A Appendix

NOT FOR PUBLICATION

A.1 Value Estimation

For each good, Swoopo publishes a visible "worth up to" price, which is essentially the manufacturer's recommended price for the item. This price is one potential measure of value, but it appears to be only useful as an upper bound. In the most extreme example, Swoopo has held nearly 4,000 auctions involving 154 types of "luxury" watches with "worth up to" prices of more than $700. However, the vast majority of these watches sell on Internet sites at heavy discounts from the "worth up to" price (20-40%). It is difficult, therefore, to justify the use of this amount as a measure of value if the auctioneer or participant can simply order the item from a reputable company at a far cheaper cost. That said, it is also unreasonable to search all producers for the lowest possible cost and use the result as a measure of value, as these producers could be disreputable or costly for either party to locate.

In order to strike a balance between these extremes, I estimate the value of items by using the average price found at Amazon.com and Amazon.de for the exact same item and using the "worth up to" price if Amazon does not sell the item. I refer to this new value estimate as the *adjusted value* of the good.[55] As prices might have changed significantly over time, I only use Amazon prices for auctions later than December 2007 and scale the value in proportion to any observable changes in the "worth up to" price over time. Amazon sells only 20% of the unique consumer goods sold on Swoopo, but this accounts for 60% of all auctions involving consumer goods (goods that are sold in Amazon are likely to occur more in repeated auctions). For the goods that are sold at Amazon, the adjusted value is 79% of the "worth up to" price without shipping costs and 75% when shipping costs are added to each price (Amazon often has free shipping, while Swoopo charges for shipping). As the adjusted value is equal to the "worth up to" price for the 40% of the auctions for consumer goods that are not sold on Amazon, it still presumably overestimates the true value.[56]

To test the validity of the measure of value, note that the equilibrium analysis (and

[55] This is a somewhat similar idea to that in Ariely and Simonson (2003), who document that 98.8 percent of eBay prices for CDs, books, and movies are higher than the lowest online price found with a 10 minute search. My search is much more simplistic (and perhaps, realistic). I only search on Amazon and only place the exact title of the Swoopo object in Amazon's search engine for a result.

[56] The main results of the paper are unchanged when run only on the subset of goods sold at Amazon.

general intuition) suggest that the winning bid of an auction should be positively correlated with the value of the object for auction. Therefore, a more accurate measure of value should show a higher correlation with the distribution of winning bids for the good. The correlation between the winning bids and the "worth up to" price is **0.522** (with a 95% confidence interval of (0.516,0.527)) for auctions with a $0.15 increment for the items I found on Amazon.[57] The correlation between the winning bids and the adjusted value is **0.694** (with a 95% confidence interval of (0.690,0.698)) for these auctions. A Fisher test of correlation equality confirms that the adjusted value is significantly more correlated with the winning bid (p-value<.0001), suggesting that it is a more accurate measure of value.

A.2 Modeling Sunk Costs

In this section, I demonstrate that the qualitative features of Eyster(2002)'s model of a naive sunk cost fallacy in a WOA hold for penny auctions. The reader is referred to that paper for technical details of the utility function. Applying Eyster's model and terminology, agents in the modified model desire "consistency" in their decisions and pay a psychological cost, which I call "regret," if they spend money on bids and do not win the auction, weighted by the parameter $\rho \in [0, \infty)$ in the utility function. As a result, agents receive less utility from exiting the auction as they pay for more bids, even though these costs are sunk. Note that this modification alone will cause agents to *underbid* as they will require a premium (in the form of a higher probability of winning) to continue at any stage to offset their (correctly predicted) future psychological losses from the sunk costs. Therefore, I follow Eyster in assuming that agents consider the effect of their current decisions on their future utility, but they naively believe that their weight on future regret will be $\rho(1-\eta)$ with $\eta \in [0, 1]$.[58] I also follow Eyster's assumption of two players for exposition purposes.

In the interest of simplicity, I deviate from the Eyster's multiple period model in one substantial way. Rather than assuming that an agent feels regret for all decisions in the game, I assume that an agent simply feels regret from his initial decision (to play or not play in the game). To elucidate this difference, consider an agent who leaves the game after bidding 10 times, with bids costing 1 unit. In Eyster's model, the agent experiences regret from each past decision to stay in the auction for a total of 55ρ units (he would have saved 10 units had he exited instead of placing the first bid, 9 units if he had exited instead of

[57] Note that I cannot compare aggregate data across auctions with different bid increments for these coorelations, as the distribution of final bids of auctions for the same item will be different. The results are robust to using the (less common) bid increments of $0.00 and $0.01.

[58] A note on terminology: I choose to use η instead of υ (Eyster's parameter of naivety) to avoid confusion with the value of the object v.

placing the second bid, 8 units...). In my model, the agent simply experiences 10ρ units of regret as he would have saved 10 units from not playing the game. As one could just rescale ρ to account for this difference, the substantial difference between the models lies in the growth of regret as the game continues. In Eyster's model, regret grows "triangularly" over time, from 1 to 3 to 6 to 10, etc. In my model, regret grows linearly over time, from 1 to 2 to 3 to 4, etc. I do not believe that there is a good reason to choose either model over the other in this application, so I proceed with the linear model in the interest of simplicity.

Specifically, consider an agent who has placed b bids up until time period t. The total utility of the agent from *never bidding again* becomes:

$$-bc - \rho bc \tag{7}$$

That is, the agent experiences the monetary loss ($-bc$) of the bids as well as regret ($-\rho bc$) from deciding to play the game in the first place. Similarly, if an agent bids in period t, does not win in the next period, and never bids again, he will receive utility of:

$$-(b+1)c - \rho(b+1)c \tag{8}$$

However, due to naivety, he (mistakenly) perceives that his feeling of regret will be lower than it really is:

$$-(b+1)c - (1-\eta)\rho(b+1)c \tag{9}$$

The case in which an agent bids and wins the auction in the next period is slightly more complicated. The level of regret depends on the situation. If the net value of the item is weakly higher than the total cost the agent, the agent does not regret his decision to enter the auction. In this case, he simply receives the utility of:

$$v - tk - (b+1)c \tag{10}$$

Notice that bc (the monetary bid cost up to period t) occurs in equations 7, 9, and 10, which is consistent with bc as a sunk cost. However, the regret term only occurs if the person exits the auction, which is consistent with the notion of the sunk cost fallacy. If the person is naive, he believes that the weight on the regret will be lower in the future than today.

Alternatively, if the net value of the auction is higher than the total cost to the agent, the agent *does* regret his decision to enter the auction. In this case, he receives utility:

$$v - tk - (b+1)c - \rho(b+1)c$$

Note that, in this situation, the regret term appears in the utility term in all situations, so the agent fully recognizes the sunk cost (as before, if the agent is naive, he perceives this term to be $v - tk - (b+1)c - (1-\eta)\rho(b+1)c$).

