Majority Rule and Utilitarian Welfare*

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Abstract

We study the welfare properties of majority and supermajority rules when voting is costly and values, costs and electorate sizes are all random. Unlike previous work, where the electorate size was either fixed or Poisson distributed and exhibited no limiting dispersion, we study general distributions which permit substantial dispersion. We identify conditions on these distributions guaranteeing that a large election under majority rule produces the utilitarian choice with probability one. Absent these conditions, non-utilitarian outcomes are possible, as we demonstrate. We also show that majority rule is the *only* voting rule with the utilitarian property—strict supermajority rules are not utilitarian.

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1 Introduction

Pre-election polls leading up to the November 2008 vote on Proposition 8, the California Marriage Protection Act, indicated that it would easily be defeated.[1] If passed, the proposition would make it illegal for same-sex couples to marry. The actual vote count sharply differed from poll predictions—Proposition 8 passed by a 52-48% margin. The results surprised most Californians and were shortly followed by mass protests and lawsuits.[2]

The intent of any referendum, including Proposition 8, is to reflect directly the will of the electorate. But of course, it can only reflect the will of those of the electorate who actually turn out to vote. The election results suggest that the preferences of those who turned out to vote were different from the preferences of the population at large, at least to the extent that the pre-election polls accurately reflected the latter. Precisely, the turnout rates of those in favor of the proposition—that is, against same-sex marriage—were greater than of those opposed. A simple explanation is that those in favor felt more strongly about the matter and turned out in greater numbers.[3] If voters on both sides had come to the polls in proportion to their numbers in the overall populace, there would have been no surprise on election day. When intensity of preference drives turnout, such surprises can, and do, happen.

This paper studies the outcomes produced by majority rule in a setting where the intensity of preference affects turnout. Our starting point is the following well-known conundrum. Suppose that 51% of the populace mildly favors one of two choices. The remainder passionately favors the alternative. If everyone voted, the choice supported by the majority would win; however, a utilitarian social planner would side with the minority since the welfare gains would more than compensate for the modest losses

[1] The three polls closest to the election had Proposition 8 losing by margins of 47-50% (Survey USA), 44-49% (Field Poll) and 44-52% (Public Policy Institute of California).

[2] The proposition was declared unconstitutional by the lower courts and this ruling was upheld by the US Supreme Court in June 2013.

[3] The vote on Proposition 8 was concurrent with the 2008 presidential election and so one may wonder whether turnout was determined by the latter. But since California voted overwhelmingly for Barack Obama in 2008, this cannot explain the "surprise" positive vote for the proposition.

of overruling the majority.[4] In such situations, majority rule would appear to be at odds with utilitarianism.

Or would it? Voting is often a choice rather than a requirement. Voters incur opportunity (or real) costs in coming to the polls and may avoid these by abstaining. This casts doubt on our earlier conclusion. Given their intensely held views, the minority may be more motivated than the majority to incur voting costs. Thus, to some degree, the *decision* to vote encodes voters' intensity of preference. But even here the connection is blunt—the turnout decision is binary while intensities are not—and, at best, indirect. Both sides are only motivated to turn out to the extent that they are likely to influence the final decision; that is, the benefits from voting are mitigated by the probability that a vote cast is *pivotal*. So even though the minority feels intensely about their favored alternative, were they sufficiently pessimistic about the prospect of being decisive, intensity alone would mean little in terms of participation.

