steady state π^L and for $\delta \approx 0$,

$$F'(\pi^L) \;\approx\; \frac{S'(1)}{S(1) + S'(1)} \; .$$

Hence, since $S(1) > 0$ and $S'(1) < 0$, π^L is unstable for $\delta \approx 0$ if $S(1) < -2S'(1)$. In the affine example, $\pi^L(\delta)$ is unstable for $\delta \approx 0$ if $a < 3b$.

9 Proofs for decreasing gain

The proof in the main paper of Proposition 6.3, which covers the current-information case, is through an application of results in Evans and Honkapohja (1999). This section also provides a direct proof of that proposition. It is merely a one-dimensional (and hence slightly simpler) version of the proof in Evans and Honkapohja (1999).

This section was written with the research assistance of Gorkem Celik.

Lemma 5 *Suppose $\{\alpha_t\}$ is a sequence in $(0,1)$ such that $\lim_{t\to\infty}\alpha_t = 0$ and $\sum_{t=1}^{\infty}\alpha_t = \infty$. Suppose $K > -1$. Let $y_s := \prod_{t=1}^{s}(1+\alpha_t K)$. Then $\lim_{s\to\infty} y_s = 0$ if $K < 0$ and $\lim_{s\to\infty} y_s = \infty$ if $K > 0$.*

PROOF: Taking logs of y_s yields

$$\log y_s \;=\; \sum_{t=1}^{s}\log(1+\alpha_t K)\ .$$

Suppose $K > 0$. Then $0 < K/2 < K = \frac{d}{d\alpha}\log(1+\alpha K)\big|_{\alpha=0}$. Therefore, there is $\epsilon > 0$ such that $\log(1+\alpha K) > \alpha K/2$ if $0 \geq \alpha < \epsilon$. Let τ be such that $\alpha_t < \epsilon$ for $t \geq \tau$. Then

$$\log y_s \;\geq\; \sum_{t=1}^{\tau-1}\log(1+\alpha_t K) + (K/2)\sum_{t=\tau}^{\infty}\alpha_t\ .$$

From $\sum_{t=1}^{\infty}\alpha_t = \infty$ it follows that $\lim_{s\to\infty}\log y_s = \infty$ and hence $\lim_{s\to\infty} y_s = \infty$.

Suppose instead $K < 0$. Since log is concave and $\frac{d}{d\alpha}\log(1+\alpha K)\big|_{\alpha=0} = K$, $\log(1+\alpha K) \leq \alpha K$ for $\alpha \in \mathbb{R}$. Therefore,

$$\log y_s \;\leq\; K\sum_{t=1}^{\infty}\alpha_t\ .$$

Hence $\lim_{s\to\infty}\log y_s = -\infty$ and $\lim_{s\to\infty} y_s = 0$. □

PROOF OF PROPOSITION 6.3: When $\pi_t^i = \pi_t$, our dynamic system can be written as the nonautonomous difference equation

$$\pi_{t+1}^e \;=\; \alpha_t F(\pi_t^e, \alpha_t) + (1-\alpha_t)\pi_t^e \;=:\; G(\pi_t^e, \alpha_t)\ , \tag{9}$$

where F is an equilibrium selection. That is, $F(\pi^e;\alpha)$ is a solution π to

$$f(\pi, \pi^e; \alpha) \;:=\; W(\alpha\pi + (1-\alpha)\pi^e, \pi^e) - \pi \;=\; 0$$

for any $\pi^e \in A$ and $\alpha \in [0,1)$.

We consider a steady state $\hat{\pi}$. Observe that f is continuously differentiable, even for negative α, as long as $\alpha\pi + (1-\alpha)\pi^e \in A$. Hence, since A is open, for any steady state $\hat{\pi}$ there is a neighborhood of $(\hat{\pi}, \hat{\pi}, 0)$ in $A \times A \times \mathbb{R}$ on which f is continuously differentiable, and

$$\begin{aligned} f_\pi(\hat{\pi}, \hat{\pi}; 0) &= -1\ , \\ f_{\pi^e}(\hat{\pi}, \hat{\pi}; 0) &= W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi})\ , \\ f_\alpha(\hat{\pi}, \hat{\pi}; 0) &= 0\ . \end{aligned}$$

By the implicit function theorem, we can choose an equilibrium selection F that is continuously differentiable in a neighborhood U of $(\hat{\pi}, 0)$, with

$$\begin{aligned} F_{\pi^e}(\hat{\pi}; 0) &= W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi}) \ , \\ F_\alpha(\hat{\pi}; 0) &= 0 \ . \end{aligned}$$

Furthermore, we can choose F so that $F(\hat{\pi}; \alpha) = \hat{\pi}$ for α such that $(\hat{\pi}, \alpha) \in U$.

Let τ be such that $(\hat{\pi}, \alpha_t) \in U$ for $t \geq \tau$. For $t = 1, \ldots, \tau - 1$, we note that

$$\begin{aligned} f_\pi(\hat{\pi}, \hat{\pi}; \alpha_t) &= \alpha_t W_1(\hat{\pi}, \hat{\pi}) - 1 \ , \\ f_{\pi^e}(\hat{\pi}, \hat{\pi}; \alpha_t) &= (1 - \alpha_t) W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi}) \ . \end{aligned}$$

By assumption, $f_\pi(\hat{\pi}, \hat{\pi}, \alpha_t) \neq 0$. Hence, we can invoke the implicit function theorem for each of these periods to choose F so that (a) $F(\hat{\pi}; \alpha_t) = \hat{\pi}$, (b) F is continuously differentiable in a neighborhood of $(\hat{\pi}, \alpha_t)$, and (c)

$$F_{\pi^e}(\hat{\pi}; \alpha_t) = \frac{(1 - \alpha_t) W_1(\hat{\pi}, \hat{\pi}) + W_2(\hat{\pi}, \hat{\pi})}{1 - \alpha_t W_1(\hat{\pi}, \hat{\pi})} \ .$$

Henceforth, normalize $\hat{\pi} = 0$. We can linearize F around $(0, 0)$, which means that we define the residual $r(\pi^e, \alpha_t)$ by

$$F(\pi^e; \alpha_t) = F_{\pi^e}(0; 0)\pi^e + r(\pi^e, \alpha_t)$$

(using $F_\alpha(0; 0) = 0$). Then we can write the difference equation (9) as

$$\pi^e_{t+1} = (1 - \alpha_t + \alpha_t F_{\pi^e})\pi^e_t + \alpha_t r(\pi^e_t, \alpha_t)) \tag{10}$$

(henceforth all derivatives are evaluated at $\hat{\pi} = 0$ and $\alpha_t = 0$).

Let $k > 0$ (the specific value will be chosen below). Because F is continuously differentiable on U and $F(0; \alpha) = 0$ for all $(0, \alpha) \in U$, there is $\epsilon > 0$ such that $|r(\pi, \alpha)| < k|\pi|$ if $|\pi| < \epsilon$ and $\alpha < \epsilon$. Let U_ϵ be the ϵ-ball around 0, and assume (w.l.o.g. by decreasing ϵ if necessary) that $U_\epsilon \times U_\epsilon \subset U$. Assume (w.l.o.g. by increasing τ if necessary) that $\alpha_t < \epsilon$ and $1 - \alpha_t + \alpha_t F_{\pi^e} > 0$ for $t \geq \tau$.