In order for this modification to affect equilibrium behavior, agents must be able to condition their strategies on the number of bids each player has made (because this now affects agents' payoffs). Following the general path of Eyster's solution (in which naive players correctly perceive other's true strategies, although they misperceive their own) yields the following *outcome* of the preferred equilibrium and the hazard rate, which is summarized in Proposition 5

Proposition 5 *Consider when $k = 0$. There is an equilibrium of the modified game in which:*

$$\widetilde{h}(t) = \left\{ \begin{array}{cc} \frac{c+c\rho-c\rho\eta(\frac{t}{2}+1)}{v+(1-\eta)c\rho(\frac{t}{2}+1)} & \text{for } t \leq \frac{2(v+c)}{c} \\ \frac{c+c\rho-c\rho\eta(\frac{t}{2}+1)}{v} & \text{for } t > \frac{2(v+c)}{c} \end{array} \right\}$$

The effect of the regret over spending fixed costs is slightly complicated. At the beginning of the auction, agents with regret are less likely to bid than agents without regret because they have no current mistakes to regret and they realize (to the extent that they are sophisticated) that they will have to pay regret costs in the future if they bet and lose. As the auction proceeds, this difference diminishes as agents amass larger sunk costs through bidding. At some point, if agents are naive, the game continues with higher probability than with normal agents because agents (incorrectly) believe that their amassed fixed costs will be lessened if they bid and then drop out in the following period. If agents are particularly naive, they can reach a point in which no one drops out, with bidders staying in the game only because they (incorrectly) believe that bidding and dropping out tomorrow will reduce the regret from their large fixed costs.

Figure 1 displays the equilibrium hazard rates for $\rho = .3$, $c =$€.50, for an increment of €.00 as η rises. Note that the curves with higher levels of naivety display the same qualitative features as those in the empirical data.

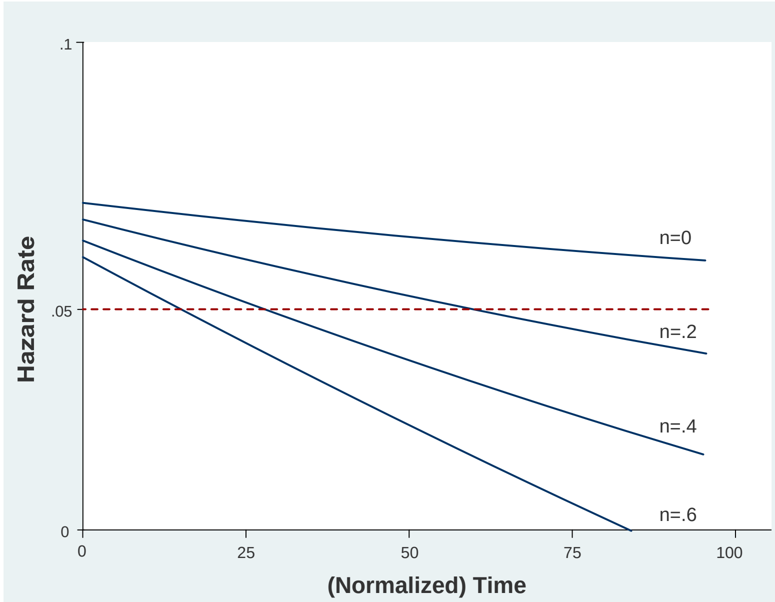

Figure 1: Equilibrium Hazard Rates in the Standard Model (dashed) vs. Equilibrium Hazard Rates for a model with a naive sunk cost fallacy. p=.3 and the different lines represent different levels of naivety.

A.3 Robustness of the model

A.3.1 mod(y-k,c)$\neq 0$

The results in the analytic section relied heavily on the assumption that mod(v-k,c)$= 0$. If this assumption does not hold, there is no equilibrium in which the game continues past period 1. However, as the following proposition shows, strategies that lead to the hazard rates in Proposition 2 form an ϵ equilibrium with ϵ very small and limiting to 0 as the size of time periods shrinks to 0:

Proposition 6 *If* $\text{mod}(v-c,k) \neq 0$, *there is no equilibrium in which the game continues past period 1. Define* $F^* = \max(t | t < \frac{v-c}{k} - 1)$. *There is an* ϵ*-perfect equilibrium which yields the same (discrete) hazard rates as those in Proposition 2 with* $\epsilon = \frac{1}{n}(1 - \frac{c}{v-F^*k})(v - (F^*+1)k - c)[\prod_{t=1}^{F^*-1}(1 - \frac{c}{v-tk})]$. *There is an contemporaneous* ϵ^c*-perfect equilibrium (Mailath (2003)) which yields the same (discrete) hazard rates as those in Proposition 2 with* $\epsilon^c = \frac{1}{n-1}(1 - \frac{c}{v-F^*k})(v-(F^*+1)k-c)$. *There is a contemporaneous* ϵ^c*-perfect equilibrium which yields the same hazard and survival rates as those in Proposition 4 with* $\epsilon^c \to 0$ *as* $\Delta t \to 0$.

To give an idea of the magnitude of the mistake of playing this equilibrium in auctions in my dataset, consider an stylized auction constructed to make ϵ as high as possible, with $v =$€9.95$, c =$€.50$, k =$€.10, and $n = 20$. In this case, $\epsilon =$ €.0000000000224 and $\epsilon^c =$€.00060. That is, even in the most extreme case and using the stronger concept of contemporaneous ϵ^c-perfect equilibrium, players lose extremely little by following the proposed strategies. This is because their only point of profitable deviation is at the end of the game, where their equilibrium strategy is to bet with low probability, there is a small chance that their bet be accepted, and the cost of the bet being accepted is small (and, ex ante, there is an extremely small chance of ever reaching this point of the game).

A.3.2 Independent Values

In the model in the main paper, I assume that players have a common value for the item. The equilibrium is complicated if players have values v_i is drawn independently from some distribution G of finite support before the game begins or $v_i(t)$ is drawn independently from G at each time t. In these equilibria, players' behavior is dependant largely on the exact form of G, with very few clear results about bidding in each individual period (which is confirmed by numerical simulation). However, if players have independent values which tend to a common value, the distribution of hazard rates approaches the bidding hazard rates in the following way:

Proposition 7 *Consider if (1) v_i is drawn independently from G before the game begins or (2) $v_i(t)$ is drawn independently from G at each time t. For any distribution G, there is a unique set of hazard rates $\{\widetilde{h}^G(1), \widetilde{h}^G(2), ...\widetilde{h}^G(t)\}$ that occur in equilibrium. Let the the support of G_i be $[v - \delta_i, \overline{v} + \delta_i]$. For any sequence of distributions $\{G_1, G_2, ...\}$ in which $\delta_i \to 0$ and the game continues past period 1 in equilibrium, $\widetilde{h}^G(t) \to \widetilde{h}(t)$ from Proposition 2 for $t > 0$. For any sequence of distributions G with $\delta \to v$ and $\Delta t \to 0$, there exist a sequence of corresponding contemporaneous ϵ^c-perfect equilibria with hazard and survival rates equal to those in Proposition 4 in which $\epsilon^c \to 0$.*

A.3.3 Leader can bid

Throughout the paper, I assume that the leader cannot bid in an auction. This assumption has no effect on the preferred equilibrium below, as the leader not bid in equilibrium even when given the option. However, the assumption does dramatically simplify the exact form of other potential equilibria, as shown below.