We show below that when voting is costly, voluntary voting under majority rule translates societal preferences into outcomes in a consistent way—it always implements the *utilitarian* outcome. Precisely, in a large election, the side with the higher aggregate willingness to pay to alter the result gets its preferred outcome in equilibrium. Moreover, majority rule is the *only* election rule with this property. Even with strategically sophisticated voters, supermajority rules—such as a rule that requires a $2:1$ vote ratio to overturn the status quo—will not deliver the utilitarian choice. That such a rule is biased in favor of the status quo is not surprising. But the magnitude of this bias is surprising. For instance, one might conjecture that a $2:1$ supermajority rule will give twice the weight to the welfare of those favoring the status quo as compared to those favoring the alternative. In fact we show that the $2:1$ rule gives *four* times the weight to the status quo over the alternative. More generally, supermajority rules favor the status quo at a weight proportional to the *square* of required vote ratio. As a consequence, even small departures from majority

[4] This, of course, presupposes that interpersonal utility comparisons are possible. We discuss this issue below.

rule disproportionately favor the status quo. For instance, the not uncommon 60% rule gives more than twice the weight to the status quo as the alternative.[5]

In this paper, we study welfare in what is, arguably, the workhorse model of voting, the pure private values model. We examine voluntary and costly voting in two-candidate elections and relate election outcomes to utilitarian choices. Most models in this class assume that the number of voters is fixed and known or Poisson distributed. While the first specification seems unrealistic in large elections, the Poisson formulation has a similar problem, since population dispersion (measured by the coefficient of variation) vanishes in the limit. Thus, once again, in large elections, voters are unreasonably sure about the size of the electorate. In contrast, we study a more general non-parametric formulation that nests these two cases, as well as allowing many other commonly used distributions. Importantly, this class permits positive population dispersion in the limit.

1. Under weak assumptions on the population distribution, majority rule selects the utilitarian candidate almost certainly in a large election (Theorem 1).

2. No other voting rule shares this property: Among all supermajority rules, majority rule is the *unique* rule having the utilitarian property (Theorem 2). Moreover, such rules disproportionately favor the status quo from a welfare perspective.

Intuition for our first result is easily seen in the following example: A finite population is to vote over two alternatives, A and B. A fraction $\lambda > 1/2$ of voters favor A and receive payoff $v_A > 0$ when it is selected. The remainder favor B and receive payoff $v_B > 0$ when it is chosen. Finally, the cost of voting is an independent draw from a uniform distribution on $[0, 1]$.

For A supporters, the benefits of voting are $v_A \Pr[Piv_A]$, where $\Pr[Piv_A]$ is the probability that an additional A vote is decisive, which depends on turnout. In

[5] A 60% rule requires the alternative to obtain a vote ratio of $3/2$ and so the implied welfare weight, its square, is $9/4$.

equilibrium, all A supporters with costs below a threshold c_A will vote. The threshold equates the costs and benefits of voting:

$$c_A = v_A \Pr[Piv_A]$$

Similarly, the cost threshold c_B for B supporters is

$$c_B = v_B \Pr[Piv_B]$$

Because voting costs are uniformly distributed, c_A also equals the turnout rate p_A of A supporters; similarly, c_B equals p_B. Using this and multiplying the fractions of voters favoring each alternative, yields expressions in terms of expected vote shares, λp_A and $(1-\lambda)\,p_B$ for A and B, respectively. Thus, in equilibrium

$$\frac{\lambda p_A}{(1-\lambda)\,p_B} = \frac{\lambda v_A}{(1-\lambda)\,v_B} \times \frac{\Pr[Piv_A]}{\Pr[Piv_B]}$$

The right-hand side is the product of two terms, the "welfare ratio" and the "pivot ratio." We claim that the vote share favors B if and only if B is utilitarian. Suppose that B is utilitarian, i.e., $\lambda v_A < (1-\lambda)\,v_B$, but that A enjoys a higher vote share. Since the welfare ratio is less than one, it must be that the pivot ratio exceeds one, i.e., an A vote is more likely to be decisive than a B vote. But this is never the case when A is ahead since a vote for the trailing candidate pushes the vote total in the direction of ties or near ties while a vote for the leading candidate pushes the total away. Thus, if the vote share favored A, then both the welfare and pivot ratios would favor B, a contradiction. Therefore, the vote share must favor B. The converse also holds.

