Information Percolation in Large Markets

Darrell Duffie — Stanford University

Gustavo Manso[1] — Massachusetts Institute of Technology

May 29, 2004. Revised January 24, 2007

We introduce a simple model of the "percolation" of information of common interest through a large market, as agents encounter each other over time and reveal information to each other, some of which they may have received earlier from other agents. We are particularly interested in the evolution over time of the cross-sectional distribution in the population of the posterior probability assignments of the various agents. We provide a market example based on privately held auctions, and obtain a relatively explicit solution for the cross-sectional distribution of posterior beliefs at each time.

Our results contribute to the literature on information transmission in markets. Hayek (1945) argues that markets allow information that is dispersed in a population to be revealed through prices. Grossman's (1981) notion of a rational-expectations equilibrium formalizes this idea in a setting with price-taking agents. Milgrom (1981), Pesendorfer and Swinkels (1997), and Reny and Perry (2006) provide strategic foundations for the rational expectations equilibrium concept in centralized markets. A number of important markets, however, are decentralized. These include over-the-counter markets and private-auction markets. Wolinsky (1990) and Blouin and Serrano (2002) study information transmission in decentralized markets.[2] In contrast to these two papers, equilibrium behavior in our market example leads to full revelation of information

[1]We happily acknowledge conversations with Manuel Amador, Frank Kelly, Jeremy Stein, and Pierre-Olivier Weill, and research assistance from Sergey Lobanov.

[2]Rubinstein and Wolinsky (1985) and Gale (1986a, 1986b) study decentralized markets without asymmetric information. Satterthwaite and Shneyerov (2003) study decentralized markets with private-value asymmetric information. Amador and Weill (2006) study a more abstract setting regarding transmission of common and private information.

through trading. We also explicitly characterize the percolation of this information through the market.

Our paper is also related to the literature on social learning. For example, our objectives are similar to those of Banerjee and Fudenberg (2004), who provide a brief survey of the literature. Like us, Banerjee and Fudenberg (2004) exploit the law of large numbers for random matching among a large population, provide a dynamic rule for updating, and show conditions for convergence. Our model allows a relatively explicit solution for the cross-sectional distribution of posterior beliefs at each time.

1 The Basic Model

A probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a "continuum" (a non-atomic finite measure space $(G, \mathcal{G}, \gamma)$) of agents are fixed. A random variable X of potential concern to all agents has two possible outcomes, H ("high") and L ("low"), with respective probabilities ν and $1-\nu$.

Each agent is initially endowed with a sequence of signals that may be informative about X. The signals $\{s_1, \ldots, s_n\}$ observed by any particular agent are, conditional on X, independent and identically distributed with outcomes 0 and 1 (Bernoulli trials). The number $n \geq 0$ of signals may vary across agents. Without loss of generality, we suppose that $\mathrm{P}(s_i = 1 \mid H) \geq \mathrm{P}(s_i = 1 \mid L)$. For any pair of agents, the sequences of signals that they observe are independent.

By Bayes' rule, given a sequence $S = \{s_1, \ldots, s_n\}$ of signals, the posterior probability that X has a high outcome is

$$\mathrm{P}(X = H \mid S) = \left[1 + \frac{1-\nu}{\nu}\left(\frac{1}{2}\right)^{\theta}\right]^{-1}, \tag{1}$$

where the "type" θ of this set of signals is

$$\theta = \sum_{i=1}^{n} s_i \log_{1/2} \frac{\mathrm{P}(s_i = 1 \mid L)}{\mathrm{P}(s_i = 1 \mid H)} + (1 - s_i) \log_{1/2} \frac{1 - \mathrm{P}(s_i = 1 \mid L)}{1 - \mathrm{P}(s_i = 1 \mid H)}. \tag{2}$$

The higher the type θ of the set of signals, the higher the posterior probability that X is high.