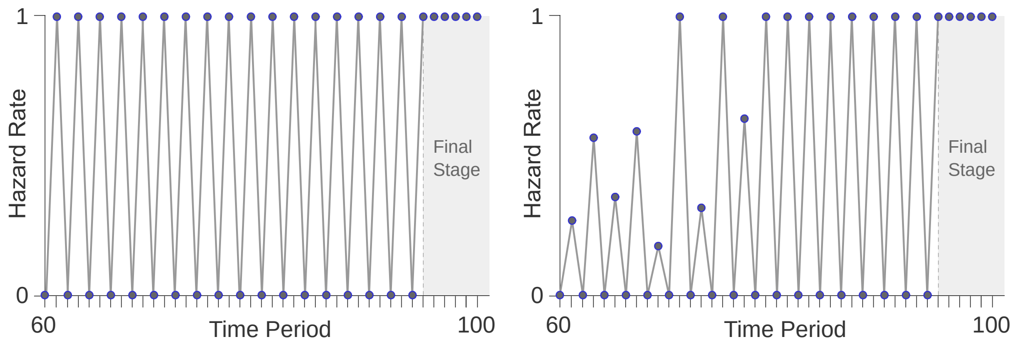

Figure 2: Change in non-bidding equilibria when the leader is allowed to bid.

Consider a modified game in which the leader can bid. Now, a (Markov) strategy for player i at period t is the probability of betting both if a non-leader ($x_t^{i,NL*}$) and, for $t > 0$, when a leader ($x_t^{i,L*}$) (there is no leader in period 0).

Proposition 8 *In the modified game, Proposition 2 still holds.*

The equilibria in the situation in which non-leaders bid becomes significantly more complicated and finding a closed form solution becomes extremely difficult. For example, consider the equilibrium in which no player bids at period F. The (numerically) solved equilibrium hazard rates are shown for $v = 10$, $c = .5$, $k = .1$ and $n = 3$ in Figure 2 Notice the obvious irregularities in the equilibrium hazard rates in the modified game. This occurs because the ability of a leader to bid in period t distorts the incentives of non-leaders in previous periods. To see this, consider the situation in which $\widetilde{h}(t+1) = 1$ and $\widetilde{h}(t) = 0$. When leaders cannot bid, there is no benefit from a non-leader bidding in period $t-1$ as he will not win the object in period t (because the game will continue with certainty) or period $t+1$ (because he cannot bid in period t and therefore will never be a leader at $t+1$), at which point the game will end. However, when leaders can bid, non-leaders in period $t-1$ can potentially benefit from bidding. Although there is still no chance that the non-leader in period $t-1$ will win the object in period t by bidding, she will be able to bid (as a leader) in period t, leading to the possibility that she will win the object in period $t+1$. Therefore, non-leaders will potentially bid in this situation in equilibrium not to win the object in the following period, but simply to keep the game going for a (potential) win in the future.

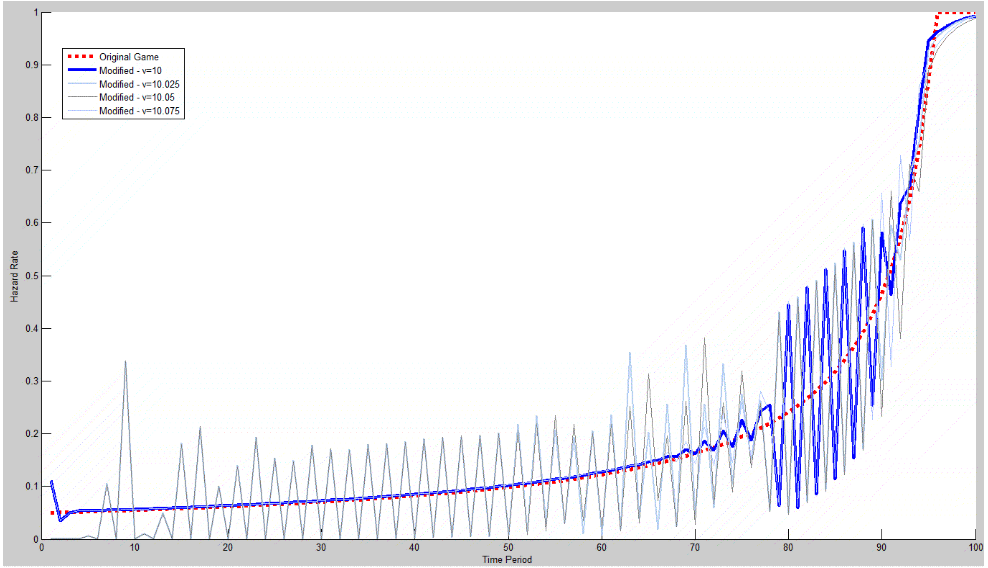

Figure 3: Numerical Analysis of the hazard rate of auctions for different values (solid lines) when the multiple bids are accepted at each time period vs. the predicted hazard rate (dotted red line) when only one bid is accepted.

A.3.4 Allowing Multiple Bids to Be Accepted

Allowing multiple bids to be accepted significantly complicates the model, especially in a declining-value auction. Consider a player facing other players that are using strictly mixed strategies. If the player bids in period t, there is a probability that anywhere from 0 to $n-2$ other non-leading players will place bids, leading the game to immediately move to anywhere from period $t+1$ to period $t+n$. In each of these periods, the net value of the object is different, as is the probability that no player will bid in that period and the auction will be won (which is dependant on the equilibrium strategies in each of the periods).

It is possible to solve the model numerically, leading to a few qualitative statements about the hazard rates. Figure 3 shows the equilibrium hazard rates (with $k = .1, c = .5, n = 10$) given small changes in the value of the good ($v = 10, 10.25, 10.5, 10.75$), as well as the analytical hazard rates from Proposition 2. These graphs demonstrate three main qualitative statements about the relationship between the equilibria in the modified model and the original model:

1. The hazard rates of the modified model are more unstable locally (from period-to-period) than those from Proposition 2, especially in later periods. As n increases, this

instability decreases (I do not present graphs for lack of space).

2. The hazard rates of the modified model closely match those from Proposition 2 when smoothed locally.

3. The hazard rates of the modified model are more stable globally to small changes in parameters in the model. Recall that the hazard rates in Proposition 2 were taken from an equilibrium when $\mathrm{mod}(y-c,k)=0$. When $\mathrm{mod}(y-c,k)\neq 0$, the hazard rates oscillated radically (although they were smooth in an ε-equilibrium with very small ε). The modified model is much more globally robust to these changes.

A.3.5 Timer

In the model in the paper, unlike that in the real world implementation of the model, there is no timer within each period. Consider a game in which, in each discrete period t, players can choose to place a bid at one sub-time $\tau \in [0,T]$ or not bid for that period. As in the original game, if no players bid, the game ends. If any players bid, one bid is randomly chosen from the set of bids placed at the smallest τ of all bids (the first bids in a period). Now, a player's (Markov) strategy set is a function for each period $\chi_t^i(\tau) : [0,T] \to [0,1]$, with $\int_0^T \chi_t^i(\tau)d\tau$ equaling the probability of bidding at some point in that period.

Proposition 9 *For any equilibrium of the modified game, there exists an equilibrium of the original game in which the distribution of the payoffs of each of the players is the same.*

This proposition demonstrates that, while the timer adds complexity to the player's strategy sets, it does not change any of the payoff-relevant outcomes.

