Raising Money *

Tingjun Liu†  Christine A. Parlour‡

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Abstract

A standard problem in finance is that of an agent, with an asset, who seeks to raise a fixed amount of money by selling part of it. For example, consider an entrepreneur who sells shares in his company to raise a fixed amount of money from venture capitalists, or a firm in financial distress that has to sell off some of its assets to settle its obligations. These sales differ from the standard auction format in which a seller tries to earn as much as possible from selling a pre-determined quantity of his good. The difference is economically important: We show many standard results do not go through in these “raising money” auctions with interdependant values. First, because symmetric and increasing pure strategy equilibria do not always exist in a first-price raising money auction, we present a condition under which they do. Second, we present conditions under which the standard seller preferences predicted by the linkage principle over auction types are reversed. Third, we characterize when a seller may not want to release information — in other words, we show that the linkage principle is again violated. Our results have implications for the choice and regulation of auctions that are designed to raise a fixed amount of money.

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†Cheung Kong Graduate School of Business, tjliu@ckgsb.edu.cn
‡Haas School, UC Berkeley parlour@haas.berkeley.edu
1 Introduction

Consider a seller who seeks to raise a fixed revenue by selling off assets. This canonical finance problem applies to an entrepreneur who sells off shares in his company to venture capitalists to raise a required investment amount; or to a firm or portfolio manager in financial distress selling off assets in order to settle its most pressing obligations; or to a land developer giving up part of his land to local governments, other developers, corporate builders, retailers, etc., in exchange for a target amount of money to fund the construction. Such sales differ from the standard auction paradigm in which a seller has a unit (or many units) of a good which he wishes to sell at the highest possible price. If an economic agent needs to raise a fixed revenue, does the intuition gleaned from fixed quantity auctions still apply?

To answer this question, we present a standard auction model based on Milgrom and Weber (1982) with interdependent values and adapt it to analyze the, common in finance, raising money auction (RMA). A close connection exists between a RMA and the standard fixed quantity auction (FQA). In the FQA, bidders receive signals on the cash value of the fixed quantity and they bid a certain cash payment in exchange for the fixed quantity. Whereas in RMAs, bidders receive signals on how many units of the good the fixed revenue is worth, and they bid a certain quantity in exchange for the fixed revenue. In light of this connection, casual intuition may suggest that the standard results in FQA should translate into RMA: This is not the case.

We show that increasing symmetric equilibria may not exist in first price raising money auctions, whereas such an equilibrium always exists in first price FQA. We then provide conditions under which they do. The equilibria may not exist because the allocation curve in a RMA is downward sloping, giving bidders an incentive to shade their bids down. This allocation effect combines with the winner’s curse effect, and may cause such significant underbidding that a bidder’s expected profit is no longer a concave function of the underlying value, rendering these type of equilibria non-existent.

Furthermore, we show that the seller’s preference over auction types and information revelation (the linkage principle) also differs from the standard fixed quantity auction. We note that there are at least two plausible objective functions for the seller in a fixed revenue auction: This is because, if he is trying to raise a fixed revenue and sell as few shares in his asset as possible, he must have some value for the retained ones. First, we consider the case in which the seller has a private value for his retained shares. This assumption corresponds to the case in which the project is run by the seller (for example in the case of an entrepreneur raising money from venture capitalists who enjoys private perquisites of control, or in the case of a developer who keeps the retained land for private use), or if the seller attaches a private value to the good (for example in the case of a financially distressed
firm selling off productive assets.)

A second possible objective is that the seller’s value depends on who wins the auction. In other words, the seller assesses his retained shares at the winner’s valuation. This assumption corresponds to the case in which the project is run by the winning bidder, and thus the cash flow depends on the winner’s value. (For example, in the case of an entrepreneur who seeks a manager to run the project.)

For both these objectives, we present conditions under which linkage principle is violated. Recall, the linkage principle, when applied to FQA, suggests that the seller’s expected revenue is larger if more information is released in the auction because information release mitigates the winner’s curse problem and encourages bidders to bid higher. It predicts a preference ordering for the seller over English, second-price and first-price auctions. Further, if the seller has private information regarding the value of the good, the linkage principle predicts that the seller is better off revealing his information. This result is a bit surprising considering that an entire regulatory structure is built around the notion that sellers will not voluntarily reveal any information.

This paradox is resolved in the case of RMA. We present conditions under which the seller’s preference over different auction forms and over the release of his own information can be completely reversed from that predicted in FQA, for both possible types of the seller’s objective. We obtain a reversal because releasing information in RMA has two competing effects on the seller’s profit. On one hand, as in FQA, releasing information reduces the winner’s curse effect and thus benefits the seller; on the other hand, releasing information introduces fluctuations in bidder’s posterior valuations of the good and this increases the expected quantity sold (because quantity allotted is the target revenue over the bidder’s value), so reducing the seller’s profit.

These two effects influence the seller’s profit in opposite ways, and the combined effect depends on their relative strength. When the dispersion in the bidders’ values is small compared with the mean of the distribution, the quantity effect is small and we show the winner’s curse effect dominates. Thus, the ordering in the seller’s preference in RMA is the same as in FQA, for both scenarios of the seller’s objective. This result can also be understood by noting that if the dispersion in bidders’ values is small compared with the mean, differences between RMA and FQA diminish because then the allotment curve in the RMA becomes almost flat, and thus the same preference ordering obtains.

On the other hand, the first effect (minimizing winner’s curse) increases in the bidders’ signal affiliation. In the limit, when bidders’ signals are almost independent, we show that the quantity effect dominates. Thus, the seller’s preference ordering in RMA is reversed from that in FQA, over both the auction formats and the release of his own information, and for both scenarios of the seller’s objective.

To our knowledge, this is the first paper to explicitly characterize fixed revenue auctions
and compare them to the standard auction form in the case of interdependent values. However, various authors have considered similar auctions under private values. Hansen (1988) studies auctions of endogenous quantity in which several producers compete for the right to sell to a market characterized by a downward sloping demand curve and producers are assumed to have private information about marginal cost, and the prices and gains from trade are compared between different auction mechanisms. We note that the setting in the above study is similar to this paper if the demand curve is of the form \( \frac{1}{p} \) where \( p \) is the unit price of the good. In a companion paper, Liu and Parlour (2014), we illustrate an equivalence result between fixed revenue auctions and fixed quantity auctions under the assumption of independent private values. To do so, we make use of a transformation of signals that does not extend to the interdependent values case.

DeMarzo, Kremer and Skrzypacz (2005) study an auction in which bidders compete for the rights to a project which requires an initial fixed amount of investment, and the bids they place are in the form of securities from the project’s cash flow. When the security used in bidding is equity, their situation is the same as in this paper. The assumption on the seller’s objective in their paper corresponds to one of our possible objective functions. In an experiment, Deck and Wilson (2008) derive bidding strategies for a raising money auction in a special case of private values. Dastidar (2008) examines procurement auctions with fixed budgets in first- and second-price auctions and derives comparative results.

Different from the above papers, we study the case of interdependent values. Our focus on interdependent values is important because many economic situations feature interdependent values. For example, in the case of an entrepreneur acquiring funds to undertake a project, the cash flows of the project under the control of different bidders will usually contain a common value component, reflecting the future market or macroeconomics conditions, etc., common to all bidders.

Finally, failure of the linkage principle has been noted for various specifications of either preferences or constraints. Perry and Reny (1999) construct an example with two bidders, each with a different marginal valuation for each unit of the good and show that the seller should not reveal his affiliated signal. The seller has two units for sale, and the winner (or winners) pay the losing bids. Each bidder has a private value for the second unit of the good which pins down their second unit bid. While releasing information may affect the first unit bids, overall revenue is determined by the two losing bids and can be lower. In contrast to their framework, in our model, each bidder has the same marginal valuation for each unit, and the seller may choose not to release information to prevent fluctuations in the quantity he sells. Fang and Parreiras (2003) illustrate that in the presence of financial constraints, the linkage principle can fail: Bidders can revise their bids downward on the release of bad news, but are constrained from increasing their bids in the wake of good news. Such asymmetry does not exist in our framework.
The focus of this paper is on the similarities and differences between auctions in which the seller raises the most revenue that he can from selling a fixed quantity of the good (which we refer to as a “fixed quantity auction,” or FQA) and those in which the seller tries to sell as little as possible, subject to raising the amount of financing that he needs (described as a “raising money auction,” or RMA). Both of these formats share common elements which we describe below.

A risk neutral seller, who owns a divisible good of size \( \kappa \) plans to sell it to \( N \) risk neutral buyers. He will either sell all of the good if he conducts a fixed quantity auction, or he will sell the amount that he needs to raise a fixed revenue. We denote the required fixed revenue by \( \mu \). In a raising money auction, there is some latitude in how to specify the seller’s payoff. We explore this in subsection 2.2 below.

Each bidder receives a signal that is informative about the per-unit value of the good for the bidder. The signals, denoted by vector \( \mathbf{X} \), are (weakly) positively affiliated with a symmetric joint probability density function \( f(x_1, ..., x_N) \) which is continuous with full support on \([\underline{x}, \bar{x}]^N\). Let \( \mathbf{X}_{-i} \) denote the vector of signals for all bidders other than bidder \( i \). Fixing a bidder, say bidder 1, let \( y_1 \) denote the highest signal among the remaining bidders’ \( N - 1 \) signals, and \( G(\cdot|x) \) and \( g(\cdot|x) \) denote the c.d.f and p.d.f. of the highest signal \( y_1 \) conditional on his own signal realization, \( x_1 = x \).

Bidder \( i \)'s value per unit of the good depends on his own signal and possibly all other bidders' signals. Specifically,

\[
v_i(\mathbf{X}) = u(x_i, \mathbf{X}_{-i}) + \omega,
\]

where the function \( u \) is the same for all bidders and is increasing in all components and symmetric in the last \( N - 1 \) arguments. Here \( \omega \geq 0 \) is a constant. This transformation will be useful to increase the mean valuation while keeping the dispersion unchanged. (When we demonstrate failures of the linkage principle, we place further restrictions on bidders’ valuations; specifically we assume that they are separable in signals.)

We define \( \underline{v} \equiv u(x, \mathbf{X}_{-i}) + \omega \) as the lowest possible value which obtains when all bidders receive the lowest signal \( x \). To ensure that the fixed revenue can always be raised by selling a fraction of the entire good, we assume that this lowest possible per-unit valuation of any buyer \( \underline{v} \) is greater than \( \mu/\kappa \) (the amount that has to be raised divided by the size of the good).

