Hedging and Competition in a Project Market*

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Abstract

We consider firms that, all else equal, wish to minimize variability in their internal capital (due to convex costs of raising external funds). These firms compete in an auction for a project generating a risky cash flow. The firms can hedge the cash flow risk of the project, but not that of winning or losing the auction. We characterize optimal hedging and bidding strategies in the presence of this friction. We show that access to financial markets makes firms bid more aggressively, possibly even above their valuation for the project. In addition, hedging increases the variance of bids and make firm values more dispersed. Further, with hedging, the covariance of internal capital changes with the risk factor is negative, and is more negative, the higher the correlation of the hedging instrument with the risk factor. We also show that the extra risk the bidders face if they have hedged and lose the auction reduces the seller’s profit, and this has both normative and positive implications.

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1 Introduction

Currently, NATO is tendering contracts to install an Active Layered Theatre Ballistic Missile Defense System. Different parts of this 680 million euro initiative are being awarded competitively for deployment in 2010. As this type of massive infrastructure investment becomes more common, firms actively compete in international markets for the rights to undertake such projects. These opportunities raise a number of theoretical and practical questions. Consider a US firm that plans to bid for a part of the ballistic defense system. First, the scale of such projects is typically large relative to existing investments and financing frictions are more relevant. Second, contracts are denominated in euros and therefore winning exposes the US firm to exchange rate risk. Third, and most importantly, firms cannot control when or if they undertake the project; the decision will be made by NATO. In light of these three characteristics how should firms value the rights to undertake these projects? If a firm is planning to bid, what is the optimal hedging strategy? What implications do these types of investments have for industry outcomes? To analyze these questions we develop a model of the relationship between competition, investment, hedging and systematic risk in an industry with indivisible projects.

We model firms in an industry competing for the rights to undertake a project. The value of the project has a private value component (a company specific synergy) and a common value component that is correlated with a common factor (such as, for example, exchange rate risk). The common value component is random and realized only if the project is undertaken; therefore winning generates uncertain internal capital. All firms in the industry face costly external financing and, in the normal course of business, invest in a concave production function. Therefore, each has an incentive to hedge cash flow risk to preserve internal capital for ongoing operations. Because the random common value component of the project is correlated with existing financial instruments, these can be used to lay off risk. However, hedging with financial instruments only eliminates cash flow risk if a firm wins the auction. Indeed, while hedging is valuable for the winner, it exposes losers to the common risk factor after the auction. This is because, in financial markets, contracts are not written contingent on who wins, therefore markets are incomplete with respect to who finally undertakes the project.

The market incompleteness with respect to contingent hedges exists for a number of reasons. First, such contingent hedges could never be provided by a competitive market. If contracts are contingent on which firms win an auction then agents’ payoffs are conditional

\footnote{While information on specific international investments is frequently confidential, aggregate overseas investment in 2004 was $222.4 billion dollars, according to U.S. Department of Commerce data at http://www.bea.gov/scb/pdf/2006/07July/0706_DIP_WEB.pdf.}
on actions they take. That is, an agent can control if he wins the auction through his bid: as in the case of competitive insurance markets moral hazard would lead to market breakdown. Second, the seller, who we will show has the highest incentive to provide such hedges, will not be able to do so if he also faces financing constraints. For example, if firms are competing to acquire a going concern, the target is usually much smaller than any potential acquirers and more financially constrained, especially if the target is already in financial distress. Also, in the case of government tendering projects, the political costs of insuring bidders would be prohibitively high.

In the practitioner literature the risk faced by firms that hedge before they know if they have won an auction is known as the “bid to award period” risk.\(^2\) This timing friction arises from two sources. First, bids for complicated projects can take a long time to evaluate because factors other than price such as political considerations, a bidder’s technical credentials, and the payment schedule are relevant. For large and complex projects, a two-stage bidding process may even be employed where bidders are first invited to submit technical offers without prices, which are then evaluated to set an acceptable technical standard for all bidders, and then bidders are asked to resubmit bids with prices.\(^3\) Second, the bid to award period also includes the time in which the auction is anticipated but has not yet taken place: For example, many of the biggest procurements follow expired contracts. Recent surveys by Yee (2000) and Espinosa (2005) find the median bid to award period connected with FX risk to be between eight and nine months. Lidbark (2003) estimates that remaining unhedged in this time period generates a substantial risk exposure. Optimally, firms should hedge partially. Yee (2000) and Espinosa (2005) find that firms that hedge the bid to award period FX risk typically cover between 40% to 70% of the full exposure.

We provide closed-form solutions for optimal hedging and bidding strategies in the presence of this timing friction. We predict that all firms should form the same hedging portfolio, but buy a dollar amount proportional to their chance of winning the project. We show that the existence of financial instruments in the presence of the timing friction makes firms compete more aggressively because by entering into a hedging position, bidders run the risk of being over hedged if they lose. We show that this effect can be so severe that a bidder may bid more than his actual synergy value for the project. In this framework, a bid translates directly into the amount that the winner invests into the project.

In addition, we show that the ability to hedge makes industry outcomes more, rather than less, variable. For example, hedging increases the variance of the bids. Bidders with larger private values have a higher probability of winning and therefore hedge more ex-

\(^2\) An excellent description of this appears in Routledge(2003).
\(^3\) More details of one such evaluation process can be found in "Procurement Guidelines,” Asian Development Bank, April 2006.
ante. The more a firm has hedged the more costly losing becomes and therefore bids are increasingly more aggressive. This effect also increases the dispersion in ex post firm values. Finally, we show that the spillover between projects can affect firms’ investment in projects that they do not compete over. With hedging, the covariance of internal capital changes with the risk factor is negative, and is more negative, the higher the correlation of the hedging instrument with the risk factor.

We also examine bidders’ and the seller’s profits. We show that any benefits to bidders from hedging instruments accrue to the seller (they compete them all away). We also show the timing friction decreases the seller’s profit: the winner is underhedged and the losers are over hedged, and the loss in social welfare is born by the seller. This has both normative and positive implications. Normatively, it is to the seller’s advantage to accelerate the evaluation process (to reduce the bid to award period) or to hedge the common value of the project himself. Positively, this suggests that the framework is most appropriate in markets in which the seller also faces financing constraints and therefore does not provide the hedge or finds it optimal for bidders to retain some risk.

A large literature studies corporate risk management. Stulz (1984), Smith and Stulz (1985), Stulz (1990), Bessembinder (1991), and Froot et al. (1993) find that hedging increases firm value in the presence of market imperfections such as costs of external financing and a progressive tax rate schedule. Given these frictions, the optimal hedging strategies are analyzed. Such studies of optimal hedging strategies posit that the firm’s cash flows are a deterministic function of some common factors and the fluctuations in cash flow come solely from the fluctuations of the factors. In our framework, a firm’s cash flow is uncertain conditional on the common factor and it depends on whether or not the firm has won the auction.

The relationship between hedging and product market competition has been analyzed by Adam, Dasgupta and Titman (2005). They characterize Cournot equilibrium in which firms may hedge input costs. They find that hedging strategy is jointly determined in equilibrium with the product market competition and that there can be asymmetric equilibria. Mello and Ruckes (2005) also consider firms with financial constraints who compete in product markets. They find that even though reducing cash flow volatility may be desirable, firms may choose not to hedge for competitive reasons. We find the reverse, that firms in active competition will always hedge and ex post those who lose the competition have too much exposure to the common factor. Loss (2002) demonstrates that a firm in competition chooses to hedge strategically, in particular, the optimal hedging depends on the correlation between firms’ internal funds and whether investments are strategic substitutes or complements. Conceptually, we differ from these papers in one important way. We consider competition
in indivisible projects. This means that only one firm will “win,” and the other firms will necessarily “lose.”

A literature has considered the interaction between hedging and bidding. Eaker and Grant (1985) model a firm with an exogenous probability of winning a project whose payoff is exposed to foreign exchange rate risk. They study the optimal exposure to a forward contract on the exchange rate. In a similar framework, Lien and Wong (2004) incorporate a single bidder facing an exogenous function that links bids to winning probabilities. We extend this literature, by rendering the bids and hedging endogenous. Therefore, we can consider the interaction between financial markets on firms’ bidding behavior.

Hedging creates an externality because it renders winning more valuable and losing more costly. Therefore, this paper relates to the literature on bidding with externalities. (See Jehiel, Moldovanu and Stacchetti (1996) and Jehiel and Moldovanu (2000), for example). However, the externalities in their work are exogenous, but in contrast the externalities in our paper are endogenous because bidders choose the amount of externality through hedging. In fact, it is precisely this endogeneity, specifically the fact that a bidder with a higher valuation will choose a larger externality by entering into a larger hedging position, that drives one of our main results that hedging makes industry outcomes more variable.

The incentive to hedge in this model arises because firms are financially constrained and face a convex cost of capital. Thus, the model also resembles auction models with risk averse bidders. Eso and White (2004) examine the effect of idiosyncratic risk on risk averse bidders. They find that idiosyncratic risk can increase agents’ expected utility. Rhodes-Kropf and Viswanathan (2005) also consider bidders who are financially constrained. In their framework, bidders with differing cash positions raise money which may be conditional on the ex post value of the firm. They show that capital markets may render the auction inefficient. By contrast to their work, as we are interested in how competition affects the investment decisions of firms, we assume that the cost of accessing the financial market either through hedging or raising extra capital does not depend on the bid or private information of the bidders.

The rest of the paper is organized as follows. We present the model in Section 2, and in Section 3, we explicitly solve the model and derive testable predictions in Section 4. We discuss the welfare implication and seller revenue in Section 5 and finally we conclude in Section 6. Most proofs are in the first appendix, Section 7. A general existence proof for the case of a concave growth opportunity appears in Section 8. Some comments on the robustness of our modeling assumptions follow in the third appendix, Section 9.
2 The Model

Consider a model with three dates, \( t = 0, 1, 2 \), no discounting, and universal risk neutrality. \( N \) identical firms, indexed by \( j = 1, \ldots, N \), with initial internal funds \( W_0 \) compete in an industry made up of two types of investment opportunities. One is indivisible and awarded competitively by an auction. We refer to this as “the project.” The other is on-going and available to all firms at time \( t = 1 \). We describe this as a “growth opportunity.” Thus our model captures firms in competition in a well-established industry.