Stability Assume $W_1 + W_2 < 1$ and hence $F_{\pi^e} < 1$. Choose k such that $F_{\pi^e} + k < 1$ and let $K := -1 + F_{\pi^e} + k < 0$. If $t \geq \tau$ and $\pi^e_t \in U_\epsilon$ then the triangle inequality applied to equation (10) yields

$$\begin{aligned} |\pi^e_{t+1}| &\leq (1 - \alpha_t + \alpha_t F_{\pi^e})|\pi^e_t| + \alpha_t |r(\pi^e_t, \alpha_t)| \\ &\leq \big(1 + \alpha_t(-1 + F_{\pi^e} + k)\big)|\pi^e_t| = (1 + \alpha_t K)|\pi^e_t| \ . \end{aligned} \tag{11}$$

Suppose $\pi^e_\tau \in U_\epsilon$. One implication of equation (11) and $\alpha_t K < 0$ is that the sequence $\{\pi_t\}_{t=\tau}^\infty$ is decreasing. Iterate the inequality in equation (11) to obtain

$$|\pi^e_{\tau+s}| \leq |\pi^e_\tau| \prod_{t=\tau}^{\tau+s-1} (1 + \alpha_t K) \ .$$

By Lemma 5, $\lim_{s\to\infty} |\pi^e_{t+s}| = 0$.

Fix any neighborhood $V \subset U_\epsilon$ of 0. We show that there is a neighborhood of 0 such that for initial conditions in this neighborhood, $\pi^e_t \in V$ for $t = 1, \ldots, \tau$. It then follows from the above that $\pi^e_t \in V$ for $t \geq \tau$ and that $\pi^e_t \to 0$. Hence, 0 is a stable steady state.

This final step follows in the usual way from the local continuity of G at 0. Specifically, let $V_\tau := V$. For $t \in 1, \ldots, \tau - 1$, given V_{t+1}, let $V_t \subset V$ be a neighborhood of 0 such that $G(V_t, \alpha_t) \subset V_{t+1}$; such a neighborhood exists because $G(\cdot, \alpha_t)$ is continuous in a neighborhood of 0 and $G(0, \alpha_t) = 0$. If $\pi^e_1 \in V_1$ then $\pi_t \in V_t$ for $t = 1, \ldots, \tau$.

Instability Assume $W_1 + W_2 > 1$ and hence $F_{\pi^e} > 1$. Choose k so that $F_{\pi^e} - k > 1$, and let $K := F_{\pi^e} - k - 1 > 0$. If $t \geq \tau$ and $\pi_t^e \in U_\epsilon$ then the triangle inequality applied to (10) yields

$$\text{(12)} \quad \begin{aligned} |\pi_{t+1}^e| \ &\geq \ (1 - \alpha_t + \alpha_t F_{\pi^e})|\pi_t^e| - \alpha_t |r(\pi_t^e, \alpha_t)| \\ &\geq \ (1 + \alpha_t(-1 + F_{\pi^e} - k))|\pi_t^e| \ = \ (1 + \alpha_t K)|\pi_t^e| \end{aligned}$$

If $|\pi_\tau^e| > 0$ and $\pi_t^e \in U_\epsilon$ for all $t \geq \tau$, we can iterating inequality (12) to obtain

$$|\pi_{\tau+s}^e| \ \geq \ |\pi_\tau^e| \prod_{t=\tau}^{\tau+s-1} (1 + \alpha_t K) \ .$$

By Lemma 5, $\lim_{s\to\infty} |\pi_{\tau+s}^e| = \infty$. Hence, $\{\pi_t^e\}_{t=\tau}^\infty$ must leave U_ϵ: a contradiction.

To conclude, we show that any neighborhood of 0 contains an open set of initial conditions such that $0 < |\pi_\tau^e|$. This holds if $G_{\pi^e}(\hat{\pi}, \alpha_t) \neq 0$ for $t = 1, \ldots, \tau$, because then $\pi_t^e \mapsto G(\pi_t^e, \alpha_t)$ is a local diffeomorphism in a neighborhood of 0 as is the composition mapping $\pi_1^e \mapsto \pi_\tau^e$.

Observe that $G_{\pi^e}(\hat{\pi}, \alpha_t) = 1 - \alpha_t + \alpha_t F_{\pi^e}$. Hence, $G_{\pi^e}(\hat{\pi}, \alpha_t) \neq 0$ if $F_{\pi^e} \neq -(1-\alpha_t)/\alpha_t$. The latter condition is

$$\begin{aligned} \frac{(1-\alpha_t)W_1(\hat{\pi},\hat{\pi}) + W_2(\hat{\pi},\hat{\pi})}{1 - \alpha_t W_1(\hat{\pi},\hat{\pi})} \ &\neq \ -\frac{1-\alpha_t}{\alpha_t} \ , \\ \alpha_t(1-\alpha_t)W_1(\hat{\pi},\hat{\pi}) + \alpha_t W_2(\hat{\pi},\hat{\pi}) \ &\neq \ -(1-\alpha_t) + \alpha_t(1-\alpha_t)W_1(\hat{\pi},\hat{\pi}) \ , \\ W_2(\hat{\pi},\hat{\pi}) \ &\neq \ -\frac{1-\alpha_t}{\alpha_t} \ , \end{aligned}$$

which we assumed to hold (for the instability result) for all t. □

10 A continuous-time model

Section 8 of the main paper discusses a continuous-time analog of our discrete-time model. Here we outline and characterize this analog.

π now denotes a *instantaneous rate* of growth of prices. That is, $\pi_t = (dp/dt)/p_t$. The demand for real money balances at time t is $S(\pi_t^e)$, where the domain of S is now $(-\infty, \infty)$. We assume that S satisfies an analog of Assumption 2.1 in the main paper:

Assumption 2

1. *There is $\pi^a \in (0, \infty)$ such that $S(\pi) = 0$ if and only if $\pi \geq \pi^a$.*
2. *S is continuous everywhere and is continuously differentiable on $(-\infty, \pi^a)$.*
3. *For $\pi \in (-\infty, \pi^a)$, $S'(\pi) < 0$. Furthermore, $S'(\pi) := \lim_{\pi \uparrow \pi^a} S'(\pi) < 0$.*
4. *$\lim_{\pi \to -\infty} S(\pi) > \delta$.*

The static equilibrium condition that money demand equals money supply is

$$\text{(13)} \qquad S(\pi_t^e) \ = \ m_t/p_t \ .$$

We take logs to obtain equation (14), differentiate to obtain equation (15), and then change notation and substitute $\pi_t = (dp/dt)/p_t$ to obtain (16).

$$\log S(\pi_t^e) = \log m_t - \log p_t \ . \tag{14}$$

$$\frac{S'(\pi_t^e)}{S(\pi_t^e)}\frac{d\pi^e}{dt} = \frac{1}{m_t}\frac{dm}{dt} - \frac{1}{p_t}\frac{dp}{dt} \tag{15}$$

$$\frac{S'(\pi_t^e)}{S(\pi_t^e)}\dot{\pi}_t^e = \frac{\dot{m}_t}{m_t} - \pi_t \ . \tag{16}$$