A.4 Proofs

The results for hazard rates hold for non-Markovian strategies (in which players condition on the leader history) with leader history H_t replacing (t,l_t) and some notational changes. Please email the author for the proofs.

Proposition 1

Proof: Assume that an equilibrium exists in which $\widetilde{h}(t^*,l_t^*)<1$ for some history (t^*,l_t^*) where $t^* > \frac{v-c}{k}-1$. Then, there must be some player $i \neq l_t^*$ with $p_t^i > 0$. Given some (t,l_t), define the probability that player i has a bid accepted at (t,l_t) as $a^i(t,l_t) \in [0,1]$ and the probability that the game ends at (t, l_t) as $q(t, l_t) \in [0, 1]$. Note that as $p_t^i > 0$, it must be that $a^i(t^*, l_t^*) > 0$. Player $i's$ continuation payoff in the proper subgame starting at (t^*, l_t^*) is then: $E[\sum_{t=t^*}^{\infty} a^i(t, l_t)(-c + q_{t+1}(t+1, i)(v - (t+1)k))] < E[\sum_{t=t^*}^{\infty} a^i(t, l_t)(-c + q_{t+1}(t, i)(v - (\frac{v-c}{k} + 1)k))] < E[\sum_{t=t^*}^{\infty} a^i(t, l_t)(-c + q_{t+1}(t+1, i)(c - k)) < 0$. But, player i could deviate to setting $p_t^i = 0$ and receive a payoff of 0. Therefore, this can not be an equilibrium.

Proposition 2

Proof: Note that the hazard function associated with the strategies matches those in the Proposition: for $t = 0$, $\widetilde{h}(t, l_t) = 0$; for $0 < t \leq F$, $\widetilde{h}(t, l_t) = (1 - (1 - \sqrt[n-1]{\frac{c}{v-tk}}))^{n-1} = \frac{c}{v-tk}$; for $t > F$, $\widetilde{h}(t, l_t) = 1$.

Claim: this set of strategies is a Markov Perfect Equilibrium.

First, consider if $k = 0$. Note that the game is stationary and strategies above are symmetric. Define the continuation payoff for every player of entering a period as the leader as π_L and a non-leader as π_{NL}. Following the strategies in the Proposition, define the probability of bidding for each non-leading player as $p = (1 - \sqrt[n-1]{\frac{c}{v}}) \in (0, 1)$ and the probability of having the bid accepted given a bid as $q \in (0, 1)$. Then, $\pi_L = \widetilde{h}(t, i)(v) + (1 - \widetilde{h}(t, i))\pi_{NL}$ which, as $\widetilde{h}(t, i) = \frac{c}{v}$ for all (t, l_t), must equal $\frac{c}{v}(v) + (1 - \frac{c}{v})\pi_{NL} = c + (1 - \frac{c}{v})\pi_{NL}$. Similarly, $\pi_{NL} = p(q(-c + \pi_L) + (1 - q)\pi_{NL}) + (1 - p)\pi_{NL}$. The only solution to these equations is $\pi_{NL} = 0$ and $\pi_L = c$. Then, the continuation payoff from bidding as a non-leader in any period must be $q(-c + \pi_L) + (1 - q)\pi_{NL} = 0$ and the continuation payoff from not bidding must be $\pi_{NL} = 0$. Therefore, no player strictly prefers to deviate from the strategies above and we have a subgame perfect equilibrium.

Second, consider if $k > 0$. Note that the game is non-stationary. I will show that, for any (t, l_t), the following statement (referred to as statement 1) is true: there is no strictly profitable deviation from the listed strategies at (t, l_t) and the continuation payoff from entering (t, l_t) as a non-leader is 0. For the subgames starting at (t, l_t) with $t > F$, refer to the proof of Proposition 1 for a proof of the statement. For the subgames starting at (t, l_t) with $t \leq F$, the proof continues using (backward) induction with the statement already proved for any (t, l_t) with $t > F$. At (t, l_t), non-leader player i will receive an expected continuation payoff of 0 from not betting (she will receive 0 at (t, l_t) and will enter some $(t+1, l_{t+1})$ as a non-leader, which has a continuation payoff of 0 by induction). By betting, there is some positive probability her bid is accepted. If this is the case, she receives $-c$ at (t, l_t), and will enter $(t+1, i)$ as the leader. The probability that she wins the auction at $(t+1, i)$ is $\widetilde{h}(t+1, i) = \frac{c}{v-(t+1)k}$, in which case she will receive $v - (t+1)k$. The probability that she loses the auction at $(t+1, i)$ is $1 - \frac{c}{v-tk}$, in which case she will enter $(t+2, l_{t+2})$

as a non-leader, which must have a continuation payoff of 0 by induction. This leads to a total continuation payoff from her bid being accepted of $-c + \frac{c}{v-(t+1)k}(v-(t+1)k) = 0$. Alternatively, if the bid is not accepted, she enters $(t+1, l_{t+1})$ as a non-leader and receives a continuation payoff of 0 by induction. Therefore, the continuation payoff from betting must be 0. Therefore, statement 1 is true for all periods and this is a Markov Perfect Equilibrium.

Proposition 3

Proof: First, consider if $k > 0$.

Consider statement (1).

I will show that, for each period t, (A) for any (t, l_t) that is reached in equilibrium, $\widetilde{h}(t, l_t)$ must match those in Proposition 2 if $t > 1$ and (B) the continuation payoff from any player $i \neq l_{t-1}$ entering any $(t-1, l_{t-1})$ that is reached in equilibrium, $\pi_i(t-1, l_{t-1})$, must be zero. By Proposition 1, the statement (A) is true for all (t, l_t) where $t > \frac{v-c}{k} - 1 = F$. Now, consider statement (B). As $\widetilde{h}(t, l_t) = 1$ for every period $t > F$, it must be that $p_i = 0$ for each player $i \neq l_t$ for every period $t > F$. Then, it must be that $\pi_i(t, l_t) = 0$ if $t > F$ for all players $i \neq l_t$ as no player bids for any $t > F$. Finally, consider $\pi_i(F, l_F)$ for any player $i \neq l_F$. There are three possible outcomes for player $i \neq l_F$ at (F, l_F), all of which lead to a continuation payoff of 0. First, the game ends. Second, another player enters period $F+1$ as the leader, where player i's continuation payoff is $\pi_i(F+1, l_{F+1})$ where $i \neq l_{F+1}$, which must be 0 by the above proof. Third, player i enters period $F+1$ as the leader, in which case her payoff must be $-c + \widetilde{h}(F+1, i)(v - Fk) + (1 - \widetilde{h}(F+1, i))\pi_i(F+2, l_{F+2}) = -c + v - (\frac{v-c}{k})k = 0$ as $\widetilde{h}(t, l_t) = 1$ for any (t, l_t) if $t > F$ by Proposition 1. Therefore, $\pi_i(F, l_F) = 0$ for $i \neq l_F$ and statement (B) is proven if $t > F$.