At \( t = 0 \), each firm observes its private value for the project, \( s_j \). This value is independently drawn across firms from distribution \( F \) on \( [\underline{s}, \bar{s}] \) and represents firm specific economies of scale (or synergies) in undertaking the project. Let \( G(s) = F^{N-1}(s) \) denote the distribution of the highest signal among \( N - 1 \) firms.

The project is awarded in a second price auction. Let \( \beta(s_j) \) denote the bidding strategy of a firm with signal \( s_j \). If a firm wins the auction, it receives its private value at \( t = 2 \) at which time it pays for the project. We denote the random payment amount by \( \bar{b} \), which is the highest losing bid.

At \( t = 0 \), after observing their signals, firms privately trade in financial contracts at the time they bid. The \( i \in \{1, \ldots, n\} \) financial contracts have \( t = 1 \) payoffs contingent on \( \bar{x} \), denoted \( h_i(\bar{x}) \). We assume that \( \bar{x} \) is a random factor that is correlated with the project’s common value, \( \bar{\omega} \). Thus, \( \bar{x} \) could represent foreign exchange. Here, \( \bar{\omega} \) is a random variable drawn from \( [\underline{\omega}, \bar{\omega}] \) with \( E[\bar{\omega}] = 0 \). This value accrues to the auction’s winner at time \( t = 1 \) and it can be interpreted as an “announcement effect”: the part of the project’s value that is independent of the firm which undertakes it. Without loss of generality, we assume that the price of each financial contract is zero, or equivalently, that the interest rate is zero.

Let \( q_i \) denote the quantity of contract \( i \) purchased by a firm. A firm’s hedging strategy is denoted by \( \{q_i(s_j)\}_{i=1}^{n} \).

**Assumption 1**

(i) The contracts are unbiased so that \( E[h_i(\bar{x})] = 0 \), for all \( i = 1, \ldots, n \).

(ii) The contracts are non-cancelable: If (i) is satisfied and \( Pr(\sum_{i=1}^{n} q_i h_i(\bar{x}) = 0) = 1 \), then \( q_i = 0 \) for all \( i \).

The first part of the assumption is that there are no transaction costs associated with the hedging instruments. That is, the expected payoff to each contract is equal to the cost. We use \( Q \) to denote the set of \( \{q_i\}_{i=1}^{n} \) satisfying assumption 1 part (i). The second part precludes degenerate solutions in which firms buy arbitrary and offsetting amounts of unbiased hedging instruments, in which case the solution is indeterminate.

The hedging instruments pay out at \( t = 1 \). At the same time, the auction winner is announced and \( \bar{\omega} \) is realized. For a firm with exposure \( q_i \) to \( n \) hedging instruments, its time
$t = 1$ internal capital is

$$W_1 = \begin{cases} W_0 + \sum_{i=1}^n q_i h_i(\bar{x}) & \text{wins} \\ W_0 + \sum_{i=1}^n q_i h_i(\bar{x}) & \text{loses.} \end{cases} \quad (1)$$

This internal capital is available for investment in the growth opportunity.

A firm’s valuation of the $t = 1$ growth opportunity determines how it competes for the project at $t = 0$. All firms face the same growth opportunity. To generate a hedging motive we adapt the framework of Froot, Scharfstein and Stein (1993) which show how a convex cost of external capital can be represented in reduced form by a concave function of internal capital. Let $I(W_1)$ denote a firm’s valuation of the growth opportunity, where $W_1$ is its internal capital at $t = 1$. To provide precise results, we assume a firm’s growth opportunity is quadratic of the form $I(W_1) = a + bW_1 - cW_1^2$ where $c > 0$. We denote the marginal value of internal capital at $W_0$ by $m \equiv I'(W_0) = b - 2cW_0$, which we assume is always positive. Empirically, Altinkilic and Hansen (2000) find that the cost of external funds is consistent with the quadratic form, and the quadratic form has also been used in Hennessy and Whited (2006). Theoretically, for any general function, $I(\cdot)$, if the curvature is small then a Taylor series expansion up to the quadratic term is a good approximation.

Given our formulation, the time $t = 2$ value of a firm with signal $s_j$, and exposure $q_i$ to $n$ hedging instruments is

$$V(s_j) = \begin{cases} I(W_0 + \sum_{i=1}^n q_i h_i(\bar{x})) + s_j - \tilde{b} & \text{firm } j \text{ wins} \\ I(W_0 + \sum_{i=1}^n q_i h_i(\bar{x})) & \text{firm } j \text{ loses} \end{cases} \quad (2)$$

As the financial contracts are not written conditional on who wins the auction, the loser (if hedged) is exposed to $\bar{x}$ risk. This increases the variability of a firm’s internal funds relative to the case in which the firm does not participate in the auction. As external funds are costly (captured by the concavity of the payoff to the growth opportunity) firms are worse off in expectation if they lose the auction.

The sequence of events is depicted in Figure 1.

<table>
<thead>
<tr>
<th>Firm observes private signal $s_j$</th>
<th>Hedges ${q_i(s_j)}_{i=1}^n$</th>
<th>Submits Bid $\beta(s_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Project awarded $\bar{\omega}$ and $\bar{x}$ realized and hedges pay out $h_i(x)$</td>
<td>Growth opportunity $I(W_1)$</td>
<td>Winner receives $s_j$ and pays $\tilde{b}$ for project</td>
</tr>
</tbody>
</table>

Figure 1: **Sequence of events**

The time between $t = 0$ and $t = 1$, is the timing friction called “the bid to award...”
period.” The longer it is, the more bidding firms are exposed to fluctuations in \( \tilde{\omega} \).\(^4\)

We have assumed that firms pay for the project and receive their private benefit in period \( t = 2 \) not \( t = 1 \). In other words, we assume that the firm’s private synergy value of the project, \( s_j \) and the cost of the project, \( b \), do not affect time \( t = 1 \) internal capital. While it is quite natural to assume that private synergies are realized over time therefore do not immediately affect internal capital,\(^5\) the assumption that payment occurs later is stronger. We primarily make this assumption for tractability. An alternate timing assumption would be that firms pay their bid and receive the private benefit at \( t = 1 \). This timing assumption complicates the analysis because firms behave as if risk averse at \( t = 0 \) (due to the concavity of the growth opportunity, \( I(W_1) \)). In a second price auction, the payment is a random variable. In a first price auction, payments are not random but there are few existence results for auctions with concave payoff functions (for example, see Reny and Zamir (2003) and McAdams (2007)), and the situation is even more complicated with hedging instruments. However, we find that our results are qualitatively robust to alternate timing assumptions. In Section 9, we show that under certain general conditions, our model delivers the leading order results, in closed forms, of an alternative model in which firms pay their bid and receive the private benefit at \( t = 1 \).

3 Characterization of Equilibrium

3.1 Benchmark Bids

For now, we abstract from the hedging positions that firms take and consider the bids. In the standard second price auction, the bidder is indifferent between losing and winning at the price of the bid. The same intuition applies with hedging: given the firm’s hedging strategy, the bid is the difference in the expected value of the growth opportunity if the firm wins the auction and the expected value if the firm loses the auction.

**Lemma 1** Consider a bidder with signal \( s_j \) and an exogenous hedging amount \( \{q_i\}_{i=1}^n \), then bidding

\[
\beta(s_j | \{q_i\}_{i=1}^n) = s_j + E[I(W_0 + \tilde{\omega} + \Sigma q_i h_i(\tilde{x}))] - E[I(W_0 + \Sigma q_i h_i(\tilde{x}))]
\]

is a (weakly) dominant strategy.

\(^4\)For example, if \( \tilde{\omega} \) evolves according to a Brownian motion then the \( t = 1 \) variance of \( \tilde{\omega} \) will be proportional to the length of the bid to award period.

\(^5\)If the private signals are unverifiable and uncontractible, the firm must wait for them to be realized before its internal capital is affected. By contrast, as the \( \tilde{\omega} \) component is the same for all firms, this affects its current cashflows.
Hedging increases the bid in two ways. It increases the value of the project conditional on winning while at the same time decreases the value upon losing.

Consider an economy in which there are no financial contracts available and therefore no hedging (NH). The bid function (by inspection of lemma 1) is just

$$\beta^{NH}(s_j) = s_j + (E[I(W_0 + \bar{\omega}) - I(W_0)]).$$

Given that $\bar{\omega}$ has mean zero, and that the investment function is concave, the bid function is less than $s_j$.

Another natural benchmark economy is one in which financial contracts can be written contingent on winning the auction. The hedging strategy in this economy is straightforward: all bidders hedge as if they would win with certainty. Since only the winner actually receives payoffs from the hedging instruments, effectively only the winner hedges in this economy. In this case the bid function with contingent hedging, $\beta^C(s_j)$, is less than the bidder’s actual signal $s_j$. However, it is greater than the bidding function with no hedging. This is because the opportunity to hedge risk increases the value of the project to the firm. Thus,

$$\beta^{NH}(s_j) \leq \beta^C(s_j) \leq s_j.$$

If there is hedging, but it is not contingent, then bids are also higher than in the no hedging case. There are two reinforcing effects: firms’ valuation for the project are higher if they can hedge the risk. In addition, a hedged firm has more to lose if it does not win the auction because losing exposes it to the factor $\bar{x}$. As the price that a firm pays for the project is the highest losing bid, this implies that the presence of financial contracts leads firms to pay more (i.e., invest more) in the project.

**Proposition 1** The equilibrium bid of any firm in an economy with hedging vehicles available is higher than an equivalent firm in an economy with no hedging instruments available. Or, $\beta(s_j) \geq \beta^{NH}(s_j)$, for all $s_j$.

### 3.2 Optimal Hedging and Bidding

In order to maximize expected profits, a firm hedges so as to optimally dampen variation in internal capital. Consider a firm with a fixed probability, $p$, of winning the auction. If $p = 0$, the firm does not purchase any financial contracts because it will lose and if it hedges it exposes itself to systematic $\bar{x}$ risk. By contrast, if $p = 1$ the firm wins for sure and optimally hedges to minimize the exposure to $\bar{\omega}$. However, if $p \in (0,1)$, the firm faces a tradeoff. Thus, the optimal hedging amounts $\{q^*_i\}_{i=1}^n \in Q$ are the solution to:

$$\{q^*_i(p)\}_{i=1}^n = \arg \max_{\{q_i\}_{i=1}^n \in Q} \left\{ pE[I(W_0 + \bar{\omega} + \sum q_i h_i(\bar{x}))] + (1 - p)E[I(W_0 + \sum q_i h_i(\bar{x}))] \right\}.$$
Firms hedge to tradeoff the payoff in states in which they win the auction and the ones in which they lose. A bidder with a higher probability of winning optimally changes hedging amounts so that the expected value of the growth opportunity conditional on winning increases, while that conditional on losing decreases.