Proposition 1 *Let $S = \{s_1, \ldots, s_n\}$ and $R = \{r_1, \ldots, r_m\}$ be independent sets of signals, with associated types θ and ϕ. Then $\theta + \phi$ is a sufficient statistic for the posterior distribution of X given S, R, and $\theta + \phi$.*

This follows from Bayes' rule, by which

$$\begin{aligned} \mathrm{P}(X = H \mid S, R, \theta + \phi) &= \left[1 + \frac{1-\nu}{\nu}\left(\frac{1}{2}\right)^{\theta+\phi}\right]^{-1} \\ &= \mathrm{P}(X = H \mid \theta + \phi). \end{aligned}$$

We will provide examples of random interaction models in which, by a particular point in time, each of the agents has met a finite number of other agents, once each, in some particular sequence. In such a setting, for a given agent α, let A_1 denote the set of agents that α directly encountered, let A_2 be the set of agents that those agents had directly encountered before encountering α, and so on, and let $A = \cup_{k \geq 1} A_k$. Let S_A denote the union of the signals of agent α and those of the agents in A, and let θ_A denote the type of the signal set S_A.

Suppose that when two agents meet, they communicate to each other their posterior probability, given all information to the point of that encounter, of the event that X is high. For example, we later provide a setting in which revelation occurs through the observation of the bids submitted in an auction.

Now, as a step of an inductive calculation of posterior beliefs of all agents, suppose that the posterior distribution of X held by a particular agent with extended encounter set A is that of type θ_A. This is certainly the case before any encounters. From (1), θ_A can be calculated from the posterior distribution of X, and is thus in the information set of the agent, and could be communicated to another agent. Suppose that two agents with disjoint extended encounter sets A and B meet, and communicate to each other θ_A and θ_B. By the previous proposition, for each of the two agents, $\theta_A + \theta_B$ is a sufficient statistic for the posterior distribution of X held by that agent, given that agent's previously held information and the information conveyed at that meeting by the other agent. This justifies, by induction, the following result.

Proposition 2 *If an agent with extended encounter set A meets an agent with a disjoint extended encounter set B, and they communicate to each other their posterior probabilities of the event that X is high, then both will hold the posterior probability of this event given the signals $S_A \cup S_B$.*

Given this result, it makes sense to extend the definition of "type" by saying that an agent with extended encounter set A has type θ_A, which leads to the following equivalent form of the last proposition.

Proposition 3 *If an agent of pre-posterior type θ meets an agent with pre-posterior type ϕ, and they communicate to each other their types, then both have posterior type $\theta + \phi$.*

2 Population Information Dynamics

Any particular agent is matched to other agents at each of a sequence of Poisson arrival times with a mean arrival rate (intensity) λ, which is common across agents. At each meeting time, the matched agent is randomly selected from the population of agents (that is, the matched agent is chosen with the uniform distribution, which we can take to be the agent-space measure γ.) We assume that, for almost every pair of agents, this matching procedure is independent.[3] At each point in time, for any particular agent, we will (almost surely) be in the setting of the previous proposition, in which all prior encounters by that agent are with agents whose extended encounter sets were disjoint with that of the given agent.

We let $g(x, t)$ denote the cross-sectional density of posterior type x in the population at time t (supposing that the posterior type distribution indeed has a density). The initial density $g(\,\cdot\,, 0)$ of types is that induced by some particular initial allocation of signals. Assuming that $g(x, t)$ is differentiable with respect to t, and relying formally on the law of large numbers,

[3] A rigorous mathematical foundation for the discrete-time analogue of this random matching model is provided by Duffie and Sun (2005a), and the associated exact law of large numbers for the matching results is provided by Duffie and Sun (2005b).

letting subscripts denote partial derivatives as usual, we have (almost everywhere)

$$g_t(x,t) = -\lambda g(x,t) + \int_{-\infty}^{+\infty} \lambda g(y,t) g(x-y,t)\, dy, \tag{3}$$

with the first term representing the rate of emigration from type x associated with meeting and leaving that type, and the second term representing the rate of immigration into type x due to type-y agents meeting, at mean rate λ, agents of type $x-y$, converting the type-y agent to one of type x. (One could easily make the mistake of multiplying the second term by 2 to reflect that both agents in the pairing become type x, but the sum is over all agents meeting someone, and integration of the right-hand side of (3) with respect to x confirms the consistency of (3) with conservation of total population mass.)

In order to compute the cross-sectional density of types at each time, we let $\hat{g}(\,\cdot\,,t)$ denote the Fourier transform of $g(\,\cdot\,,t)$. By linearity of the transform and integrability, for each z in $\mathbb{R}$,

$$\hat{g}_t(z,t) = -\lambda \hat{g}(z,t) + \lambda \hat{g}^2(z,t), \tag{4}$$

using the fact that the transform of a convolution $g * h$ is the product $\hat{g}\hat{h}$ of the transforms.