We further define the following expressions:

\[
v(x, y) \equiv E[v_1|x_1 = x, y_1 = y],
\]
and
\[ \hat{v}(x, y) \equiv E[v_1|x_1 = x, y_1 < y]. \tag{3} \]

Let \( b_i \) denote the bid submitted by bidder \( i \), and the set of bids by all \( N \) bidders as \( b \). We restrict attention to auctions in which bidders do not submit demand schedules. That is, they do not submit bids that are conditional on their allocation. We therefore avoid complications with auctioning divisible goods identified in, for example, Wilson (1979).

The outcome of any auction can be characterized by a payment rule and an allocation rule. We denote the payment made by bidder \( i \) as \( \theta_i(b) \). Similarly, we denote his allocation by \( \alpha_i(b) \), which specifies how much of the good bidder \( i \) receives. These enable us to distinguish between the two auctions types:

**Definition 1** a **Fixed Quantity Auction (FQA)**, is one in which the total allocation sums to \( \kappa \), or,
\[ \sum_{i=1}^{N} \alpha_i(b) = \kappa \text{ for all } b_1, ..., b_N, \tag{4} \]
and

A **Raising Money Auction (RMA)** is one in which the total revenue sums up to \( \mu \), or,
\[ \sum_{i=1}^{N} \theta_i(b) = \mu \text{ for all } b_1, ..., b_N. \tag{5} \]

Thus, in a fixed quantity auction, the seller always sells the entire good (of size \( \kappa \)) and receives a revenue that is dependent on the bids and the auction’s payment rule, whereas in a raising money auction, the seller always receives \( \mu \) and sells a quantity that depends on the bids and the auction’s allocation rule.

### 2.1 Standard auction formats

We illustrate the standard first- and second-price auction formats for raising money auctions and use superscripts to denote these different auctions. In a standard FQA, each bidder receives a signal about his valuation of the good. The bid that he submits can be interpreted as the price he is willing to pay per unit of the good. Thus, if bidder \( i \) submits a bid, \( b_i \), then he is offering to pay \( b_i \kappa \) in exchange for receiving the entire allotment. For raising money auctions, we adopt the same interpretation: The bid is the price the bidder is willing to pay per unit of the good, which implies that if bidder \( i \) submits a bid \( b_i \), then he is asking for \( \frac{\mu}{b_i} \) units of the good in exchange for providing the required revenue of \( \mu \).

**First Price, Sealed Bid RMA**

In this case, the highest bidder gets a quantity determined by his own bid. Specifically, the
allocation in a first-price RMA is

\[ \alpha_i^I(b) = \begin{cases} \frac{\mu}{b_i} & \text{if } i = \arg \max_j \{b_j\} \\ 0 & \text{otherwise,} \end{cases} \]

while the payment is

\[ \theta_i^I(b) = \begin{cases} \mu & \text{if } i = \arg \max_j \{b_j\} \\ 0 & \text{otherwise.} \end{cases} \]

Notice, that conditional on being the highest bidder, the allocation the winner receives is decreasing in his bid. Contrast this to the standard FQA, in which the allocation the winning bidder receives is fixed and independent of his own bid. This property is shared with a Dutch auction which is strategically equivalent to a first price one.

**Second Price, Sealed Bid RMA**

In this case, the highest bidder gets a quantity determined by the second highest bid. Specifically, the allocation rule in a second-price RMA is:

\[ \alpha_i^{II}(b) = \begin{cases} \frac{\mu}{\max_{j \neq i}(b_j)} & \text{if } i = \arg \max_j \{b_j\} \\ 0 & \text{otherwise,} \end{cases} \] \hspace{1cm} (6)

while the payment rule is

\[ \theta_i^{II}(b) = \begin{cases} \mu & \text{if } i = \arg \max_j \{b_j\} \\ 0 & \text{otherwise.} \end{cases} \] \hspace{1cm} (7)

Notice, in this case, the winner’s allocation is independent of his bid. Thus, the second price RMA and second price FQA share the characteristics that the allocation is not affected by the winner’s bid.

### 2.2 Seller’s Objective

Defining a seller’s payoff in a RMA is not immediate, because the seller must be raising funds for some reason, and implicitly has a positive value for the asset (else he would be willing to sell all of it). There are thus two plausible values he could attribute to the good: His value could either be independent of the bidders’, or it could reflect bidders’ valuations.

First, suppose that the seller attaches a private value to the good. We refer to this as a “private sale.” In this case, the seller’s payoff is simply his per unit private value multiplied by the retained quantity. As his private value is known to him, the seller optimally maximizes the expected quantity he retains, or minimizes the expected quantity that he sells.
Second, the value of the retained good to the seller could be the same as the winner’s. A natural example is the sale of a project to owner/managers, so that the winner runs the project. In this case, the winner has the highest value because he will be the most efficient at running the project and will generate the highest cash flows. We refer to this case as a “project sale.” In this case, if the seller retains some shares, then he maximizes the expected value of the product of his retained quantity and the winner’s value.

**Definition 2**: In a raising money auction (RMA), the seller makes a  
*Private Sale*: if his valuation for the retained amount is independent of any bidder’s valuation, and a  
*Project Sale*: if his valuation for the retained amount is equal to the highest bidder’s valuation.

As we have indicated, the project sale case was part of the analysis in DeMarzo, Kremer and Skrzypacz (2005). They consider an auction for a project which requires a fixed investment and is paid for by securities contingent on the ensuing cash flows. The project sale case is equivalent to an equity auction. In a companion paper, which characterizes the RMA in the IPV case, we illustrate a quantity equivalence result and exhibit the optimal auction.  

### 3 Equilibrium

The easiest outcome to characterize is that of a second price auction: Bidding strategies in the second-price RMA are identical to those in a corresponding FQA with an identical signal structure. (The proofs are the same as those for FQA in Krishna (2002).) That is, a symmetric equilibrium strategy in a second-price RMA is for each buyer to bid his expected valuation of the allotment given that his signal is the highest and tied to the second highest one. Or,

\[ \beta_{II}(x) = E[v \mid x, y_1 = x]. \]

As it is the case with the FQA, the second-price and English RMA are different because bidders’ signals in the English auction are revealed as they drop out. However, bidding strategies in the English RMA are also identical to the corresponding FQA.

More generally, the winner’s allocation may depend on his bid, and bidding strategies from a FQA typically do not constitute an equilibrium for the corresponding RMA. In particular, symmetric and increasing pure strategy equilibria in first-price RMA may not exist, whereas they always do in first-price FQA. This complication does not arise in second-price or English auctions because (as we observed above) the winner’s allotment is independent

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1 A survey of the recent literature on auctions with contingent payments appears in Skrzypacz(2013).
of his bid and only depends on the second highest bid (or other losing bids). To proceed, we first derive a necessary condition for a symmetric and increasing pure strategy equilibrium in a first price auction, and compare it to the standard results of Milgrom and Weber (1982). We then illustrate why it might not be sufficient, and then provide a condition under which it is.

3.1 Symmetric increasing Pure Strategy Equilibrium in a First Price RMA

Let \( \beta^I(x) \) denote the symmetric and increasing equilibrium strategy in a first-price RMA. For tractability, we perform a change of variable and define \( Q^I(z) \equiv \frac{1}{\beta^I(x)} \). Notice, \( \beta^I(x) \) is the per unit valuation of the bidder with signal \( x \); thus the variable \( Q^I(z) \) corresponds to the quantity demanded by the bidder in return for a unit payment, or, \( \mu Q^I(z) \) is the quantity demanded by the bidder in return for the fixed payment of \( \mu \). The variable \( Q^I \) is somewhat more convenient to work with than \( \beta^I \).

Assume all but one bidder follow \( \beta^I(\cdot) \) and let \( \Pi^I(z,x) \) be the bidder’s expected profit when his signal is \( x \) but bids \( \beta^I(z) \) instead. It is straightforward that

\[
\Pi^I(z,x) = \int_0^z (\mu Q^I(z) v(x,y) - \mu) g(y|x) dy. \tag{8}
\]

Taking the derivative with respect to \( z \), we obtain

\[
\frac{1}{\mu} \frac{\partial \Pi^I(z,x)}{\partial z} = (Q^I(z)v(x,z) - 1) g(z|x) + \frac{d}{dz} Q^I(z) \int_0^z v(x,y) g(y|x) dy \tag{9}
\]

\[
= (Q^I(z)v(x,z) - 1) g(z|x) + \frac{d}{dz} Q^I(z) G(z|x) \hat{v}(x,z). \tag{10}
\]

The first-order condition implies that

\[
\frac{d}{dx} Q^I(x) = \left( \frac{1}{\hat{v}(x,x)} - \frac{v(x,x)}{\hat{v}(x,x)} Q^I(x) \right) \frac{g(x|x)}{G(x|x)}. \tag{11}
\]

This has a unique solution, which yields:

**Proposition 1** If a first-price RMA has a symmetric and increasing pure strategy equilibrium, then it is given by

\[
Q^I(x) = \frac{1}{\beta^I(x)} = \int_x^x \frac{1}{v(y,y)} dL(y|x) \tag{12}
\]

where

\[
L(y|x) \equiv e^{-\int_y^x s(t) dt}, \tag{13}
\]
and
\[ s(t) = \frac{v(t,t) g(t|t)}{\bar{v}(t,t) G(t|t)}. \] (14)

It is instructive to compare the solution presented in Proposition 1, with the general solution due to Milgrom and Weber (1982) for the standard FQA. Recall, they characterize the equilibrium bid (differentiated in what follows by a tilde) as

\[ \tilde{\beta}^I(x) = \int_x^x v(y,y) d\tilde{L}(y|x), \] (15)
\[ \tilde{L}(y|x) = e^{-\int_x^x \frac{g(t|t)}{\bar{v}(t,t)} dt}. \] (16)

On inspection, there are two differences between the bid functions \( \beta^I(x) \) and \( \tilde{\beta}^I(x) \) (equations (12) and (15)). First, \( L(\cdot|x) \) and \( \bar{L}(\cdot) \) are defined differently. Note that both \( L(\cdot|x) \) and \( \bar{L}(\cdot) \) can be thought of as cumulative distribution functions on \([x,x]\). To see this, note that \( \bar{L}(x|x) = 0 \) and \( \bar{L}(x|x) = 1 \) (Milgrom and Weber 1982). Similarly, we can show \( L(x|x) = 0 \) and \( L(x|x) = 1 \) (see proof of Proposition 1). Because \( \frac{v(t,t)}{\bar{v}(t,t)} \geq 1 \), \( L(y|x) \leq \bar{L}(y) \) and thus \( L(y|x) \) first order stochastically dominates \( \bar{L}(y) \). This effect works to make \( \beta^I(x) \) larger than \( \tilde{\beta}^I(x) \).