**Lemma 2** Let \( \{q'_i\}_{i=1}^n \) be the optimal hedging amounts for \( p' \), and let \( \{\tilde{q}_i\}_{i=1}^n \) be the optimal hedging amount for \( \tilde{p} \). If \( \tilde{p} > p' \), then

(i) \( E[I(W_0 + \tilde{\omega} + \Sigma\tilde{q}_i h_i(\tilde{x}))] \geq E[I(W_0 + \hat{\omega} + \Sigma q'_i h_i(\tilde{x}))] \) and

(ii) \( E[I(W_0 + \Sigma\tilde{q}_i h_i(\tilde{x}))] \leq E[I(W_0 + \Sigma q'_i h_i(\tilde{x}))] \).

If there are many hedging instruments available, then a firm forms a portfolio to achieve the optimal hedge. Let \( \theta(\tilde{x}) \) be the portfolio with the highest possible correlation with \( \tilde{\omega} \) out of all possible portfolios that has the same variance as \( \tilde{\omega} \). We dub this the maximum correlation portfolio.

**Definition 1** The maximum correlation portfolio, \( \theta(\tilde{x}) = \Sigma_{i=1}^n z_i h_i(\tilde{x}) \), is the portfolio of financial instruments that is maximally correlated with \( \tilde{\omega} \) so that

(i) \( \text{corr}(\tilde{\omega}, \Sigma z_i h_i(\tilde{x})) \geq \text{corr}(\tilde{\omega}, \Sigma z'_i h_i(\tilde{x})) \) for \( \forall \{z'_i\}_{i=1}^n \)

(ii) \( \text{var}(\Sigma_{i=1}^n z_i h_i(\tilde{x})) = \sigma^2_\omega \).

Let \( \rho \equiv \frac{\sigma^2_{\omega, \theta(\tilde{x})}}{\sigma_{\theta} \sigma_\omega} \), the correlation coefficient between the maximum correlation portfolio and the risk factor \( \tilde{\omega} \). Different industries face different common factors in their project cash flows. An industry with a larger \( \rho \) has project cash flows that are more correlated with financial instruments. If there is no hedging vehicle available, then \( \rho = 0 \). An industry’s financial capacity also affects \( \rho \). For example, given margins, a firm with high leverage may not have the ability to put on futures contracts, rendering \( \rho \) low.

We show that each firm acquires the maximum correlation portfolio, but the dollar value that each firm invests in it depends on the probability that it wins the project. Specifically, the position that a firm takes depends on the probability that it has the highest value conditional on the private signal, \( s_j \). Recall, \( G(\cdot) \) is the distribution of the highest signal among the other \( N-1 \) bidders. Thus \( G(s_j) \) is the probability that a firm with signal \( s_j \) has the highest signal. As this auction is efficient, the firm with the highest signal wins the auction, therefore \( G(s_j) \) is the probability that \( s_j \) wins. We adhere to the convention that firm 1 has the highest realized signal and the firm with \( s_1 \) wins the auction.

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6Mello and Parsons (2000) argue that hedging transfers wealth across states therefore increases firm’s financial flexibility.

7This follows from our assumption that the random component of the cash flow is the same for all firms.
Figure 2: The solid line plots the bids as a function of the bidder’s signal assuming that $N = 2$, $\sigma^2 = 0.4$, $\rho^2 = 0.7$ and that signal is uniformly distributed on $[0, 1]$. The bottom dashed line plots the bids in an economy without hedging instruments, the middle dashed line plots the bids in an economy with contingent hedging and the top dashed line plots the bidder’s actual signal.

**Proposition 2** If the $n$ hedging instruments are unbiased and $I(\cdot)$ is quadratic, then a firm with signal $s_j$

(i) hedges $q(s_j) = -G(s_j) \rho$ of portfolio $\theta(\bar{x})$

(ii) and bids $\beta(s_j) = s_j - \sigma^2 [1 - 2G(s_j)\rho^2]$

whereas if hedging is contingent then a firm with signal $s_j$

(iii) hedges $-\rho$ of portfolio $\theta(\bar{x})$ if he wins and 0 otherwise

(iv) and bids $\beta^C(s_j) = s_j - \sigma^2 [1 - \rho^2]$.

Figure 2 plots bids as a function of the signals assuming that $N = 2$, $\sigma^2 = 0.4$, $\rho^2 = 0.7$ and that signal is uniformly distributed on $[0, 1]$. The solid line is the optimal bid. The bottom dashed line plots the bids in an economy without hedging instruments (NH), the middle dashed line plots the bids in an economy with contingent hedging (C) and the top dashed line plots the bidder’s actual signal.

Notice, that in this example, $\beta(s_j) > \beta^C(s_j)$ for $s_j > \frac{1}{2}$. More generally, it follows from proposition 2 that $\beta(s_j) > \beta^C(s_j)$ for $G(s_j) > \frac{1}{2}$. In addition, in this example, a bidder with a signal large enough will bid even more than his actual value, or $\beta(s_j) > s_j$. 

Once again, access to non-contingent hedging instruments makes the bids more aggressive, especially for bidders with large signals, because hedged bidders have more to lose.

4 Predictions

To quantify the industry wide effects of hedging, we define a sample variance for $N$ realized variables as

**Definition 2** The sample variance for $\{x_i\}_{i=1}^N$ is

$$\Sigma_x = \frac{1}{N-1} \sum_{i=1}^N \left( x_i - \frac{1}{N} \sum_{j=1}^N x_j \right)^2.$$ 

If the sample is an i.i.d draw, the expected sample variance corresponds to the distribution variance. Under this definition, we show formally that the ability to hedge increases the expected sample variance for the bids. In other words, an econometrician estimating the variance of bids in an industry, on average would report a higher sample variance if firms have access to hedging instruments.

**Proposition 3** If the $n$ hedging instruments are unbiased and $I(\cdot)$ is quadratic, then the ability to hedge increases the expected sample variance for the bids. Or, $E[\Sigma_x] > E[\Sigma_{xH}]$.

The industry wide dispersion in bids is larger if firms can hedge. This is because ex-ante a bidder with a larger signal has a higher probability of winning and hence will hedge more. As a result, he has more to lose and his bid will be increasingly more aggressive.

From Proposition 2 the optimal hedging strategy for a firm is to scale the hedge in an economy in which the hedging instruments are contingent by the probability that it will win the auction. Thus, each firm hedges with the same portfolio and the amount is proportional to its equilibrium probability of winning.\(^8\) Thus, the firm that wins the auction hedges the most, and among firms that lose, the amount each firm hedges differs. This leads to an observed dispersion of hedging positions in the industry.

**Proposition 4** The expected sample variance of relative hedging positions in an industry is

$$E[\Sigma_h] = \rho^2 \frac{(N-1)^2}{N^2(2N-1)}.$$ 

\(^8\)This is consistent with Eaker and Grant (1985) who consider optimal hedging in the quadratic case with exogenous probabilities of winning.
On average, the dispersion in hedging positions depends on both the number of firms in the industry and the availability of highly correlated financial instruments. If \( \rho \) is high, then the dispersion is higher. Or, the easier it is to lay off risk, the more diverse the expected outcomes.

After the project has been awarded, ex ante identical firms hold different internal funds. Thus, each firm raises different amount of external capital and invests different amounts in the growth opportunity. Let \( \Delta W(s_j) \) denote the change in a firm’s internal capital, i.e., \( \Delta W(s_j) = W_1(s_j) - W_0 \). We have

\[
\Delta W(s_j) = \begin{cases} 
    \bar{w} - G(s_j)\rho \theta(\bar{x}) & \text{firm } j \text{ wins} \\
    -G(s_j)\rho \theta(\bar{x}) & \text{firm } j \text{ loses.}
\end{cases}
\]

The internal funds of the auction’s winner are only partially hedged while all losers are over hedged. Furthermore, among the losing firms, the one with a larger signal \( s \) is more over hedged. Therefore the winner’s internal capital will tend to be either the largest or smallest among all firms depending on the realization of \( \tilde{\omega} \). For example, if \( \tilde{\omega} \) is positive, the winner’s internal capital is likely to be above \( W_0 \) while the losers’ is likely to be below \( W_0 \). Hence, a plot of the investment in the growth opportunity against the private signal \( s \) for all the losing firms will be monotonic with the slope depending on the realization of \( \tilde{\omega} \). If we proxy the firm’s private benefit by firm size or quality, then the losing firm’s investment varies monotonically with firm size or firm quality.

We can calculate the sample variance of changes in internal capital \( \Sigma \Delta W \).

**Proposition 5** The expected sample variance of changes in internal capital is

\[
\frac{\sigma^2}{N} \left( 1 - \rho^2 \frac{(\frac{N^2}{N-1})}{N} \right),
\]

where

(i) \( \frac{\sigma^2}{N} \) is the sample variance if there are no hedging instruments available,

(ii) \( \frac{\sigma^2}{N} (1 - \rho^2) \) is the expected sample variance if hedging is contingent.

Figure 3 plots the expected sample variance of changes in internal capital as a function of \( N \) in the middle curve, and the top and bottom curves are for the benchmark cases of no hedging and contingent hedging respectively. Parameters \( \sigma_\omega = 1 \) and \( \rho^2 = \frac{1}{2} \) are used. The variance of internal cash flows is highest if there are no hedging instruments available, and lowest if the hedging instruments are contingent.

Even though only one firm (the winner) takes on the project, as all firms hedge, there is an industry-wide negative covariance between \( \tilde{\omega} \) and changes in internal capital: hedging introduces a negative component in a regression of innovations in internal capital on systematic risk. Further, the size of the reduction depends on the correlation of the hedging
The middle curve is the general case, and the top and bottom curves are for no hedging and contingent hedging respectively. Parameters $\sigma_\omega = 1$ and $\rho^2 = \frac{1}{2}$ are used.

Firms’ internal capital becomes more negatively correlated to the common factor if $\rho$ is higher.