For $1 < t \leq F$, the proof continues using (backward) induction with the statement already proved for all periods t with $t > F$. First consider statement (A). Taking the other players' strategies as fixed, define the probability of each player $i \in \{1, 2, ..n\}$ being chosen as the leader in $t+1$ at (t, l_t) as $q_i^{j=B}(t, l_t)$ if player j bids and $q_i^{j=NB}(t, l_t)$ if player j does not bid. Note that $q_i^{i=B}(t, l_t)$ must be strictly positive. For part (1) of the statement, consider some (t^*, l_t^*) which is reached in equilibrium in which $\widetilde{h}(t^*, l_t^*) \neq \frac{c}{v-tk}$. Consider any (t^*-1, l_{t-1}^*) that proceeds (t^*, l_t^*) and any (t^*-2, l_{t-2}^*) that proceeds (t^*-1, l_{t-1}^*). Note that $l_t^* \neq l_{t-1}^*$ and $l_{t-1}^* \neq l_{t-2}^*$. The expected difference in continuation payoff from player l_t^* in period $t-1$ for history (t^*-1, l_{t-1}^*) from bidding and not bidding is:

$q_{l_t^*}^{l_t^*=B}(t^*-1, l_{t-1}^*)(-c+\widetilde{h}(t^*, l_t^*)(v-tk)) + (1-q_{l_t^*}^{l_t^*=B}(t^*-1, l_{t-1}^*))\sum_{j\neq l_t^*} q_j^{l_t^*=B}(t^*-1, l_{t-1}^*) * \pi_{l_t^*}(t, j) - \sum_{j\neq l_t^*} q_j^{l_t^*=NB}(t^*-1, l_{t-1}^*) * \pi_{l_t^*}(t, j)$. By induction, $\pi_{l_t^*}(t, j) = 0$ for any $j \neq l_t^*$. Therefore, the above equation simplifies to $q_{l_t^*}^{l_t^*=B}(t^*-1, l_{t-1}^*)(-c + \widetilde{h}(t^*, l_t^*)(v - tk))$. Now, consider the situation in which $\widetilde{h}(t^*, l_t^*) < \frac{c}{v-tk}$. In this case, the difference in continuation payoff is negative, and therefore player l_t^* must strictly prefer to not bid at any (t^*-1, l_{t-1}^*) that proceeds (t^*, l_t^*). But then (t^*, l_t^*) will not be reached in equilibrium and we have a contradiction. Next, consider the situation in which $\widetilde{h}(t^*, l_t^*) > \frac{c}{v-tk}$. In this case, the difference in continuation payoff is positive, and therefore player l_t^* must strictly prefer to bid in period $t-1$ at any (t^*-1, l_{t-1}^*) that proceeds (t^*, l_t^*). This implies that $\widetilde{h}(t^*, l_t^*) = 0$ in equilibrium. However, now consider l_{t-1}^* in period $t-2$ in any (t^*-2, l_{t-2}^*) that proceeds (t^*-1, l_{t-1}^*). Claim: in each potential state of the world at (t^*-2, l_{t-2}^*) (the other players' bids and the auctioneer's choice of leader are unknown), player l_{t-1}^* weakly prefers to not bid and, in at least one state of the world, l_{t-1}^* strictly prefers to not bid. First, consider the states of the world in which no other player is bidding. Here, a bid from player l_{t-1}^* leads to an expected continuation payoff of $-c+\widetilde{h}(t^*-1, l_{t-1}^*)(v-(t-1)k)+(1-\widetilde{h}(t^*-1, l_{t-1}^*))\pi_{l_{t-1}^*}(t, l_t) = -c$ as $\widetilde{h}(t^*-1, l_{t-1}^*) = 0$ in equilibrium and $\pi_{l_{t-1}^*}(t, l_t) = 0$ by induction. The expected continuation payoff from not bidding in these states of the world is 0, as the game ends. Therefore, in these states, player l_{t-1}^* strictly prefers to not bid. Second, consider the states of world in which another player bids and player l_{t-1}^*'s bid will be accepted. Here, the expected continuation payoff from bidding is $-c$ (as above) and the expected continuation payoff from not bidding is $\pi_{l_{t-1}^*}(t-1, l_{t-1})$ for some $l_{t-1} \neq l_{t-1}^*$. $\pi_{l_{t-1}^*}(t-1, l_{t-1})$ much be weakly greater than 0, as a player could guarantee an expected payoff of 0 from never bidding. Therefore, in these states, player l_{t-1}^* strictly prefers to not bid. Note that one state from these first two categories of states must occur, so player l_{t-1}^* strictly prefers to not bid in at least one state. Finally, consider the states of the world in which another player bids and player l_{t-1}^*'s bid will not be accepted. Here, (t, l_{t-1}) is constant if player l_{t-1}^* bids or not, and therefore player l_{t-2}^* weakly prefers to not bid. Therefore, in equilibrium, player l_{t-1}^* must not bid at any (t^*-2, l_{t-2}^*) that proceeds any (t^*-1, l_{t-1}^*) that proceeds any (t^*, l_t^*). But, then we have a contradiction as (t^*, l_t^*) cannot occur in equilibrium.

Next, I will prove statement (B) for period t. Consider $\pi_i(t-1, l_{t-1})$ for any player $i \neq l_{t-1}$ in any $(t-1, l_{t-1})$ that is reached in equilibrium. There are three possible outcomes for player i at $(t-1, l_{t-1})$, all of which lead to a continuation payoff of 0. First, the game ends. Second, another player enters period t as the leader, in which case player i's continuation payoff is $\pi_i(t, l_t)$ for some $l_t \neq i$, which must be 0 by induction. Third, player i enters period t as the leader, in which case her payoff must be $-c+\widetilde{h}(t, i)(v-tk)+(1-\widetilde{h}(t, i))\pi_i(t+1, l_{t+1}) = -c + \frac{c}{v-tk}(v-tk) + (1 - \frac{c}{v-tk})\pi_i(t+1, l_{t+1}) = 0$ as $\widetilde{h}(t, i) = \frac{c}{v-tk}$ for period t by above and $l_{t+1} \neq i$, so $\pi_i(t+1, l_{t+1}) = 0$ by induction. Therefore, it must be that $\pi_i(t-1, l_{t-1})$ for any player $i \neq l_{t-1}$ in any $(t-1, l_{t-1})$ that is reached in equilibrium and the statement is proved.

Consider Statement (2):

Assume there is an equilibrium in which player i uses Markov Strategies and $p_0^i > 0$, $p_1^i > 0$ for all i.

For each period $t > 1$, I will prove that player i must follow the strategies listed in the Proposition 2. First, note that by Proposition 1, $\widetilde{h}(t, l_t) = 1$ where $t > \frac{v-c}{k} - 1 = F$, so it must be that $p_t^i = 0$ for each player i for every period $t > F$. For periods $1 < t \leq F$, the proof is by induction with period 2 as the initial period. Period 2: As $p_0^i > 0$, $p_1^i > 0$ for all i, it must be true that for each player i, $(2, l_2 = i)$ occurs on the equilibrium path. Suppose that $p_t^i \neq p_t^j$ for some players i and j for $t = 2$. Then, $\widetilde{h}(t, l_t = i) = \frac{\prod_{k=1}^{n}(1-p_t^k)}{(1-p_t^i)} \neq \frac{\prod_{k=1}^{n}(1-p_t^k)}{(1-p_t^j)} = \widetilde{h}(t, l_t = j)$ for $t = 2$. But, by Statement (1) of Proposition **3**, it must be that $\widetilde{h}(t, l_t = i) = \frac{c}{v-2k} = \widetilde{h}(t, l_t = j)$ for $t = 2$ so we have a contradiction. Therefore, $p_t^i = p_t^j$ for all i and j and therefore $p_t^i = \sqrt[n-1]{1 - \frac{c}{v-tk}}$ for all i when $t = 2$. Period t: Suppose the statement is true for periods prior to t. Then, it must be true that for each player i, $(t, l_t = i)$ occurs on the equilibrium path. Now, follow the rest of the proof for $t = 2$ for any $t \leq F$ to show that the statement holds for any period $1 < t \leq F$. Therefore, in any Markov Perfect Equilibrium in which play continues past period 1,strategies must match these after period 1.