The simplicity of the above argument is deceptive and relies essentially on heterogeneous voting costs (with a lower support at 0). If, instead, all voters faced the same cost, $c > 0$, to vote, along the lines of Palfrey and Rosenthal (1983) and Campbell (1999), then participation rates (in this case, the probability of voting) would again be determined by equating the costs and benefits (in an interior equilibrium), replacing c for c_A and c_B in the expression above. When A receives a greater vote share than B, the "underdog" principle again implies that pivotality considerations favor B. Since voters face the same costs, so too must they receive the same expected benefits. Therefore, A enjoys a higher vote share if and only if $v_A > v_B$. But this pays no attention to the fraction of voters of each type, so the outcome is not utilitarian. Formally, with a fixed cost of voting, votes shares favor B if and only if the utility of a typical B voter, v_B, exceeds that of an A voter—irrespective of the population fractions of each type of voter.

The argument that the vote shares must favor the utilitarian candidate holds independent of the distribution of the voter population. But vote share differences alone do not guarantee that the utilitarian candidate will win almost surely in large elections. If the limiting population dispersion swamps vote share differences, the "wrong" candidate may be elected with positive probability (see Example 3 below). Lacking any limiting dispersion, the fixed and Poisson voter models obscure this possibility. Theorem 1 identifies some (weak) conditions where vote share differences dominate, and the utilitarian candidate is elected almost surely, even in the face of positive population dispersion.

In addition to our main results, we make two technical contributions to the analysis of voting that enable us to do away with parametric assumptions as to the size of the electorate. The first tool is to formulate pivot probabilities using complex roots of unity, which permits the study of voting rules in a way that avoids the complicated, and distribution specific, combinatorics inherent in their calculation (see Appendix A). The second tool is a "new" approximation result due to Roos (1999), discussed in Appendix D, that allows us to closely approximate pivot probabilities of any population distribution as a mixture of Poisson distributions.

Related Literature We also studied the welfare properties of voluntary voting in Krishna and Morgan (2011), but only under majority rule and only in a Poisson context. Our Result 1 identifies a weak condition on population distributions that guarantees the utilitarianism result. This condition allows for realistic levels of limiting population dispersion and goes well beyond, but includes, the Poisson model. We also show by example that the condition cannot be dispensed with—some pathological kinds of population uncertainty can upset the utilitarian property of majority rule. Result 2 shows that the utilitarian property is unique to majority rule—all supermajority rules produce severely biased, non-utilitarian outcomes.

The connection between majority voting and utilitarianism originates with Ledyard (1984). In a fixed voter model, Ledyard studies ideological positioning of candidates when faced with voters with Hotelling-type preferences and privately known voting costs. If voting were costless, both candidates would co-locate at the preferred point of the median voter. With costly voting, Ledyard shows that candidates still co-locate, but at the welfare maximizing (utilitarian) ideology. Thus, there is no incentive to participate, and, in equilibrium, the first-best outcome obtains without any actual voting! Myerson (2000) reproduces this result when the number of voters is Poisson distributed. Using the convenient asymptotic formulae for the Poisson model, he links majority rule and utilitarianism as a stepping stone to obtaining the co-location/nobody votes outcome. Our model is a more general version of Ledyard allowing for general population distributions, supermajority voting rules, and doing away with purely vote maximizing objectives on the part of candidates.

When candidates have concerns other than merely winning the election, they will not co-locate, which is our starting point. We study a situation where candidates' ideological positions are given and different. Here, turnout is positive as the supporters of both sides vie to obtain their preferred choice; nonetheless, the chosen candidate maximizes societal welfare—the utilitarian choice enjoys higher vote share and, in large elections, wins with certainty. Unlike Ledyard, we also examine supermajority rules and show that the utilitarian property is *unique* to majority rule. In a closely related model with *fixed* voting costs, Campbell (1999) finds that majority rule is *not* utilitarian.[6] As the two examples in our introduction illustrate, the utilitarian property of majority rule relies essentially on *random* voting costs.