**Proposition 6** The covariance of innovations in internal capital with the risk factor is smaller, the more effective the hedging instruments. Or, for $\rho > \rho'$,

$$E[\Delta W(s_j \mid \rho)\omega] < E[\Delta W(s_j \mid \rho')\omega].$$

**5 Social Welfare**

The two benchmark cases of no hedging and contingent hedging represent the lower and upper bounds on social welfare in the economy. Ex ante social efficiency is obtained when the firm with the highest synergy value wins the project and no firm has uncertain cash flows at $t = 1$.

While hedging makes bids more aggressive, it does not increase expected firm profits. Specifically, firms bid more aggressively because hedging increases (conditional on winning) the value of the growth opportunity. Thus, while the value of winning is higher, so is the cost as all firms adjust their bids accordingly. In equilibrium, these two effects are exactly
offsetting. This arises because of the linearity of a bidder’s profit in the private synergy value and the bid payment.

**Proposition 7** Consider an economy with a fixed number of hedging instruments. If an additional hedging instrument is added, then each firm’s expected profit remains unchanged at $E[V(s_j)] = I(W_0) + \int_2^{s_j} G(z)dz$.

Therefore, a firm’s ex-ante expected value is independent of the availability of hedging instruments. However, this is not true for firms’ values ex-post. In particular, the ability to hedge increases the winner’s profit and decreases a loser’s. This is because hedging makes the losers of the auction much worse off, but the winner is much better off because ex-ante a bidder anticipates the expected loss conditional on losing, and therefore he demands a higher premium upon winning. Therefore hedging increases the dispersion in the ex-post firm values.

**Proposition 8** In an $N$ firm industry with quadratic investment opportunities, the winner’s expected value is larger than in an economy without hedging instruments, and any loser’s expected value is smaller. Specifically, firm value differences are given by

$$E[V] - E[V^{NH}] = \begin{cases} \frac{N(N-1)}{(2N-1)(3N-2)} c\sigma_x^2 \rho^2 & \text{for the winning firm} \\ -\frac{N}{(2N-1)(3N-2)} c\sigma_x^2 \rho^2 & \text{for a losing firm} \end{cases}$$

Recall that if any new instruments are added to the economy, bidders’ expected profits do not increase. However, indifference to the number of hedging instruments is not true for the seller: a seller is better off if there are extra instruments available. This follows immediately from the fact that social welfare must weakly increase if there are more hedging instruments, but bidders’ profits are unchanged. The expected revenue to the seller is the expected value of the second highest bid:

$$\pi_s = \int_2^{s_2} \beta(s_2)dF_2(s_2),$$

where $s_2$ is the second highest order statistic and $F_2(\cdot)$ is its distribution. Thus, if bidders bid more aggressively, then the seller garners extra revenue. On the other hand, the seller’s expected profit is still less than in the case of contingent hedging.

**Proposition 9** If an additional hedging instrument is added, the seller’s expected revenue (weakly) increases, but is less than with contingent hedging, or, $\pi_{NH}^{s} \leq \pi_{s} \leq \pi_{s}^{C}$. The reduction in the seller’s profit because the hedging instruments are not contingent is $\pi_{s}^{C} - \pi_{s} = c\sigma_x^2 \rho^2 \frac{N-1}{2N-1}$.
We can put bounds on the seller’s profits. Specifically, the seller’s profit lies between $\pi_s^{NH}$, the no hedging profit, and $\pi_s^C$, the contingent hedging profit. Intuitively, this is because the social welfare lies between the two poles, and any changes in the social welfare accrue entirely to the seller. The reduction in the seller’s profit because the hedging instruments are not contingent is proportional to $\sigma^2_\omega$, the variance of the common factor, which increases in the bid to award period. If the common factor follows a random walk, then the loss in the seller’s profit is directly proportional to the length of the bid to award period.

The seller’s lower profit because hedging instruments are not contingent has both normative and positive implications. Normatively, it is to the seller’s advantage to accelerate the evaluation process or to hedge the common value of the project himself, if the cost of raising capital is lower for the seller than for the buyers. Positively, this suggests that the framework is most appropriate in markets in which the seller also faces financing constraints and therefore does not provide the hedge or finds it optimal for bidders to retain some risk. For example, if firms are competing to acquire a going concern, the target is usually much smaller than any potential acquirers and more financially constrained, especially if the target is already in financial distress. Also, in the case of government tendering projects, the political costs of insuring bidders would be prohibitively high. Finally, in procurement auctions, there may be moral hazard reasons that make it optimal for a seller to ensure buyers retain some risk.

6 Conclusion

We have exhibited equilibrium bidding and hedging strategies in a second price auction with random payoffs and a bid-to-award lag. Bidders know they will face a convex cost of capital in the future, and have an incentive to hedge the cash flow as they bid. Because the bidders’ hedging decisions are made prior to the award announcement, they face a basic trade off between maximally hedging the cash flow in the event of winning the auction and not hedging any amount in the case of losing. As a result, ex-ante hedging is optimal for a firm, but ex-post the losers are exposed to a common risk factor.

The ability to hedge makes firms compete more aggressively because by entering into a hedging position, bidders run the risk of being over hedged if they lose. Indeed, this effect can be strong enough such that the bid may even exceed that which obtains when contracts can be written contingent on winning the auction. Furthermore, the ability to hedge may make certain outcomes more rather than less variable. For example, the variance of the bids is increased. This is because a bidder with a larger private value for the project has a larger winning probability and will hedge more ex-ante, and this results in larger losses for him.
upon losing and therefore his bid is increasingly more aggressive. In addition, hedging also
increases the dispersion in the ex-post firm values because hedging makes the loser much
worse off, and that in turn makes the winner much better off. Furthermore, the spillover
between projects can affect firms’ investment in projects that they do not compete over.
With hedging the covariance of internal capital changes with the risk factor is negative, and
is more negative, the higher the correlation of the hedging instrument with the risk factor.

Because bidders have to make hedging decisions prior to knowing the auction outcome,
social welfare is lower. This loss is born by the seller, and therefore reduces the seller’s
profit. This result has both normative and positive implications. Normatively, it is to the
seller’s advantage to accelerate the evaluation process or to hedge the common value of
the project himself, if the cost of raising capital is lower for the seller than for the buyers.
Positively, this suggests that the framework is most appropriate in markets in which the
seller also faces financing constraints and therefore does not provide the hedge or finds it
optimal for bidders to retain some risk.

Overall, this paper presents a framework in which the effect of a financing friction is
magnified in an industry due to competition. All firms hedge as they anticipate future cash
flow constraints. Due to competition, firms bid away benefits to hedging and in aggregate
exacerbate the industry’s constraints.
7 Proofs

Proof of Lemma 1

Suppose that a bidder with signal \( s \), and hedging instruments, \( \{q_i\}_{i=1}^n \) bids \( b \). Let \( K(\cdot) \) denote the distribution of the highest bid among the remaining \( N-1 \) bidders. Then, his expected profit, \( \mathbb{E}[V(s)] \) is

\[
\int_0^b [s - z + \mathbb{E}[I(W_0 + \tilde{\omega} + \sum q_i h_i(\tilde{\omega}))]] dK(z) + (1 - K(b)) \mathbb{E}[I(W_0 + \sum q_i h_i(\tilde{\omega}))].
\]

Hence,

\[
\frac{\partial \mathbb{E}[V(b)]}{\partial b} = k(b) [s - b + \mathbb{E}[I(W_0 + \tilde{\omega} + \sum q_i h_i(\tilde{\omega}))]] - \mathbb{E}[I(W_0 + \sum q_i h_i(\tilde{\omega}))].
\] (4)

where \( k(b) = \frac{dK(b)}{db} \). Equation 4 is (weakly) decreasing in \( b \) and zero when the bid is as in the lemma.

Proof of Proposition 1

First, observe that the optimal holding of hedging instruments is zero if \( s = \underline{s} \). Thus, for any \( s > \underline{s} \), from lemma 2

\[
\mathbb{E}[I(W_0 + \tilde{\omega} + \sum q_i (s) h_i(\tilde{\omega}))] \geq \mathbb{E}[I(W_0 + \tilde{\omega})],
\]

and

\[
\mathbb{E}[I(W_0 + \sum q_i (s) h_i(\tilde{\omega}))] \leq I(W_0).
\]

Therefore, equation 3 implies

\[
\beta(s) \geq s + \mathbb{E}[I(W_0 + \tilde{\omega})] - \mathbb{E}[I(W_0)] = \beta^{NH}(s).
\]

Proof of Lemma 2

Recall that \( \{q'_i\}_{i=1}^n \) is the optimal hedging amounts if the probability of winning is \( p' \), and \( \{\tilde{q}_i\}_{i=1}^n \) is the optimal hedging amounts for \( \tilde{p} \) where \( \tilde{p} > p' \). By the definition of optimality;

\[
p'\mathbb{E}[I(W_0 + \tilde{\omega} + \sum q'_i h_i(\tilde{\omega}))] + (1 - p') \mathbb{E}[I(W_0 + \sum q'_i h_i(\tilde{\omega}))] \]

\[
\geq p'\mathbb{E}[I(W_0 + \tilde{\omega} + \sum \tilde{q}_i h_i(\tilde{\omega}))] + (1 - p') \mathbb{E}[I(W_0 + \sum \tilde{q}_i h_i(\tilde{\omega}))]
\] (5)
and
\[
\tilde{p} \mathbb{E}[I(W_0 + \bar{\omega} + \Sigma q_i h_i(\bar{x}))] + (1 - \tilde{p}) \mathbb{E}[I(W_0 + \Sigma q_i h_i(\bar{x}))] 
\geq \tilde{p} \mathbb{E}[I(W_0 + \bar{\omega} + \Sigma q'_i h_i(\bar{x}))] + (1 - \tilde{p}) \mathbb{E}[I(W_0 + \Sigma q'_i h_i(\bar{x}))].
\]
Adding the two inequalities, and using the fact that \( \tilde{p} > p' \) yields
\[
\mathbb{E}[I(W_0 + \bar{\omega} + \Sigma q_i h_i(\bar{x}))] + \mathbb{E}[I(W_0 + \bar{\omega} + \Sigma q'_i h_i(\bar{x}))] 
\geq \mathbb{E}[I(W_0 + \bar{\omega} + \Sigma q_i h_i(\bar{x}))] + \mathbb{E}[I(W_0 + \Sigma q'_i h_i(\bar{x}))].
\]
We prove part (i) of the lemma by contradiction. Suppose it is not true so that
\[
\mathbb{E}[I(W_0 + \bar{\omega} + \Sigma q'_i h_i(\bar{x}))] > \mathbb{E}[I(W_0 + \bar{\omega} + \Sigma q_i h_i(\bar{x}))].
\]
Then, Inequality 6 implies that
\[
\mathbb{E}[I(W_0 + \Sigma q_i h_i(\bar{x}))] > \mathbb{E}[U(W_0 + \Sigma q_i h_i(\bar{x}))].
\]
Summing inequalities 8 and 9, yields
\[
\mathbb{E}[I(W_0 + \Sigma q_i h_i(\bar{x}))] + \mathbb{E}[I(W_0 + \Sigma q'_i h_i(\bar{x}))] > \mathbb{E}[U(W_0 + \Sigma q'_i h_i(\bar{x}))] + \mathbb{E}[I(W_0 + \bar{\omega} + \Sigma q_i h_i(\bar{x}))],
\]
which contradicts Inequality 7. Part (i) is therefore true. A similar argument can be constructed to demonstrate part (ii).