A second difference, is that in equation (12), both \( \beta^I(x) \) and \( v(y,y) \) appear in the denominator; whereas \( \tilde{\beta}^I(x) \) and \( v(y,y) \) appear in the numerator. This has implications on the bid ranking because:
\[ \beta^I(x) = \frac{1}{\int_x^x \frac{1}{v(y,y)} dL(y|x)} \] (17)
\[ < \frac{1}{\int_x^x \frac{v(y,y)}{dL(y|x)}} \text{ by Jensen’s Inequality} \] (18)
\[ = \int_x^x v(y,y) dL(y|x). \] (19)

Equation (19) shows that, under the (counterfactual) assumption that \( L(\cdot) = \bar{L}(\cdot) \), \( \beta^I(x) \) is strictly less than \( \tilde{\beta}^I(x) \). Or, the stated per unit valuation of the good is lower in a RMA. This is because bidders have an incentive to shade their bids to increase the quantity that they get. (Recall, the allocation curve is downward sloping in a RMA.) We therefore refer to this as the “allocation effect.” This effect is large: In Section 3.4, we show the bid (the per unit price the bidder offers) in a RMA is always lower than that in the standard
FQA with the same signal structure.

We have alluded to the fact that an equilibrium in symmetric and increasing bids in a RMA might not exist. In the case of a first-price FQA, the necessary condition for a symmetric and increasing equilibrium is also sufficient (Milgrom and Weber 1982). In other words, a symmetric and increasing pure strategy equilibrium always exists in a first-price FQA. However, this is not true in RMA. The necessary condition in equation (12) is not a sufficient condition in general, and thus a symmetric and increasing equilibria may not exist in RMA.

3.2 Example illustrating that the necessary condition is not sufficient

For simplicity, assume that there are two bidders. To maximize the effect of the winner’s curse, we assume a pure common value so that $u = \frac{1}{2}x_1 + \frac{1}{2}x_2 + v$, where $v > 0$. Signals are independent with marginal distribution:

$$f(x) = \begin{cases} 
\epsilon & \text{if } \Delta x < x < 1 - \epsilon \Delta x \\
\frac{1 - \epsilon (1 - \Delta x)}{\Delta x} & \text{if } 0 < x < \Delta x.
\end{cases} \quad (20)$$

We set the amount that the seller wishes to raise, $\mu = 1$ and set the quantity that he has to sell, $\kappa = 11$. This ensures that if both bidders get the smallest possible valuation, the seller could still raise $1. (The appendix contains more details on the calculations we report on below.)

We choose parameters so that there is a strong incentive to underbid. Intuitively, since the allocation curve is $\mu$ over the price in RMA, the curvature of the allocation curve is largest when the price is close to zero. Thus, if the distribution of the bidders’ value has a large component near zero, underbidding will be severe. Specifically, we choose $\Delta x, \epsilon$ and $v$ to be small. In this example, we choose $\epsilon = 0.1$ and $v = 0.1$, and we take the limit $\Delta x \to 0$ which simplifies the calculation.

We proceed by assuming that an increasing and symmetric equilibrium does exist so that the equilibrium bidding strategy is $\beta^I(x)$ (given by equation (12)) and consider $\Pi^I(z,x)$, a bidder’s expected profit when he has signal $x$ but follows the equilibrium strategy of an agent with signal $z$. We illustrate the bid function in Figure 1. Notice that the bids are very low and $\beta^I(1) = 0.114$ which is only $0.014$ above $v$. The severe underbidding is generated by the combination of the large concentration (90%) of the signal distribution at zero and the steeply downward sloping allocation curve with a low $v$ as we explained above.

One can infer from Figure 1 that $\beta^I(x)$ cannot be the equilibrium strategy. Suppose bidder 2 follows $\beta^I(\cdot)$ and bidder 1 has a signal $x_1 = 0$. If bidder 1 follows the equilibrium strategy by bidding $\beta^I_1(0) = v = 0.1$, then his expected profit is zero because his winning probability is zero. Now suppose he deviates and bids $\beta^I_1(1) = 0.114$ instead, then his
winning probability is 1 and his expected profit is:

$$\Pi^I(1, 0) = \frac{E[v|x_1 = 0, x_2 < 1]}{\beta^I(1)} - 1 = \frac{v + E[\frac{1}{2}x_2]}{0.114} - 1 = \frac{0.1 + \frac{1}{2} \times 0.1 \times 0.5}{0.114} - 1 = 0.096$$

which is positive and is thus greater than the equilibrium profit.

Intuitively, the equilibrium fails because the downward sloping allocation curve and the winner’s curse effect both make bidders underbid (relative to their signal). If the combined effect is strong enough, underbidding can be severe and the function $\Pi^I(z, x)$ may not be concave in $z$. If everyone else is underbidding, then a bidder benefits if he deviates and bids as if he has a higher signal. He only has to increase his bid slightly to increase his winning probability. If the benefit of deviation outweighs the cost, the hypothesized equilibrium cannot be sustained.

Figure 2 explicitly demonstrates this non-concavity of the payoff function. It plots the value of $\Pi^I(z, x)$ as a function of $z$ for $x = 0$, and we see that $\Pi^I(z, x)$ indeed is not a concave function of $z$. Even though $z = 0$ is still a local maximum, the function increases with $z$ after an initial decrease, and it attains a maximum value of $0.096$ at $z = 1$ which is consistent with our earlier calculation.

This example allows us to conclude that:

**Lemma 1** Symmetric and increasing pure strategy equilibria may not exist in first price
Figure 2: A Plot of $\Pi^I(z, x)$ as a function of $z$ for $\epsilon = 0.1$, $\varphi = 0.1$ and $x = 0$

raising money auctions.

3.3 When the necessary condition is sufficient

To construct the previous example in which symmetric and increasing equilibria do not exist, two effects were important. First, the standard underbidding due to the “winner’s curse.” Second, underbidding that comes about because the allocation curve is downward sloping for the bidder. If these effects were large enough, then, combined, each bidder’s profit function is not concave. This suggests that a condition that mitigates the allocation effect will prove sufficient.

Recall,

$$v_i(X) = u(x_i, X_{-i}) + \omega, \quad (21)$$

for all $i$, where the constant $\omega \geq 0$. This transformation increases the mean in the value while keeping the dispersion unchanged. Therefore the larger is $\omega$, the less important the downward sloping allocation curve. For a fixed $u(x_i, X_{-i})$, let $\beta^I(x)$ be the equilibrium bidding strategy in the first-price RMA, and let $\beta^I_{\omega=0}(x)$ be the equilibrium bidding strategy in a corresponding first-price FQA with $\omega = 0$.

**Proposition 2** (i) There exists a $\hat{\omega}$ such that for all $\omega > \hat{\omega}$, a unique symmetric and
increasing pure strategy equilibrium exists in first-price RMA.

(ii) For any $\epsilon > 0$, there exists a $\bar{\omega}(\epsilon)$ such that for all $\omega > \bar{\omega}(\epsilon)$ and all $x$, $|\beta^I(x) - \tilde{\beta}^I_{\omega=0}(x) - \omega| < \epsilon$.

To summarize, the combined effect of a downward sloping allocation curve and the winner’s curse may result in severe underbidding and render the bidder’s profit function non-concave and symmetric and increasing equilibria non-existent in a first-price RMA. This is in contrast to a first-price FQA in which a symmetric and increasing equilibrium always exists. However, if bidder’s valuations are sufficiently high, this mitigates the allocation effect and the necessary condition is sufficient.

3.4 Comparing RMA and FQA bids

The bid (the per unit price the bidder offers) in the RMA is always lower than that in the standard FQA with the same signal structure because of the downward sloping allocation curve. For $\omega$ sufficiently large, the per unit price in the RMA approaches that of the FQA.

**Lemma 2** If a symmetric and increasing equilibrium exists in a first-price RMA, then $\beta^I(x) \leq \tilde{\beta}^I(x)$ for all $x$, where $\tilde{\beta}^I(x)$ denotes the symmetric and increasing bidding strategy in the corresponding first-price FQA with the same signal structure.

From Lemma 2 and the earlier results on second-price and English auctions, we see that in standard auction formats (first-price, second-price, and English), the bids are (weakly) lower in RMA than in the corresponding FQA. Further, note the allocation curve in RMA is (strictly) downward sloping while it is flat in FQA. Therefore, the quantity weighted transaction price is (strictly) lower in a RMA that the corresponding FQA for the standard auction types.

**Proposition 3** The quantity weighted transaction price is lower for RMA than for the corresponding FQA with the same signal structure, for Dutch, English, second-price or first-price auctions (if a symmetric and increasing equilibrium exists in the first-price RMA).

The implication of Proposition 3 is that, ceteris paribus, bidders’ returns calculated from a RMA will be higher than those calculated from a FQA. It is natural to impute return differences to either risk exposure or the presence of value-destroying frictions such as moral hazard. While not ruling out these frictions as possible determinants of returns, our results reveal that it is important to distinguish the nature of the auction type (fixed quantity versus raising money) in order to understand the return patterns.
4 The Linkage Principle

One of the most important ideas in single unit auction theory is the linkage principle. According to the linkage principle, the seller in a FQA is better off if more information is released in the auction because information release minimizes the winner’s curse and encourages the bidders to bid higher. The linkage principle has two major implications. First, it implies that the seller’s expected revenue in an English FQA is greater than that in a second-price FQA, which is still greater than that in a first-price auction. Second, if the seller also has information concerning the value of the good, then the expected revenue to the seller is larger if he always releases his information than always concealing it.

In the case of RMA, for both private and project sales (so our results are not driven by assumptions about the seller’s payoffs), we show that the linkage principle breaks down. Specifically, we show that a seller’s preference over different auction forms and over the release of his own information can be completely reversed from that predicted in FQA. This has implications for regulators and those interested in transparency.

