Bargaining with Deadlines and Private Information

VERY PRELIMINARY AND INCOMPLETE

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Abstract

1 Introduction

In this paper we study dynamic bargaining with private information in the presence of a deadline. Many negotiations have a preset deadline by which an agreement must be reached. For example, with a known trial date looming ahead, parties engage in pretrial negotiations. Before international summits, countries bargain over the terms of the accords to be signed at the summit. Broadcasters selling advertising space for some live event have until the event takes place to reach an agreement with the advertisers. Negotiations to renew labor contracts have until the expiration date of the current contract or the pre-set strike date if conflicts are to be avoided.

Financial considerations might also act as an effective deadline. Countries that have large debt repayments ahead of them bargain with international agencies such as the IMF for financing that would help them avoid default. Private companies also face refinancing deadlines or deadlines to obtain financing in order to be able to invest in a given venture.

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1For example, the current negotiations between Greece the EU and IMF are carried under the looming refinancing needs due to loans maturing. "Greece must refinance 54 billion euros in debt in 2010, with a crunch in the second quarter as more than 20 billion euros becomes due." http://www.reuters.com/article/idUSTRE63F2ZR20100416
Finally, negotiations can be affected by regulatory deadlines. For example, to take advantage of the home buyer credit program, buyers and sellers of homes had to close their transactions by a given deadline to qualify for the subsidy.

Empirical literature has documented that a large fraction of agreements are reached in the "eleventh hour" that is at or very close to the deadline. For example, Cramton and Tracy (1992) study a sample of 5002 labor contract negotiations involving large bargaining units they claim a "clear ‘deadline effect’ exists in the data" since 31% of agreements are reached on the deadline. Williams (1983) in a sample of civil cases from Arizona has found that 70% cases were settled in the last 30 days before trial and 13% were settled on the day of the trial. Such strong deadline effects have been also observed in experimental studies (see for example Roth, Murnighan and Schoumaker (1988)).

So why is it that parties only reach an agreement on the morning of the trial date or in the wee hours of the night before the labor contract expires?

We study a model of bargaining with a deadline based on a classic paper by Sobel and Takahashi (1983) (henceforth ST). In the model a seller makes an offer that can be either accepted or refused. If rejected, the process continues until a deadline is reached. The buyer has private information about the value of the good for sale (i.e. we have one-sided private information). Both parties prefer earlier trade which is modeled as discounting costs.

We add to the ST model in two important ways. First, ST consider only the case that if trade does not take place by the deadline, the trade opportunity and all surplus is lost. We consider different amounts of lost surplus and different splits of the remaining surplus if parties fail to reach an agreement before the deadline. It allows us to capture better the post-deadline continuation game and to make sure that not all bargaining power resides with the seller at the deadline. Second, we take the non-commitment limit (i.e. taking the time between periods to zero) and show that the limit of equilibria is very simple and displays a clear deadline effect: in the limit there is smooth trade (probability flow) before deadline and an atom (probability mass) at the deadline. Moreover, prices in the limit have

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2 They interpret agreements reached even a day after the contract expiration as reached "on" the deadline.
3 There are other bargaining models with deadlines but they are quite different. See for example Ma and Manove (1993) and Fershtman and Seidman (1993).
4 Admittedly, each of the bargaining environments provided as examples above has idiosyncratic and potentially important details that would affect the way negotiations are carried forward. Our model abstracts from many of those details, yet it is rich enough to capture the effect of deadlines and the consequence of not reaching an agreement on the bargaining outcome. The disagreement payoff can also be thought as the expected payoffs the agents would get from the continuation game that would start at T + 1. For example, if the private information is revealed at T + 1 (possibly at some cost) and then the players bargain with full information.
a natural economic interpretation related to the Coase conjecture intuition (despite trade being inefficient and trade happening with delay). The tractability of the continuous-time limit is surprising because non-stationary models (not only beliefs evolve over time but also time till the deadline changes) are usually much more difficult to analyze (as can be seen in discrete time by comparing ST to Stokey (1981) and Bulow (1982)).

Dynamic bargaining with asymmetric information has been studied extensively since the classic results on the Coase conjecture Stokey (1981), Bulow (1982), Fudenberg, Levine and Tirole (1985) and Gul, Sonnenschein and Wilson (1986). In such games, the seller becomes more and more pessimistic over time (higher types trade sooner than lower types) and asks for lower and lower prices. The famous Coase conjecture result is that without a deadline, as commitment disappears, the seller reduces its prices faster and faster and in the limit trade is efficient (no delay). We show that deadlines dramatically change the outcomes. A deadline provides the seller with a lower payoff bound that she can achieve by making unacceptable offers until the deadline. Our main result (Proposition 2) is that as commitment disappears, the seller’s equilibrium payoff converges to this lower bound: she obtains a payoff equal to the outside option of just waiting for the deadline. However, trade and prices do not converge to the Coase conjecture (immediate trade all types at one price) or the outside option (trade only at the deadline), but rather in the limit trade happens gradually over time. The price paid by each type is equal to the discounted price that this type would pay at the deadline if the seller adopted from now on the wait-till-deadline strategy. This is an important property of the equilibrium prices: if prices were higher, the seller would like to speed up trade; if they were lower, she would prefer to wait for the deadline. Finally, the speed at which equilibrium prices drop over time (which is one-to-one related to the speed at which the seller screens the types) assures that no buyer type wants to delay or speed up trade.