**Proof of Proposition 2**

We first demonstrate parts (i) and (ii).

Let \( \{q_i^*(s)\}_{i=1}^{n} \) be a firm’s optimal hedging strategy and therefore a solution to Equation 24. For any unbiased hedging instrument \( h(\bar{x}) \), we define a firm’s expected profits
\[
H(h) \equiv G(s) \mathbb{E}[I(W_0 + \bar{\omega} + h(\bar{x}))] + (1 - G(s)) \mathbb{E}[I(W_0 + h(\bar{x}))]
= I(W_0) - cG(s) \sigma_{\omega}^2 - c^2 \sigma_{h(\bar{x})}^2 - 2cG(s) \sigma_{\omega,h(\bar{x})}^2.
\]
Where the second line follows from observing that \( I(W) = a + bW - cW^2 \), and \( Eh(\bar{x}) = E\bar{\omega} = 0 \).

Consider an agent choosing a quantity of hedging instruments \( z \), to maximize \( H(\cdot) \). Thus, he chooses
\[
z^* = \arg \max_z I(W_0) - cG(s) \sigma_{\omega}^2 - c^2 \sigma_{h(\bar{x})}^2 - 2cG(s) z \sigma_{\omega,h(\bar{x})}^2.
\]
The first order condition is necessary and sufficient, and therefore,
\[
-2cz^* \sigma_{\omega}^2 - 2cG(s) \sigma_{\omega,h(\bar{x})}^2 = 0
\]
\[
z^* = \frac{G(s) \sigma_{\omega,h(\bar{x})}^2}{\sigma_{h(\bar{x})}^2}.
\]
Thus, the maximized value of $H$ is

$$H = I(W_0) - cG(s)\sigma_\omega^2 - c\frac{G(s)^2\sigma_{\omega,h(x)}}{\sigma_h(x)} + 2cG^2(s)\frac{\sigma_{\omega,h(x)}}{\sigma_h(x)}$$

$$= I(W_0) - cG(s)\sigma_\omega^2 + c\frac{G(s)^2\sigma_{\omega,h(x)}}{\sigma_h(x)}$$

$$= I(W_0) - cG(s)\sigma_\omega^2 + cG(s)^2\sigma_{\omega,h(x)}^2,$$

where $\rho_{\omega,h(x)}$ is the correlation coefficient between $\omega$ and $h(x)$. Observe that this maximum value of $H$ is increasing in $\rho_{\omega,h(x)}$. By definition, $\theta(\bar{x})$ is the maximum correlation portfolio, and therefore the optimal hedge position is

$$= -G(s)\frac{\sigma_{\omega,\theta(\bar{x})}}{\sigma^2(\theta(\bar{x}))}$$

$$= -G(s)\rho.$$

Using Lemma 1 and the definition of $\rho$, yields the bidding function in the lemma:

$$\beta(s_j) = s_j - c\sigma_\omega^2 (1 - 2G(s_j)\rho^2).$$

Part (iii) and (iv) follow from arguments in the text.

**Proof of Proposition 3**

Because the signals are drawn from an iid distribution, equation (ii) of proposition 2 shows that the bids are also drawn from an iid distribution. Therefore the expected sample variance equals the distribution variance of $\beta(s_j)$. From equation (ii) of proposition 2, it is straightforward to have for the distribution variance

$$\text{var}(\beta) = \text{var}(s) + (2\rho^2c\sigma_\omega^2)^2\text{var}(G(s)) + (4\rho^2c\sigma_\omega^2)\text{cov}(s, G(s))\quad(11)$$

In the case of NH, the corresponding distribution variance is simple the variance of the signal (or equivalently it is given by plugging $\rho = 0$ into equation 11.)

$$\text{var}(\beta^{NH}) = \text{var}(s)\quad(12)$$

we now prove the following claim.

Claim 1: $\text{cov}(s, G(s)) > 0.$
To prove the claim, first note that $G(s) = F_{N}^{-1}(s)$ and $E[G(s)] = \frac{1}{N}$. Then $\text{cov}(s, G(s)) =$

$$
E[sG(s)] - E[s]E[G(s)]
= E[sG(s)] - \frac{1}{N}E[s]
= \int_{\frac{1}{2}}^{\hat{s}} sF_{N}^{-1}(s) dF(s) - \frac{1}{N} \int_{\frac{1}{2}}^{\hat{s}} sdF(s)
= \frac{1}{N} \left[ \int_{\frac{1}{2}}^{\hat{s}} sdF_{N}(s) - \int_{\frac{1}{2}}^{\hat{s}} sdF(s) \right]
$$

Because the distribution $F_{N}(s)$ first order stochastically dominates over $F(s)$ (because $F_{N}(s) \leq F(s)$), the above expression is positive. Therefore $\text{cov}(s, G(s)) > 0$. This proves the claim.

We now compare equations 11 and 12. Making use of claim 1 and notice that $\text{var}(G(s)) > 0$, we have $\text{var}(\beta) > \text{var}(\beta^{NH})$. This gives that $E[\Sigma_{\beta}] > E[\Sigma_{\beta^{NH}}].$

**Proof of Proposition 4**

To facilitate the calculation, observe that, if $s$ is drawn from $F(\cdot)$, then

$$
E[G(s)] = \int_{\frac{1}{2}}^{\hat{s}} F_{N}^{-1}(s)f(s)ds
= \frac{1}{N} \quad (13)
$$

and

$$
E[G^{2}(s)] = \int_{\frac{1}{2}}^{\hat{s}} F_{2N}^{-2}(s)f(s)ds
= \frac{1}{2N-1} \quad (14)
$$

and

$$
E\left[\left(\Sigma_{i=1}^{N} G(s_{i})\right)^{2}\right] = E\left[\Sigma_{i=1}^{N} G^{2}(s_{i})\right] + E\left[\Sigma_{i \neq j}^{N} G(s_{i}) G(s_{j})\right]
= NE\left[G^{2}(s)\right] + N(N-1)E\left[G(s)\right]E\left[G(s)\right]
= \frac{N}{2N-1} + \frac{N-1}{N} \quad (15)
$$

We then have for the sample variance of relative hedging positions:

$$
\Sigma_{q} = \frac{1}{N-1} \Sigma_{i=1}^{N} (-G(s_{i}) \rho)^{2} - \frac{N}{N-1} \left( \frac{1}{N} \Sigma_{i=1}^{N} G(s_{i}) \rho \right)^{2}
$$
The expectation of the first term is
\[
\frac{1}{N-1} E \left[ \sum_{i=1}^{N} (-G(s_i) \rho)^2 \right] = \rho^2 E [G^2(s)] = \frac{N}{N-1} \frac{1}{2N-1} \rho^2
\]
and for the second term
\[
\frac{1}{N(N-1)} E \left[ \left( \sum_{i=1}^{N} G(s_i) \rho \right)^2 \right] = \frac{\rho^2}{N(N-1)} E \left[ \left( \sum_{i=1}^{N} G(s_i) \right)^2 \right] = \frac{\rho^2}{(N-1)(2N-1)} + \frac{\rho^2}{N^2}
\]
where Equation 15 was used. Combining both terms, we get $E[\Sigma_h]$ as claimed.

Proof of Proposition 5

Let $s_1$ be the highest signal. Then
\[
E[G(s_1)] = \int_{-\infty}^{\infty} F^{N-1}(s) dF_N(s) = N \int_{-\infty}^{\infty} F^{2N-2}(s) dF(s) = \frac{N}{2N-1}.
\]

The expected value of the second term in Equation 17 is
\[
\Sigma_{\Delta W} = \frac{1}{N-1} \sum_{i=1}^{N} \left( \Delta W(s_i) \right)^2 - \frac{N}{N-1} \left( \frac{1}{N} \sum_{i=1}^{N} \Delta W(s_i) \right)^2.
\]

The expected value of the first term is
\[
\frac{1}{N-1} E \left[ \sum_{i=1}^{N} \left( \Delta W(s_i) \right)^2 \right] = \frac{\sigma^2}{N-1} E \left[ G^2(s_1) \rho^2 + 1 - 2\rho^2 G(s_1) + \sum_{i=2}^{N} \rho^2 G^2(s_i) \right]
= \frac{\sigma^2}{N-1} \rho^2 E \left[ \sum_{i=1}^{N} G^2(s_i) \right] + \frac{\sigma^2}{N-1} \rho^2 E \left[ G^2(s_1) \right] + \frac{\sigma^2}{2N-1} \rho^2 E \left[ G^2(s) \right] + \frac{\sigma^2}{2N-1} \rho^2 E \left[ G^2(s) \right] - \frac{2\rho^2 \sigma^2}{2N-1} E \left[ G(s_1) \right] - \frac{2\rho^2 \sigma^2}{2N-1} E \left[ G(s) \right]
= \frac{\sigma^2}{N-1} \rho^2 E \left[ \sum_{i=1}^{N} G^2(s_i) \right] + \frac{\sigma^2}{N-1} \rho^2 E \left[ G^2(s_1) \right] + \frac{\sigma^2}{2N-1} \rho^2 E \left[ G^2(s) \right] - \frac{2\rho^2 \sigma^2}{2N-1} E \left[ G(s_1) \right] - \frac{2\rho^2 \sigma^2}{2N-1} E \left[ G(s) \right]
= \frac{\sigma^2}{N-1} \left( 1 - \rho^2 \frac{N}{2N-1} \right)
\]
where Equations 15 and 16 were used.