Why does the linkage principle fail in RMA? Intuitively, releasing information in a RMA has two competing effects on the seller’s profit. On one hand, as in a standard FQA, releasing information reduces bidders’ fear of the winner’s curse and thus benefits the seller as bidders are emboldened to bid more aggressively. On the other hand, releasing information introduces fluctuations in bidder’s post-information-release valuations of the good, and hence fluctuations in the bids. Because the quantity the seller has to sell is the target revenue divided by the bid, fluctuations in the bid increase the expected quantity sold due to Jensen’s inequality (note that the function one over the bid is convex in the bid): Specifically, the quantity $Q$ allocated is $\mu$ over the bid $\beta$, or $Q = \frac{\mu}{\beta}$. Therefore, $E[Q] = E\left[\frac{\mu}{\beta}\right] > \frac{\mu}{E[\beta]}$. This “quantity risk” effect reduces the expected quantity the seller retains, and therefore his payoff.

These two effects work in opposite ways on the seller’s payoff, and the combined effect depends on their relative strength. When the dispersion in the bidders’ values is small compared with the mean of the distribution, the “quantity risk” effect from Jensen’s inequality is small and we expect the “winner’s curse” effect to dominate. In this case, the seller’s preference ordering will be the same in both a RMA and FQA. This result can also be understood by noting that, if the dispersion in bidders’ values is small compared with the mean, differences between RMA and FQA diminish because then the allocation curve in the RMA becomes almost flat (this is the logic behind Proposition 2), and thus the preference ordering is the same.

On the other hand, the “winner’s curse” effect increases in the signal affiliation. Therefore we expect that when affiliation is weak, the quantity effect will dominate and the seller’s preference ordering in RMA will be reversed from that in FQA.
4.1 Seller’s Preference over Common Auction Formats

In the limit when \( \omega \) goes to infinity, the winner’s curse effect dominates. In this case, we show that the seller has the same preference ordering among different forms of RMA as predicted by the linkage principle for FQA, for both scenarios of the seller’s objective.

**Proposition 4 (The Linkage Principle)** Suppose there are three or more bidders and their signals have strictly positive affiliation. For any \( u(x_i, X_{-i}) \), there exists a \( \hat{\omega} \) such that for all \( \omega \geq \hat{\omega} \), and for both project and private sales, the seller’s expected profit in English auction is larger than that in second-price auction, which is still larger than that in first-price auction.

Next we show that the ranking is completely reversed when the "quantity risk" effect dominates (which obtains when signal affiliation is weak). We present the extreme case in which signals are independent.

**Proposition 5 (The Failure of Linkage Principle)** Suppose that there are three or more bidders and their signals are independent, and bidder’s values are separable, or, \( u(x_i, X_{-i}) = u_1(x_i) + u_2(X_{-i}) \), where \( u_1 \) and \( u_2 \) are two weakly increasing functions. Then, for both project and private sales, the seller’s expected profit in an English auction is smaller than that in a second-price auction, which is still smaller than that in a first-price auction.

The separable form of the bidder’s valuation presented in Proposition 5 is consistent with finance applications in which a bidder privately interprets common information (i.e., he knows how he will use the asset, but in addition can learn something about market demand from others’ valuations.)

Propositions 4 and 5 present completely reversed preference orderings over auction forms. When the “winner’s curse” effect is sufficiently large, then the linkage principle holds. As in the standard FQA, an open outcry English auction garners the largest payoff, because this auction format reveals the most information and hence minimizes bidders’ fear of the winner’s curse. In sharp contrast, if the “quantity risk” effect is sufficiently large, a seller maximizes his payoff with a first-price sealed bid auction. This is because a first-price auction reveals the least amount of information, which minimizes the quantity risk faced by the seller. This then, represents a failure of the linkage principle. The other way in which the linkage principal fails is that in a RMA, a seller might prefer not to reveal information.

4.2 Seller’s Preference over Release of Public Information

The standard prediction of the FQA is that a seller will always commit to reveal his private information. To investigate if this implication survives, we enhance our model to incorporate the seller’s information. Specifically, let the random variable \( s \in [\underline{s}, \bar{s}] \) denote seller’s
information which is positively affiliated with bidder’s signals \( X \), and we assume that bidder \( i \)'s value per unit of the good is

\[ v_i(s,X) = u(s,x_i,X_{-i}) + \omega, \]

where the function \( u \) is the same for all bidders and is increasing in all components and symmetric in the last \( N-1 \) components; we also assume \( u(s,x_i,X_{-i}) + \omega > \mu/\kappa \).

We first show when the dispersion in bidder’s value is negligible compared with its mean value, the seller is better off by revealing his information in RMA, the same as in FQA, for both project and private sales.

**Proposition 6** *(The Linkage Principle)* Suppose the seller’s signal is strictly positively affiliated with bidders’ signals. Then, for any \( u(x_i,X_{-i}) \), there exists a \( \hat{\omega} \) such that for all \( \omega \geq \hat{\omega} \), for English, second-price and first-price RMA, and for both formulations of the seller’s objective, the seller’s expected profit is larger if he always reveals his information than always hides it.

Next we show that the ranking can be reversed when signal affiliation is weak. Indeed, we show that in the extreme case when signals are independent, the ranking is completely reversed and the seller is better off hiding information in all auction types, and for both forms of the seller’s objective.

**Proposition 7** *(The Failure of Linkage Principle)* Suppose the seller’s information is independent from bidders’ and that the bidder’s value is separable, i.e., \( u(s,x_i,X_{-i}) = u_1(x_i,X_{-i}) + u_2(s) \), where \( u_1 \) and \( u_2 \) are two functions.

(i) In English and second-price RMA;

(ii) in first-price RMA if \( u_1(x_i,X_{-i}) = u_1(x_i), \)

then, for both private and project sales, a seller’s expected profit is less if he always reveals his information than always hides it.

The conditions in Proposition 7 suggest that whenever a seller has information about the level value of the underlying asset, and he is trying to raise money, he will keep this information private. This validates common intuition and provides a basis for most of the regulations that mandate disclosure.

5 Conclusion

In this paper we have investigated some general properties of raising money auctions (RMA) and compared them with the more familiar case of fixed quantity auctions (FQA). RMA
and FQA differ because bidders face a downward sloping allocation curve in RMA. This difference has several implications. First, symmetric and increasing pure strategy equilibria sometimes do not exist in first-price RMA when values are interdependent. This is because the downward sloping allocation curve and the winner’s curse effect combine to induce significant underbidding. In these cases, expected bidder profit is not concave in the underlying signal and increasing equilibria fail to exist.

Second, the linkage principle breaks down. This breakdown is because releasing information in a RMA introduces ”quantity risk” that is absent in the standard FQA: because the quantity allocated in a RMA is the target revenue over the transaction price (unlike the flat allotment curve in the FQA), releasing information induces fluctuations in the price and this increases the expected quantity sold, so reducing the seller’s profit. We show that when such quantity risk is sufficiently high, the linkage principle breaks down entirely: the seller’s preference over different auction forms and over the release of his own information are completely reversed from that predicted in FQA.

All of our main results also extend to equity auctions in which the seller sells a fixed share of a project and accepts equity instead of cash as payment. In particular, our results show that for equity auctions under interdependent values, symmetric and increasing equilibria sometimes do not exist for first-price auctions and the linkage principle breaks down. More generally, our findings suggest for general-security-bid auctions (in which bidders pay with securities) the standard intuition derived from cash auctions may no longer apply under interdependent valuations.
References


6 Calculation Details in the Numerical Example of Non-existence of Increasing Symmetric Equilibria in First-Price RMA.

For \( x > 0 \) and \( y > 0 \), equations 13 and 14 yield

\[
L(y|x) = e^{-\int_0^x \frac{t+y}{0.5t+0.25t^2/(1-\epsilon+\epsilon t)+\frac{\epsilon}{2}} \frac{1-\epsilon+\epsilon t}{1-\epsilon+\epsilon t} dt},
\]

where we have used the relation

\[
\hat{v}(t, t) = 0.5t + 0.25\epsilon t^2 / (1 - \epsilon + \epsilon t) + v.
\]

Notice in the above expression, \( L(0|x) > 0 \) which seems to contradict our earlier assertion that \( L(0|x) = 0 \). This is because we have taken the limit \( \Delta x \to 0 \), and in fact \( L(\cdot|x) \) should abruptly drop to zero at zero. We will take care of this complication appropriately in the following calculation.

From equation 12 we have:

\[
Q^I_I(z, x) = \frac{1}{\beta^I_I(x)} \int_0^x \frac{1}{v(y, y)} d\left(e^{-\int_y^x \frac{t+y}{0.5t+0.25t^2/(1-\epsilon+\epsilon t)+\frac{\epsilon}{2}} \frac{1-\epsilon+\epsilon t}{1-\epsilon+\epsilon t} dt} + \frac{1}{v(0, 0)} e^{-\int_0^x \frac{t+y}{0.5t+0.25t^2/(1-\epsilon+\epsilon t)+\frac{\epsilon}{2}} \frac{1-\epsilon+\epsilon t}{1-\epsilon+\epsilon t} dt},
\]

where the second term is the contribution from the discontinuity of \( L(\cdot|x) \) at zero which we alluded to earlier. Next, we compute the payoff function from equation 8:

\[
\Pi^I_I(z, x) = Q^I_I(z) \int_0^z v(x, y) f(y) dy - F(z)
\]

\[
= Q^I_I(z) \int_0^z (0.5x + 0.5y + v) f(y) dy - [1 - \epsilon + \epsilon z] \\
= \left[ \int_0^z \frac{1}{y + v} d\left(e^{-\int_y^z \frac{t+y}{0.5t+0.25t^2/(1-\epsilon+\epsilon t)+\frac{\epsilon}{2}} \frac{1-\epsilon+\epsilon t}{1-\epsilon+\epsilon t} dt} + \frac{1}{v} e^{-\int_0^z \frac{t+y}{0.5t+0.25t^2/(1-\epsilon+\epsilon t)+\frac{\epsilon}{2}} \frac{1-\epsilon+\epsilon t}{1-\epsilon+\epsilon t} dt},
\right]
\]

\[
\left[(0.5x + v)(1 - \epsilon + \epsilon z) + 0.25\epsilon z^2\right] - [1 - \epsilon + \epsilon z].
\]

This completes the calculation.
7 Proofs

Proof of Proposition 1.