Once the deadline is reached and the last take it or leave it offer is made, there is a large probability it is accepted but there is also the possibility that the offer is not accepted. In that case, the players get their disagreement payoffs. Cramton and Tracy (1992) report that 57 percent of the labor negotiations in their sample end in disputes (strikes (10%), holdouts (47%) and lockouts(0.4%)) This seems very surprising since the failure to reach an agreement could be very inefficient. For example, when the existing contract between the NHL and its players expired on September 15, 2004 without having reached an agreement, the entire season was cancelled. The tractable characterization of the continuous time limit allows us to perform comparative statics. We are able obtain a series of testable predictions

See Ausubel, Cramton, and Deneckere (2001) for a survey of the literature.
that resonate very well with some of the available evidence. For example, in Proposition 5 we show that the likelihood that an agreement will be reached is increasing in the loss of surplus from not doing so. This is consistent with the findings of Gunderson, Kervin and Reid (1986) that strikes are more likely to occur (in Canada) when the efficiency losses from a strike are lower. This might also be an explanation of why strikes in professional sports leagues are very infrequent relative to other activities.

A closely related paper to ours is Hart (1989). That paper studies a model similar to ours to try to explain duration of strikes and labor disputes. In that model there is also an exogenous deadline \( T \) after which the firm is facing a "crunch". The main difference from our paper is the way the crunch is modeled: we assume that at \( T \) some of the value of the firm is lost - for example, because if the firm does not resolve the dispute by \( T \), a major supplier is lost. In Hart (1989) at \( T \) the discounting increases. That leads to a major difference: in a model like Hart (1989) as the commitment to make offers disappears, disputes are again resolved immediately and efficiently; in a model with a discrete cost at \( T \), inefficient delay persists even in the non-commitment limit.

Spier (1992) studies a model of pretrial negotiations and her section with exogenous deadlines is also relevant to our paper. The main difference from our model is that the social cost of delay is independent of type because legal costs are independent of the type of the defendant. Our model applies to pre-trial negotiations if the defendant’s legal costs are proportional to his type. That would be the case, for example, if the plaintiff would restrict a use of an asset (a patent or a real estate) until the dispute is resolved. Similarly, the defendant may not be able to sell an asset or secure outside investment until the case is over (and the deadline may represent a loss of such outside opportunity). The difference in results is that in Spier (1992) offers are increasing over time while in our model they are decreasing (although in both models the distribution gets weaker over time). Also, we get a unique equilibrium in which each buyer type has a uniquely optimal time to trade. In her model there are multiple equilibria and the all defendant types that settle are completely indifferent over the time to settle. The similarity is that both models can deliver a deadline effect (however, in case of a binary distribution Spier (1992) obtains a U-shaped distribution of agreement times, while in our model there is no atom at \( t = 0 \)).

Cramton and Tracy (1992) construct a more detailed model of wage bargaining. They seek

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6Our model should be interpreted as capturing the labour negotiations before the end of the current contractual agreement or the preset strike date.

7To avoid the Coase conjecture, Hart (1989) explicitly focuses on the discrete time model.

8Also, it may be that the higher the defendant’s type the more it costs him to prevent the plaintiff discovering damning evidence.
to explain why in many instances when contracts are not renegotiated in time, unions choose to continue working under the old contract instead of starting a strike. This is referred to as holdout. They assume a holdout is inefficient (for example, as a result of working to rule). Their model starts with the old contract already expired and the choice of the unions of what regime they want to be in while they continue negotiating. Our model should rather be interpreted as capturing the negotiation before the old contract expires. The ex-post choice of threat by the union would still be relevant in our model since it would affect the disagreement payoffs which could be modeled as arising from the game described in Cramton and Tracy (1992). In this sense, we see our paper as complementary to theirs since it allows to determine when the firm would reach a holdout.

On the more technical side, our paper is also related to bargaining models with interdependent valuations, as in Olsen (1992) and Deneckere and Liang (2006). The reason is that although the buyer value is independent of seller cost, by trading today the seller gives up the option of trading at the deadline. That opportunity cost is correlated with the value of the buyer. The main difference is that in our model the interdependence is created endogenously by the deadline and that the game is necessarily non-stationary.

The next section presents the general model and a characterization of the unique equilibrium of the game. Section 3 characterizes the limit of the equilibria as offers can be revised continuously. We then look at the uniform distribution case in Section 4 which allows us to perform some additional comparative statics analysis. Finally, Section 5 studies two benchmark cases for general distributions. First we look at the case when the opportunity to trade and hence all surplus disappears at \( T \). Then we look at opposite case in which there is efficient trade after reaching the deadline (for example following the release of the private information).

2 The Model

There is a seller (a she) and a buyer (a he). The seller has an indivisible good (or asset) to sell. The buyer has a privately known type \( v \in [0, 1] \) that represents his valuation of the asset. Types are distributed according to the c.d.f. \( \mu(v) = v^a \) with full support in \([0, 1]\). We denote its density by \( f(v) \). The seller’s value of the asset is zero.\(^9\)

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\(^9\)This assumption guarantees that truncated versions of the distribution between \([0, k]\) have the same shape as the original distribution. This gives some stationarity to the problem and helps simplify the analysis. In addition, it satisfies the decreasing marginal revenue condition for the seller’s problem.

\(^{10}\)The only non-trivial assumption about the range of \( v \) and the seller’s value is that the seller’s value is no lower than the lowest buyer’s value - i.e. the "no-gap case". The rest is a normalization.
There is a total amount of time $T < \infty$ for the parties to try to reach an agreement. The seller is able to commit to the current offer for a discrete period of length $\Delta > 0$. The timing within periods is as follows. In the beginning of the period the seller makes a price offer $p$. The buyer then decides whether to accept or reject this price. If he accepts, the game ends. If he rejects, the game moves to the next period. If time $T$ is reached the game ends.

A strategy for the seller for a given period $P(p^{t-1}, T - t; \Delta)$ is a mapping from the histories of rejected prices $p^{t-1}$ and the remaining time $(T - t)$ to a current period price offer $p_t$. Similarly, in each period a strategy of the buyer of type $v$, which we denote $A_v(p_t, T - t; \Delta)$ is a mapping from the history of prices (rejected plus current) and the remaining time to a choice whether to accept or reject the current offer.