The expected value of the second term in Equation 17 is
\[
\frac{N}{N-1} E \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \Delta W(s_i) \right)^2 \right] = \frac{1}{N(N-1)} E \left[ (\tilde{\omega} - \sum_{i=1}^{N} G(s_i) \rho \theta(\tilde{x}))^2 \right]
\]
\[
= \frac{\sigma^2_\omega}{N(N-1)} E \left[ 1 + \rho^2 (\sum_{i=1}^{N} G(s_i))^2 - 2\rho \sum_{i=1}^{N} G(s_i) \right]
\]
\[
= \frac{\sigma^2_\omega}{N(N-1)} \left[ 1 + \rho^2 \left( \frac{N}{2N-1} + \frac{N-1}{N} \right) - 2\rho^2 \right]
\]
\[
= \frac{\sigma^2_\omega}{N(N-1)} \left( 1 - \rho^2 \left( \frac{N-1}{2N-1} + \frac{1}{N} \right) \right)
\]

Combining both terms in Equation 17, we get \( E[\Sigma \Delta W] \) as claimed.

Further, the corresponding result for no hedging instruments is obtained from the general case by setting \( \rho = 0 \), and that for contingent hedging is obtained from that of no hedging instruments by replacing \( \sigma^2_\omega \) with \( \sigma^2_\omega (1 - \rho^2) \).

**Proof of Proposition 6**

If the firm wins the auction, then
\[
E[\Delta W(s_1)\tilde{\omega}] = E\tilde{\omega} - G(s_1)\rho E\tilde{\omega} \theta(\tilde{x})
\]
\[
= \sigma^2_\omega - G(s_1)\rho^2 \sigma^2_\omega.
\]

If the firm loses the auction, then
\[
E[\Delta W(s_j)\tilde{\omega}] = -G(s_j)\rho^2 \sigma^2_\omega.
\]

The result follows.

**Proof of Proposition 7**

We use an argument similar to that used to prove the standard revenue equivalence principle and demonstrate that this implies an agent’s payoff is independent of the number of hedging vehicles.

We proceed by constructing an agent’s incentive compatibility constraint. Let \( \pi(s, z) \) denote the expected utility of a bidder when his signal is \( s \) but bids and hedges as type \( z \). We compare \( \pi(s, z) \) with \( \pi(z, z) \), the expected utility of a bidder who has a signal \( z \) and bids and hedges as type \( z \).

Since the agent in both these cases has the same probability of winning: \( G(z) \), and are bidding the same and hedging optimally, the only difference in the expected profits is in their private valuations. Therefore,
\[
\pi(s, z) - \pi(z, z) = (s - z) G(z)
\]
Taking a partial derivative with respect to \( z \) and evaluating at \( z = s \) yields:
\[
\frac{\partial \pi (s, z)}{\partial z} \bigg|_{z=s} - \frac{d}{dz} \bigg|_{z=s} \pi (z, z) = -G (s).
\]

The first order condition of the bidding strategy implies that \( \frac{\partial \pi (s, z)}{\partial z} \bigg|_{z=s} = 0 \). Thus,
\[
\frac{d}{ds} \pi (s, s) = G (s).
\]

Thus, integrating
\[
\pi (s, s) = \pi (s, s) + \int_s^s G (z) \, dz.
\]

(18)

Since bidder with \( s \) has zero chance of winning the auction and holds zero hedging instruments, \( \pi (s, s) = I(W_0) \). Therefore,
\[
\pi (s, s) = I(W_0) + \int_s^s G (z) \, dz.
\]

(19)

This is independent of the hedging vehicles available. Note that \( E[V (s)] = \pi (s, s) \), this establishes the proposition.

**Proof of Proposition 8**

Let \( s_1 \) and \( s_2 \) denote the highest and second highest signals respectively. Then the ex-post value for a firm with signal \( s_j \) is

\[
V(s_j) = \begin{cases} 
[\Delta f + I(W_0)] + \sigma^2 \left( 1 - 2G(s_2) \rho^2 \right) + m \Delta W(s_1) - c \left( \Delta W(s_1) \right)^2 & \text{wins} \\
 m \Delta W(s_j) - c \left( \Delta W(s_j) \right)^2 + I(W_0) & \text{loses}.
\end{cases}
\]

where \( \Delta f \equiv s_1 - s_2 \).

We first calculate the expectation \( E[V(s_j)] \) conditional on the signal \( s_j \). Note that \( E[\Delta W(s_j)] = 0 \). Further, if firm \( j \) wins, or equivalently \( j = 1 \), then
\[
E \left[ (\Delta W(s_1))^2 \right] = \sigma^2 \left( 1 + G^2(s_1) \rho^2 - 2G(s_1) \rho^2 \right).
\]
Recall that \( G(s) = F^{N-1}(s) \), and the distribution of \( s_2 \) conditional on the highest signal being \( s_1 \) is \( \frac{1}{F^{N-1}(s_1)} F^{N-1}(s_2) \), we then have

\[
E \left[ G(s_2) \mid s_1 \right] = \int_{s_2}^{s_1} F^{N-1}(s_2) \frac{1}{F^{N-1}(s_1)} d \left( F^{N-1}(s_2) \right)
= \frac{N-1}{F^{N-1}(s_1)} \int_{s_2}^{s_1} F^{2N-3}(s_2) \, dF \left( s_2 \right)
= \frac{1}{2} F^{N-1}(s_1)
= \frac{1}{2} G(s_1).
\]

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On the other hand, if firm \( j \) loses, then

\[
E \left[ (\Delta W(s_j))^2 \right] = \rho^2 \sigma^2 \omega G^2(s_j)
\]

Combining the above relations, we get

\[
E[V(s_j)] = \begin{cases} 
E[\Delta f|s_1] + I(W_0) + c\sigma^2 \rho^2 \left( G(s_1) - G^2(s_1) \right) & \text{if firm } j \text{ wins} \\
-c\rho^2 \sigma^2 \omega G^2(s_j) + I(W_0) & \text{if firm } j \text{ loses.}
\end{cases}
\]

(20)

To calculate the unconditional expectation, note that

\[
E[E[\Delta f|s_1]] = E[\Delta f],
\]

and

\[
E[G(s_1)] = \int_{\frac{1}{2}}^{s} F^{N-1}(s_1) dF^N(s_1) = N \int_{\frac{1}{2}}^{s} F^{2N-2}(s_1) dF(s_1) = \frac{N}{2N-1}
\]

and

\[
E[G^2(s_1)] = \int_{\frac{1}{2}}^{s} F^{2N-2}(s_1) dF^N(s_1) = N \int_{\frac{1}{2}}^{s} F^{3N-3}(s_1) dF(s_1) = \frac{N}{3N-2}
\]

This allows us to compute \( E[V] \). Note that \( E[V^{NH}] \) can be derived by plugging \( \rho = 0 \) into \( E[V] \), we have \( E[V] - E[V^{NH}] \) as in the proposition.

To get the corresponding value for the losing firm, let \( L(\cdot) \) denote the distribution of \( s_j \) conditional on \( s_j \) not being the highest, we then have from Bayes’ rule:

\[
dL(s_j) = \frac{N}{N-1} (1 - G(s_j)) dF(s_j)
\]

where \( 1 - G(s_j) \) and \( \frac{N-1}{N} \) are the conditional and unconditional probabilities of \( s_j \) not being the highest respectively. We then have

\[
E_{s_j} \left[ G^2(s_j) | s_j < s_1 \right] = \int_{\frac{1}{2}}^{s} F^{2N-2}(s_j) \frac{N}{N-1} (1 - F^{N-1}(s_j)) dF(s_j) = \frac{N}{(2N-1)(3N-2)}
\]

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Again note that $E[V^{NH}]$ can be derived by plugging $\rho = 0$ into $E[V]$, which gives us $E[V] - E[V^{NH}]$ for the losing firm as in the proposition.

This completes the proposition. 

Proof of Proposition 9

Let $s_2$ be the second highest signal. From Lemma 2, we have

$$\beta(s_2) = s_2 - c\sigma^2 \left[ 1 - 2G(s_2)\rho^2 \right],$$

and

$$\beta^C(s_2) = s_2 - c\sigma^2 \left[ 1 - \rho^2 \right]$$

By setting $\rho = 0$ in the above equation, we have

$$\beta^{NH}(s_2) = s_2 - c\sigma^2$$

Since $\pi_s = E[\beta(s_2)]$, we have

$$\pi^C_s - \pi^{NH}_s = c\sigma^2 \rho^2$$

Further, let $F_2(s_2)$ denote the distribution of $s_2$, we then have:

$$F_2(s_2) = NF(s_2)^{N-1}(1 - F(s_2)) + F(s_2)^N = NF(s_2)^{N-1} - (N - 1)F(s_2)^N$$

and thus we have

$$\pi_s - \pi^{NH}_s = c\sigma^2 \int^s_2 2G(s_2)\rho^2 dF_2(s_2)$$

$$= c\sigma^2 \int^s_2 2F(s_2)^{N-1}\rho^2 N(N - 1)(F(s_2)^{N-2} - F(s_2)^{N-1})dF(s_2)$$

$$= c\sigma^2 \rho^2 \int_0^1 2x^{N-1}\rho^2 N(N - 1)(x^{N-2} - x^{N-1})dx$$

$$= c\sigma^2 \rho^2 \frac{N}{2N - 1} \geq 0$$

we then have

$$\pi^C_s - \pi_s = (\pi^C_s - \pi^{NH}_s) - (\pi_s - \pi^{NH}_s)$$

$$= c\sigma^2 \rho^2 \frac{N - 1}{2N - 1} \geq 0$$
8 Appendix: Existence

We characterize the existence of equilibrium under somewhat weaker assumptions than those presented in the text. Specifically, we assume that

**Assumption 2**

(i) The contracts are costly, so that \( q_i E[h_i(\bar{x})] \leq 0 \), for all \( i = 1, ..., n \).

(ii) The contracts are non-cancelable: If (i) is satisfied and \( Pr(\sum_{i=1}^{n} q_i h_i(\bar{x}) = 0) = 1 \), then \( q_i = 0 \) for all \( i \).

(iii) The payoff to each firm’s growth opportunity is profitable, or \( I(W_1) \geq W_1 \) for all \( W_1 \).

(iv) The payoff to each firm’s growth opportunity is sufficiently concave, so that there exists \( W_1' \neq \tilde{W}_1 \), so that \( I'(W_1') \neq I'(\tilde{W}_1) \).

To characterize Bayesian Nash equilibrium, we first determine the optimal hedging strategy for a given probability of winning. Then, given the hedging strategy, we determine optimal bid.