To solve equation 11, multiply both side of equation 11 by $e^{\int_{y}^{x}s(t)dt}$ to get

$$\frac{d}{dx} \left[ e^{\int_{y}^{x}s(t)dt} Q^I (x) \right] = e^{\int_{y}^{x}s(t)dt} \frac{s(x)}{v(x,x)}$$

where $s(x)$ is defined in equation 14. The above equation gives upon integration

$$e^{\int_{y}^{x}s(t)dt} Q^I (x) = \int_{y}^{x} e^{\int_{t}^{y}s(t)dt} \frac{s(y)}{v(y,y)} dy + \frac{1}{v}$$

where we have used $Q^I (x) = \frac{1}{v}$. Therefore we have

$$Q^I (x) = \int_{y}^{x} e^{-\int_{t}^{y}s(t)dt} \frac{s(y)}{v(y,y)} dy + \frac{1}{v} e^{-\int_{y}^{x}s(t)dt} \quad (22a)$$

$$= \int_{y}^{x} \frac{1}{v(y,y)} dL(y|x) + \frac{1}{v} L(x|x) \quad (22b)$$

where $L(y|x)$ is defined in equation 13. We note that $L(y|x)$ can be thought of as a distribution function on $[x,x]$. On one hand, we readily have $L(x|x) = e^0 = 1$; on the other hand we have $\frac{g(t|x)}{G(t|x)} \geq \frac{g(y|x)}{G(y|x)}$ for $t \geq x$ because of affiliation and $\frac{v(x,x)}{v(x,x)} \geq 1$, therefore

$$-\int_{x}^{x} s(t) dt \leq -\int_{x}^{x} \frac{g(t|x)}{G(t|x)} dt$$

$$= - \ln \frac{G(x|x)}{G(x|x)}$$

$$= -\infty$$

which gives that $L(x|x) = 0$. Therefore equation 22b becomes equation 12.

Note that we multiplied the term $e^{\int_{y}^{x}s(t)dt}$ to both sides of equation 11, and this term is infinite as we have just shown. Alternatively, we could have used a finite term $e^{\int_{y}^{x+\epsilon}s(t)dt}$ instead, where $\epsilon$ is a small and positive number. Following the same procedure as in the above derivations and taking the limit that $\epsilon$ goes to zero, we could also arrive at equation 12.

Proof of Proposition 2

We first prove part (i) of the proposition by showing a $\omega$ exists such that for all $\omega \geq \hat{\omega}$, equation 12 is sufficient for the equilibrium.

Let $\beta^I (\cdot)$ and $Q^I (\cdot)$ be given by equation 12. Making use of equation 11, we rewrite
Consider two cases. Case (1): $z < x$. Note that $v(z, z) \geq \beta^I(z)$ for all $z \in [\underline{z}, \bar{x}]$ by virtue of equation 35. Thus $\frac{\beta^I(z)}{\beta^J(z)} = 0$. Thus, for all $\omega \geq v(\bar{x}, \bar{x})$, one has

$$
\frac{1}{\mu} \frac{\partial \Pi^I(z, x)}{\partial z} \geq g(z|x) \left[ \frac{v(x, z) - v(z, z)}{\beta^I(z)} - \frac{\bar{v}(x, z) - \bar{v}(z, z)}{\bar{v}(z, z)} \right] \left( \frac{v(z, z)}{\beta^I(z)} - 1 \right)
$$

(26)

$$
= g(z|x) \left[ \frac{v(x, z) - v(z, z)}{\beta^I(z)} - \frac{\bar{v}(x, z) - \bar{v}(z, z)}{\bar{v}(z, z)} \right] \left( \frac{v(z, z)}{\beta^I(z)} - 1 \right)
$$

(27)

$$
\geq g(z|x) \left[ \frac{v(x, z) - v(z, z)}{v(z, z)} - \frac{\bar{v}(x, z) - \bar{v}(z, z)}{\bar{v}(z, z)} \right] \left( \frac{v(z, z)}{\beta^I(z)} - 1 \right)
$$

(28)

$$
\geq g(z|x) \left[ \frac{\nu_{\omega=0}(x, z) - \nu_{\omega=0}(z, z)}{\omega} - \frac{\bar{\nu}_{\omega=0}(x, z) - \bar{\nu}_{\omega=0}(z, z)}{\omega} \right] \nu_{\omega=0}(\bar{x}, \bar{x}),
$$

(29)

where we have used the relation that $v(x, z) - v(z, z) = \nu_{\omega=0}(x, z) - \nu_{\omega=0}(z, z) \geq 0$ and $\bar{v}(x, z) - \bar{v}(z, z) = \bar{\nu}_{\omega=0}(x, z) - \bar{\nu}_{\omega=0}(z, z) \geq 0$ because $z < x$. Then, for all $\omega \geq v(\bar{x}, \bar{x})$, one has

$$
\frac{1}{\mu} \frac{\partial \Pi^I(z, x)}{\partial z} \geq g(z|x) \left[ \frac{\nu_{\omega=0}(x, z) - \nu_{\omega=0}(z, z)}{2\omega} - \frac{\bar{\nu}_{\omega=0}(x, z) - \bar{\nu}_{\omega=0}(z, z)}{\omega} \nu_{\omega=0}(\bar{x}, \bar{x}) \right]
$$

(23)

$$
= g(z|x) \left[ \frac{(\nu_{\omega=0}(x, z) - \nu_{\omega=0}(z, z)) - (\bar{\nu}_{\omega=0}(x, z) - \bar{\nu}_{\omega=0}(z, z))}{\omega} \right] \frac{2\nu_{\omega=0}(\bar{x}, \bar{x})}{\omega}
$$

(24)

Next, let $c$ be a large enough quantity which satisfies

$$
c |(\nu_{\omega=0}(x, z) - \nu_{\omega=0}(z, z))| \geq |(\bar{\nu}_{\omega=0}(x, z) - \bar{\nu}_{\omega=0}(z, z))|
$$
for all $x, z$ (such a quantity exists under general conditions). Define $\hat{\omega} = v(\bar{x}, \bar{x}) \max (2c, 1)$.

Then, for all $\omega \geq \hat{\omega}$, $\frac{\partial \Pi^I(z, x)}{\partial z} \geq 0$. Next, consider Case (2): $z > x$. Note that $\frac{\partial \Pi^I(z, x)}{\partial z} \geq 1$ due to signal affiliation. Thus equation 25 holds with the ”$\geq$” relation replaced by ”$\leq$”; i.e., the following holds:

$$
\frac{1}{\mu} \frac{\partial \Pi^I(t, z, x)}{\partial z} \leq g(z|x) \left[ \left( \frac{v^t(x, z)}{\beta^t(z)} - 1 \right) - \frac{\hat{v}^t(x, z)}{\hat{v}^t(z, z)} \left( \frac{v^t(z, z)}{\beta^t(z)} - 1 \right) \right]
$$

Following the same steps as in Case (1), all the relations we derived there hold with ”$\geq$” replaced by ”$\leq$” (due to the fact that $z > x$ here whereas $z < x$ in Case (1)). In particular, with the same $\hat{\omega}$ as defined in Case (1), we have that for all $\omega \geq \hat{\omega}$, $\frac{\partial \Pi^I(z, x)}{\partial z} \leq 0$. Summarizing these two cases, we have that for all $\omega \geq \hat{\omega}$, $\frac{\partial \Pi^I(z, x)}{\partial z} \geq 0$ if $z < x$ and $\frac{\partial \Pi^I(z, x)}{\partial z} \leq 0$ if $z > x$, thus $\Pi^I(z, x)$ obtains its maximum at $z = x$. Therefore, part (i) of the proposition follows.

We next prove part (ii) of the proposition. Applying equation 36, we have

$$
\frac{d}{dx} \beta^I(x) = \frac{\beta^I(x)}{\hat{v}(x, x)} \left( v(x, x) - \beta^I(x) \right) \frac{g(x|x)}{G(x|x)}
$$

(30)

$$
= \left( v(x, x) - \beta^I(x) \right) \frac{g(x|x)}{G(x|x)} + M(x) \frac{g(x|x)}{G(x|x)}
$$

(31)

where

$$
M(x) = \frac{\beta^I(x)}{\hat{v}(x, x)} \left( v(x, x) - \beta^I(x) \right).
$$

Note that $v(\bar{x}, \bar{x}) \leq \beta^I(x) \leq v(\bar{x}, \bar{x})$ (see equation 35). Then

$$
|M(x)| \leq \frac{(v_{\omega=0}(\bar{x}, \bar{x}) - v_{\omega=0}(\bar{x}, \bar{x}))^2}{\omega + v_{\omega=0}(\bar{x}, \bar{x})} \leq \frac{(v_{\omega=0}(\bar{x}, \bar{x}) - v_{\omega=0}(\bar{x}, \bar{x}))^2}{\omega}.
$$

(32)

Note $\beta^I(x) - \omega$ and $\beta^I_{\omega=0}(x)$ satisfy the same differential equation (compare equations 31 and 38) expect for the $M(\cdot)$ term. We can ”solve” equation 31 by using the same techniques as for solving equation 38:

$$
\beta^I(x) = \omega + \int_{\bar{x}}^{x} v_{\omega=0}(y, y) d\bar{L}(y \mid x) + \int_{\bar{x}}^{x} M(y) d\bar{L}(y \mid x)
$$

$$
= \omega + \beta^I_{\omega=0}(x) + \int_{\bar{x}}^{x} M(y) d\bar{L}(y \mid x).
$$

(33)

22
Using (32), one has

\[ |\beta^I (x) - \omega - \tilde{\beta}^I_{\omega=0} (x) | \leq \frac{(v(\bar{x}, \bar{x}) - v(x, x))^2}{\omega} \]

for all \( x \) and \( \omega \). Let \( \tilde{\omega}(\epsilon) = \frac{(v(x, x) - v(x, x))^2}{\epsilon} \), part (ii) of the proposition follows.

**Proof of Lemma 2**

We first show that \( \beta^I (x) \leq \hat{\nu} (x, x) \). Equation 8 can be rewritten as

\[ \Pi^I (z, x) = \mu \left( \frac{\hat{\nu}(x, z)}{\beta^I (z)} - 1 \right) G(z|x) \] (34)

and we have \( \Pi^I (x, x) = \max_z \Pi^I (z, x) \). Since \( \Pi^I (0, x) = 0 \), we have \( \Pi^I (x, x) \geq 0 \), and this gives \( \beta^I (x) \leq \hat{\nu} (x, x) \) due to equation 34. Since \( \hat{\nu} (x, x) \leq v (x, x) \), we also have

\[ \beta^I (x) \leq v (x, x) \] (35)

Next we plug \( Q^I (x) = \frac{1}{\beta^I (x)} \) into equation 11 to get:

\[ \frac{d}{dx} \beta^I (x) = \frac{\beta^I (x)}{\hat{\nu}(x, x)} (v(x, x) - \beta^I (x)) \frac{g(x|x)}{G(x|x)} \] (36)

\[ \leq \left( v(x, x) - \beta^I (x) \right) \frac{g(x|x)}{G(x|x)} \] (37)

with the boundary condition of \( \beta^I (x) = v \). From [6], the equation for \( \tilde{\beta}^I (x) \) is

\[ \frac{d}{dx} \tilde{\beta}^I (x) = \left( v(x, x) - \tilde{\beta}^I (x) \right) \frac{g(x|x)}{G(x|x)} \] (38)

with the same boundary condition \( \tilde{\beta}^I (x) = v \). Therefore we have \( \beta^I (x) = \tilde{\beta}^I (x) \) and \( \frac{d}{dx} \beta^I (x) \leq \frac{d}{dx} \tilde{\beta}^I (x) \) for all \( x > 0 \), establishing the lemma.