δ is a rate at which the government consumes resources financed by expansion of the money supply. Hence, the government budget constraint is

$$\delta = (dm/dt)/p_t \ . \tag{17}$$

We define $\sigma(\pi_t^e) = \delta/S(\pi)$. Using again equation (13), we have $\sigma(\pi_t^e) = \dot{m}_t/m_t$. We also define $\rho(\pi_t^e) := -S'(\pi_t^e)/S(\pi_t^e)$ and make these two substitutions in equation (16):

$$-\rho(\pi_t^e)\dot{\pi}_t^e = \sigma(\pi_t^e) - \pi_t \tag{18}$$

We assume a standard linear expectations rule

$$\frac{d\pi^e}{dt} = \beta(\pi_t - \pi_t^e) \ , \tag{19}$$

which can be rearranged as $\pi_t = \frac{1}{\beta}\dot{\pi}_t^e + \pi_t^e$. We substitute this into equation (18) to obtain equation (20) and then rearrange to obtain equation (21). (Henceforth we drop the t subscript for simplicity.)

$$-\rho(\pi^e)\dot{\pi}^e = \sigma(\pi_t^e) - \frac{1}{\beta}\dot{\pi}^e - \pi^e \tag{20}$$

$$\dot{\pi^e} = \frac{\beta}{1-\beta\rho(\pi^e)}(\sigma(\pi_t^e) - \pi^e) =: g(\pi^e) \ . \tag{21}$$

A steady state $\hat{\pi}$ is defined by $g(\hat{\pi}) = 0$. We first characterize low-inflation and high-inflation steady states, in an analog to Proposition 2.2 in the main paper.

Proposition 4 *There is $\hat{\delta} > 0$ and a continuously differentiable function $\pi^H(\delta)$ defined on $[0, \hat{\delta})$ such that (a) for $\delta \in (0, \hat{\delta})$, $\pi^H(\delta)$ is a steady state; (b) $\pi^H(0) = \pi^a$; and (c) for $\delta \in [0, \hat{\delta})$, $d\pi^H/d\delta < 0$.*

If $\beta\rho(0) \neq 1$ and $\rho(0) \neq 1$, there is a continuously differentiable function $\pi^L(\delta)$ defined on $[0, \hat{\delta})$ such that (a) for $\delta \in (0, \hat{\delta})$, $\pi^L(\delta)$ is a steady state; (b) $\pi^L(0) = 0$; and (c) for $\delta \in [0, \hat{\delta})$, $d\pi^L/d\delta > 0$ if $\rho(0) < 1$ whereas $d\pi^L/d\delta < 0$ if $\rho(0) > 1$.

Proof: The condition $g(\pi) = 0$ can be written $\beta\rho(\pi) \neq 1$ and $\sigma(\pi) = \pi$. The latter can be rewritten (as long as $S(\pi) \neq 0$)

$$\eta(\pi, \delta) := \pi S(\pi) - \delta = 0 \ .$$

Then $\eta(0, 0) = 0$ and $\eta(\pi^a, 0) = 0$. Furthermore,

$$\begin{aligned}
\frac{\partial \eta}{\partial \pi} &= S(\pi) + \pi S'(\pi) \\
\frac{\partial g}{\partial \pi}(0, 0) &= S(0) + S'(0) \\
\frac{\partial g}{\partial \pi}(\pi^a, 0) &= \pi^a S'(\pi^a) \ .
\end{aligned}$$

Consider first the high-inflation steady state. Since $\lim_{\pi \uparrow \pi^a} \rho(\pi) = \infty$, the condition $\beta\rho(\pi) \neq 1$ is satisfied for $\pi \approx \pi^a$ and hence $\eta(\pi, \delta) = 0$ is necessary and sufficient for a steady state π in a neighborhood of π^a. Since also $\frac{\partial g}{\partial \pi}(\pi^a, 0) < 0$, by the implicit function theorem there is a neighborhood U of 0 and a continuously differentiable function π^H defined on U such that $\pi^H(0) = \pi^a$ and $\pi^H(\delta)$ is a steady state for $\delta \in U$. Furthermore, since $\frac{\partial g}{\partial \delta}(\pi^a, 0) = -1$, we can choose U so that $d\pi^H/d\delta < 0$ for $\delta \in U$.

Consider now the low-inflation steady state. We need the additional assumptions so that $\beta\rho(0) \neq 1$ and $S(0)+S'(0) \neq 0$, which are required for the IFT. Then there is a neighborhood U of 0 and a continuously differentiable function π^L defined on U such that $\pi^L(0) = 0$ and $\pi^L(\delta)$ is a steady state for $\delta \in U$. Furthermore, $d\pi^L/d\delta$ at $\delta = 0$ is $1/(S(0) + S'(0))$. We note that $S(0) + S'(0) > 0$ if $\rho(0) < 1$ and $S(0) + S'(0) < 0$ if $\rho(0) > 1$. □

The usual sufficient condition for stability (resp., instability) of a steady state $\hat{\pi}$ is that $g'(\hat{\pi}) < 0$ (resp., $g'(\hat{\pi}) > 0$). In a steady state $\hat{\pi}$, $\sigma(\hat{\pi}) = \hat{\pi}$. Hence, $g'(\hat{\pi}) = \frac{\beta}{1-\beta\rho(\hat{\pi})}(\sigma'(\hat{\pi}) - 1)$. Observe that

$$\sigma'(\pi^e) = -\delta S'(\pi^e)/S(\pi^e)^2 = \rho(\pi^e)\sigma(\pi^e) \; . \tag{22}$$

Hence,

$$g'(\hat{\pi}) = \frac{\beta}{1-\beta\rho(\hat{\pi})}(\rho(\hat{\pi})\sigma(\hat{\pi}) - 1) \; .$$

Consider first the steady state π^H. Observe that $\rho(\pi)\sigma(\pi) = -\delta S'(\pi)/S(\pi)^2$. Because $S(\pi^a) - 0$ and $d\pi^H/d\delta$ is finite at $\delta = 0$, one can show (e.g., using l'Hôpital's rule) that $\lim_{\delta \downarrow 0} \rho(\pi^H(\delta))\,\sigma(\pi^H(\delta)) = \infty$. Furthermore, since $\lim_{\delta \downarrow 0} \rho(\hat{\pi}) = \infty$, $g'(\pi^H) < 0$ for $\delta \approx 0$. Hence, π^H is stable for $\delta \approx 0$. This is the same stability property we found for the discrete-time, constant-gain model with *current* information.

Consider now the steady state π^L. At $\delta = 0$, $\rho(\pi^L)$ is finite whereas $\sigma(\pi^L) = 0$. Hence, the sign of $g'(\pi^L)$ is the sign of $\beta\rho(\pi^L) - 1$. For example, if $\rho(0) < 1$, we have $g'(\pi^L) < 0$ if $\beta < 1$. Furthermore, for fixed β, π^L is stable if $\rho(0)$ is close enough to zero, whereas for fixed π^L, π^L is unstable if β is large enough (enough weight is put on recent information). Qualitatively, these properties are shared by the discrete-time model whether lagged or current information is used.

11 The case of no government debt and price expectations

The main paper studies a model with constant deficit $\delta > 0$. This section extends the model and some results to case in which there is no government debt, i.e., $\delta = 0$, which we assume for the rest of this section. This was treated in an earlier version of the paper, but we deleted it it because of the notational tediousness of dealing with infinite prices and having to define an autarkic equilibrium as a special case that, for technical reasons, is not covered by the equilibrium conditions in the paper.