Second, consider if $k = 0$.

Consider Statement (1):

Assume that players use symmetric strategies: $p_t^i = p_t^j = p_t$. Note that this implies that $\widetilde{h}(t, l_t = i) = \widetilde{h}(t, l_t = j) = \widetilde{h}(t)$. Define the continuation payoff for every player of entering period t as the leader as $\pi_L(t)$ and a non-leader as $\pi_{NL}(t)$. Claim: $\pi_{NL}(t) = \pi_L(t) - c$ for any period $t > 1$ that appears on the equilibrium path. First, suppose that there exists t on the equilibrium path such that $\pi_{NL}(t) > \pi_L(t) - c$. Using notation from the Proof to Proposition **3**, the difference in the expected payoff from bidding and not bidding for player i at period $t-1$ is $(1 - q_i^{i=B}(t-1))\pi_{NL}(t) + q_i^{i=B}(t-1)(-c + \pi_L(t)) - \pi_{NL}(t) < 0$. Therefore, all non-leading bidders must strictly prefer to not bid in period $t - 1$. However, this implies that t cannot be reached on the equilibrium path, a contradiction. Second, suppose that there exists t on the equilibrium path such that $\pi_{NL}(t) < \pi_L(t) - c$. The difference in the expected payoff from bidding and not bidding for player i at period $t - 1$ is $(1 - q_i^{i=B}(t-1))\pi_{NL}(t) + q_i^{i=B}(t-1)(-c + \pi_L(t)) - \pi_{NL}(t) > 0$. Therefore, all non-leading bidders must strictly prefer to bid in period $t - 1$. This implies that $\pi_L(t-1) = \pi_{NL}(t)$ as a leader in period $t-1$ will necessarily become a non-leader in period t. It also implies that

$\pi_{NL}(t-1) \geq \pi_{NL}(t)$ as a non-leader in period $t-1$ could not bid and guarantee $\pi_{NL}(t)$. Therefore, the difference in the expected payoff from bidding and not bidding for player i at period $t-2$ i

$$(1-q_i^{i=B}(t-2))\pi_{NL}(t-1)+q_i^{i=B}(t-1)(-c+\pi_L(t-1))-\pi_{NL}(t-1)=$$
$$(1-q_i^{i=B}(t-2))\pi_{NL}(t-1)+q_i^{i=B}(t-1)(-c+\pi_{NL}(t))-\pi_{NL}(t-1)<$$
$$(1-q_i^{i=B}(t-2))\pi_{NL}(t)+q_i^{i=B}(t-1)(-c+\pi_{NL}(t))-\pi_{NL}(t)=$$
$$q_i^{i=B}(t-1)(-c)<0$$

Therefore, players in $t-2$ must strictly prefer to not bid. This implies that period t is not on the equilibrium path, a contradiction. Therefore, it must be that $\pi_{NL}(t)=\pi_L(t)-c$ for any period $t>1$ that appears on the equilibrium path. Now, note in equilibrium $\pi_L(t)=H(t)v+(1-H(t))\pi_{NL}(t+1)$ and $\pi_{NL}(t)=H(t)(0)+(1-H(t))(\frac{1}{n}(-c+\pi_L(t+1))+\frac{n-2}{n-1}\pi_{NL}(t+1))$. Suppose $t>1$ and t is on the equilibrium path. Note that $t+1$ must also be on the equilibrium path: If $H(t)=1$, then $\pi_L(t)=v>\pi_{NL}(t)+c$, a contradiction. Therefore, it must be that $\pi_L(t)=\pi_{NL}(t)+c$ and $\pi_L(t+1)=\pi_{NL}(t+1)+c$. Imposing this on the equations for $\pi_{NL}(t)$ and $\pi_L(t)$ yields the unique solution: $H(t)=\frac{c}{v}$. Therefore, the Proposition is true.

Consider Statement (2):

Assume there is an equilibrium in which players uses Symmetric Markov Strategies and $p_0>0$, $p_1>0$.

For each period $t>1$, I will prove that players must follow the strategies listed in the Proposition 2. The proof is by induction with period 2 as the initial period. Period 2: As $p_0>0$, $p_1>0$, it must be true that period t occurs on the equilibrium path. By Statement (1) of Proposition **3**, it must be that $\widetilde{h}(t)=\frac{c}{v}$ and therefore $p_t=\sqrt[n-1]{1-\frac{c}{v-tk}}$ when $t=2$. Period t: Suppose the statement is true for periods prior to t. Then, it must be true that period t occurs on the equilibrium path. Now, follow the rest of the proof for $t=2$ for any t to show that the statement holds for any period $1<t$. Therefore, in any Symmetric Markov Perfect Equilibrium in which play continues past period 1,strategies must match these after period 1.

Proposition 4

Proof:

(For Non-Normalized Time): Let $S(t)=p$. As $\widetilde{h}(t)=\frac{c\Delta t}{v-(t)k}$ for $t\leq F$, $S(t+\Delta t)=(1-\frac{c\Delta t}{v-(t+\Delta t)k})p$. Therefore, $S(t)=\lim_{\Delta t\to 0}\frac{S(t)-S(t+\Delta t)}{\Delta t\cdot S(t)}=\lim_{\Delta t\to 0}\frac{p(\frac{c\Delta t}{v-(t+\Delta t)k})}{\Delta t\cdot p}=\lim_{\Delta t\to 0}\frac{c}{v-(t+\Delta t)k}=$

$\frac{c}{v-tk}$. As $H(t) = \int_0^t \frac{c}{v-\widetilde{t}k} d\widetilde{t}$, $H(t) = \frac{c(\ln(v)-\ln(v-tk))}{k}t$ if $k > 0$ and $t \leq F$, while $H(t) = \frac{c}{v}t$ if $k = 0$. Note that $H(t) = \int_0^t \lim_{\Delta t \to 0} \frac{S(\widetilde{t})-S(\widetilde{t}+\Delta t)}{\Delta t \cdot S(\widetilde{t})} d\widetilde{t} = -\int_0^t \frac{1}{S(\widetilde{t})}(\frac{d}{d\widetilde{t}}S(\widetilde{t}))d\widetilde{t} = -\ln S(t)$. Therefore $S(t) = e^{-H(t)}$ and $S(t) = (1-\frac{tk}{v})^{\frac{c}{k}}$ if $k > 0$ and $t \leq F$, while $S(t) = e^{-(\frac{c}{v})t}$ if $k = 0$.