**Proof of Proposition 3.**

First note that Lemma 2 shows in first-price auctions, the bidding strategy in RMA is no more than that in the corresponding FQA with the same signal structure. Note also that Dutch auction is strategically equivalent to first-price auction for both RMA and FQA. Further note that, for English and second-price auctions, the bidding strategy is the same for RMA and the corresponding FQA with the same signal structure. Therefore, if we let the random variables \( p \) and \( \tilde{p} \) denote the transaction prices (for per unit quantity of the good) for RMA and the corresponding FQA respectively, we have \( p \leq \tilde{p} \) at all realizations
of synergies, whether the auction format is first-price, second-price or English auction. Now let \( p^* \) and \( \tilde{p}^* \) denote the quantity weighted transaction prices for the RMA and the corresponding FQA respectively. Note that the quantity weighted price is the expected revenue divided by expected quantity: to see this, let random variables \( p \) and \( q \) denote the (unit) price and quantity, then the quantity-weighted price is 
\[
E\left[ \frac{pq}{q} \right] = \frac{E[p]}{E[q]},
\]
where \( E[pq] \) is the expected revenue. Noting that the revenue in RMA is constant at \( \mu \) while the quantity in FQA is constant at \( \kappa \), we have:
\[
p^* = \mu \frac{1}{E[\frac{1}{p}]} \leq \frac{1}{E[p]} = \frac{\kappa E[\frac{1}{p}]}{\kappa} = \tilde{p}^*,
\]
where we have used the relation \( p \leq \tilde{p} \) and Jensen’s inequality. Therefore, the proposition is proved.

**Proof of Proposition 4**

Let random variables \( Q^E, Q^{II}, \) and \( Q^I \) denote \( \frac{1}{\mu} \) of the quantity sold (i.e., \( \mu Q^E, \mu Q^{II}, \) and \( \mu Q^I \) are the actual quantities) in English, second-price, and first-price RMA respectively, and let \( p^E, p^{II}, \) and \( p^I \) denote the corresponding transaction price per unit quantity of the good (i.e., \( p^I \)’s are the inverse of the \( Q^I \)’s). Without loss assume bidder 1 has the highest signal. Thus, for the first-price auction, we have \( p^I = \beta I(x_1) \) and \( \tilde{p}^I_{\omega=0}(x_1) \), where the tilda with the subscript \( \omega=0 \) denote variables associated with the corresponding FQA under the same signal structure with \( \omega = 0 \).

Define \( \Delta I = p^I - \tilde{p}^I_{\omega=0} - \omega \). Then

\[
E[Q^I] = E\left[ \frac{1}{p^I} \right] = E\left[ \frac{1}{\tilde{p}^I_{\omega=0} + \omega + \Delta I} \right] = \frac{1}{\omega} E\left[ \frac{1}{1 + \frac{p^I_{\omega=0} + \Delta I}{\omega}} \right]
\]

\[
= \frac{1}{\omega} E\left[ 1 - \frac{p^I_{\omega=0} + \Delta I}{\omega} + \left( \frac{p^I_{\omega=0} + \Delta I}{\omega} \right)^2 \right] = \frac{1}{\omega} E\left[ 1 - \frac{p^I_{\omega=0} + \Delta I^*}{\omega} \right],
\]

where \( \Delta I^* = \Delta I - \left( \frac{p^I_{\omega=0} + \Delta I}{\omega + p^I_{\omega=0} + \Delta} \right)^2 \). Similarly we have for second-price RMA:

\[
E[Q^{II}] = E\left[ \frac{1}{p^{II}} \right] = E\left[ \frac{1}{\omega + \tilde{p}^{II}_{\omega=0}} \right] = \frac{1}{\omega} E\left[ \frac{1}{1 + \frac{p^{II}_{\omega=0}}{\omega}} \right]
\]

\[
= \frac{1}{\omega} E\left[ 1 - \frac{\tilde{p}^{II}_{\omega=0}}{\omega} + \left( \frac{\tilde{p}^{II}_{\omega=0}}{\omega} \right)^2 \right] = \frac{1}{\omega} E\left[ 1 - \frac{\tilde{p}^{II}_{\omega=0} + \Delta^{II*}}{\omega} \right],
\]
where $\Delta^{I*} \equiv \frac{(\hat{p}^I_{\omega=0})^2}{\omega + \hat{p}^E_{\omega=0}}$, and for English RMA:

$$E[Q^E] = E \left[ \frac{1}{p^E} \right] = E \left[ \frac{1}{\omega + \hat{p}^E_{\omega=0}} \right] = \frac{1}{\omega} E \left[ 1 - \frac{\hat{p}^E_{\omega=0} + \Delta^{E*}}{\omega} \right],$$

where $\Delta^{E*} \equiv \frac{(\hat{p}^E_{\omega=0})^2}{\omega + \hat{p}^E_{\omega=0}}$.

From part (ii) of proposition 2, we have $\lim_{\omega \to \infty} \Delta^I = 0$. Further, because $\hat{p}^I_{\omega=0}$ is finite ($\leq u(\bar{x}, \bar{x}, ..., \bar{x})$), we also have $\lim_{\omega \to \infty} \frac{(\hat{p}^I_{\omega=0} + \Delta^I)^2}{\omega + \hat{p}^I_{\omega=0} + \Delta^I} = 0$. Thus, $\lim_{\omega \to \infty} \Delta^{I*} = 0$. Similarly, we have $\lim_{\omega \to \infty} \Delta^{II*} = \lim_{\omega \to \infty} \Delta^{E*} = 0$. Next, note $E[\hat{p}^E_{\omega=0}] > E[\hat{p}^{II}_{\omega=0}] > E[\hat{p}^I_{\omega=0}]$ under strict affiliation for three or more bidders due to the linkage principle. Define $\delta \equiv E[\hat{p}^{II}_{\omega=0}] - E[\hat{p}^I_{\omega=0}] > 0$. Thus

$$E[Q^{II}] - E[Q^I] = \frac{1}{\omega^2} E \left[ \hat{p}^{II}_{\omega=0} - \hat{p}^I_{\omega=0} + (\Delta^{I*} - \Delta^{II*}) \right].$$

Because $\lim_{\omega \to \infty} \Delta^{I*} = \lim_{\omega \to \infty} \Delta^{II*} = 0$, there exists a $\omega^*$ such that $|\Delta^{I*} - \Delta^{II*}| < \delta$ for all $\omega \geq \omega^*$. Therefore, for all $\omega \geq \omega^*$, $E[Q^{II}] < E[Q^I]$ (i.e., the seller is better off in second- than in first-price auctions). Similarly we can show there exists $\omega^{**}$ such that for all $\omega \geq \omega^{**}$, $E[Q^E] < E[Q^{II}]$. Thus we have proved the proposition for the case of private sale.

Next we consider the case of project sale. In all these auctions formats, the seller’s expected profit takes the form

$$\pi_s = E \left[ v_1 \left( \kappa - \frac{\mu}{p} \right) \right] = \kappa E[v_1] - \mu E\left[ \frac{v_1}{p} \right]. \quad (41)$$

Note the first term is independent of the auction format, it is only necessary to compare the second term. For the first-price RMA, we have

$$E\left[ \frac{v_1}{p^I} \right] = \frac{1}{\omega} E \left[ \left( 1 - \frac{\hat{p}^I_{\omega=0} + \Delta^{I*}}{\omega} \right) v_1 \right]
= \frac{1}{\omega} E[v_1] - \frac{1}{\omega} E \left[ \frac{\hat{p}^I_{\omega=0} + \Delta^{I*}}{\omega} v_1 \right]
= \frac{1}{\omega} E[v_1] - \frac{1}{\omega} E \left[ \hat{p}^I_{\omega=0} + \phi^I \right], \quad (42)$$

where $\Delta^{I*}$ was defined earlier, similar algebra as in the derivation of equation 40 has been used, and $\phi^I \equiv \Delta^{I*} + \frac{(\kappa - \omega)(\hat{p}^I_{\omega=0} + \Delta^{I*})}{\omega}$. Because $\hat{p}^I_{\omega=0}$ and $(\kappa - \omega)$ are finite ($\leq u(\bar{x}, \bar{x}, ..., \bar{x})$), and that $\lim_{\omega \to \infty} \Delta^{I*} = 0$ as shown earlier, we have $\lim_{\omega \to \infty} \phi^I = 0$.

Similarly, we have for second-price RMA:
\[
E \left[ \frac{v_1}{p^{II}} \right] = \frac{1}{\omega} E [v_1] - \frac{1}{\omega} E \left[ \tilde{p}^{II} + \phi^{II} \right],
\]
where \( \phi^{II} \equiv \Delta^{II*} + \frac{(v_1 - \omega)}{\omega} (\tilde{p}^{II} + \Delta^{II*}) \), and for English RMA:
\[
E \left[ \frac{v_1}{p^{E}} \right] = \frac{1}{\omega} E [v_1] - \frac{1}{\omega} E \left[ \tilde{p}^{E} + \phi^{E} \right],
\]
where \( \phi^{E} \equiv \Delta^{E*} + \frac{(v_1 - \omega)}{\omega} (\tilde{p}^{E} + \Delta^{E*}) \). We also similarly have \( \lim_{\omega \to \infty} \phi^{II} = \lim_{\omega \to \infty} \phi^{E} = 0 \).