11.1 Extension of the model

We must expand the definition of an equilibrium. The definition of an equilibrium in the main paper (Definition 2.1) is still coherent and defines, for this case, a *non-autarkic* *equilibrium*. There is also an equilibrium, called an *autarkic* equilibrium, in which money has no value and there is no trade. Given our choice of money as the numeraire, this means that $p_t = p^e_{t+1} = \infty$ for all t. (It is plausible that money suddenly becomes worthless because, even if past prices have been finite, $p_t = \infty$ is a market clearing price if it induces $p^e_{t+1} = \infty$. As an ad-hoc restriction, we exclude from our definition of an equilibrium those price paths with both finite and infinite prices.) As a convention, we define the inflation factors and expected inflation factors to be π^a in an autarkic equilibrium.

There are exactly two steady-state REE in this case. One is defined by the same condition as in the main paper, which for $\delta = 0$ becomes $S(\pi) = S(\pi)/\pi$ and hence implies $\pi = 1$. There is no inflation and prices are constant in this steady state. The other steady state is the autarkic equilibrium, in which $p_t = \infty$ and $\pi_t = \pi^a$ in each period. Thus, one steady state is $\pi^L(0)$ and the other steady state is $\pi^H(0)$, where π^L and π^H are the low and high-inflation steady states defined in Section 4 the main paper. It is because of this and because the economy is approximately in autarky when π^e_t is close to π^a that we adopted the convention that $\pi_t = \pi^a$ in the autarkic equilibrium.

The equilibrium conditions that define non-autarkic equilibria, including the steady state $\pi_t = 1$, are the same as in the main paper even when $\delta = 0$. Hence, the analysis of stability of the lower inflation steady state in the main paper applies to this case, without modification. In the earlier version, we also verified that the stability properties of π^H with constant-gaih expectation rules, which are derived in Section 5 of the main paper, also hold when $\delta = 0$ and $\pi^H = \pi^a$. (Stability of π^a means that for any $\pi^1 < \pi^a$, there is $\pi^2 \in (\pi^1, \pi^a)$ such that if the initial conditions are in (π^2, π^a), then the equilibrium inflation is bounded below by π^1 and converges to π^a.) The proofs require minor modifications, because at $\delta = 0$ certain formulae appearing in the proofs in the main paper would involve division by 0.

11.2 Learning about prices

As an additional example that stability results can be sensitive to small details of the expectations rules, we study, for the case of $\delta = 0$, a class of expectations rules for which the expected price in each period is in the convex hull of the past prices. As a matter of convention, we will refer to these rules as "learning about prices", and we write $p^e_{t+1} := P^e_t(H_t)$. In contrast, the adaptive expectations rules in the main paper have the property that the expected inflation rate lies in the convex hull of past inflation rates.

The learning-about-prices rules include the expectations rules in Fuchs and Laroque (1976) and Tillman (1983), and the rule

$$\frac{1}{p^e_{t+1}} = \frac{1}{t+1}\sum_{s=-1}^{t-1}\frac{1}{p_s} = \frac{1}{t+1}\frac{1}{p_{t-1}} + \frac{t}{t+1}\frac{1}{p^e_t} \tag{23}$$

(where p_{-1} is an initial condition) in Lucas (1986), which sets the expected value of money to a weighted average of the past values of money. It also includes

$$p^e_{t+1} = \frac{1}{t+1}\sum_{s=-1}^{t-1} p_s = \frac{1}{t+1}p_{t-1} + \frac{t}{t+1}p^e_t\ , \tag{24}$$

or simpler rules such as $p^e_{t+1} = p_t$ and $p^e_{t+1} = p_{t-1}$. We show that the zero-inflation REE is stable, whereas the autarkic REE is not. This is a summary and generalization of the some of the results in the papers cited above; we refer mainly to Lucas (1986), but see Remark 2 at the end of Section 11.3 of these notes for a comparison with other papers.

When $\delta = 0$, the market clearing condition, equation (3), becomes

$$p_t S(p^e_{t+1}/p_t) \; = \; m_0 \; , \tag{25}$$

and the low-inflation stationary REE has a constant price given by

$$p^L \; = \; m_0/S(1) \; .$$

Therefore, it is possible for this stationary REE price to be a steady state when agents forecast that the future price is some average of past and current prices.

For the case where $U(c_{1t}, c_{2t}) = (c_{1t})^{1/2} + 2(c_{2t})$, Lucas (1986) finds that this steady state is stable for the adaptive rule in equation (23). More generally, we need only require that the expectations rule P^e have the following properties. Assumption 3 is the "expected prices lie in the convex hull of past realized and expected prices" condition by which we defined "learning about prices" expectations rules.

Assumption 3 *There are $0 < p^\perp \leq p^\top < \infty$ such that $P^e(p_0, \ldots, p_t)$ lies in the convex hull of $\{p^\perp, p^\top, p_0, \ldots, p_t\}$ for all t.*

Assumption 4 states that agents do not ignore persistent information forever; e.g., if the price is always above 15, eventually the agents believe the price will always be near or above 15.

Assumption 4 *For all sequences $\{p_t\}_{t=0}^\infty$ in $(0, \infty]$:*

$$\limsup_{t\to\infty} P^e(p_0, \ldots, p_t) \; \leq \; \limsup_{t\to\infty} p_t \; , \; \textit{and}$$
$$\liminf_{t\to\infty} P^e(p_0, \ldots, p_t) \; \geq \; \liminf_{t\to\infty} p_t \; .$$

Under these assumptions, we can show that in every equilibrium except the autarkic steady state, the price converges to p^L.

Proposition 5 *Under Assumptions 3 and 4, for any non-autarkic equilibrium $\{p_t\}_{t=0}^\infty$, $\lim_{t\to\infty} p_t = p^L$. Furthermore, if $P^e(H_t)$ does not depend on p_t, then there is a unique equilibrium and it is non-autarkic.*

PROOF: See Section 11.3. □

Section 11.3 provides a detailed explanation of this result. A brief summary of the intuition is as follows. Prices can only increase if price expectations, i.e., if inflation is expected. Since agents form their expectations by averaging past prices, they will expect prices do go down after the price has risen in the past. Thus prices cannot rise forever and inflation has to converge to unity. This, in turn, means that prices converge to p^L.

To illustrate the difference between price expectations and inflation expectations, it helps to think of the stochastic models that could justify the expectation rules. For example, the price expectation rules $p^e_{t+1} = p_t$ and $p^e_{t+1} = p_{t-1}$ satisfy $p^e_{t+1} = E[p_{t+1} \,|\, H_t]$ and $p^e_{t+1} = E[p_{t+1} \,|\, H_{t-1}]$, respectively, if $\{p_t\}_{t=0}^\infty$ follows a random walk. If $p^e_{t+1} = (1/t+1)\sum_{s=-1}^{t-1} p_t$ or $p^e_{t+1} = (1/t+1)\sum_{s=0}^{t} p_s$, then p^e_{t+1} is the OLS estimate of p_{t+1} given $\{p_{-1}, \ldots, p_{t-1}\}$ or $\{p_0, \ldots, p_t\}$, respectively, for the model $p_t = \bar{p} + \epsilon_t$. For inflation expectations, analogous to $p^e_{t+1} = p_t$ and $p^e_{t+1} = p_{t+1}$, we have the examples $\pi^e_{t+1} = \pi_t$ and $\pi^e_{t+1} = \pi_{t-1}$. These are the Bayesian forecasts if $\{\pi_t\}_{t=1}^\infty$, rather than $\{p_t\}_{t=0}^\infty$, follows a random walk. Other examples are the OLS estimates for the models $\pi_t = \bar{\pi} + \epsilon_t$ and $p_{t+1} = \bar{\pi} p_t + \epsilon_t$, considered in Section 7 of the main paper. The "learning about prices" rules are plausible enough for the case where there is no government debt. However, when we think of this as just a special case of a more general model in which $\delta \geq 0$, inflation expectations seem more plausible. E.g., there is always a steady state that is consistent with inflation expectations, but if $\delta > 0$ there is no steady state consistent with price expectations.