Börgers (2004) compares compulsory and voluntary voting in a completely symmetric special case of our model. His main concern is with the cost of participation,

[6] Campbell's reformulation of situations where voters have heterogeneous costs and values into a unidimensional ratio of the two is only valid when either costs or values are degenerate.

where he shows that voluntary voting, by economizing on voting costs, Pareto dominates compulsory voting. The model's symmetry precludes questions like those we pose. Moreover, Krasa and Polborn (2009) show that Börgers' result may not hold when the symmetry is broken.

Palfrey and Rosenthal (1985) characterize equilibrium properties of large elections with random voting costs that, with strictly positive probability, are zero or negative. They find that, in large elections, only voters with negative (or zero) costs vote. Taylor and Yildirim (2010) study a similar model but where voting costs are bounded above zero. In this setting, they establish the so-called "underdog principle"—a vote for the minority candidate is more likely to be pivotal than a vote for the majority candidate. In both papers preference intensities are identical for both sides and so welfare considerations of election outcomes are not investigated.

In a Poisson setting, Feddersen and Pesendorfer (1999) examine majority rule elections in a hybrid model in which voter preferences have both private and common components with differing preference intensities. Even though voting is costless, the "swing voter's curse" leads some voters to abstain. They show that, in large elections, information aggregates in the sense of full information equivalence—the outcome corresponds to what would be obtained were all voters informed about the underlying state. Full information equivalence is entirely separate from utilitarianism. For instance, the outcome under compulsory voting in our setting, which is not utilitarian, also satisfies full information equivalence.

Limiting population dispersion is a form of aggregate uncertainty, a topic growing in importance. In a private values model with identical preference intensities, Myatt (2012) introduces aggregate uncertainty about the fraction of voters supporting each candidate. He studies the effect of aggregate uncertainty on percentage turnout. Welfare is not considered owing to the absence of diverse voter preferences.

In a fixed voter model, Schmitz and Tröger (2012) study the welfare properties of voting rules in symmetric settings, i.e. situations where both voters and candidates are ex ante identical. They show that, when voter values are independently distributed, majority rule is *second-best*—it maximizes utilitarian welfare among all incentive compatible, anonymous and neutral voting rules.[7] By contrast, our setting permits asymmetries across candidates and uncertainty as to the number of potential voters. We show that, in large elections under costly voting, majority rule is *first-best*—it maximizes utilititarian welfare.

The remainder of the paper proceeds as follows. We sketch the model in Section 2. Section 3 establishes that vote shares favor the utilitarian choice when voting costs are uniform. Sections 4 and 5 study large elections. In Section 4, we generalize the vote share result to arbitrary voting costs. Section 5 identifies mild conditions on the distribution of eligible voters such that majority rule is utilitarian. Section 6 studies supermajority rules and shows that they do not satisfy the utilitarian property. Indeed, supermajority rules are "over-biased" towards the status quo. Finally, Section 7 concludes.

2 The Model

We study a general version of the familiar "private values" voting model in electoral settings where ideology is the main driver of voter decisions and where there is uncertainty about the size and preferences of the voting populace at large. As discussed above, a key distinction between our model and the extant literature is a non-parametric specification of the process determinning the number of eligible voters and permitting limiting dispersion.

Two candidates, named A and B, who differ only in ideology, compete in an majority election with ties resolved by the toss of a fair coin.[8] Voters differ both in the *direction* and *intensity* of their preferences. With probability $\lambda \in (0,1)$ a voter supports A and with probability $1-\lambda$, supports B. Next, each A supporter draws a *value* v from the distribution G_A over $[0,1]$ which is her value of electing A over B. Similarly, each B supporter draws a value v from the distribution G_B over $[0,1]$ which is her value of electing B over A. A voter's *type* is the combination of the direction

[7] Also in a symmetric setting, Kim (2013) shows that, when there are three or more candidates, no ordinal rule is second-best.