Thus, equation 41 yields
\[
\pi^{II}_s - \pi^{I}_s = \frac{\mu}{\omega} E [\tilde{p}^{II} - \tilde{p}^{I} + (\phi^{II} - \phi^{I})].
\]

Because \( \lim_{\omega \to \infty} \phi^{I} = \lim_{\omega \to \infty} \phi^{II} = 0 \), there exists \( \omega^* \) such that \( |\phi^{II} - \phi^{I}| < \delta \) (where \( \delta \) was defined earlier) for all \( \omega \geq \omega^* \). Therefore, for all \( \omega \geq \omega^* \), \( \pi^{II}_s > \pi^{I}_s \). Similarly we can show there exists a \( \omega^{**} \) such that for all \( \omega \geq \omega^{**} \), \( \pi^{E}_s > \pi^{II}_s \). Thus we have proved the proposition for the case of project sale.

**Proof of Proposition 5**

Let random variables \( \tilde{Q}^{E}, \tilde{Q}^{II}, \) and \( \tilde{Q}^{I} \) denote \( \frac{1}{\mu} \) of the quantity sold in English, second-price, and first-price RMA respectively, and let \( \tilde{p}^{E}, \tilde{p}^{II} \) and \( \tilde{p}^{I} \) be the corresponding transaction price per unit quantity of the good (i.e., \( \tilde{p}^{\cdot} \)'s are the inverse of the \( \tilde{Q}^{\cdot} \)'s). Let \( X_1 \) denote the highest signal. We first show \( E[\tilde{Q}^{E}] > E[\tilde{Q}^{II}] \). We have \( \tilde{p}^{E} = u(y_1, y_1, x_3, ..., x_N) \) and \( \tilde{p}^{II} = v(y_1, y_1) \), where \( y_1 \) denotes the second highest signal, and \( x_3 \) through \( x_N \) refer to the third largest to smallest signals. Since signals are independent, we have:

\[
E \left[ \tilde{p}^{E} | y_1 = y \right] = E \left[ u(y_1, y_1, x_3, ..., x_N) | X_1 = y, y_1 = y \right]
\]
\[
= v(y, y)
\]
\[
= \tilde{p}^{II} (y_1 = y)
\]

Note that conditional on \( y_1 = y \), \( \tilde{p}^{E} \) is still random but \( \tilde{p}^{II} \) is deterministic. Jensen’s
inequality and the law of iterated expectation give

\[
E \left[ \tilde{Q}^E \right] = E \left[ \frac{1}{P^E} \right] \\
= E \left[ E \left[ \frac{1}{P^E} \mid y_1 = y \right] \right] \\
> E \left[ \frac{1}{E [P^E \mid y_1 = y]} \mid y_1 = y \right] \\
= E \left[ \frac{1}{P^I} \mid y_1 = y \right] \\
= E \left[ \tilde{Q}^I \right].
\]

Next we prove \( E[\tilde{Q}^{II}] > E[\tilde{Q}^I] \). From equation 12 we have

\[
E \left[ \tilde{Q}^I \mid X_1 = x \right] = \int_0^x \frac{1}{v(y,x)} dL(y \mid x)
\]

On the other hand, we have

\[
E \left[ \tilde{Q}^{II} \mid X_1 = x \right] = \int_0^x \frac{1}{v(y,x)} dK(y \mid x)
\]

where \( K(y \mid x) = \frac{G(y)}{F(x)} \). Notice both \( L(\cdot \mid x) \) and \( K(\cdot \mid x) \) are distribution functions on \([x, x]\), we next show that \( L(\cdot \mid x) \) first order stochastically dominates over \( K(\cdot \mid x) \). Using equation 14 and since \( \frac{v(x,x)}{v(x,x)} > 1 \) for \( x > 0 \), we have that for \( x > y > 0 \):

\[
- \int_y^x s(t) dt < - \int_y^x \frac{g(t)}{G(t)} dt = - \ln \frac{G(x)}{G(y)}.
\]

Then from equation 13 we have that \( L(y \mid x) < \frac{F(y)}{F(x)} = K(y \mid x) \) for \( x > y > x \), therefore \( L(\cdot \mid x) \) first order stochastically dominates over \( K(\cdot \mid x) \). Since \( \frac{1}{v(y,x)} \) is decreasing in \( y \), we have \( E[\tilde{Q}^{II} \mid X_1 = x] > E[\tilde{Q}^I \mid X_1 = x] \). Using the law of iterated expectation, we have that \( E[\tilde{Q}^{II}] > E[\tilde{Q}^I] \). This proves the proposition for the case of private sale.

Now we prove for the case of project sale. For any one of the three auction formats, denote by \( \pi_i(z, x) \) bidder \( i \)'s expected profit when his signal is \( x \) but bids as if he has signal
\[
\frac{d}{dx} \pi_i(x, x) = \frac{\partial}{\partial z} \pi_i(z, x) \bigg|_{z=x} + \frac{\partial}{\partial x} \pi_i(z, x) \bigg|_{z=x}
\]
\[
= \frac{\partial}{\partial x} \pi_i(z, x) \bigg|_{z=x}
\]
\[
= G(x) \frac{du_i(x)}{dx} E \left[ \bar{Q} | X_1 = x \right].
\]

As \( \pi_i(y, y) = 0 \), one has \( \pi_i(x, x) = \int_x^y G(t) \frac{du_i(t)}{dt} E \left[ \bar{Q} | X_1 = t \right] dt \). Using similar arguments as above for the case of private sale, it is straightforward to show
\[
E \left[ \bar{Q}^E | X_1 = x \right] > E \left[ \bar{Q}^{II} | X_1 = x \right] > E \left[ \bar{Q}^I | X_1 = x \right].
\]

Thus, \( \pi_{E}^E(x, x) > \pi_{E}^{II}(x, x) > \pi_{E}^I(x, x) \).

Denote by \( \pi_s \) the seller’s expected profit. Note that the sum of all bidders’ and the seller’s expected profit \( \{ \Sigma_i \left[ \pi_i(x, x) \right] + \pi_s \} \) is the same across the three auction formats. Thus, \( \pi_{E}^E < \pi_{E}^{II} < \pi_{E}^I \), proving the proposition for the case of project sale.

**Proof of Proposition 6**

We use the same notation as well as some of the results in the proof of Proposition 4. Additionally, we use subscript “reveal” and “hide” to denote cases in which the seller always reveals or hides his information, respectively. First consider first-price format. Note that Proposition 2 still holds when the seller possesses information. In the case in which the seller always reveals his information, define \( \Delta^l_{reveal} \equiv p^l_{reveal} - \bar{p}^l_{\omega=0,reveal} - \omega \). Then
\[
E \left[ Q_{reveal}^l \right] = E \left[ \frac{1}{\bar{p}^l_{\omega=0,reveal} + \omega + \Delta^l_{reveal}} \right] = \frac{1}{\omega} E \left[ 1 - \frac{\bar{p}^l_{\omega=0,reveal} + \Delta^l_{reveal}}{\omega} \right],
\]
where \( \Delta^l_{reveal} \equiv \Delta^l_{reveal} - \frac{(\bar{p}^l_{\omega=0,reveal} + \Delta^l_{reveal})^2}{\omega + \bar{p}^l_{\omega=0,reveal} + \Delta^l_{reveal}} \). Similarly, when the seller always hides the information, we have:
\[
E \left[ Q_{hide}^l \right] = \frac{1}{\omega} E \left[ 1 - \frac{\bar{p}^l_{\omega=0,hide} + \Delta^l_{hide}}{\omega} \right],
\]
where \( \Delta^l_{hide} \equiv \Delta^l_{hide} - \frac{(\bar{p}^l_{\omega=0,hide} + \Delta^l_{hide})^2}{\omega + \bar{p}^l_{\omega=0,hide} + \Delta^l_{hide}}\) and \( \Delta^l_{hide} \equiv p^l_{hide} - \bar{p}^l_{\omega=0,hide} - \omega \). We then have
\[
E \left[ Q_{hide}^l \right] - E \left[ Q_{reveal}^l \right] = \frac{1}{\omega^2} E \left[ \bar{p}^l_{\omega=0,reveal} - \bar{p}^l_{\omega=0,hide} + (\Delta^l_{reveal} - \Delta^l_{hide}) \right].
\]
Next, define $\delta = \mathbb{E}[\tilde{p}_{\omega=0,\text{reveal}}] - \mathbb{E}[\tilde{p}_{\omega=0,\text{hide}}]$. We have $\delta > 0$ due to strict affiliation. Because $\lim_{\omega \to \infty} \Delta_{\text{reveal}}^{I^*} = \lim_{\omega \to \infty} \Delta_{\text{hide}}^{I^*} = 0$, there exists a $\omega^*$ such that $|\Delta_{\text{reveal}}^{I^*} - \Delta_{\text{hide}}^{I^*}| < \delta$ for all $\omega \geq \omega^*$. Therefore, for all $\omega \geq \omega^*$, $\mathbb{E}[Q_{\text{hide}}] > \mathbb{E}[Q_{\text{reveal}}]$. Similarly, we can show there exists a $\omega^{**}$ such that $\mathbb{E}[Q_{\text{II}}_{\text{hide}}] > \mathbb{E}[Q_{\text{II}}_{\text{reveal}}]$ for all $\omega \geq \omega^{**}$, and that there exists a $\omega^{***}$ such that $\mathbb{E}[Q_{E_{\text{I}}}^I_{\text{hide}}] > \mathbb{E}[Q_{E_{\text{I}}}^I_{\text{reveal}}]$ for all $\omega \geq \omega^{***}$. Thus, we have proved the proposition for the case of private sale.

Now consider the case of project sale. We first examine first-price RMA. Following similar procedures as for deriving equations 41 and 42, in the case in which the seller always reveals information, we have

$$\pi_{I_s,\text{reveal}} = \kappa \mathbb{E}[v_1] - \mu \mathbb{E}\left[\frac{v_1}{p_{\text{reveal}}^I}\right]$$

and

$$\mathbb{E}\left[\frac{v_1}{p_{\text{reveal}}^I}\right] = \frac{1}{\omega} \mathbb{E}[v_1] - \frac{1}{\omega} \mathbb{E}[\tilde{p}_{\omega=0,\text{reveal}} + \phi_{\text{reveal}}^I],$$

where $\phi_{\text{reveal}}^I = \Delta_{\text{reveal}}^{I^*} + \frac{v_1 - \omega}{(\tilde{p}_{\omega=0,\text{reveal}} + \Delta_{\text{reveal}}^{I^*})}$ ($\Delta_{\text{reveal}}^{I^*}$ has been defined earlier). Following similar arguments as in the proof of Proposition 4 we have $\lim_{\omega \to \infty} \phi_{\text{reveal}}^I = 0$.