The payoffs are as follows. If the game ends in disagreement the buyer gets a disagreement discounted payoff of $e^{-rT} \beta v$ and seller gets $e^{-rT} \alpha v$ where $r$ is the common discount rate. We assume $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$. For example, the case $\alpha = \beta = 0$ represents that the opportunity to trade disappears after the deadline. This is the case analyzed by ST. In contrast, the case $\alpha = 1 - \beta$ could represent the revelation of information at time $T$ leading to bargaining with perfect information and no efficiency loss. Intermediate cases could for example capture the partial (or probabilistic) loss of efficiency if two parties do not reach an agreement before a trial and have to incur litigation costs. For example, if the buyer does not obtain the asset by $T$, it may lose with some probability the opportunity of using it. Also, the model in Cramton and Tracy (1992) would deliver some intermediate values for a setting of wage negotiations.

If the game ends with the buyer accepting price $p$ at time $t$, then the seller’s payoff is $e^{-rT} p$ and the buyer’s payoff is $e^{-rT}(v - p)$.

### 2.1 Equilibrium Definition

A complete strategy for the seller $P = \{ P(p^{t-1}, T - t; \Delta) \}_{t=0}^{t=T}$ determines the prices to be offered in every period after any possible price history. As usual (in dynamic bargaining...
games), in any equilibrium the buyer types remaining after any history are a truncated sample of the original distribution (even if the seller deviates from the equilibrium prices). This is due to the skimming property which states that in any sequential equilibrium after any history of offered prices $p^{t-1}$ and for any current offer $p_t$, there exists a cutoff type $\kappa(p_t, p^{t-1}, T-t; \Delta)$ such that buyers with valuations exceeding $\kappa(p_t, p^{t-1}, T-t; \Delta)$ accept the offer $p_t$ and buyers with valuations less than $\kappa(p_t, p^{t-1}, T-t; \Delta)$ reject it. Best responses satisfy the skimming property because it is more costly for the high types to delay trade than it is for the low types. We can hence summarize the buyers’ strategy by $\kappa = \{\kappa(p_t, p^{t-1}, T-t; \Delta)\}_{t=0}^T$.

**Definition 1** A pair of strategies $(P, \kappa)$ constitute a Perfect Bayesian Nash equilibrium of the game if they are mutual best responses after every history of the game. That requires:

1) Given the buyers’ acceptance strategy every period (after every history) the seller chooses her current offer as to maximize her current expected discounted continuation payoff.

2) For every history, given the seller’s future offers which would follow on the continuation equilibrium path induced by $(P, \kappa)$, the buyers’ acceptance choice of the current offer is optimal.

Being a finite horizon game, the equilibrium of the game can be solved by backward induction. As we show formally in the proof, with our distributional assumptions, given any period and any cutoff $k$ induced by $(P, \kappa)$ and the history so far, the seller problem has a unique solution. Therefore, continuation equilibrium is unique and depends on the history only via the induced cutoff (and the remaining time), $k$ and $(T-t)$. That allows us to simplify notation: let the current equilibrium price be denoted by $p = P(k, T-t; \Delta)$. Then the next period price is $P(\kappa(p, k, T-t; \Delta), T-(t+\Delta); \Delta)$ and so on.

The equilibrium $(P, \kappa)$ induces a decreasing step function $K(t; \Delta)$ which specifies the highest remaining type in equilibrium as a function of time (with $K(0; \Delta) = 1$) and a decreasing step function $Y(v; \Delta)$ (with $Y(1; \Delta) = 0$) which specifies the time at which each type $v$ trades (if it trades at all in equilibrium). For notational purposes, we let $k_+ = \kappa(P(k, T-t; \Delta), k, T-t; \Delta)$ denote the highest remaining type at the beginning of the next period given current cutoff $k$ and the strategies $(P, \kappa)$.

Let $V(k, T-t; \Delta)$ be the expected continuation payoff of the seller given a cutoff $k$ with $(T-t)$ time left and the strategy pair $(\kappa, P)$. For $t < T$ we can express $V(k, T-t; \Delta)$ recursively as:
\[ V(k, T - t; \Delta) = \left( \frac{F(k) - F(k_+)}{F(k)} \right) P(k, T - t; \Delta) + \frac{F(k_+)}{F(k)} e^{-\Delta r} V(k_+, T - (t + \Delta); \Delta) \]

(1)

and for \( t = T \) we have:

\[ V(k, 0; \Delta) = \left( \frac{F(k) - F(k_+)}{F(k)} \right) P(k, 0; \Delta) + \frac{F(k_+)}{F(k)} \int_0^{k_+} \alpha v f(v) dv \]

(2)

For \( t < T \), the seller’s strategy is a best response to the buyer’s strategy \( \kappa(p, t; \Delta) \) if:

\[ P(k, T - t; \Delta) \in \arg \max_p \left( \frac{F(k) - F(\kappa(p, T - t; \Delta))}{F(k)} e^{-\Delta r} V(\kappa(p, T - t; \Delta), T - (t + \Delta); \Delta) \right) \]

(3)

At \( t = T \), the seller’s strategy is a best response to the buyer’s strategy \( \kappa(p, T; \Delta) \) if:

\[ P(k, 0; \Delta) \in \arg \max_p \left( \frac{F(k) - F(\kappa(p, 0; \Delta))}{F(k)} p^+ + \frac{F(\kappa(p, 0; \Delta))}{F(k)} \int_0^{\kappa(p, 0; \Delta)} \alpha v f(v) dv \right) \]

(4)

These best response problems capture the seller’s lack of commitment: in every period she chooses the price to maximize her current value (instead of committing to a whole sequence of prices at time 0).