**Lemma 3** For any probability, \( p \), of winning the auction, there exists a unique set of finite hedging quantities \( \{q_i^*\}_{i=1}^{n} \in Q \) that solve

\[
\{q_i^*(p)\}_{i=1}^{n} = \arg \max_{\{q_i\}_{i=1}^{n}} \left\{ pE[I(W_0 + \tilde{W} + \Sigma q_i h_i(\bar{x}))] + (1 - p)E[I(W_0 + \Sigma q_i h_i(\bar{x}))] \right\}.
\]

(22)

**Proof of Lemma 3**

We first establish the following claims before proceeding.

*Claim 1:* For any random variable \( \tilde{r} \), \( E[\tilde{r}] = Pr(\tilde{r} < 0)E[\tilde{r}|\tilde{r} < 0] + Pr(\tilde{r} > 0)E[\tilde{r}|\tilde{r} > 0] \).

We have from the law of iterated expectations:

\[
E[\tilde{r}] = Pr(\tilde{r} < 0)E[\tilde{r}|\tilde{r} < 0] + Pr(\tilde{r} = 0)E[\tilde{r}|\tilde{r} = 0] + Pr(\tilde{r} > 0)E[\tilde{r}|\tilde{r} > 0]
\]

Note that the second term is zero, we readily have the claim.

*Claim 2:* For any \( \{q_i\}_{i=1}^{n} \in Q \) and \( \{q_i\}_{i=1}^{n} \neq 0 \), \( Pr(\Sigma q_i h_i(\bar{x}) < 0) > 0 \)

We prove the claim by contradiction. Suppose that \( Pr(\Sigma q_i h_i(\bar{x}) < 0) = 0 \). Then from Claim 1,

\[
E[\Sigma q_i h_i(\bar{x})] = Pr(\Sigma q_i^* h_i(\bar{x}) < 0)E[\Sigma q_i h_i(\bar{x}) | \Sigma q_i h_i(\bar{x}) < 0] + Pr(\Sigma q_i h_i(\bar{x}) > 0)E[\Sigma q_i h_i(\bar{x}) | \Sigma q_i h_i(\bar{x}) > 0]
\]

\[
= Pr(\Sigma q_i^* h_i(\bar{x}) > 0)E[\Sigma q_i^* h_i(\bar{x}) | \Sigma q_i^* h_i(\bar{x}) > 0]
\]

Further, from part (i) of the Assumption, the above expression must be non-positive. This implies that \( Pr(\Sigma q_i h_i(\bar{x}) > 0) = 0 \). However, under the earlier assumption that
Pr(Σqihi(\(\bar{x}\)) < 0) = 0, then we must have Pr(Σqihi(\(\bar{x}\)) = 0) = 1, which contradicts part (ii) of the Assumption that the hedging instruments are non-cancelable.

Claim 3: Define \(q_m \equiv \max_{\{q_i\}_{i=1}^n} \{q_i\} \in C_n \text{ Pr}(\Sigma q_i h_i(\(\bar{x}\)) < 0)E[\Sigma q_i h_i(\(\bar{x}\)) | \Sigma q_i h_i(\(\bar{x}\)) < 0]\) where \(C_n \equiv \{\{q_i\}_{i=1}^n | \{q_i\}_{i=1}^n \in Q \text{ and } \Sigma_{i=1}^n q_i^2 = 1\}\). Then \(q_m < 0\).

Notice that the set \(C_n\) is compact, and the maximization argument is a continuous function of \(\{q_i\}_{i=1}^n\), therefore there exists \(\{q_i\}_{i=1}^n \in C_n\) such that

\[q_m = \text{ Pr}(\Sigma q_i^* h_i(\(\bar{x}\)) < 0)E[\Sigma q_i^* h_i(\(\bar{x}\)) | \Sigma q_i^* h_i(\(\bar{x}\)) < 0].\]

Note that \(\text{ Pr}(\Sigma q_i^* h_i(\(\bar{x}\)) < 0) > 0\) from Claim 2, and \(E[\Sigma q_i^* h_i(\(\bar{x}\)) | \Sigma q_i^* h_i(\(\bar{x}\)) < 0] < 0\) by construction, therefore \(q_m < 0\).

Claim 4: For any \(\{q_i\}_{i=1}^n \in Q\) and \(\sqrt{\sum_{i=1}^n q_i^2} = r\), we have \(\text{ Pr}(\Sigma q_i h_i(\(\bar{x}\)) < 0)E[\Sigma q_i h_i(\(\bar{x}\)) | \Sigma q_i h_i(\(\bar{x}\)) < 0] \leq rq_m\).

This readily follows from Claim 3 and the observation that \(\Sigma q_i h_i(\(\bar{x}\))\) is linear in \(\{q_i\}_{i=1}^n\).

Claim 5: For any \(\epsilon \in (-\infty, \infty)\), there exists \(r^* > 0\) such that \(E[I(W_0 + \bar{\omega} + \Sigma q_i h_i(\(\bar{x}\))) < \epsilon\) for any \(\{q_i\}_{i=1}^n \in Q\) and \(\sqrt{\sum_{i=1}^n q_i^2} > r^*\).

From part (iii) of the Assumption, we can assume \(I'(W_b) > I'(W_a) > 0\) without loss of generality. Concavity of \(I(\cdot)\) gives that \(I(w) \leq I'(W_b) w + a_1\) and \(I(w) \leq I'(W_a) w + a_2\) for all \(w\), where \(a_1\) and \(a_2\) are some constants.

Define \(\Delta \equiv I'(W_b) - I'(W_a) > 0\) and \(W_m \equiv W_0 + \bar{\omega}\) where \(\bar{\omega}\) is the upper bound of \(\bar{\omega}\).

Further, for \(\{q_i\}_{i=1}^n \in Q\), let \(\bar{\sigma}\) denote \(\sum_{i=1}^n q_i h_i(\(\bar{x}\))\) for notational convenience. We have

\[E[I(W_0 + \bar{\omega} + \bar{\sigma})]
\leq E[I(W_m + \bar{\sigma})]
\leq \text{ Pr}(\bar{\sigma} > 0) E[I(W_m + \bar{\omega}) | \bar{\sigma} > 0] + \text{ Pr}(\bar{\sigma} < 0) E[I(W_m + \bar{\omega}) | \bar{\sigma} < 0]
\leq \text{ Pr}(\bar{\sigma} > 0) E[I'(W_a)(\bar{\sigma} + W_m) + a_1 | \bar{\sigma} > 0] + \text{ Pr}(\bar{\sigma} < 0) E[I'(W_b)(\bar{\sigma} + W_m) + a_2 | \bar{\sigma} < 0]
\leq \text{ Pr}(\bar{\sigma} < 0) E[\bar{\sigma} | \bar{\sigma} < 0] + \Delta \text{ Pr}(\bar{\sigma} < 0) + \Delta \text{ Pr}(\bar{\sigma} < 0) + a_3
\leq \Delta q_m r + a_3\]

where \(r \equiv \sqrt{\sum_{i=1}^n q_i^2}, a_3 \equiv I'(W_b) W_m + |a_1| + |a_2|\), and we have used Claims 1 and 4 and the fact that \(E[\bar{\sigma}] \leq 0\). Since \(\Delta > 0\) and \(q_m < 0\), \(E[I(W_0 + \bar{\omega} + \sum_{i=1}^n q_i h_i(\(\bar{x}\)))]\) becomes arbitrarily small as \(r\) increases. This establishes the claim.

Claim 6: For any \(\epsilon \in (-\infty, \infty)\), there exists \(r^* > 0\) such that \(E[I(W_0 + \sum_{i=1}^n q_i h_i(\(\bar{x}\))) < \epsilon\) for any \(\{q_i\}_{i=1}^n \in Q\) and \(\sqrt{\sum_{i=1}^n q_i^2} > r^*\).

The proof is similar to that in Claim 5.

Claim 7: For any \(\epsilon \in (-\infty, \infty)\), there exists \(r^* > 0\) such that \(pE[I(W_0 + \bar{\omega} + \sum_{i=1}^n q_i h_i(\(\bar{x}\))) + (1 - p)E[I(W_0 + \sum_{i=1}^n q_i h_i(\(\bar{x}\))) < \epsilon\) for any \(p \in [0, 1]\), any \(\{q_i\}_{i=1}^n \in Q\) and \(\sqrt{\sum_{i=1}^n q_i^2} > r^*\).

The proof follows directly from Claims 5 and 6 and is thus omitted.
We are now ready to prove the lemma. Utilizing Claim 7, there exists \( r^* > 0 \) such that
\[
pE [I(W_0 + \bar{\omega} + \Sigma q_i h_i(\bar{x}))] + (1 - p)E [I(W_0 + \Sigma q_i h_i(\bar{x}))] < pE [I(W_0 + \bar{\omega})] + (1 - p)I(W_0)
\]
for any \( \{q_i\}_{i=1}^n \in Q \) and \( \sqrt{\Sigma_{i=1}^n q_i^2} > r^* \). Further define a compact and convex space \( S_n = \{ \{q_i\}_{i=1}^n \in Q \mid \sqrt{\Sigma_{i=1}^n q_i^2} \leq r^* \} \). Notice that the argument in Equation 22 is continuous and concave in \( \{q_i\}_{i=1}^n \), therefore it obtains a unique maximum on \( S_n \) at some finite \( \{q_i^*\}_{i=1}^n \). Since \( \{q_i = 0\}_{i=1}^n \in S_n \), we have
\[
pE [I(W_0 + \bar{\omega} + \Sigma q_i^* h_i(\bar{x}))] + (1 - p)E [I(W_0 + \Sigma q_i^* h_i(\bar{x}))] \geq pE [I(W_0 + \bar{\omega})] + (1 - p)I(W_0)
\]
In light of Inequality 23, \( \{q_i^*\}_{i=1}^n \) is thus the unique solution for Equation 22 where the optimization is over the entire set \( Q \). This establishes the lemma.

Using Lemmas 2 and 1, we can show the existence of a unique increasing equilibrium. Specifically, each firm’s probability of winning the auction is the probability that it has the highest bid, or that its private value to winning the auction is higher than the second highest: \( G(s_j) \).

