This appendix material solves the problem faced by liquidity supplying investors in the infinite-horizon version of the simultaneous trade model. It appears in the paper "Inventory Information," by H. Cao, M. Evans, and R. Lyons (March 2002).

A.1. Proof of Proposition 1: Public Investors

The liquidity-supplying (LS) investors have the following utility defined over intertemporal consumption c_t (per equation 1):

(1)
$$U_t = \sum_{t=t}^{\infty} -d^{t} \exp\left(-g\, c_t\right)$$

We begin by conjecturing a value function, which we show below is consistent with optimizing behavior on the part of LS investors:

$$V_t = -a \exp\left(-\hat{g} W_t - y\, h_t^2\right)$$

where W_t is the LS investors' nominal wealth at the end of day t and h_t is the total holding of the risky asset at the end of day *t* (defined in equation 9). We need to determine the conditions under which h_t is willingly held by the LS investors.

Given the proposed price function $P_{4t} = -a h_t$ in text proposition 1, our task is to begin with the Bellman equation corresponding to the maximum of equation (1) and derive explicit expressions for the three coefficient values $\hat{g}$, y, and a, in this conjectured value function, as well as an expression for the parameter a in the price function. We shall show that we must have:

$$\hat{g} = \left(\frac{r}{1+r}\right) g$$

$$y = \left(\frac{1}{2ns_x^2}\right) + (1+r)a\hat{g} - \left(\frac{(a\hat{g} + \frac{1}{ns_x^2})[(1+r)a - s_R^2\hat{g}]}{2a}\right) - \left(\frac{s_R^2\hat{g}}{2}\right)$$

$$a = \left(\frac{\hat{g}}{g} a \sqrt{1+2ns_x^2 y}\right)^{\frac{1}{1+r}} + \left(\frac{\hat{g}}{g} a \sqrt{1+2ns_x^2 y}\right)^{-\frac{r}{1+r}} \left(\frac{a}{1+r}\sqrt{1+2ns_x^2 y}\right)$$

and $$a^2\hat{g} - a\left(\frac{r}{ns_x^2} + 2y + 2ry\right) + \left(\frac{1}{ns_x^2} + 2y\right)\hat{g}s_R^2 = 0$$

To prove these conditions, write down the Bellman equation:

$$\mathrm{V_t} = \underset{\{c_t, D_t\}}{\mathrm{Max}} \; -\exp\left(-g\,\mathrm{c_t}\right) - d\,\mathrm{E_t}\left[a\,\exp\left(-\hat{g}\,\mathrm{W_{t+1}} - y\,\mathrm{h_{t+1}^2}\right)\right]$$

where

$$W_{t+1} = (1+r)\left(W_t - c_t\right) + D_t\left(P_{t+1} + R_{t+1} - (1+r)P_t\right)$$

and where we have used D_t to denote the LS investors' demand for the risky asset. The first order condition with respect to c_t is:

$$g\,\exp\left(-g\,c_t\right) = d\hat{g}(1+r)E_t\left[a\,\exp\left(-\hat{g}W_{t+1} - y\,h_{t+1}^2\right)\right]$$

Notice that the consumption decision is unaffected by the investment decision D_t, due to CARA utility. To get an explicit expression for the right-hand side, we calculate the following expectation with respect to the two random variables R_{t+1}, and x_{t+1}, both of which are normally distributed with mean zero and respective variances s_R^2 and s_x^2:

(A1) $$E_t\left[-\exp\left(-\hat{g}W_{t+1} - y\,h_{t+1}^2\right)\right] =$$

$$-\left(\sqrt{1+2ns_x^2 y}\right)\exp\left(-\hat{g}(1+r)(W_t - c_t)\right) + \frac{\left(agD_t + \frac{h_t}{ns_x^2}\right)^2}{4y + 2/(ns_x^2)} + \left(\frac{\hat{g}^2 D_t^2 s_R^2}{2}\right) - (1+r)\hat{g}aD_t h_t - \left(\frac{h_t^2}{2ns_x^2}\right)$$

Maximizing with respect to the choice of risky asset demand D_t, we get

$$D_t = \left(\frac{a(1+r) - \left(\frac{a}{ns_x^2}\right)\left(\frac{1}{ns_x^2} + 2y\right)^{-1}}{\hat{g}s_R^2 + a^2\hat{g}\left(\frac{1}{ns_x^2} + 2y\right)^{-1}}\right)h_t$$

For market clearing we must have $D_t = h_t$, so:

$$1=\frac{a(1+r)-\left(\frac{a}{ns_{x}^{2}}\right)\left(\frac{1}{ns_{x}^{2}}+2y\right)^{-1}}{\hat{g}s_{R}^{2}+a^{2}\hat{g}\left(\frac{1}{ns_{x}^{2}}+2y\right)^{-1}}$$

which equates to the expression above that pins down the pricing parameter "a". Now, collecting terms in equation (A1) involving h_t^2, we get the coefficient y on h_t^2 in the value function:

$$y=\left(\frac{1}{2ns_{x}^{2}}\right)+(1+r)a\hat{g}-\frac{\left(a\hat{g}+\frac{1}{ns_{x}^{2}}\right)\left(a(1+r)-s_{R}^{2}\hat{g}\right)}{2a}-\left(\frac{s_{R}^{2}\hat{g}}{2}\right)$$

Substituting the expected value function in the next period back to the Bellman equation, we get the expression for a:

$$a=\left(\frac{\hat{g}}{g}a\sqrt{1+2ns_{x}^{2}y}\right)^{\frac{1}{1+r}}+\left(\frac{\hat{g}}{g}a\sqrt{1+2ns_{x}^{2}y}\right)^{-\frac{r}{1+r}}\left(\frac{a}{1+r}\right)\sqrt{1+2ns_{x}^{2}y}$$

It is easy to show that when γ is sufficiently small, there exists a positive solution for the parameters. Finally, that a value function with this simple exponential form exists ensures that the linear equilibrium pricing rule described in proposition 1 also exists (recall that the mean payoff on the risky asset is zero). Q.E.D.