Necessary\(^{16}\) conditions for the buyer’s strategy \( \kappa \) to be a best response is that given the expected path of prices the cutoffs satisfy:

For \( t < T \):

\[ k_+ - P(k, T - t; \Delta) = e^{-\Delta r} (k_+ - P(k_+, T - (t + \Delta); \Delta)) \]

(5)

For \( t = T \):

\[ k_+ - P(k, 0; \Delta) = \beta k_+ \]

(6)

Following the proof strategy of Theorem 6 in ST we can establish the following result:

\(^{16}\)They turn out to be also sufficient.
Proposition 1  The game has a unique Perfect Bayesian Nash equilibrium. The equilibrium pricing function \( P(p^{t-1}, T-t; \Delta) \) depends only on the history of past prices via the cutoff type \( k \) that they induce and it is linear in \( k \). The seller’s value \( V(k, T-t; \Delta) \) is also linear in \( k \):

\[
P(p^{t-1}, T-t; \Delta) = P(k, T-t; \Delta) = \gamma_t k
\]

where:

\[
\gamma_T = (1 - \beta) \left( \frac{(1 - \beta)}{(1 + a)(1 - \beta) - a \alpha} \right)^{\frac{1}{a}}
\]

and for \( t < T \)

\[
\gamma_t = (1 - e^{-r \Delta} + e^{-r \Delta \gamma_{t+\Delta}}) \left( \frac{(1 - e^{-r \Delta} + e^{-r \Delta \gamma_{t+\Delta}})}{(a + 1)(1 - e^{-r \Delta} + e^{-r \Delta \gamma_{t+\Delta}})} \right)^{\frac{1}{a}}
\]

(7)

The proof is by induction and is delegated to the appendix. Even though we are able to describe the equilibrium in close form, it is quite complicated.

3 Limit of Equilibria as \( \Delta \to 0 \).

We now take the limit of equilibria described in the previous section as \( \Delta \to 0 \). We show that the limit expressions are much simpler. While much of the previous literature refers to \( \Delta \) as the "bargaining friction" we prefer to refer to it as measure of the seller’s ability to commit to the current offer.

Since \( \Delta \) affects equilibrium strategies only via its effect on \( \gamma_t \) and that the difference equation for \( \gamma_t \), (7), converges to a simple differential equation as \( \Delta \to 0 \), we are able to obtain:

Proposition 2  As \( \Delta \to 0 \):

\[
\gamma_t \to e^{-r(T-t)} \gamma_T
\]

\[
\gamma_T = (1 - \beta) \left( \frac{(1 - \beta)}{(1 + a)(1 - \beta) - a \alpha} \right)^{\frac{1}{a}}
\]

(1)
\[ (2) \quad V(k, T - t; \Delta) \rightarrow V(k, T - t) = \frac{\alpha}{\alpha + 1} \gamma_t k = e^{-r(T-t)} V(k, 0) \]

\[ (3) \quad P(k, T - t; \Delta) \rightarrow P(k, T - t) = \gamma_t k = e^{-r(T-t)} P(k, 0) \]

\[ (4) \quad K(t, T - t; \Delta) \rightarrow K(t, T - t) = \exp \left( \frac{1}{\gamma_T} e^{rt} (e^{-rt} - 1) \right) \]

and an atom \( m(T) \) of trade at time \( t = T \).

\[
    m(T) = K(T, 0)^a - (\kappa (P(K(T, 0), 0), K(T, 0); 0))^a
    = \exp \left( \frac{1}{\gamma_T} (1 - e^{rt}) \right)^a \left( 1 - \left( \frac{\gamma_T}{1 - \beta} \right)^a \right)
\]

**Corollary 1 (Atom at the deadline)** If \( \alpha + \beta < 1 \) then trade is continuous for \( t \in [0, T) \) but there is a positive measure of trade at time \( T \).

Beyond being more tractable, the limiting expressions are also quite intuitive. Consider first the value for the seller. \( V(k, T - t) \) is simply what she would get from just waiting and making the last offer.\(^{17}\) This implies that the seller’s value is driven down to her outside option of simply waiting to make her last offer. The uniformed party cannot capture more than her reservation value once her ability to commit to an offer disappears.\(^{18}\)

Second, the equilibrium prices display a no-regret property: the seller is indifferent between collecting \( P(k, T - t) \) from type \( k \) today or getting what this type would contribute to her value upon reaching the deadline, \( e^{-r(T-t)} P(k, 0) \). The equation for \( K(t, T - t) \) is a bit more complicated but it is a solution to an intuitive differential equation. It follows from taking the continuous time limit of the buyers best response condition given in (5):

\[
    r (k - P(k, T - t)) = -P_k(k, T - t) \dot{K} - P_t(k, T - t) (8)
\]

where \( P_k \) and \( P_t \) are the derivatives of \( P(k, T - t) \) with respect to \( k \) and \( t \), respectively. The RHS represents the change in price that results from the horizon getting closer and the seller

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\(^{17}\) Even though this is not what she actually does in equilibrium.

\(^{18}\) This is similar to the result in Fuchs and Skrzypacz (2010), which is a stationary bargaining problem with outside options that arrive stochastically over time.
updating her beliefs downwards. The LHS captures the costs of delay for the buyer in terms of interest lost on the profit.

Below we plot an example of the path of cutoff types and prices over time.\(^\text{19}\)

\[
\begin{array}{c}
\text{k \& p} \\
\text{0.0} \\
\text{0.5} \\
\text{1.0} \\
\end{array}
\]

\[
\begin{array}{c}
0.2 \\
0.4 \\
0.6 \\
0.8 \\
1.0 \\
\end{array}
\]

Note that in the limit trade takes place smoothly over time except for the last instant before the deadline; at this point there is an atom of trade \(m (T)\). In the example plotted above the atom of trade at \(T\) approximately includes types between 0.6 and 0.4. This feature of the equilibrium captures the "deadline effect" which is found across a wide variety of bargaining situations. Our tight characterization also allows us to derive testable predictions about how changes in the bargaining environment affect this "deadline effect" and other features of the equilibrium.