Note that, as $\widetilde{h}(F+\Delta t) = 1$, $\lim_{\Delta t \to 0} \Pr(T > F) = 0$ and therefore $S(F) = 0$.

(For Normalized Time): $S(tv_1; v_1) = (1-\frac{tv_1 y}{v_1})^{\frac{c}{y}} = (1-ty)^{\frac{c}{y}} = (1-\frac{tv_2 y}{v_2})^{\frac{c}{y}} = S(tv_2; v_2)$ if $k > 0$ and $t \leq F$. $S(tv_1; v_1) = 0 = S(tv_2; v_2)$ if $k > 0$ and $t > F$. $S(tv_1; v_1) = e^{-(\frac{c}{v_1})tv_1} = e^{-ct} = e^{-(\frac{c}{v_2})tv_2} = S(tv_2; v_2)$ if $k = 0$. $S(t\frac{v_1}{c_1}; v_1, c_1) = e^{-(\frac{c_1}{v_1})t\frac{v_1}{c_1}} = e^{-t} = e^{-(\frac{c_2}{v_2})t\frac{v_2}{c_2}} = S(t\frac{v_2}{c_2}; v_2, c_2)$ if $k = 0$.

Proposition 5

Proof:

The proof follows Eyster (2002) directly. Players must be indifferent between bidding and not bidding in order to sustain the equilibrium in which $\widetilde{h}(t) \in (0,1)$. Consider an player in even period t. As there are two players, this player must have made $\frac{t}{2}$ bids up to this point. As noted in the text, player's incentives change when the net value of the item drops below the total bid costs spent on the item if the player wins, which is when $\frac{t}{2}c - c = v \Rightarrow t^* = \frac{2(v+c)}{c}$. Consider if $t \leq t^*$. Given the hazard rate in the Proposition $\widetilde{h}^*(t+1) = \frac{c+c\rho-c\rho\eta(\frac{t+1}{2}+1)}{v+(1-\eta)c\rho(\frac{t+1}{2}+1)}$, the player is indifferent between bidding today or not as $(-\frac{t}{2}c-(1-\eta)\rho\frac{t}{2}c) = \widetilde{h}^*(t+1)(v-(\frac{t+2}{2})c) + (1-\widetilde{h}^*(t+1))(-(\frac{t+2}{2})c-(1-\eta)\rho(\frac{t+2}{2})c)$. The argument is similar if $t > t^*$. Therefore, no player has a strict incentive to deviate from choosing $p_i^t = 1- h^*(t)$ and the equilibrium holds.

Proposition 6

Proof: Consider the strategies noted in the proof of Proposition 2 with $F = F^*$. For the standard ϵ-perfect equilibrium, we consider the ex ante benefit of deviating to the most profitable strategy, given that the other players continue to follow this strategy. Following the proof of Proposition 2, it is easy to show that there is no profitable deviation in periods $t > F^*$ and $t < F^*$. Therefore, the only profitable deviation is to not bet in $t = F^*$. This will yield a continuation payoff of 0 from period F^*. The ex ante continuation payoff from betting is $\epsilon = \frac{1}{n}(1-\frac{c}{v-F^*k})(v-(F^*+1)k-c)[\prod_{t=1}^{F^*-1}(1-\frac{c}{v-tk})]$. (To see this, note that there is a $\prod_{t=1}^{F^*-1}(1-\frac{c}{v-tk})$ change that the game reaches period F^*. In period F^*, there is a $(1-\frac{c}{v-F^*k})$ probability that at least one player bets. As strategies are symmetric, this means that, ex ante, a player has a $\frac{1}{n}(1-\frac{c}{v-F^*k})$ probability of her bet being accepted in this period, given that the game reaches this period. If the bet is accepted, the player will receive $(v - (F^* + 1)k - c))$. Therefore, the ex ante benefit from deviating to the most profitable strategy is ϵ. For the contemporaneous ϵ^c-perfect equilibrium, we consider the benefit of deviating to the most profitable strategy once period F^* is reached, given that the other players continue to follow this strategy. This is $\epsilon^c = \frac{1}{n-1}(1 - \frac{c}{v-F^*k})(v - (F^* + 1)k - c)$ (To see this, note that in period F^*, there is a $(1 - \frac{c}{v-F^*k})$ probability that at least one player bets. As strategies are symmetric, this means that, ex ante, a non-leader has a $\frac{1}{n-1}(1 - \frac{c}{v-F^*k})$ probability of her bet being accepted in this period (as there are only $n - 1$ non-leaders). If the bet is accepted, the player will receive $(v - (F^* + 1)k - c))$.

Proposition 7

Proof: In case 1, I will refer to $v_i(t) = v_i$. The proof is simple (backward) induction on the statement that there is a unique hazard rate that can occur in each period in equilibrium. By the same logic in the proof to Proposition 1, $\widetilde{h}(t) = 1$ for all $t > \frac{v+\delta_i-c}{k} - 1$. Consider periods $t \leq F^* = \max\{t|t \leq \frac{v+\delta_i-c}{k} - 1\}$ where $\widetilde{h}(t+1)$ is unique in equilibrium by induction. If $\widetilde{h}(t + 1) = 0$, then $\widetilde{h}(t) = 1$ as any player with finite $v_i(t)$ strictly prefers to not bid. If $\widetilde{h}(t + 1) > 0$, a player with cutoff type $v^*(t) = \frac{c}{\widetilde{h}(t+1)} + (t + 1)k$ is indifferent to betting at time t given $\widetilde{h}(t + 1)$. Therefore, $\widetilde{h}(t) = G(\max(\min(v^*, v + \delta), v - \delta))$ and the statement is true. Suppose G_i is such that the game continues past period 1. Claim 1: If $\delta < k$,then (1) $\widetilde{h}(t) = 0$ for $t \leq F \Rightarrow \widetilde{h}(t - 1) = 1$ and (2) $\widetilde{h}(t) = 1$ for $t \leq F \Rightarrow \widetilde{h}(t - 1) = 0$. Statement (1) is true as a bidding leads to $-c$, a lower payoff than not bidding. Statement (2) is true as if $\widetilde{h}(t) = 1$, then the payoff of bidding for a player with value $\widetilde{v}$ at period $t - 1$ is Pr[Bid Accepted]$(\widetilde{v}- tk - c)$. Note that Pr[Bid Accepted] > 0 if a player bids. Note that $t \leq F \Rightarrow t \leq \frac{v-c}{k} - 1 \Rightarrow 0 \leq v - c - (t + 1)k \Rightarrow 0 < v - \delta - c - tk$ as $\delta < v$. Therefore, $0 <$ Pr[Bid Accepted]$(\widetilde{v}- (t + 1)k - c)$ for every $\widetilde{v} \in [v - \delta, \overline{v} + \delta]$ and therefore $\widetilde{h}(t) = 0$ and the claim is proved. Claim 2: If $\delta < k, \widetilde{h}(t) \in (0, 1)$ for every $0 < t \leq F$. Suppose that $\widetilde{h}(t) = \{0, 1\}$ for some $0 < t \leq F$. If $\widetilde{h}(1) = 1$, then game ends at period 1, leading to a contradiction. If $\widetilde{h}(1) = 0$, then $\widetilde{h}(0) = 1$, and game ends at period 0, leading to a contradiction. If $\widetilde{h}(t) = 1$ (alt: 0) for $0 < t \leq F$, then $\widetilde{h}(t-1) = 0$ (alt: 1), $\widetilde{h}(t-2) = 1$ (alt: 0) by claim 1. But, then $\widetilde{h}(1) = \{0, 1\}$, which is leads to a contradiction as above. Claim 3: By the same logic in the proof to Proposition 1, $\widetilde{h}(t) = 1$ for all $t > \frac{v+\delta_i-c}{k} - 1$. Therefore, $\widetilde{h}^G(t) = 1$ for $t > \frac{v-c}{k} - 1 = F$ as $\delta_i \to 0$. For $0 < t \leq F$, note that for some i^*, $\delta_i < k$ for all $i > i^*$ and therefore claim 1 holds for all $i > i^*$. If claim 1 holds, $\widetilde{h}(t - 1) \in (0, 1)$ implies a cutoff value $v^*(t) \in (v - \delta, v + \delta)$ from above, which by the definition of $v^*(t)$ implies that $\widetilde{h}(t) \in (\frac{c}{v-\delta-(t+1)k}, \frac{c}{v+\delta-(t+1)k})$, and therefore $\widetilde{h}^{G_i}(t) \to \frac{c}{v-tk}$ for periods $0 < t \leq F$ as $\delta_i \to 0$. Therefore, $\widetilde{h}^G(t) \to \widetilde{h}(t)$ from Proposition 2 for $t > 0$.