**Proposition 10** There exists a unique symmetric Bayesian Nash equilibrium in which the bidding strategy is increasing in \( s \). Let \( \{q_i^* (s_j)\}_{i=1}^n \) and \( \beta(s_j) \) be a firm’s hedging and bidding strategies, then,

\[
(i) \{q_i^* (s_j)\}_{i=1}^n = \text{argmax}_{(q_i)_{i=1}^n \in Q} \left\{ \frac{G(s_j)E[I(W_0 + \bar{\omega} + \Sigma q_i h_i(\bar{x}))]}{1 - G(s_j)E[I(W_0 + \Sigma q_i h_i(\bar{x}))]} \right\}
\]

\[
(ii) \quad \beta(s_j) = s_j + E[I(W_0 + \bar{\omega} + \Sigma q_i^* h_i(\bar{x}))] - E[I(W_0 + \Sigma q_i^* h_i(\bar{x}))].
\]

**Proof of Proposition 10**

We first show that, if \( \{q_i^* (s)\}_{i=1}^n \) and \( \beta(s) \) are a firm’s hedging and bidding strategies in a symmetric equilibrium and that \( \beta(s) \) is increasing, then equations \((i)\) and \((ii)\) must be satisfied. Note that, when a firm with signal \( s \) plays the equilibrium bidding strategy, the winning probability is then \( G(s) \), and this gives equation \((i)\) through Lemma 3. Then Lemma 1 gives equation \((ii)\).

Next we assume that \( \{q_i^* (s)\}_{i=1}^n \) and \( \beta(s) \) are solutions to equations \((i)\) and \((ii)\), and we know from Lemma 3 that such solutions exist and are unique. Since \( G(s) \) increases in \( s \), we have from Lemma 2 that \( E[I(W_0 + \bar{\omega} + \Sigma q_i h_i(\bar{x}))] \) weakly increases in \( s \). Similarly \( E[I(W_0 + \Sigma q_i h_i(\bar{x}))] \) weakly decreases in \( s \). Therefore \( \beta(s) \) strictly increases in \( s \). We now show that \( \{q_i^* (s)\}_{i=1}^n \) and \( \beta(s) \) constitute an equilibrium.
We examine the best response for bidder 1 assuming all other bidders follow the strategies \( q_i(s) \) for \( i = 1, \ldots, n \) and \( \beta(s) \). Assume bidder 1 has signal \( s \). Since \( \beta(s) \) strictly increases in \( s \) and there is no need to bid above \( \beta(s) \) or below \( \beta(s) \), we assume she bids \( \beta(z) \) for some \( z \in [\hat{s}, \bar{s}] \). Thus the winning probability is \( G(z) \). Following Lemma 3, the optimal hedging amounts are therefore \( \{q_i^*(z)\}_{i=1}^n \) where \( \{q_i^*(z)\}_{i=1}^n \) solves equation (i). Let \( \pi(s, z) \) be bidder 1’s expected profit when bidding as type \( z \) and hedging optimally, then

\[
\pi(s, z) = \int_2^z (s - \beta(Y_1)) dG(Y_1) + G(z) E[I(W_0 + \bar{\omega} + \Sigma q_i^* h_i(\tilde{x}))] + (1 - G(z)) E[I(W_0 + \Sigma q_i^* h_i(\bar{x}))]
\]

where \( Y_1 \) is the highest signal among the other \( N - 1 \) firms. Since \( \{q_i^*(z)\} \) is optimal for a bidder with winning probability \( G(z) \), we apply the envelope theorem and have

\[
\frac{\partial \pi(s, z)}{\partial z} = g(z) [s - \beta(z) + E[I(W_0 + \bar{\omega} + \Sigma q_i^* h_i(\tilde{x}))] - E[I(W_0 + \Sigma q_i^* h_i(\bar{x}))]]
\]

where the last line follows from Equation 3. Since \( \frac{\partial \pi(s, z)}{\partial z} > 0 \) for \( z < s \) and \( \frac{\partial \pi(s, z)}{\partial z} < 0 \) for \( z > s \), \( \pi(s, z) \) has a unique maximum at \( z = s \), and therefore \( \{q_i(s)\}_{i=1}^n \) and \( \beta(s) \) are indeed the best response.

9 Appendix: Robustness

The timing in our model is such that the payment is made at \( t = 2 \) and thus does not affect the internal funds. In this section we compare our model with an alternative model in which firms pay their bid and receive the private benefit at \( t = 1 \). We show that these two models are equivalent up to the leading orders under certain general conditions.

Note that an affine transformation does not change the preference ordering. Therefore, for the alternative model with a growth function \( I^A(\cdot) \), we try to find a (potentially different) growth function \( I(\cdot) \) in our model such that the following approximation holds:

\[
E[I^A(W_0 + \bar{\omega} + \sum_{i=1}^n q_i h_i(\tilde{x}) + s - \tilde{b})] \approx c_1 \left\{ E[I(W_0 + \bar{\omega} + \sum_{i=1}^n q_i h_i(\bar{x}))] + E \left[ s - \tilde{b} \right] \right\} + c_2 \tag{26}
\]

where \( c_1, c_2 \) are some constants, and \( \tilde{b} \) is the random payment that the winner makes conditional on \( s \) being the highest signal, or \( \tilde{b} = \beta(s_2) \) conditional on \( s_2 < s \), where \( s_2 \) is the second highest signal. If the approximation in equation 26 holds, then a model under the alternative timing assumption with the growth function \( I^A(\cdot) \) is approximately equivalent to our model with a growth function \( I(\cdot) \). We next show that, under the following conditions, a function \( I(\cdot) \) exists such that the approximation in equation 26 holds:
Assumption 3  

(i) $I^A(\cdot)$ is quadratic, or, $I^A(x) = a + bx + cx^2$.

(ii) $c^2 \sigma_s^2 \ll 1$, where $\sigma_s$ is the dispersion in the private benefit.

(iii) The hedging instruments are unbiased.

These assumptions are not restrictive. Assumption (i) assumes that the growth function is quadratic. As is discussed in Section 3, this is consistent with empirical evidence in Altinkilic and Hansen (2000). Furthermore, as long as the curvature of the growth function is sufficiently small, for example, if the marginal cost of raising capital is small, then Taylor expansion up to the quadratic term is a good approximation. Furthermore, the value of $c$ is small based on the findings in Altinkilic and Hansen (2000), and Assumption (ii) generally holds.

We first prove the following lemma.

Lemma 4  For any quadratic function $I(\cdot)$, for any independent random variables $\bar{x}$ and $\bar{y}$ and constant $z$, if $E[\bar{x}] = 0$, then $E[I(z + \bar{x} + \bar{y})] = E[I(z + \bar{x})] + E[I(z + \bar{y})] - I(z)$.

Proof. Assume $I(z) = a + bz + cz^2$. By direct calculation, we have

$$E[I(z + \bar{x} + \bar{y})] = a + bz + bE[\bar{y}] + cz^2 + cE[\bar{x}^2] + cE[\bar{y}^2] + 2zE[\bar{y}]$$

and

$$E[I(z + \bar{x})] = a + bz + cz^2 + cE[\bar{x}^2]$$

and

$$E[I(z + \bar{y})] = a + bz + bE[\bar{y}] + cz^2 + cE[\bar{y}^2] + 2zE[\bar{y}]$$

The lemma can be readily verified.

We now establish the approximate equivalence between the two models.

Proposition 11  Under Assumption 3, the alternative model with a growth function $I^A(\cdot)$ is approximated by our model with a growth function $I(\cdot) \equiv \frac{1}{I^W(W_0)} I^A(\cdot)$ where $I^W(\cdot)$ is the derivative of $I^A(\cdot)$.

Proof. Note that $\tilde{b}$ is independent of $\bar{x}$ and $\bar{w}$, therefore $(s - \tilde{b})$ is independent of $(\bar{w} + \Sigma z_i h_i(\bar{x}))$. Thus, if $I^A(\cdot)$ is quadratic and hedging instruments are unbiased, we have from lemma 4 that,

$$E[I^A(W_0 + \bar{w} + \Sigma z_i h_i(\bar{x}) + s - \tilde{b})]$$

$$= E[I^A(W_0 + \bar{w} + \Sigma z_i h_i(\bar{x}))] + E[I^A(W_0 + s - \tilde{b})] - I^A(W_0) \quad (27)$$

$$= E[I^A(W_0 + \bar{w} + \Sigma z_i h_i(\bar{x}))] + I^W(W_0) E[s - \tilde{b}] + cE[(s - \tilde{b})^2] \quad (28)$$
We next show that under part (ii) of Assumption 3, the last term in equation 28 is negligible compared with the 2nd term and thus can be dropped as an approximation. To proceed, we consider the term \( s - \tilde{b} \) which is the winner’s surplus. The magnitude of this term is proportional to the dispersion in private benefit and is inversely proportional to the number of bidders, or \( \left| \mathbf{E} \left[ s - \tilde{b} \right] \right| \sim \frac{\sigma_s}{n} \). Similarly \( \left| \mathbf{E} \left[ (s - \tilde{b})^2 \right] \right| \sim \left( \frac{\sigma_s}{n} \right)^2 \). In addition, note that \( I^A(W_0) \) is not too different from one by orders of magnitude. Therefore we have for the ratio

\[
\frac{c \mathbf{E} \left[ (s - \tilde{b})^2 \right]}{I^A(W_0) \mathbf{E} \left[ s - \tilde{b} \right]} \sim \frac{c \sigma_s}{n} << 1
\]

where part (ii) of Assumption 3 has been used. Therefore the term \( c \mathbf{E} \left[ (s - \tilde{b})^2 \right] \) is negligible compared with \( I^A(W_0) \mathbf{E} \left[ s - \tilde{b} \right] \) and thus can be safely neglected in equation 28. We then have

\[
E \left[ I^A \left( W_0 + \tilde{\omega} + \Sigma z_i h_i(\bar{x}) + s - \tilde{b} \right) \right]
\]

\[
\approx E \left[ I^A \left( W_0 + \tilde{\omega} + \Sigma z_i h_i(\bar{x}) \right) \right] + I^A(W_0) E \left[ s - \tilde{b} \right] \quad (29)
\]

\[
= I^A(W_0) \left\{ E \left[ \frac{1}{I^A(W_0)} I^A \left( W_0 + \tilde{\omega} + \Sigma z_i h_i(\bar{x}) \right) \right] + E \left[ s - \tilde{b} \right] \right\} \quad (30)
\]

We therefore have shown that the alternative model is approximated by our model with a growth function \( I(\cdot) = \frac{1}{I^A(W_0)} I^A(\cdot) \). □
References