Despite this last rush of agreements, there is still a probability that the parties will not reach an agreement. Types below 0.4 do not reach an agreement in the example above. In general the probability of agreement is given by:

\[
\Pr (\text{agreement}) = 1 - K_{T+}^a
\]

where \(K_{T+}\) is the cutoff after the last equilibrium offer.

**Time to the Deadline \(T\).**

**Proposition 3** \(V (k, T - t)\) and \(P (k, T - t)\) are decreasing in \(rT\) and go to 0 uniformly as \(rT \to \infty\). The probability of agreement is increasing in \(T\) and goes to 1 as \(rT \to \infty\). The last atom \(m (T)\) is decreasing in \(rT\) and goes to 0 as \(rT \to \infty\). \(K (t, T - t)\) is decreasing in \(rT\) and goes to 0 as \(rT \to \infty\).

**Proof.** Follows from the characterization of the equilibrium objects and the fact that \(\gamma_t\) is decreasing in \(rT\) and for any finite \(t\) (and in particular for \(t = 0\)) \(\lim_{rT \to \infty} \gamma_t = 0\). \(\blacksquare\)

\(^{19}\)The parameters used are \(T = 1\), \(r = 10\%\), \(a = 1\), \(\alpha = \beta = \frac{1}{4}\).
Corollary 2

(i) (Delay): For all $0 < rT < \infty$ the expected time to trade is strictly positive.

(ii) (Coase conjecture): As $rT \to \infty$, the expected time to trade and transaction prices converge to 0 for all types (i.e. $\Upsilon(v) \to 0$ and $P(k,T) \to 0$).

(i) Follows directly from our characterization, but the intuition is as follows: for there to be no delay in equilibrium the transaction prices for all types have to be close to zero, implying a seller’s payoff close to zero, in particular, less than $e^{-rT}V(k,0) > 0$. But that leads to a contradiction since the seller can guarantee himself that by just waiting for the last period. That indirect proof establishes that trade is necessarily inefficient for any equilibrium not only for our family of distributions, but for any distribution without a gap.\(^{20}\)

Part (ii) shows that our limit of equilibria converges to the equilibria in GSW and FLT: as we make the horizon very long (convergence of the model) trade takes place immediately and the buyer captures all the surplus (convergence of equilibrium outcome). Note that ST had established that the same holds when one reverses the order of limits, first taking the limit as $T \to \infty$ (the FLT and GSW models) and then $\Delta \to 0$. Actually, like a lot of the previous literature ST use a per period discount rate $\delta = e^{-r\Delta}$ and take the limit as $\delta \to 1$. In the finite horizon case it actually matters if $\delta \to 1$ as a result of $\Delta \to 0$ or $r \to 0$. The latter would make the whole game essentially a static game since $rT \to 0$.

Below we graph the seller’s value at time zero (solid) and the initial price demanded (dashed) for the uniform ($\alpha = 1$) case for $\alpha = \beta = .25$. As we can see from the plot, as the (normalized) horizon until the deadline extends we converge to the Coasian results, initial prices and seller’s value are very close to 0, for $rT > 1$, but substantially away from the Coase conjecture for $rT < 0.2$.

\[^{20}\text{By the same argument, even if there is a gap between the lowest buyer type, } \underline{v}, \text{ and the seller cost (normalized to zero), trade must be inefficient in equilibrium even as } \Delta \to 0 \text{ if } \underline{v} \ll e^{-rT}V(\eta,0; \Delta) \text{ where } \eta \text{ is the highest type.}\]
Below we graph the probability of agreement (solid) and atom size (dots) and percentage of trades that take place at the deadline (dashed). The plot shows how although for short horizons most of the trade takes place in the last offer, as the horizon increases this changes. Furthermore, when $rT$ is greater than one, trade takes place with very high probability and it mostly takes place before the deadline.

It is worth noting that it is quite different if we face a fixed deadline date rather than a stochastic deadline that arrives at a Poisson rate $\lambda$ after which the opportunity to trade disappears, even if from time zero perspective the expected time available to reach an agreement is the same ($\frac{1}{\lambda} = T$). This difference arises because knowing that the game ends at $T$ allows the seller to make a credible last take it or leave it offer at $T$. This possibility allows him to extract a positive amount of surplus. Instead, a stochastic loss of the opportunity of trade that arrives as a surprise is equivalent to having a higher discount rate $\hat{r} = r + \lambda$ and would lead to immediate trade with the buyer capturing all the surplus. This is a difference between our model and that discussed in Hart (1989).

If we instead allowed the seller to make a last take it or leave it offer when the stochastic deadline materialized, the outcome would be much closer (but not the same) to what we would obtain with a deterministic deadline. In fact, the seller would prefer the stochastic deadline. In this case her value would be $\frac{\lambda}{\lambda + r} V (1, 0)$ instead of $e^{-\tilde{r}} V (1, 0)$ and $\frac{\lambda}{\lambda + r} > e^{-\tilde{r}}$. This follows simply because the present value of a dollar is a convex function of the time at which it is generated. Therefore, a mean preserving spread increases value.