Note that if constant-gain expectations rules use current information, then the stability properties under "learning about prices" rules (π^L is stable and π^H is unstable) are the opposite of those under "learning about inflation" (π^L is unstable and π^H is stable). Thus, we find again that a simple or innocuous change can reverse stability properties. For example, Marimon et al. (1993) point out that the inclusion of current information does not make a difference for stability when agents forecast price levels. In contrast, we have shown (in the main paper and in Section 11 of these notes) that this does matter when agents forecast inflation factors as averages of past inflation factors.

11.3 Proof of stability under learning about prices

The following three lemmas and example are preliminary results for the proof of Proposition 5. Recall that the market clearing condition when $\delta = 0$ is

$$p_t S(p^e_{t+1}/p_t) \;=\; m_0 \;. \tag{26}$$

Lemma 6 *For $p^e_{t+1} \in (0, \infty)$, there is a unique $p_t \in (0, \infty)$ that solves (26). Let $P(p^e_{t+1})$ be this solution. The function $P\colon \mathbb{R}_{++} \to \mathbb{R}_{++}$ has the following properties:*

1. *P is continuous and strictly increasing.*
2. *The stationary monetary equilibrium price p^L is the unique fixed point of P.*
3. *If $p^e_{t+1} > p^L$, then $p^L < P(p^e_{t+1}) < p^e_{t+1}$.*
4. *If $p^e_{t+1} < p^L$, then $p^e_{t+1} < P(p^e_{t+1}) < p^L$.*
5. *$\lim_{n\to\infty} P^n(p) = p^L$ for all $p > 0$.*

Proof: Let $p^e_{t+1} > 0$. Since $S'(\cdot) < 0$, $p_t S(p^e_{t+1}/p_t)$ is strictly increasing in p_t, and it is continuous. Hence, there is a unique solution to (26) if $p_t S(p^e_{t+1}/p_t)$ takes on values less than and greater then m_0. When $p_t = p^e_{t+1}/\pi^a$, $p_t S(p^e_{t+1}/p_t) = p_t S(\pi^a) = 0$. Since $S'(\cdot) > 0$, $S(p^e_{t+1}/p_t) > 0$ for $p_t > p^e_{t+1}/\pi^a$ and $\lim_{p_t\to\infty} p_t S(p^e_{t+1}/p_t) = \infty$.

Differentiating (26),

$$\frac{\partial p_t}{\partial p^e_{t+1}} \;=\; \frac{-S'(p^e_{t+1}/p_t)}{S(p^e_{t+1}/p_t) - S'(p^e_{t+1}/p_t)(p^e_{t+1}/p_t)} \;>\; 0 \;. \tag{27}$$

Therefore, P is continuous and strictly increasing.

From (26), a fixed point p of P satisfies $pS(p/p) = m_0$, or $p = m_0/S(1)$, and hence is the unique stationary monetary equilibrium price p^L. From (27),

$$P'(p^L) \;=\; -S'(1)/(S(1) - S'(1)) \;<\; 1 \;.$$

Hence, the graph of P crosses the 45° line once at p^L, from above, which implies the remaining three properties of P. □

Example 4 For example, if $S(\pi) = a - b\pi$, then P is linear: $P(p^e_{t+1}) = (m_0 + bp^e_{t+1})/a$. The equation of motion for the REE is $p_t = P^{-1}(p_{t-1})$. From the properties of P, we can see that if $p_t > p^L$, then the path increases monotonically and the inflation factor converges monotonically to π^H. There is no equilibrium path with $p_t < p^L$, because the path would decrease monotonically until $P^{-1}(p_{t-1})$ is not defined. Hence, p^L is unstable, and all other equilibria converge to autarky. However, when prices are predicted as the average of past prices, the dynamics are inverted. This is the most obvious if we look at the adaptive rule

$$p^e_{t+1} \;=\; p_{t-1}\ . \tag{28}$$

Then, starting with a price expectation $p^e_1 = p_{-1}$, we get $p_0 = P(p^e_1)$. This then becomes the next price expectation, and so $p_1 = P(p^e_2) = P^2(p_{-1})$. For any $t \geq 0$, $p_t = P^{t+1}(p_{-1})$, whereas with perfect foresight expectations $p_t = (P^{-1})^{t+1}(p_{-1})$. The model has the same stationary equilibria, but the stability is reversed. Most paths converge to p^L with this adaptive rule, and a single perturbation from the autarkic equilibrium begins a path converging to p^L.

Lemma 7 *Under Assumption 4, if* $\liminf p_t > 0$ *and* $\limsup_{t\to\infty} p_t < \infty$*, then* $\lim_{t\to\infty} p_t = p^L$*.*

Proof: Suppose that $\liminf_{t\to\infty} p_t > 0$. Then

$$\liminf p_t \;\underset{(a)}{=}\; \liminf P(p^e_{t+1}) \;\underset{(b)}{=}\; P(\liminf p^e_{t+1}) \;\underset{(c)}{\geq}\; P(\liminf p_t)\ .$$

Equality (a) follows from $p_t = P(p^e_{t+1})$. Equality (b) follows from continuity of P. Inequality (c) follows from $\liminf p^e_{t+1} \geq \liminf p_t$ (according to Assumption 4) and the fact that P is increasing. That P is increasing and that $\liminf p_t \geq P(\liminf p_t)$ imply that $P(\liminf p_t) \geq P^2(\liminf p_t)$, and by induction $\liminf p_t \geq P^n(\liminf p_t)$ for all $n \in \mathbb{N}$.

Since $\lim_{n\to\infty} P^n(p) = p^L$ for all $p > 0$, $\liminf_{t\to\infty} p_t \geq p^L$. By a mirror argument, $\limsup_{t\to\infty} p_t \leq p^L$, and hence $\lim p_t = p^L$. □

Lemma 8 *Under Assumption 3,*

$$\min\{p^L, p^\perp\} \;\leq\; p_t \;\leq\; \max\{p^L, p^\top\}$$

for $t \geq 0$*.*

Proof: Assumption 3 implies that $p_0 \leq p^\top$, hence $\max\{p_0, \dots, p_t\} \leq \max\left\{p^L, p^\top\right\}$ holds for $t = 1$. Let $t \geq 2$ and suppose that $\max\{p_0, \dots, p_{t-1}\} \leq \max\left\{p^L, p^\top\right\}$. In the former case, $p_t \leq p^L$ and hence $\max\{p_0, \dots, p_t\} \leq \max\left\{p^L, p^\top\right\}$. In the latter case, since by Assumption 3 $p^e_{t+1} \leq \max\left\{p^\top, p_0, \dots, p_t\right\}$ and since $p^e_{t+1} > P_t$, $p^e_{t+1} \leq \max\left\{p^\top, p_0, \dots, p_{t-1}\right\}$. Since also $p^e_{t+1} > p^L$ and $\max\{p_0, \dots, p_{t-1}\} \leq \max\left\{p^L, p^\top\right\}$, it must be that $p^\top > p^L$ and $p^e_{t+1} \leq p^\top$. Since also $p_t < p^e_{t+1}$, $\max\{p_0, \dots, p_t\} \leq \max\left\{p^L, p^\top\right\}$. By induction, $\max\{p_0, \dots, p_t\} \leq \max\left\{p^L, p^\top\right\}$ for all $t \geq 1$. □

Now we complete the proof of the proposition:

Proof of Proposition 5: 1. Lemma 8 implies that the condition in Lemma 7 is satisfied, and hence $\lim_{t\to\infty} p_t = p^L$.