[8] Supermajority rules are considered in Section 6.

and intensity of preferences. Types are distributed independently across voters and independently of the number of eligible voters. A citizen knows her own type and that the types of the others are distributed according to λ, G_A and G_B.

To facilitate welfare analysis, which inherently relies on making interpersonal comparisons, we measure voters' values v on a common money-metric scale, say in dollars. Thus, for an A supporter, her value v represents the monetary amount she would be willing to pay to switch the outcome of the election from B to A. Later, we will introduce voting costs, which we also measure in dollars. So effectively, voters have quasi-linear preferences.

The exact number of eligible voters may also vary. The size of the electorate is a random variable $N \in \{0, 1, ...\}$ distributed according to the *objective* probability distribution function π^*, with finite mean m. Thus, the probability that there are exactly n eligible voters (or *citizens*) is $\pi^* (n)$. From an individual voter's perspective, the probability that there are exactly $n-1$ *other* eligible voters is given by

$$\pi (n-1) = \pi^* (n) \times \frac{n}{m} \tag{1}$$

The *total* number of eligible voters from the perspective of a participant is thus the realized number of other voters plus the voter herself.[9] An individual voter thus believes that this population is distributed according to the *subjective* probability distribution π^{**} defined on $\{1, 2, ...\}$ by

$$\pi^{**} (n) = \pi (n-1) = \pi^* (n) \times \frac{n}{m} \tag{2}$$

As long as there is some population uncertainty (π^* is non-degenerate), the two probability distributions differ—the subjective distribution first-order stochastically

[9] To derive (1), suppose that there is a large pool of M identical potential voters from which n eligible voters are drawn according to π^*. All potential voters are equally likely to be eligible. Conditional on the event that a particular voter has been chosen to be eligible, the probability that there are $n-1$ other eligible voters is

$$\frac{\pi^* (n) \frac{n}{M}}{\sum_{k=1}^{M} \pi^* (k) \frac{k}{M}} = \frac{\pi^* (n)\, n}{\sum_{k=1}^{M} \pi^* (k)\, k}$$

and as $M \to \infty$, the denominator on the right-hand side converges to m.

dominates the objective distribution. Thus, an individual voter perceives herself as voting in a stochastically larger election than is warranted from the perspective of an outside observer.[10] As we show in Section 4, this difference in perception matters since it affects turnout rates and, consequently, election results. Stochastically large majority elections perform well when subjective and objective beliefs are in rough concordance with one another and can perform poorly when they are not.

Utilitarianism

Before analyzing elections, we establish a benchmark where a utilitarian social planner selects the candidate. Since voters' valuations are all measured on the same (monetary) scale, simply summing values on each side produces an aggregate "willingness to pay" to switch the outcome from B to A and vice-versa. Reflecting this idea, suppose that the planner's choice is made *ex ante*, before types are realized, and gives equal weight to each potential voter. The expected welfare of A supporters from electing A over B is $v_A = \int_0^1 v dG_A(v)$ while the expected welfare of B supporters from electing B over A is $v_B = \int_0^1 v dG_B(v)$. Since, with probability λ, a voter is an A supporter (otherwise she is a B supporter), ex ante *utilitarian* welfare is higher from electing A rather than B if and only if $\lambda v_A > (1 - \lambda) v_B$. When this inequality holds, we say that A is the *utilitarian choice* (and if it is reversed then B will be referred to as such). We will say that a voting rule is *utilitarian* if the candidate elected is the same as the utilitarian choice.