Proposition 8

Proof: Set $x_t^{i,NL*} = x_t^{i*}$ from the proof of Proposition 2 and set $x_t^{i,L*} = 0$ for all i and all t. Note that, as in the proof of Proposition 2. these strategies yield the hazard rates listed in the Proposition 2. The same proof for Proposition 2 shows that, if strategies are followed, the continuation payoff from entering period t as a non-leader is 0 and there is no profitable deviation for a non-leader. Now, consider if there is a profitable deviation for a leader. For the subgames starting in periods $t > F$, refer to the proof of Proposition 1 for a proof that there is no profitable deviation for a leader in these periods. For the subgames starting in period $0 < t \leq F$, the proof continues using (backward) induction with the lack of profitable deviation already proved for all periods $t > F$. In period t, by not bidding, the leading player will receive v with probability $\frac{c}{v-tk}$ (with the game ending) and 0 as a continuation probability as a non-leader in period $t+1$ with probability $1 - \frac{c}{v-tk}$, yielding an expected payoff of $v\ (\frac{c}{v-tk}) > 0$. By bidding, the game will continue to period $t+1$ with certainty, with some positive probability that her bid is accepted. If her bid is accepted, she receives $-c$ in period t and receives $v - (t+1)k$ in $t+1$ with probability $\frac{c}{v-(t+1)k}$ and 0 as a continuation probability as a non-leader in period $t+2$ with probability $1 - \frac{c}{v-tk}$, leading to a continuation payoff of $-c+\ (v-(t+1)k)(\frac{c}{v-tk}) = 0$. If her bid is not accepted, she will receive a continuation probability of 0 as a non-leader in period $t+1$. Therefore, the payoff from not bidding in period t is strictly higher than the payoff from bidding.

Proposition 9

Proof: Consider a vector of bidding probabilities $x = [x^1, x^2, ...x^n] \in [0,1]^n = X$ in some period. Let $\Psi : X \to \Delta^n$ be a function that maps x_t into a vector of probabilities of each player's bid being accepted, which I will denote $a = [a^1, a^2, ...a^n]$. Claim: For any $a^* \in \Delta^n$, $\exists\ x \in X$ such that $\Psi(x) = a^*$.

Consider the following sequence of betting probabilities, indexed by $j = \{1, 2, 3...\}$. Let $x^i(1) = 0$. Define $a(j) = \Psi(x(j)) = \Psi([x^1(j), x^2(j), ...x^n(j)])$. Define $\widetilde{a}_i(j) = \Psi([x^1(j-1), x^2(j-1), ...x^i(j), ..., x^n(j-1)])$ and let $x^i(j)$ be chosen such that $\widetilde{a}_i^i(j) = a^{i*}$. Claim: $x(j)$ exists, is unique, $x(j-1) \leq x(j)$ and $a_i(j) \leq a^*$ for all j. This is a proof by induction, starting with $t = 2$. As $x(1) = 0$, $x(2) = a^*$ by the definition of $\widetilde{a}_i(j)$. Therefore, $x(2)$ exists, is unique, $x(1) \leq x(2)$ and $a_i(2) \leq a^*$ as $\frac{\partial \Psi_i}{\partial x^k} < 0$ for $k \neq i$. Now, consider $x^i(j)$. Note (1) $x^i(j) = 0 \Rightarrow \widetilde{a}_i^i(j) = 0$, (2) $x^i(j) = 1 \Rightarrow \widetilde{a}_i^i(j) \geq 1 - \sum_{k \neq i} a^k(j-1) \geq 1 - \sum_{k \neq i} a^{k*} \geq a^{i*}$ where $1 - \sum_{k \neq i} a^k(j-1) \geq 1 - \sum_{k \neq i} a^{k*}$ follows by $a_i(j-1) \leq a^*$, which follows by induction (3) $\widetilde{a}_i(j)$ is continuous in $x^i(j)$ and $\frac{\partial \Psi_i}{\partial x^i} > 0$. Therefore, there is a unique solution $x^i(j)$ such that $\widetilde{a}_i^i(j) = a^{i*}$. As $a_i(j-1) \leq a^*$ by induction, it must be that $x_i(j) \geq x_i(j-1)$ as $\frac{\partial \Psi_i}{\partial x^i} > 0$. Finally, note that if $\widetilde{a}_i^i(j) = a^{i*}$,then as $a_i^i(j) \leq a^{i*}$ as $\frac{\partial \Psi_i}{\partial x^k} < 0$ for $k \neq i$ and

$x_k(j) \geq x_k(j-1)$ for $k \neq i$.

Set $x^* = \lim_{j\to\infty} x(j)$. Claim: x^* exists and $\Psi(x^*) = a^*$. First, $\lim_{j\to\infty} x(j)$ must exist as $x^i(j)$ is bounded above by 1 and weakly increasing. Next, note that $\sum_i x^{i*}$ must also exist with $\sum_i x^{i*} \leq n$. Now, suppose that $\Psi(x^*) \neq a^*$. Then, as $\Psi(x(j)) = a(j) \leq a^*$ for all j, $\Psi(x^*) \leq a^*$ and there must be some i such that $\Psi^i(x^*) - a^{i*} = z > 0$. Choose L such $|\sum_i x^{i*} - \sum_i x^i(j)| < \frac{z}{2}$ for all $j > L$. By the definition of $x^i(j+1)$, it must be that $x^i(j+1) \geq x^i(j) + z$. But, as $x(j+1) \geq x(j)$, then $\sum_i x^i(j+1) \geq \sum_i x^i(j) + z \geq \sum_i x^{i*} + \frac{z}{2}$, which is a contradiction of L. Therefore, the claim is proved.