2. If $p^\perp \geq p^L$ (resp., $p^\top \leq p^L$), then Lemma 8 implies that $p_t \geq p^L$ (resp., $p_t \leq p^L$) for all t.

3. If P^e_{t+1} does not depend on p_t, the equilibrium in each period is determined uniquely by $p_t = P(P^e_{t+1}(\cdot))$ and price expectations are then uniquely determined by past prices. Hence, there can be only one equilibrium path. (If p^e_{t+1} depends on p_t, then p_t is on both the left and right side of the equation $p_t = P(P^e_{t+1}(p_0, \dots, p_t))$, and there could be multiple solutions.) □

Remark 2 Fuchs and Laroque (1976) study a version of this OLG model (with $\delta = 0$), allowing for any finite number of goods and consumers (with heterogeneous expectations). Part (i) of Corollary 1 in Section 3 in their paper shows that the monetary steady state is (*locally*) stable, which is analogous to our (nearly) global stability result. They assume that $\hat{S}'(\pi) \leq 0$ rather than $\hat{S}'(\pi) < 0$. Expectations are a stationary function of the current price and the past T prices. I.e., there is $\psi\colon \mathbb{R}^{T+1} \to \mathbb{R}$ such that $P^e_t(H_t) = \psi(p_{t-T}, \dots, p_t)$ for $t \geq T + 1$. Their assumption that ψ is continuously differentiable, that $\psi(p, \dots, p) = p$ for all p, and that $\frac{\partial \psi}{\partial p_{t-s}}(p_{t-T}, \dots, p_t) \geq 0$ for $s = 0, \dots, T$ imply that P^e_t satisfies our Assumptions 3 and 4.

Tillman (1983) studies a single good OLG model (with $\delta = 0$) with multiple consumers. Part (a) of Proposition 3 is a global stability result analogous to ours, but with more restrictive assumptions on the expectations rule. He assumes there is $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ such that $P^e_t(H_t) = \psi(p_{t-1}, p_t)$ for $t \geq 2$, with $\psi(p, p) = p$ for all p and assumptions on the derivatives of ψ that imply that our Assumptions 3 and 4 are satisfied. In one version of the result, he assumes a specific form of the utility function that implies that $\hat{S}'(\pi) < 0$, and in another this assumption is replaced by an additional restriction on the derivatives of ψ.

Appendix A Mathematica input/output for test of tatônnement stability

Tatonnement Stability of Affine Example

This is numerical test of tatonnment stability of lower equilibrium inflation rate for affine example, when system is at or near high steady state. See also the section on tatonnement stability in the supplement.

Tatonnement stability is given by sign of following function:

```
z[piE_, pi_, a0_, a1_] := -a0 + a1 ( piE + pi^2)
```

The high steady state is:

```
piH[a0_, a1_, d_] :=
 (a0 + a1 - d + Sqrt[(a0 + a1 - d)^2 - 4 a0 a1] ) / (2 a1)
```

The low equilibrium inflation rate is:

```
F[piE_, a0_, a1_, d_] :=
 (a0 - d - Sqrt[(a0 - d)^2 - 4 a1 (a0 - a1 piE)] ) / (2 a1)
```

So we have to check the sign of:

```
X[a0_, a1_, d_] :=
 z[piH[a0, a1, d], F[piH[a0, a1, d], a0, a1, d], a0, a1]
```

It is also useful to know highest value of delta:

```
dmax[a0_, a1_] := (a0 + a1) - 2 Sqrt[a0 a1]
```

Now we can randomly test. The absolute value of a0 does not matter and is normalized to 1 (this just sets the units by which the real good is measured). Then a1 is drawn uniformly from (0,1), and d is drawn uniformly from (0,dmax[1,a1]).

```
RandomTest :=
 (a1 = Random[]; d = dmax[1, a1] Random[]; X[a0, a1, d])
```

This does n tests, and the reports values with positive sign:


```
DoTest[n_] := Module[{list, out},
  list = {};
  For[i = 1, i < n, i++,
   out = RandomTest;
   If[Sign[out] == 1, Append[list, {a0, a1, d, out}]]
  ];
  Return[list]
 ]
```

None found!

```
DoTest[100000]

{}
```

Appendix B Mathematica input/output for proof of Proposition 6.4 in main paper

Stability with decreasing gain and lagged information

We linearize the difference equation, obtaining $\theta_{t+1} = M_t\, \theta_t$, and diagonalize the matrix, writing $M_t = S_t\, \Lambda_t\, S_t^{-1}$. Hence, $\theta_{t+k+1} = S_{t+k}\, \Lambda_{t+k}\, \Gamma_{t+k-1} \cdots \Gamma_t\, S_t^{-1}\, \theta_t$, where $\Gamma_t = S_{t+1}^{-1}\, S_t\, \Lambda_t$. Our task is to obtain a bound on the norm of Γ_t.

Definitions:

```
M[α_] := (1 - α + α W1   α W2
              1            0 )

EigVal[α_] := Eigenvalues[M[α]]

Λ[α_] := DiagonalMatrix[EigVal[α]]

S[α_] := Transpose[Eigenvectors[M[α]]]
```

Let's see what these look like:

```
TableForm[{
  {"M[α]:", MatrixForm[M[α]]},
  {"M[0]:", MatrixForm[M[0]]}
 }]
```

M[α]: $\begin{pmatrix} 1-\alpha+\alpha W_1 & \alpha W_2 \\ 1 & 0 \end{pmatrix}$

M[0]: $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

```
TableForm[{
  {"EigVal[α]:", MatrixForm[EigVal[α]]},
  {"EigVal[0]:", MatrixForm[EigVal[0]]}
 }]
```

EigVal[α]: $\begin{pmatrix} \frac{1}{2}\left(1-\alpha+\alpha W_1-\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}\right) \\ \frac{1}{2}\left(1-\alpha+\alpha W_1+\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}\right) \end{pmatrix}$

EigVal[0]: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

```
TableForm[{{"Λ[α]:", MatrixForm[Λ[α]]}, {"Λ[0]:", MatrixForm[Λ[0]]}}]
```

Λ[α]: $\begin{pmatrix} \frac{1}{2}\left(1-\alpha+\alpha W_1-\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}\right) & 0 \\ 0 & \frac{1}{2}\left(1-\alpha+\alpha W_1+\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}\right) \end{pmatrix}$

Λ[0]: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

```
TableForm[{{"S[α]:", MatrixForm[S[α]]}, {"S[0]:", MatrixForm[S[0]]}}]
```

S[α]: $\begin{pmatrix} \frac{1}{2}\left(1-\alpha+\alpha W_1-\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}\right) & \frac{1}{2}\left(1-\alpha+\alpha W_1+\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}\right) \\ 1 & 1 \end{pmatrix}$

S[0]: $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

```
TableForm[
 {{"S-1[α]:", MatrixForm[Inverse[S[α]]]}, {"S-1[0]:", MatrixForm[Inverse[S[0]]]}}]
```