Compulsory Voting

We now study elections under compulsory voting where the penalties for not voting are severe enough that all eligible voters turn out at the polls. Once there, a voter may still abstain by submitting a blank or spoilt ballot. In equilibrium, all voters turn out and vote for their preferred candidate, and, in large elections, the outcome only depends on λ, the fraction of A voters. Candidate A wins if and only if $\lambda > 1/2$, but this is obviously not utilitarian since the outcome is independent of the

[10] Myerson (1998b) defines "environmental equivalence" as $\pi(n) = \pi^*(n)$, a property of the Poisson distribution. But this says that a voter's subjective belief that there are n *other* voters is the same as the objective probability that there are a *total* of n voters.

intensity of preferences. Thus, A wins in circumstances where a majority of voters tepidly support it but a minority intensely prefer B, so that $\lambda v_A < (1-\lambda) v_B$. To summarize: *Under compulsory voting, majority rule is not utilitarian.*

Voluntary and Costly Voting

We now turn to voluntary voting. A citizen's voting cost is determined by an independent realization from a continuous probability distribution F, which does not vary by the type or number of voters, satisfying $F(0) = 0$ and with a strictly positive density over the support $[0, 1]$.

Prior to the voting decision, each citizen has two pieces of private information—her type and her cost of voting. Each voter compares the costs and benefits of voting in deciding whether to turn out. The benefits from voting depend on the chances that a given vote will be *pivotal*, i.e., swing the election outcome in favor of the voter's preferred candidate either from a loss to a tie or from a tie to a win.

Pivotal Events

An *event* is a pair of vote totals (j, k) such that there are j votes for A and k votes for B. An event is *pivotal* for A if a single additional A vote will affect the outcome of the election, i.e., where there is a tie or when A has one less vote than B. We denote the set of such events by Piv_A. It consists of $T = \{(k, k) : k \geq 0\}$, the set of ties, and $T_{-1} = \{(k-1, k) : k \geq 1\}$, the set of events in which A is one vote short. Similarly, Piv_B is defined to be the set of events which are pivotal for B. It consists of the set T of ties together with $T_{+1} = \{(k, k-1) : k \geq 1\}$, the set of events where A is ahead by one vote.

To determine the chances of pivotal events, suppose that voting behavior is such that, *ex ante*, each voter casts a vote for A with probability q_A and a vote for B with probability q_B. Then $q_0 = 1 - q_A - q_B$ is the probability that a voter abstains. Fix a voter, say 1. Consider an event where the number of *other* voters is exactly n and among these, there are k votes in favor of A and l votes in favor of B. The remaining $n - k - l$ voters abstain. If voters make decisions independently, the probability of this event is

$$\Pr\left[(k,l) \mid n\right] = \binom{n}{k,l} (q_A)^k (q_B)^l (q_0)^{n-k-l}$$

where $\binom{n}{k,l} = \binom{n}{k+l}\binom{k+l}{k}$ denotes the trinomial coefficient.[11] For a realized number of eligible voters, n, the chance of a tie is simply the probability of events of the form (k,k). Formally,

$$\Pr\left[T \mid n\right] = \sum_{k=0}^{n} \binom{n}{k,k} (q_A)^k (q_B)^k (q_0)^{n-2k} \tag{3}$$

Since an individual voter is unaware of the realized number of potential voters, the probability of a tie from that voter's perspective is

$$\Pr\left[T\right] = \sum_{n=0}^{\infty} \pi(n) \Pr\left[T \mid n\right]$$

where the formula reflects a voter's uncertainty about the size of the electorate.

Similarly, for fixed n, the probability that A falls one vote short is

$$\Pr\left[T_{-1} \mid n\right] = \sum_{k=1}^{n} \binom{n}{k-1,k} (q_A)^{k-1} (q_B)^k (q_0)^{n-2k+1} \tag{4}$$

and, from the perspective of a single voter, the overall probability of this event is

$$\Pr\left[T_{-1}\right] = \sum_{n=0}^{\infty} \pi(n) \Pr\left[T_{-1} \mid n\right]$$

The probabilities $\Pr\left[T_{+1} \mid n\right]$ and $\Pr\left[T_{+1}\right]$ are analogously defined.