### 3.1 Heuristic Derivation of the limit

Our proof of Proposition 2 uses the explicit closed form construction of equilibria in discrete time. To obtain this strong characterization we limited our analysis to a family of distributions that has the nice property that any truncation has the same shape as the original
distribution. We conjecture that the result is more general, that for any well-behaved \( f(v) \), taking a limit of any sequence of PBNE of the discrete time games as \( \Delta \to 0 \), we would get

\[
V(k, T - t) = e^{-r(T-t)}V(k, 0), \quad P(k, T - t) = e^{-r(T-t)}P(k, 0)
\]

and \( K(t, T - t) \) is a solution to (8) with a boundary condition \( K(0, T) = 1 \). This conjecture is likely to be true if in the limit the equilibria become stationary (the continuation payoffs depend on the history only via \( k \) and \( T - t \)) and in the limit there are no atoms of trade before \( T \). In that case, heuristically the seller best response problem would be to choose the speed with which to skim buyer types, \( \dot{K} \) and the value of that strategy would be:

\[
rV(k, T - t) = \left( P(k, T - t) - V(k, T - t) \right) \frac{f(k)}{F(k)} \left( -\dot{K} \right) + V_k(k, T - t) \dot{K} + V_t(k, T - t)
\]

If an interior \( \dot{K} \) is optimal in the limit, then because the RHS is linear in \( \dot{K} \), it must be that the coefficients on \( \dot{K} \) add up to zero:

\[
(P(k, T - t) - V(k, T - t)) \frac{f(k)}{F(k)} = V_k(k, T - t)
\]

If that is satisfied then all terms with \( \dot{K} \) drop out of 9 and we get:

\[
rV(k, T - t) = V_t(k, T - t)
\]

Together with the boundary condition \( V(k, 0) \), this differential equation has a unique solution \( V(k, T - t) = e^{-r(T-t)}V(k, 0) \); the seller value at any moment of the game and after any history is equal to the value of the outside option of waiting for the deadline. Finally, plugging it into (10) we get \( P(k, T - t) = e^{-r(T-t)}P(k, 0) \) as claimed (note that by the envelope theorem, \( \frac{\partial F(k)\psi(k,0)}{\partial k} = P(k, 0) f(k) \)). The economic intuition behind this heuristic reasoning is that prices higher than \( e^{-r(T-t)}P(k, 0) \) would make the seller want to speed up trade, while prices lower than that would make him want to stop it and wait. The Coase conjecture forces manifest themselves in the linearity of the limit objective function in \( \dot{K} \).

---

21 Our distributional assumption also makes the seller’s problem quadratic and hence easier to solve in closed form.

22 If the limit \( K(t, T - t) \) is continuous but not differentiable at \( t \), then the reasoning applies to the right derivative.
4 The Uniform Distribution Case

In this section we will assume \( a = 1 \) which is the uniform case \( F(v) = v \). This allows us to obtain simpler expressions for all the objects of interest and carry out some comparative static analysis. We study the effects of the disagreement payoffs on the trade dynamics and division of surplus. We decompose the analysis in two parts. First we look at the effect of different degrees of inefficiency upon not reaching an agreement by changing \( \alpha + \beta \) (lower values \( \Rightarrow \) more inefficiency) and keeping \( \frac{\alpha}{\beta} \) constant. Then we consider the converse, for a certain amount of inefficiency how does the relative disagreement payoffs affect the equilibrium of the game.

**Constant Efficiency with Varying Disagreement Positions.**

**Proposition 4** Keeping \( \alpha + \beta \) constant, \( V(k, T - t) \) and \( P(k, T - t) \) are increasing in \( \alpha/\beta \) and for every \( t > 0 \) \( K(t, T - t) \) will be higher with higher \( \alpha/\beta \). Additionally, the size of the atom at time \( T \), \( m(T) \), and the probability trade takes place on or before \( T \) are decreasing in \( \alpha/\beta \).

**Proof.** First note that keeping \( \alpha + \beta \) constant while increasing \( \alpha/\beta \) leads to a higher value of \( P(k, 0) \) for all \( k \). Letting \( x = [1 - (\alpha + \beta)] \)

\[
\frac{\partial P(k, 0)}{\partial \beta} = \partial \left[ \frac{(1-\beta)^2 \exp\left(\frac{x(1-\beta)}{(1-\beta)^2 + (1-\beta)^2}(1-e^{T \gamma})\right)}{\partial \beta} \right] < 0
\]

That in turn implies \( P(k, T - t) \) is higher as well. Similarly, a higher value of \( P(k, 0) \) implies a higher value for \( V(k, T - t) \). Regarding the size of the last atom of trade note that

\[
\frac{\partial m(T)}{\partial \beta} > 0
\]

therefore, the size of the atom is decreasing in \( \alpha/\beta \).

Finally,

\[
\frac{\partial \Pr(\text{agreement})}{\partial \beta} > 0
\]

**Hence**, the probability of agreement is decreasing in \( \alpha/\beta \)

\[
\frac{\partial K(t, T - t)}{\partial \beta} < 0
\]

15
Hence, as we increase $\alpha/\beta$ trade takes place at a slower pace. That is for all $t$, $K(t, T - t)$ will be higher with higher $\alpha/\beta$.

Summarizing, for a given amount of disagreement surplus. As the fraction of that surplus that goes to the buyer increases (higher $\beta$) the lower the prices the seller charges. There is also more trade before the deadline is reached. When the deadline is reached the atom of trade is also larger since the seller is decreasing the price both because of the decrease in her disagreement payoff and because of the increase in the seller’s reservation value. Finally, more and faster trade implies that the surplus of the trading relationship will be higher. The decrease in the sellers value is (on average) more than offset by the increase in the buyer’s value.

One way to relate this result to the data for the firms (buyers) bargaining with the unions (sellers) is to think of $\alpha/\beta$ as a function of the unemployment rate. When unemployment is higher the firm could have an easier time replacing workers and workers a harder time finding new employment that is higher $\beta$ lower $\alpha$. Thus we would expect that when bargaining their yearly contracts unions would settle for worse terms and there would a lower likelihood of not reaching and agreement (going on strike) when the unemployment rate is higher. Capturing this effect in the data is not simple since the unemployment rate is not independent of the distribution of firm profitability $a$.\textsuperscript{23}

Below we graph the probability of agreement (solid) and atom size (dashed) for $\alpha + \beta = .5$ $r = 10\%$ and $T = 1$.