$S^{-1}[\alpha]$: $\begin{pmatrix} -\frac{1}{\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}} & \frac{1-\alpha+\alpha W_1+\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}}{2\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}} \\ \frac{1}{\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}} & -\frac{1-\alpha+\alpha W_1-\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}}{2\sqrt{(-1+\alpha-\alpha W_1)^2+4\alpha W_2}} \end{pmatrix}$

$S^{-1}[0]$: $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$

- **Here we check that $M_t = S_t\,\Lambda_t\,S_t^{-1}$:**

```
TableForm[{
  {"M[α]:", MatrixForm[M[α]],
   "St Λt St-1:", MatrixForm[Simplify[S[α] . Λ[α] . Inverse[S[α]]]]},
  {"M[0]:", MatrixForm[M[0]],
   "S∞ Λ∞ S∞-1:", MatrixForm[Simplify[S[0] . Λ[0] . Inverse[S[0]]]]}
 }]
```

M[α]:	$\begin{pmatrix} 1-\alpha+\alpha W_1 & \alpha W_2 \\ 1 & 0 \end{pmatrix}$	$S_t\ \Lambda_t\ S_t^{-1}$:	$\begin{pmatrix} 1-\alpha+\alpha W_1 & \alpha W_2 \\ 1 & 0 \end{pmatrix}$
M[0]:	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$S_\infty\ \Lambda_\infty\ S_\infty^{-1}$:	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

- **Also curious whether $S_{t+1}^{-1} S_t$ is approximately the identity:**

```
SS = Inverse[S[α_{t+1}]] . S[α_t]
```

$$\left\{\left\{-\frac{1-\alpha_t+W_1\alpha_t-\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}+\frac{1-\alpha_{1+t}+W_1\alpha_{1+t}+\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}},\right.\right.$$
$$\left.-\frac{1-\alpha_t+W_1\alpha_t+\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}+\frac{1-\alpha_{1+t}+W_1\alpha_{1+t}+\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}\right\},$$
$$\left\{\frac{1-\alpha_t+W_1\alpha_t-\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}-\frac{1-\alpha_{1+t}+W_1\alpha_{1+t}-\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}},\right.$$
$$\left.\left.\frac{1-\alpha_t+W_1\alpha_t+\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}-\frac{1-\alpha_{1+t}+W_1\alpha_{1+t}-\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}\right\}\right\}$$

```
Simplify[%]
```

$$\left\{\left\{\frac{-(-1+W_1)\alpha_t+\sqrt{4W_2\alpha_t+(1+(-1+W_1)\alpha_t)^2}+(-1+W_1)\alpha_{1+t}+\sqrt{4W_2\alpha_{1+t}+(1+(-1+W_1)\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(1+(-1+W_1)\alpha_{1+t})^2}},\right.\right.$$
$$\left.\frac{-(-1+W_1)\alpha_t-\sqrt{4W_2\alpha_t+(1+(-1+W_1)\alpha_t)^2}+(-1+W_1)\alpha_{1+t}+\sqrt{4W_2\alpha_{1+t}+(1+(-1+W_1)\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(1+(-1+W_1)\alpha_{1+t})^2}}\right\},$$
$$\left\{\frac{(-1+W_1)\alpha_t-\sqrt{4W_2\alpha_t+(1+(-1+W_1)\alpha_t)^2}-(-1+W_1)\alpha_{1+t}+\sqrt{4W_2\alpha_{1+t}+(1+(-1+W_1)\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(1+(-1+W_1)\alpha_{1+t})^2}},\right.$$
$$\left.\left.\frac{(-1+W_1)\alpha_t+\sqrt{4W_2\alpha_t+(1+(-1+W_1)\alpha_t)^2}-(-1+W_1)\alpha_{1+t}+\sqrt{4W_2\alpha_{1+t}+(1+(-1+W_1)\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(1+(-1+W_1)\alpha_{1+t})^2}}\right\}\right\}$$

- **Define $\Gamma_t = S_{t+1}{}^{-1} S_t \Lambda_t$**

```
Γt = Inverse[S[αt+1]] . S[αt] . Λ[αt]
```

$$\left\{\left\{\frac{1}{2}\left(1-\alpha_t+W_1\alpha_t-\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}\right)\left(-\frac{1-\alpha_t+W_1\alpha_t-\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}+\frac{1-\alpha_{1+t}+W_1\alpha_{1+t}+\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}\right),\right.\right.$$
$$\left.\frac{1}{2}\left(1-\alpha_t+W_1\alpha_t+\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}\right)\left(-\frac{1-\alpha_t+W_1\alpha_t+\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}+\frac{1-\alpha_{1+t}+W_1\alpha_{1+t}+\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}\right)\right\},$$
$$\left\{\frac{1}{2}\left(1-\alpha_t+W_1\alpha_t-\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}\right)\left(\frac{1-\alpha_t+W_1\alpha_t-\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}-\frac{1-\alpha_{1+t}+W_1\alpha_{1+t}-\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}\right),\right.$$
$$\left.\left.\frac{1}{2}\left(1-\alpha_t+W_1\alpha_t+\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}\right)\left(\frac{1-\alpha_t+W_1\alpha_t+\sqrt{4W_2\alpha_t+(-1+\alpha_t-W_1\alpha_t)^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}-\frac{1-\alpha_{1+t}+W_1\alpha_{1+t}-\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}{2\sqrt{4W_2\alpha_{1+t}+(-1+\alpha_{1+t}-W_1\alpha_{1+t})^2}}\right)\right\}\right\}$$

- **Check the norm of Γ_t:**

We need to check the norm of Γ_t. We use the norm that is the maximum of the sum of the absolute values of the columns (because it is the simplest in this case). Assume first that all the terms of Γ_t are positive. Then the sums of the first and second columns are quite simple:

```
norm1 = Simplify[Γt[[1]][[1]] + Γt[[2]][[1]] ]
```

$$\frac{1}{2}\left(1+(-1+W_1)\alpha_t-\sqrt{4W_2\alpha_t+(1+(-1+W_1)\alpha_t)^2}\right)$$

```
norm2 = Simplify[Γt[[1]][[2]] + Γt[[2]][[2]] ]
```

$$\frac{1}{2}\left(1+(-1+W_1)\alpha_t+\sqrt{4W_2\alpha_t+(1+(-1+W_1)\alpha_t)^2}\right)$$

For small α_t, norm1 is close to zero and norm2 is close to 1. Hence, the overall norm is norm2. We can derive an upper bound on norm2 that allows us to complete the proof.

Let us consider the assumption that terms of Γ_t are positive. We assume that α_t and α_{t+1} are close to zero.

The top-left and bottom-left terms are close to zero because $\left(1 - \alpha_t + W_1 \alpha_t - \sqrt{4 W_2 \alpha_t + (-1 + \alpha_t - W_1 \alpha_t)^2}\right)$ is close to zero. Hence, even if either of these terms is negative, the sum of their absolute values is close to zero.

The top-right term is close to zero because $\left(-\frac{1-\alpha_t+W_1 \alpha_t+\sqrt{4 W_2 \alpha_t+(-1+\alpha_t-W_1 \alpha_t)^2}}{2 \sqrt{4 W_2 \alpha_{1+t}+(-1+\alpha_{1+t}-W_1 \alpha_{1+t})^2}} + \frac{1-\alpha_{1+t}+W_1 \alpha_{1+t}+\sqrt{4 W_2 \alpha_{1+t}+(-1+\alpha_{1+t}-W_1 \alpha_{1+t})^2}}{2 \sqrt{4 W_2 \alpha_{1+t}+(-1+\alpha_{1+t}-W_1 \alpha_{1+t})^2}}\right)$ is close to -1+1=0.