Proceeding formally and ignoring the potential role of singularities, we can let $G(z,t) = \hat{g}(z,t)^{-1}$, and by the chain rule obtain

$$G_t(z,t) = \lambda G(z,t) - \lambda, \tag{5}$$

with the usual solution

$$G(z,t) = e^{\lambda t}(G(z,0) - 1) + 1, \tag{6}$$

and, again only formally,

$$\hat{g}(z,t) = \frac{\hat{g}(z,0)}{e^{\lambda t}(1 - \hat{g}(z,0)) + \hat{g}(z,0)}. \tag{7}$$

While technical conditions on the initial type distribution might be needed to justify this calculation, we have confirmed the result in special cases by explicit calculation and by Monte Carlo simulation.

A particular agent who is assigned an initial type that is randomly drawn with density $\pi(\,\cdot\,,0)$ at time zero has a posterior type at time t that is a Markov process with a probability density $\pi(\,\cdot\,,t)$ at time t that evolves according to

$$\pi_t(x,t) = -\lambda\pi(x,t) + \int_{-\infty}^{+\infty} \lambda\pi(y,t)g(x-y,t)\,dy. \tag{8}$$

The probability density $\pi(\,\cdot\,,t)$ of the agent's type at time t therefore has the explicit transform $\hat{\pi}(t)$ given by

$$\hat{\pi}(z,t) = \hat{\pi}(z,0)e^{-\lambda\int_0^t(1-\hat{g}(z,s))\,ds}.$$

3 A Market Example

In order to provide a specific example in which agents have an incentive to completely reveal their information to the agents that they encounter, we study a private-auction setting in which, at each meeting, agents learn the types of the other agents encountered at that meeting by observation of bids submitted in an auction conducted at that meeting. This information is not revealed to agents that do not participate in the auction.

At a given time $T > 0$, it is revealed whether X is high or low. Before that time, uninformed agents that wish to hedge the risk associated with X arrive at the market at a total rate of 2λ per unit of time. (For example, there may be a continuum of uninformed hedgers that arrive independently, at total rate of 2λ.) Whenever an uninformed agent arrives at the market, he contacts two informed agents that are randomly chosen from the continuum of agents. In light of our extension in the next section to encounters of more than two informed agents each, we could consider an auction in which the uninformed agent contacts multiple agents.

The uninformed agent conducts a second-price auction with the two chosen informed agents. The lower bidder sells the uninformed agent a forward financial contract that pays 1 at time T if X is high and 0 otherwise. In return, the contract specifies payment of the winning (low) bid to the informed agent at time T. The informed agents, who are assumed to be risk-neutral, tender bids that are then revealed to the two bidders (only). After purchasing the contract, the uninformed agent leaves the market. For concreteness, informed agents maximize the expected discounted sum of auction sales net of contract payoffs, with a constant discount factor.

These second-price common-value auctions are known as "wallet games," and are discussed by Klemperer (1998). In the unique symmetric Nash equilibrium of each auction, each agent's bid is the posterior probability that X is high. From the one-to-one mapping between an agent's type and the agent's posterior probability distribution of X, informed agents learn each others' types from their bids.[4] The dynamics of information transmission are therefore as described in Section 2.

Informed agents earn positive expected payoffs by participating in this market, while uninformed agents earn negative expected profits. This is consistent with the hedging motive for trade of the uninformed agents.

For a numerical example, we let $\lambda = 1$, so that one unit of time is the mean time inter-contact time for agents, and we let $\nu = 1/2$. We assume that each agent initially observes a signal s such that $P(s = 1|H) + P(s = 1|L) = 1$ and $P(s = 1|H)$ is drawn from a uniform distribution over the interval $[1/2, 1]$. It is easy to show that, on the event $\{X = H\}$ of a high outcome, this initial allocation of signals induces an initial cross-sectional density $f(p) = 2p$ for the prior likelihood p of a high state, for $p \in [0, 1]$. From a simple change of variables using (1), the initial cross-sectional density of types on the event $\{X = H\}$ of a high outcome is

$$g(\theta, 0) = \frac{2^{1+2\theta} \log 2}{(1 + 2^{\theta})^3}.$$

We then use the explicit solution (7) for the transform $\hat{g}(\,\cdot\,, t)$ of the density of posterior types to characterize the dynamics of information transmission. The evolution of the cross-sectional densities of type and the associated posterior probability are illustrated in Figures 1 and 2, respectively. Figure 3 shows that the evolution of the mean of the cross-sectional distribution of

[4]Because we have a continuum of agents, when two agents meet they know that in the future they will almost surely not meet someone that has directly or indirectly met the other agent. Therefore, agents are not strategic about the information they reveal in a meeting. In a market with a small number of agents, this would obviously be a concern.

posterior probability of a high state, and the evolution of the cross-sectional standard deviation of this posterior probability.