In the figure below we plot the first period price (solid line) and seller’s value at time 0 (dashed).

\textsuperscript{23}Times of high unemployment might also correspond to low values of $a$. See Section 5 below for comparative statics analysis w.r.t. $a$. 
Equal Disagreement Payoffs with Varying Efficiency:

**Proposition 5** Letting \( \alpha = \beta < \frac{1}{2} \) we have that for \( rT \leq 1 \) the overall probability of trade and the amount of trade at \( T \) are decreasing in the disagreement payoffs. Furthermore, conditional on reaching an agreement the likelihood it will happen at \( T \) is also decreasing in \( \alpha + \beta \).

The bleaker the prospects if they do not reach an agreement the more likely they will reach one. The last atom of trade is larger with worse disagreement payoffs. Furthermore, proportionally, more of the trade now takes place in the last instant when it is clear that not agreeing will be bad for both. The buyer will be more eager to accept a given offer. Also, if the seller were facing the same demand function she would ask for less. Now, since the seller is more eager to trade, she might even raise her demands but never to an extent that would reduce the likelihood of trade.

Below we graph the probability of agreement (solid) and atom size (dotted) and the percentage of agreements at \( T \) (dashed) for \( \alpha = \beta \), \( r = 10\% \) and \( T = 1 \).

Studying the incidence of strikes from Canadian data Gunderson, Kervin and Reid (1986) find that strikes are more likely to occur (reaching \( T \)) when the cost of not reaching an
agreement by the strike date are lower (higher \( \alpha \) and \( \beta \)).\(^{24}\)

5 Two benchmark cases

We now return to the richer class of distributions, \( F(v) = v^a \) and discuss two benchmark cases.

**Information Revealed at \( T \)**

Suppose that at time \( T \) the private information is revealed and hence there can be efficient trade i.e. \( \alpha + \beta = 1 \). This case has some interesting properties. The first thing to note is that there will no longer be an atom of trade in the last instant before the deadline. This is simply a manifestation of the no trade theorem. The buyer will not be willing to accept any price lower than \((1 - \beta)k\) on the other hand the seller would not want to offer any price lower than this since the next instant she expects to get that amount. Before the last instant there will be trade since there is a cost of waiting until the deadline and hence the highest-type buyer and seller can find mutually beneficial terms of trade.

**Proposition 6** Suppose \( \alpha + \beta = 1 \). Then, \( V(k; T-t), K(t; T-t) \) and \( P(k; T-t) \) are increasing in \( \alpha \) and for every \( t \geq 0 \) the probability trade takes place on or before \( T \) is decreasing in \( \alpha \).

**Proof.** Follows from noting that when \( \alpha + \beta = 1 \) we have that \( \gamma_T = \alpha \). Using this fact some simple computations lead to the results. \( \blacksquare \)

**Corollary 3 (Coasian Extreme)** If \( \alpha = 0 \) and \( \beta = 1 \) then trade is immediate at a price of \( 0 \) and the buyer captures all the surplus.

Note that the key to this result is not the fact that \( \alpha = 0 \) but rather that \( \beta = 1 \). Even if the seller cannot capture anything upon disagreement the fact that it can make a last take it or leave it offer at \( T \) gives him a lot of power. Yet, this power is proportional to \( (1 - \beta) \) and, when \( \beta = 1 \) it effectively has no power since the buyer can capture all the surplus when it does not take the seller’s last offer.

**Corollary 4** The ex-ante expected efficiency from trade is decreasing in \( \alpha \) and all buyer types are worse as \( \alpha \) increases.

\(^{24}\)Also in line with our results, anecdotal evidence suggests that pre-trial settlements are more likely when the legal system is more inefficient.
The loss of efficiency follows because trade is slower since \( K(t, T - t) \) is increasing in \( \alpha \) and since there is no atom at the end more types trade after the information is revealed. The inefficiency arises from the discounting cost of delayed trade. In addition to trading later, buyers also suffer from paying higher prices.

**Remark 1** Note that the fact that \( T \) is independent of the distribution implies that the equilibrium strategies are also independent of the distribution.

**Proposition 7** Suppose \( \alpha + \beta = 1 \). Then, \( V(k, T - t) \) and expected transaction prices are increasing in \( a \). Expected time to trade is decreasing in \( a \).

**Proof.** This follows from noting that the equilibrium pricing and acceptance decisions are independent of the distribution in this case. The only effect of a higher \( a \) is that there are relative more high types and since trades happens sooner and at higher prices with these types the result follows. ■

Note that for a fixed buyer type \( v \) differences in \( a \) would have no impact on its outcome. This is only true in the extreme with no efficiency loss upon reaching \( T \). More generally, strategies do depend on \( a \); a fixed buyer type would then face different outcomes when \( a \) is high than when it is low.\(^{25}\)

**Lost Opportunity of Trade at** \( T \)

Lastly we look at the special case in which \( \alpha = \beta = 0 \). If players do not reach an agreement the potential surplus from doing business is lost forever. In this case we have:

\[
\gamma_T = \left( \frac{1}{1 + a} \right)^{\frac{1}{\beta}}
\]

Note in particular that:

\[
\frac{\partial \gamma_T}{\partial a} > 0 \quad \text{and} \quad \frac{\partial}{\partial a} \left( \frac{a}{a+1} \right) > 0
\]

which lead to the following comparative statics results:

**Proposition 8** Suppose \( \alpha = \beta = 0 \). Then, \( V(k, T - t) \), \( K(t, T - t) \) and \( P(k, T - t) \) are increasing in \( a \).