Similarly, in the case in which the seller hides information, we have

$$\pi_{I_s,\text{hide}} = \kappa \mathbb{E}[v_1] - \mu \mathbb{E}\left[\frac{v_1}{p_{\text{hide}}^I}\right]$$

and

$$\mathbb{E}\left[\frac{v_1}{p_{\text{hide}}^I}\right] = \frac{1}{\omega} \mathbb{E}[v_1] - \frac{1}{\omega} \mathbb{E}[\tilde{p}_{\omega=0,\text{hide}} + \phi_{\text{hide}}^I],$$

where $\phi_{\text{hide}}^I$ is defined with a similar structure as $\phi_{\text{reveal}}^I$ and it has the same limiting property that $\lim_{\omega \to \infty} \phi_{\text{hide}}^I = 0$. Thus, there exists a $\omega^*$ such that $|\phi_{\text{reveal}}^I - \phi_{\text{hide}}^I| < \delta$ for all $\omega \geq \omega^*$. Therefore, for all $\omega \geq \omega^*$, $\pi_{I_s,\text{reveal}} > \pi_{I_s,\text{hide}}$. Similarly, we can show in second-price and English RMA, the seller is also better off always revealing than hiding information. Thus we have established the proposition for the case of project sale.

**Proof of Proposition 7**

Let random variable $\tilde{Q}$ denote $\frac{1}{p}$ of the quantity sold and let $\tilde{p} \equiv \frac{1}{\tilde{Q}}$ denote the transaction price per unit of the good. We use subscript “reveal” and “hide” to denote the situations in which the seller always reveals or hides his information, respectively. For expositional ease and without loss of generality, we assume $\omega = 0$ and that bidder 1 has the highest signal.

**Part (i)**
Define
\[ v(s, x, y) \equiv E[u|S = s, X_1 = x, y_1 = y], \tag{48} \]
where \( X_1 \) denote bidder 1’s signal, and \( y_1 \) is the highest signal among the other \( N-1 \) bidders (hence \( y < x \)).

We first prove part (i) of the proposition. Consider the private sale. In second-price auction, we have
\[ \tilde{Q}^I_{\text{reveal}}(S = s, X_1 = x, y_1 = y) = \frac{1}{v(s, y, y)} \]
and
\[ \tilde{Q}^I_{\text{hide}}(X_1 = x, y_1 = y) = \frac{1}{v(y, y)}, \]
where \( v(y, y) = E_s[v(s, y, y)] \) with \( E_s \) denoting expectation over the seller’s information \( s \). Thus,
\[ E_s \left[ \tilde{Q}^I_{\text{reveal}}(S = s, X_1 = x, y_1 = y) \right] = E_s \left[ \frac{1}{v(s, y, y)} \right] > \frac{1}{E_s[v(s, y, y)]} = \frac{1}{v(y, y)} = \tilde{Q}^I_{\text{hide}}(X_1 = x, y_1 = y), \]
where the inequality comes from the Jensen’s inequality (note the function \( \frac{1}{v} \) is convex is \( v \)). Then the law of iterated expectations gives:
\[ E \left[ \tilde{Q}^I_{\text{reveal}} \right] = E \left[ E_s \left[ \tilde{Q}^I_{\text{reveal}}(S = s, X_1 = x, y_1 = y) \right] \right] > E \left[ \tilde{Q}^I_{\text{hide}}(X_1 = x, y_1 = y) \right] = E \left[ \tilde{Q}^I_{\text{hide}} \right]. \]
Similarly, one can show \( E \left[ \tilde{Q}^E_{\text{reveal}} \right] > E \left[ \tilde{Q}^E_{\text{hide}} \right] \). Thus the seller is better off hiding information in both second-price and English auctions.

Next, consider the project sale. Whether the seller reveals or hides information, the seller’s expected profit takes the form:
\[ \pi_s = E \left[ v_1 \left( \kappa - \frac{\mu}{p} \right) \right] = \kappa E[v_1] - \mu E \left[ \frac{v_1}{p} \right]. \tag{49} \]
Note the first term \( \kappa E[v_1] \) is the same whether the seller reveals or hides information. Therefore it is only necessary to compare the second term between revealing and hiding
information. Consider second price auction. Define
\[ v_1^*(x, y) \equiv \mathbb{E}[u_1|X_1 = x, y_1 = y]. \]

Then
\[ \frac{v_{1,\text{hide}}^I}{p_{\text{hide}}^I}(X_1 = x, y_1 = y) = \frac{v_1^*(x, y) + \mathbb{E}[u_2(s)]}{v_1^*(y, y) + \mathbb{E}[u_2(s)]} = 1 + \frac{v_1^*(x, y) - v_1^*(y, y)}{v_1^*(y, y) + \mathbb{E}[u_2(s)]} \]
and
\[ \frac{v_{1,\text{reveal}}^I}{p_{\text{reveal}}^I}(S = s, X_1 = x, y_1 = y) = \frac{v_1^*(x, y) + u_2(s)}{v_1^*(y, y) + u_2(s)} = 1 + \frac{v_1^*(x, y) - v_1^*(y, y)}{v_1^*(y, y) + u_2(s)}. \]
Note that \( \mathbb{E}_s \left[ \frac{v_1^*(x, y) - v_1^*(y, y)}{v_1^*(y, y) + \mathbb{E}[u_2(s)]} \right] > \frac{v_1^*(x, y) - v_1^*(y, y)}{v_1^*(y, y) + \mathbb{E}[u_2(s)]} \) due to Jensen’s inequality (because the function \( \frac{1}{v_1(x, y)} \) is convex in \( u_2(s) \)) and the fact that \( v_1^*(x, y) > v_1^*(y, y) \). Thus,
\[ \mathbb{E}_s \left[ \frac{v_{1,\text{reveal}}^I}{p_{\text{reveal}}^I}(S = s, X_1 = x, y_1 = y) \right] > \frac{v_{1,\text{hide}}^I}{p_{\text{hide}}^I}(X_1 = x, y_1 = y), \]
which, upon using the law of iterated expectations, yields \( \mathbb{E} \left[ \frac{v_{1,\text{reveal}}^I}{p_{\text{reveal}}^I} \right] > \mathbb{E} \left[ \frac{v_{1,\text{hide}}^I}{p_{\text{hide}}^I} \right] \). Equation (49) then yields \( \pi_{s,\text{reveal}}^I < \pi_{s,\text{hide}}^I \). Next, consider English auction. Then
\[ \frac{v_{1,\text{hide}}^E}{p_{\text{hide}}^E}(X_1 = x, y_1 = y, x_3...x_N) = \frac{u_1(x, y, x_3...x_N) + \mathbb{E}[u_2(s)]}{u_1(y, y, x_3...x_N) + \mathbb{E}[u_2(s)]}, \]
where \( x_3 \) through \( x_N \) denote the N-2 signals other than the highest two. We also have:
\[ \frac{v_{1,\text{reveal}}^E}{p_{\text{reveal}}^E}(S = s, X_1 = x, y_1 = y, x_3...x_N) = \frac{u_1(x, y, x_3...x_N) + u_2(s)}{u_1(y, y, x_3...x_N) + u_2(s)} \]
\[ = 1 + \frac{u_1(x, y, x_3...x_N) - u_1(y, y, x_3...x_N)}{u_1(y, y, x_3...x_N) + u_2(s)}. \]
Following the same logic as in the case of second-price auctions, we have \( \pi_{s,\text{reveal}}^E < \pi_{s,\text{hide}}^E \). The above establishes part (i) of the proposition.

**Part (ii)**

Now we prove part (ii) of the proposition. When the seller always reveals information,
equation 12 becomes

\[ \tilde{Q}_{\text{reveal}} (S = s, X_1 = x) = \int_0^x \frac{1}{u_1(y) + u_2(s)} dL(y|x). \]

When the seller always hides information, equation 12 gives

\[ \tilde{Q}_{\text{hide}} (X_1 = x) = \int_0^x \frac{1}{u_1(y) + E[u_2(s)]} dL(y|x), \]

Thus,

\[
\begin{align*}
\mathbb{E}_s \left[ \tilde{Q}_{\text{reveal}} | S = s, X_1 = x \right] &= \int_0^x \mathbb{E}_s \left[ \frac{1}{u_1(y) + u_2(s)} \right] dL(y|x) \\
&> \int_0^x \frac{1}{u_1(y) + \mathbb{E}_s [u_2(s)]} dL(y|x) \\
&= \mathbb{E} \left[ \tilde{Q}_{\text{hide}} | X_1 = x \right].
\end{align*}
\]

Thus, \( \mathbb{E}[\tilde{Q}_{\text{reveal}}] > \mathbb{E}[\tilde{Q}_{\text{hide}}] \). Hence, in first-price RMA, the seller is better off hiding information in the case of private sale. In the case of project sale, refer to equation 49 and note that

\[
\begin{align*}
u_{1, \text{reveal}}^f (S = s, X_1 = x) &= \int_0^x \frac{u_1(x) + u_2(s)}{u_1(y) + u_2(s)} dL(y|x) \\
&= 1 + \int_0^x \frac{u_1(x) - u_1(y)}{u_1(y) + u_2(s)} dL(y|x),
\end{align*}
\]

and

\[
\begin{align*}
u_{1, \text{hide}}^f (X_1 = x) &= \int_0^x \frac{u_1(x) + E[u_2(s)]}{u_1(y) + E[u_2(s)]} dL(y|x) \\
&= 1 + \int_0^x \frac{u_1(x) - u_1(y)}{u_1(y) + E[u_2(s)]} dL(y|x).
\end{align*}
\]
Thus,

\[
E_s \left[ \frac{v_1^{r,\text{reveal}}}{p_{\text{reveal}}^r}(S = s, X_1 = x) \right] = 1 + E_s \left[ \int_0^x \frac{u_1(x) - u_1(y)}{u_1(y) + u_2(s)} dL(y|x) \right] \\
> 1 + \int_0^x \frac{u_1(x) - u_1(y)}{u_1(y) + E_s[u_2(s)]} dL(y|x) \\
= \frac{v_1^{r,\text{hide}}}{p_{\text{reveal}}^r}(X_1 = x). 
\]

The law of iterated expectations then gives

\[
E \left[ \frac{v_1^{r,\text{reveal}}}{p_{\text{reveal}}^r} \right] > E \left[ \frac{v_1^{r,\text{hide}}}{p_{\text{hide}}^r} \right],
\]

which, upon using equation 49, yields \( \pi_{s,\text{reveal}}^r > \pi_{s,\text{hide}}^r \). The above establishes part (ii) of the proposition. \[\blacksquare\]