The bottom-left term is close to 1.

Hence, the possibility we need to consider is that the top-right term is negative. This complicates our lives because then the sum of the absolute values of the right-hand column is

```
norm2alt = -Γt[[1]][[2]] + Γt[[2]][[2]]
```

$$\frac{1}{2}\left(1 - \alpha_t + W_1 \alpha_t + \sqrt{4 W_2 \alpha_t + (-1 + \alpha_t - W_1 \alpha_t)^2}\right)\left(\frac{1 - \alpha_t + W_1 \alpha_t + \sqrt{4 W_2 \alpha_t + (-1 + \alpha_t - W_1 \alpha_t)^2}}{2\sqrt{4 W_2 \alpha_{1+t} + (-1 + \alpha_{1+t} - W_1 \alpha_{1+t})^2}} - \frac{1 - \alpha_{1+t} + W_1 \alpha_{1+t} - \sqrt{4 W_2 \alpha_{1+t} + (-1 + \alpha_{1+t} - W_1 \alpha_{1+t})^2}}{2\sqrt{4 W_2 \alpha_{1+t} + (-1 + \alpha_{1+t} - W_1 \alpha_{1+t})^2}}\right) - \frac{1}{2}\left(1 - \alpha_t + W_1 \alpha_t + \sqrt{4 W_2 \alpha_t + (-1 + \alpha_t - W_1 \alpha_t)^2}\right)\left(-\frac{1 - \alpha_t + W_1 \alpha_t + \sqrt{4 W_2 \alpha_t + (-1 + \alpha_t - W_1 \alpha_t)^2}}{2\sqrt{4 W_2 \alpha_{1+t} + (-1 + \alpha_{1+t} - W_1 \alpha_{1+t})^2}} + \frac{1 - \alpha_{1+t} + W_1 \alpha_{1+t} + \sqrt{4 W_2 \alpha_{1+t} + (-1 + \alpha_{1+t} - W_1 \alpha_{1+t})^2}}{2\sqrt{4 W_2 \alpha_{1+t} + (-1 + \alpha_{1+t} - W_1 \alpha_{1+t})^2}}\right)$$

Even simplified this is ugly:

```
Simplify[norm2alt]
```

$$\Bigg(2 (-1 + W_1)^2 \alpha_t^2 - \left(1 + \sqrt{4 W_2 \alpha_t + (1 + (-1 + W_1) \alpha_t)^2}\right)(-1 + (-1 + W_1) \alpha_{1+t}) + \alpha_t \Big(-3 + 4 W_2 - 2\sqrt{4 W_2 \alpha_t + (1 + (-1 + W_1) \alpha_t)^2} - \alpha_{1+t} - W_1^2 \alpha_{1+t} + W_1 \left(3 + 2\sqrt{4 W_2 \alpha_t + (1 + (-1 + W_1) \alpha_t)^2} + 2 \alpha_{1+t}\right)\Big)\Bigg) \Big/ \left(2\sqrt{4 W_2 \alpha_{1+t} + (1 + (-1 + W_1) \alpha_{1+t})^2}\right)$$

Let's decompose the upper-right term of Γ_t as g1[α_]g2[α_]

```
Γ12 = Γt[[1]][[2]]
```

$$\frac{1}{2}\left(1 - \alpha_t + W_1 \alpha_t + \sqrt{4 W_2 \alpha_t + (-1 + \alpha_t - W_1 \alpha_t)^2}\right)\left(-\frac{1 - \alpha_t + W_1 \alpha_t + \sqrt{4 W_2 \alpha_t + (-1 + \alpha_t - W_1 \alpha_t)^2}}{2\sqrt{4 W_2 \alpha_{1+t} + (-1 + \alpha_{1+t} - W_1 \alpha_{1+t})^2}} + \frac{1 - \alpha_{1+t} + W_1 \alpha_{1+t} + \sqrt{4 W_2 \alpha_{1+t} + (-1 + \alpha_{1+t} - W_1 \alpha_{1+t})^2}}{2\sqrt{4 W_2 \alpha_{1+t} + (-1 + \alpha_{1+t} - W_1 \alpha_{1+t})^2}}\right)$$

We can write $\Gamma_{12} = \frac{1}{4}\ \frac{g(\alpha_t)}{h(\alpha_{t+1})}\ (g(\alpha_{t+1}) - g(\alpha_t))$, where

```
g[α_] := 1 + (W1 - 1) α + Sqrt[4 W2 α + (-1 + α - W1 α)^2]

h[α_] := Sqrt[4 W2 α + (-1 + α - W1 α)^2]
```

$$g[\alpha\_] := 1 + (W_1 - 1)\,\alpha + \sqrt{4\,W_2\,\alpha + (-1 + \alpha - W_1\,\alpha)^2}$$

$$h[\alpha\_] := \sqrt{4\,W_2\,\alpha + (-1 + \alpha - W_1\,\alpha)^2}$$

One can show (separate notes??) that $g'\ (0) < 0$. Thus, $\Gamma_{12} \geq 0$ if $\alpha_{t+1} \leq \alpha_t$.

If not, what can we say about norm2? Recall:

```
norm2alt
```

$$\frac{1}{2}\left(1 - \alpha_t + W_1\,\alpha_t + \sqrt{4\,W_2\,\alpha_t + (-1 + \alpha_t - W_1\,\alpha_t)^2}\right)\left(\frac{1 - \alpha_t + W_1\,\alpha_t + \sqrt{4\,W_2\,\alpha_t + (-1 + \alpha_t - W_1\,\alpha_t)^2}}{2\sqrt{4\,W_2\,\alpha_{1+t} + (-1 + \alpha_{1+t} - W_1\,\alpha_{1+t})^2}} - \frac{1 - \alpha_{1+t} + W_1\,\alpha_{1+t} - \sqrt{4\,W_2\,\alpha_{1+t} + (-1 + \alpha_{1+t} - W_1\,\alpha_{1+t})^2}}{2\sqrt{4\,W_2\,\alpha_{1+t} + (-1 + \alpha_{1+t} - W_1\,\alpha_{1+t})^2}}\right) -$$
$$\frac{1}{2}\left(1 - \alpha_t + W_1\,\alpha_t + \sqrt{4\,W_2\,\alpha_t + (-1 + \alpha_t - W_1\,\alpha_t)^2}\right)\left(-\frac{1 - \alpha_t + W_1\,\alpha_t + \sqrt{4\,W_2\,\alpha_t + (-1 + \alpha_t - W_1\,\alpha_t)^2}}{2\sqrt{4\,W_2\,\alpha_{1+t} + (-1 + \alpha_{1+t} - W_1\,\alpha_{1+t})^2}} + \frac{1 - \alpha_{1+t} + W_1\,\alpha_{1+t} + \sqrt{4\,W_2\,\alpha_{1+t} + (-1 + \alpha_{1+t} - W_1\,\alpha_{1+t})^2}}{2\sqrt{4\,W_2\,\alpha_{1+t} + (-1 + \alpha_{1+t} - W_1\,\alpha_{1+t})^2}}\right)$$

This is equal to $\frac{1}{2}\ \frac{g(\alpha_t)}{h(\alpha_{t+1})}\ (g(\alpha_t) - 1 - (W_1 - 1)\,\alpha_{t+1})$

... Giving up here. Just assume $\alpha_{t+1} \leq \alpha_t$.

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