4 New Private Information

Suppose that, independently across agents as above, each agent receives, at Poisson mean arrival rate ρ, a new private set of signals whose type outcome y has a probability density $h(y)$. Then (3) is extended to

$$g_t(x,t) = -(\lambda+\rho)g(x,t) + \int_{-\infty}^{+\infty} \lambda g(y,t)g(x-y,t)\,dy + \rho\int_{-\infty}^{+\infty} h(y)g(x-y,t)\,dy. \tag{9}$$

In this case, (4) is extended to $\hat{g}_t(z,t) = -(\lambda+\rho)\hat{g}(z,t) + \lambda\hat{g}^2(z,t) + \rho\hat{g}(z,t)\hat{h}(z)$. As for the dynamics of the transform, we can further collect terms in $\hat{g}(z,t)$ to obtain

$$\hat{g}_t(z,t) = -\gamma(z)\hat{g}(z,t) + \lambda\hat{g}^2(z,t), \tag{10}$$

where $\gamma(z) = \lambda + \rho(1-\hat{h}(z))$, and extend (7) to obtain

$$\hat{g}(z,t) = \frac{\hat{g}(z,0)}{e^{\gamma(z)t}(1-\hat{g}(z,0)) + \hat{g}(z,0)}. \tag{11}$$

5 Multi-Agent Information Exchanges

Suppose that, at Poisson arrival intensity λ, three agents are drawn at random and share their information. Then (3) is extended to

$$g_t(x,t) = -\lambda g(x,t) + \lambda\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x-y-u,t)g(u,t)g(y,t)\,dy\,du. \tag{12}$$

Letting $w = y+u$, this can be expressed as

$$g_t(x,t) = -\lambda g(x,t) + \lambda\int_{-\infty}^{+\infty} g(x-w,t)\int_{-\infty}^{+\infty} g(w-y,t)g(y,t)\,dy\,dw. \tag{13}$$

Letting $r(w,t) = \int_{-\infty}^{+\infty} g(w-y,t)g(y,t)\,dy$, we have

$$g_t(x,t) = -\lambda g(x,t) + \lambda\int_{-\infty}^{+\infty} g(x-w,t)r(w,t)\,dw. \tag{14}$$

Now, because $\hat{r}(z,t) = \hat{g}^2(z,t)$, we see that (4) is extended to

$$\hat{g}_t(z,t) = -\lambda \hat{g}(z,t) + \lambda \hat{g}^3(z,t). \tag{15}$$

Similarly, if n agents are drawn at random to exchange information at each encounter, then (4) is extended to

$$\hat{g}_t(z,t) = -\lambda \hat{g}(z,t) + \lambda \hat{g}^n(z,t). \tag{16}$$

In order to solve for $\hat{g}(z,t)$ given $\hat{g}(z,0)$, we let $H(z,t) = \hat{g}(z,t)^{1-n}$. Formally at least, we have

$$H_t(z,t) = (n-1)\lambda H(z,t) - (n-1)\lambda, \tag{17}$$

with the usual solution $H(z,t) = e^{(n-1)\lambda t}(H(z,0)-1)+1$, and

$$\hat{g}(z,t) = H(z,t)^{\frac{1}{1-n}}. \tag{18}$$

If private information is also learned over time, as in the previous section, then (16) is extended to

$$\hat{g}_t(z,t) = -(\lambda+\rho)\hat{g}(z,t) + \lambda \hat{g}^n(z,t) + \rho \hat{g}(z,t)\hat{h}(z). \tag{19}$$

In this case, we have (18), where $H(z,t) = e^{(n-1)\gamma(z)t}(H(z,0)-1)+1$.

6 Conclusion

We introduce a simple model of the "percolation" of information through a large market. Our model allows a relatively explicit solution for the cross-sectional distribution of posterior beliefs at each time t. We applied our model to study information transmission in a decentralized market and, in contrast to previous models, obtained full revelation of information through trading.