It then follows that $\Pr\left[Piv_A\right] = \frac{1}{2}\Pr\left[T\right] + \frac{1}{2}\Pr\left[T_{-1}\right]$, where the coefficient $\frac{1}{2}$ reflects the fact that, in the first case, the additional vote for A breaks a tie while, in the second, it leads to a tie. Likewise, $\Pr\left[Piv_B\right] = \frac{1}{2}\Pr\left[T\right] + \frac{1}{2}\Pr\left[T_{+1}\right]$.

Our next proposition shows that when others are more likely to choose B than A, a vote cast for the "underdog" is more likely to be pivotal, and vice-versa. Such underdog results appear in various forms in the literature. For instance, Ledyard (1984) showed it for the fixed population private values model. The result below is a simple generalization to our setting.

Proposition 1 *Under majority rule,* $\Pr\left[Piv_A\right] > \Pr\left[Piv_B\right]$ *if and only if* $q_A < q_B$.

[11] We follow the convention that if $m < k + l$, then $\binom{m}{k+l} = 0$ and so $\binom{m}{k,l} = 0$, as well.

Proof. Note that

$$\Pr[Piv_A] - \Pr[Piv_B] = \tfrac{1}{2}\left(\Pr[T_{-1}] - \Pr[T_{+1}]\right)$$

and since

$$\begin{aligned} q_A \Pr[T_{-1}] &= \sum_{n=0}^{\infty} \pi(n) \sum_{k=0}^{n} \binom{n}{k, k+1} (q_A)^{k+1} (q_B)^{k+1} (q_0)^{n-2k-1} \\ &= q_B \Pr[T_{+1}] \end{aligned}$$

$\Pr[T_{-1}] > \Pr[T_{+1}]$ if and only if $q_A < q_B$. ■

3 Equilibrium

We now study equilibrium voting and participation decisions. Voting behavior is very simple—A supporters vote for A and B supporters for B. For both, voting for their preferred candidate is a weakly dominant strategy. Thus, it only remains to consider the participation behavior of voters, where we study *type-symmetric* equilibria. In these equilibria, all voters of the same type and same realized cost follow the same strategy. Myerson (1998) showed that in voting games with population uncertainty, *all* equilibria are type-symmetric.[12] Thus, when we refer to equilibrium, we mean type-symmetric equilibrium.

An equilibrium consists of two functions $c_A(v)$ and $c_B(v)$ such that (i) an A supporter (resp. B supporter) with cost c votes if and only if $c < c_A$ (resp. $c < c_B$); (ii) the participation rates $p_A(v) = F(c_A(v))$ and $p_B(v) = F(c_B(v))$ are such that the resulting pivotal probabilities make an A supporter (resp. B supporter) with value v and costs $c_A(v)$ (resp. $c_B(v)$) indifferent between voting and abstaining. An equilibrium is thus defined by the equations:

$$\begin{aligned} c_A(v) &= v \Pr[Piv_A] \\ c_B(v) &= v \Pr[Piv_B] \end{aligned}$$

[12] For the degenerate case where the number of eligible voters is fixed and commonly known, type asymmetric equilibria may arise; however, such equilibria are not robust to the introduction of even a small degree of uncertainty about the number of eligible voters.