\(^{25}\)See Proposition (8) below which looks at the effect of changes in the distribution for the case \( \alpha = \beta = 0 \).
Note that this implies for example that a firm of a fixed type (profitability) \( v \) is less likely to reach an agreement with the unions when the overall economy is stronger (higher \( a \)) than when the economy is in a recession. As stated by Kennan and Wilson (1989) although hard to obtain in a model this is consistent with some of the empirical studies on strikes.

"Other aspects of the incidence and duration of strikes pose particularly difficult challenges. There is a well-established body of evidence (summarized in Kennan, 1986), showing that there are more strikes in good economic times than in bad times, and S. Vroman (1989) and Gunderson et al. (1986) have recently sharpened this result for US and Canadian data respectively, showing that the incidence of contract strikes is also procyclical."

6 Conclusions

To be added.

7 Appendix

Proof of Proposition 1. The proof is by induction. At time \( T \) from equation (6) we get the buyer will not accept any price higher than \((1 - \beta) k\). Hence, the last price solves:

\[
\max_p p \left( k^a - \left( \frac{p}{1 - \beta} \right)^a \right) + \frac{1}{k^a} \frac{a}{a + 1} \alpha \left( \frac{p}{1 - \beta} \right)^{a+1}
\]

which has a unique solution at:

\[
P(k, T; T; \Delta) = k(1 - \beta) \left( \frac{1 - \beta}{(a - \beta - a\alpha - a\beta + 1)} \right)^{\frac{1}{\gamma_T}}
\]

The seller’s payoff then is:

\[
V(k, 0; \Delta) = \gamma_T k \left( \frac{k^a - k^a (1 - \beta)}{k^a (a - \beta - a\alpha - a\beta + 1)} \right)^a + \frac{1}{k^a} \frac{a}{a + 1} \alpha k^a \left( \frac{1 - \beta}{(a - \beta - a\alpha - a\beta + 1)} \right) \left( \frac{\gamma_T k}{1 - \beta} \right)
\]

\[
= \gamma_T^\prime \frac{k a}{a + 1}
\]
For a general $t$ the seller problem is:

$$V(k, T-t; \Delta) = \max_p p \left( \frac{k^a - \kappa(p, T-t; \Delta)^a}{k^a} \right) + e^{-r\Delta} V(k, T-(t+\Delta); \Delta) \frac{\kappa(p, T; \Delta)^a}{k^a}$$

where:

$$\kappa(p, T-t; \Delta) - p = e^{-r\Delta} (\kappa(p, T-t; \Delta) - P(\kappa, T-(t+\Delta); \Delta))$$  \hspace{1cm} (11)

is the buyer’s best response (necessary) condition.

In order to complete the proof by induction, assume that:

$$V(\kappa, T-(t+\Delta); \Delta) = \frac{a}{a+1} \gamma_{t+\Delta} \kappa \quad \text{and} \quad P(\kappa, T-(t+\Delta); \Delta) = \gamma_{t+\Delta} \kappa$$

Substituting the buyer’s best response into the objective function and assuming the induction hypothesis above, we can re-write the seller problem as choosing the next cutoff:

$$V(k, T-t; \Delta) = \max_\kappa \left( 1 - e^{-r\Delta} + e^{-r\Delta} \gamma_{t+\Delta} \right) \kappa \left( \frac{k^a - \kappa^a}{k^a} \right) + e^{-r\Delta} \frac{a}{a+1} \gamma_{t+\Delta} \frac{\kappa^{a+1}}{k^a}$$

The F.O.C. implies that the unique maximum is attained when:

$$((a+1)(1-e^{-r\Delta}) + e^{-r\Delta} \gamma_{t+\Delta}) \kappa^a = (1-e^{-r\Delta} + e^{-r\Delta} \gamma_{t+\Delta}) k^a$$

which yields the optimal cutoff:

$$\kappa^*(k) = \left( \frac{(1-e^{-r\Delta} + e^{-r\Delta} \gamma_{t+\Delta})}{((a+1)(1-e^{-r\Delta}) + e^{-r\Delta} \gamma_{t+\Delta})} \right)^\frac{1}{2} k$$

hence:

$$V(k, T-t; \Delta) = (1-e^{-r\Delta} + e^{-r\Delta} \gamma_{t+\Delta}) \kappa \left( \frac{k^a - \kappa^a}{k^a} \right) + e^{-r\Delta} \frac{a}{a+1} \gamma_{t+\Delta} \frac{\kappa^{a+1}}{k^a}$$

$$= (1-e^{-r\Delta} + e^{-r\Delta} \gamma_{t+\Delta}) \left( \frac{(1-e^{-r\Delta} + e^{-r\Delta} \gamma_{t+\Delta})}{((a+1)(1-e^{-r\Delta}) + e^{-r\Delta} \gamma_{t+\Delta})} \right)^\frac{1}{2} \frac{a}{a+1} k$$

$$= \frac{a}{a+1} \gamma_t k$$
Finally, the equilibrium price at $t$ has to satisfy the buyer best response condition (11):

$$P(k, T-t; \Delta) = \kappa^*(k) - e^{-r\Delta} (\kappa^*(k) - \gamma_{t+\Delta}\kappa^*(k))$$

$$= \kappa^*(k)\left(1 - e^{-r\Delta} + e^{-r\Delta}\gamma_{t+\Delta}\right) = \gamma_t k$$

which completes the proof.

The proposition does not specify the buyer strategy, but it can be derived from the seller strategy. Given any price $p$ at time $t$ (on or off the equilibrium path) the cutoff type accepting this price is the unique solution to:

$$k - p = e^{-r\Delta} (k - \gamma_t k)$$

so that $\kappa(p, T-t; \Delta) = \frac{p}{1-e^{-r\Delta} + e^{-r\Delta}\gamma_t}$. ■

8 References (to be completed)

References


23