References

Amador, Manuel, and Pierre-Olivier Weill. 2006. "Learning from Private and Public Observation of Other's Actions." UCLA Working Paper.

Banerjee, Abhijit, and Drew Fudenberg. 2004. "Word-of-Mouth Learning." *Games and Economic Behavior*, **46:** 1-22.

Duffie, Darrell, and Yeneng Sun. 2005a. "Existence of Independent Random Matching." forthcoming in *Annals of Applied Probability.*

Duffie, Darrell, and Yeneng Sun. 2005b. "The Exact Law of Large Numbers for Independent Random Matching."Stanford University Working Paper.

Gale, Douglas. 1986a. "Bargaining and Competition Part I: Characterization." *Econometrica*, **54:** 785-806.

Gale, Douglas. 1986b. "Bargaining and Competition Part II: Existence." *Econometrica*, **54:** 807-818.

Hayek, Friedrich. 1945. "The Use of Knowledge in Society." *American Economic Review*, **35:** 519-530.

Klemperer, Paul. 1998. "Auctions with Almost Common Values: The "Wallet Game" and its Applications." *European Economic Review*, **42:** 757-769.

Milgrom, Paul. 1981. "Rational Expectations, Information Acquisition, and Competitive Bidding." *Econometrica*, **50:** 1089-1122.

Pesendorfer, Wolfgang and Jeroen Swinkels. 1997. "The Loser's Curse and Information Aggregation in Common Value Auctions." *Econometrica*, **65:** 1247-1281.

Reny, Philip and Motty Perry. 2006. "Toward a Strategic Foundation for Rational Expectations Equilibrium." *Econometrica*, **74:** 1231-1269.

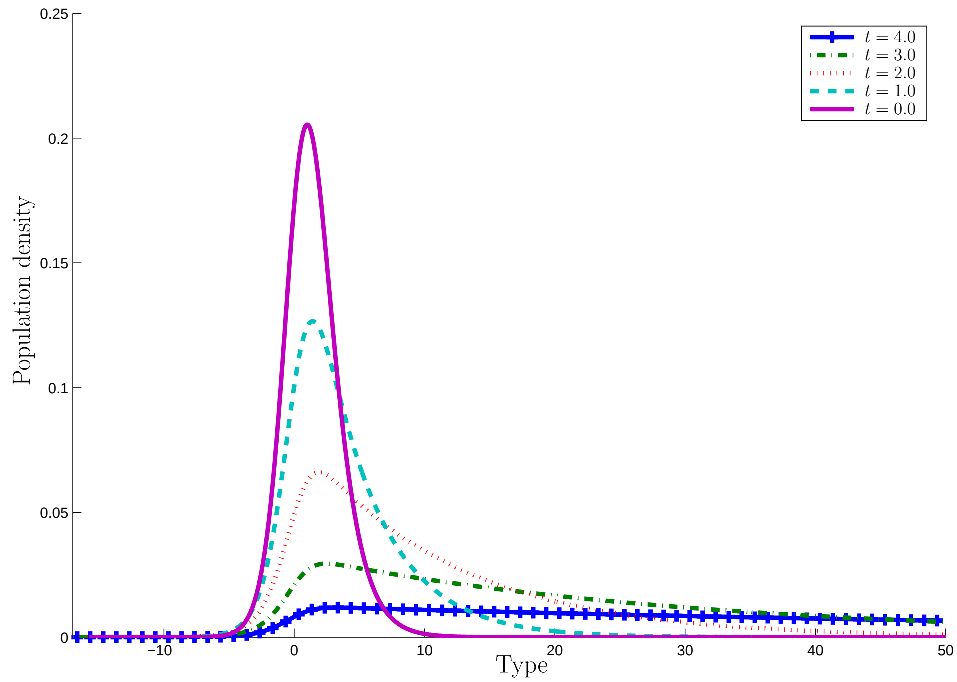

Figure 1: Evolution of cross-sectional population density of type, on the event $\{X = H\}$.

Rubinstein, Ariel and Asher Wolinsky. 1985. "Equilibrium in a Market With Sequential Bargaining." *Econometrica*, **53:** 1133-1150.

Satterthwaite, Mark, and Artyom Shneyerov. 2003. "Convergence of a Dynamic Matching and Bargaining Market with Two-Sided Incomplete Information to Perfect Competition." Northwestern University Working Paper.

Stein, Jeremy. 2006. "Conversations Among Competitors." Harvard University Working Paper.

Wolinsky, Asher. 1990. "Information Revelation in a Market With Pairwise Meetings." *Econometrica*, **58:** 1-23.

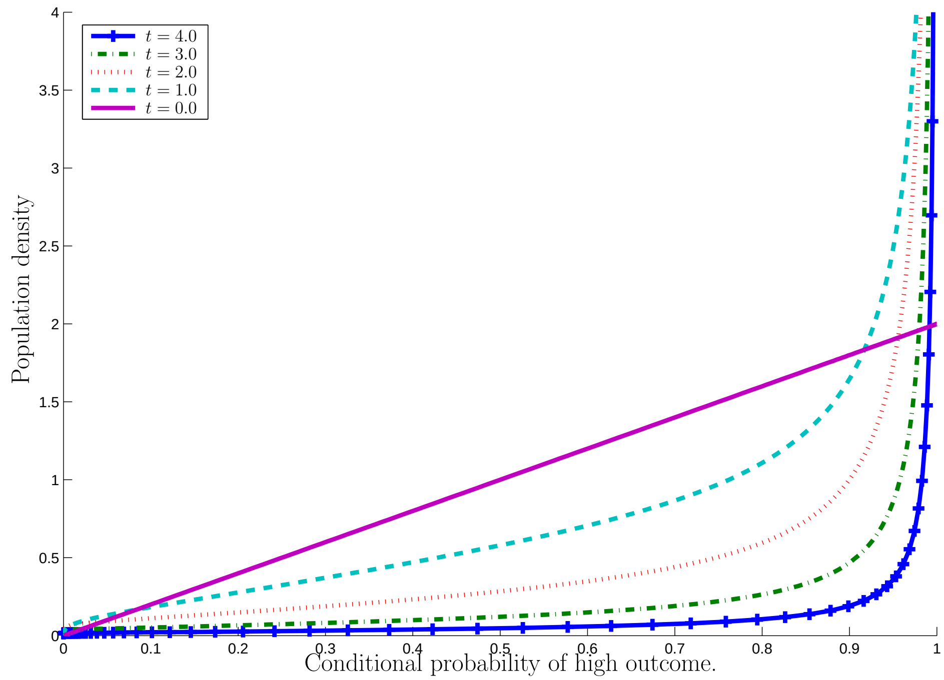

Figure 2: On the event $\{X = H\}$, the evolution of the cross-sectional population density of posterior probability of the event $\{X = H\}$.

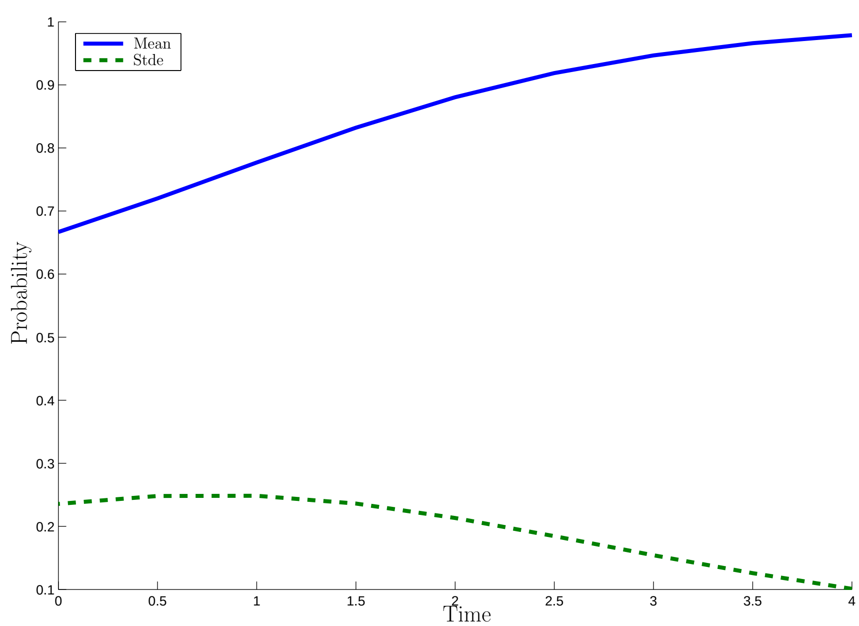

Figure 3: On the event $\{X = H\}$, the evolution of the cross-sectional mean and standard deviation of the posterior probability of the event $\{X = H\}$.