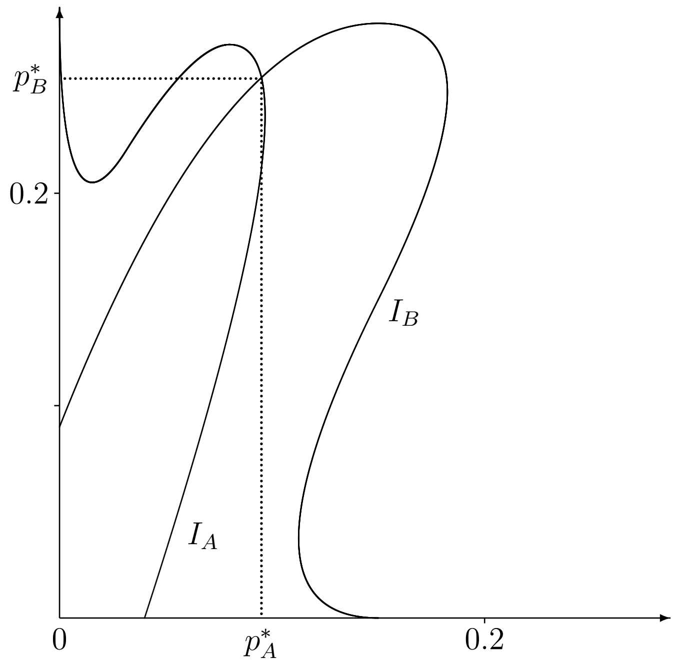

Figure 1: Equilibrium in Example 1

which must hold for all $v \in [0,1]$. Equilibrium may be equivalently expressed in terms of participation rates

$$\begin{aligned} p_A(v) &= F(v \Pr[Piv_A]) \\ p_B(v) &= F(v \Pr[Piv_B]) \end{aligned}$$

To obtain the *ex ante* probability of an A vote, $q_A = \lambda p_A$, integrate the function $p_A(v)$ over $[0,1]$ to obtain p_A, the ex ante probability that a given A-voter will vote for A, and multiply this by the fraction of A supporters λ. An analogous procedure produces the *ex ante* probability of an A vote, $q_B = (1-\lambda) p_B$. In terms of voting propensities, the equilibrium conditions are

$$q_A = \lambda \int_0^1 F(v \Pr[Piv_A]) \, dG_A(v) \tag{5}$$

$$q_B = (1-\lambda) \int_0^1 F(v \Pr[Piv_B]) \, dG_B(v) \tag{6}$$

It is now straightforward to establish:

Proposition 2 *With costly voting, there exists an equilibrium. In every equilibrium, all types of voters participate with a probability strictly between zero and one.*

Proof. Since both $\Pr[Piv_A]$ and $\Pr[Piv_B]$ are continuous functions of q_A and q_B, Brouwer's Theorem ensures that there is a solution $(q_A, q_B) \in [0,1]^2$ to (5) and (6) with associated participation rates, p_A and p_B. Neither p_A nor p_B can equal 1. If $p_A = 1$, then for all v, $p_A(v) = 1$ and hence for all v, $c_A(v) = 1$ as well. But $c_A(v) \leq v < 1$ almost everywhere, so this is impossible. Second, neither p_A nor p_B can equal 0. If $p_A = 0$, then, for an A supporter, there is a strictly positive probability, of at least λ^n with n other voters, that no one else shows up. Thus, $\Pr[Piv_A] > 0$, which implies that, for all v, $c_A(v) > 0$ and, in turn, $p_A(v) > 0$ as well. ■

Example 1 *Suppose that the population is distributed according to a Poisson distribution with mean $m = 100$. Suppose also that $\lambda = \frac{2}{3}$, $v_A = \frac{1}{3}$, $v_B = 1$ and that voting costs are distributed according to $F(c) = 3c$ over $\left[0, \frac{1}{3}\right]$. Figure 1 depicts the equilibrium conditions (5) and (6).*

Notice that in the example, for a given p_B there may be multiple values of p_A that solve (5). This is because for fixed p_B, $\Pr[Piv_A]$ is a non-monotonic function of p_A while $F^{-1}(p_A)$ is monotone. Despite the fact that both curves "bend backwards," there is a unique equilibrium.

Uniform Costs

We now study relative participation rates temporarily assuming that voting costs are uniformly distributed. The advantage of this specification is that equilibrium cost thresholds and participation rates are identical. In this case, $F(c) = c$, and so the equilibrium conditions (5) and (6) can be rewritten as

$$\begin{aligned} q_A &= \lambda \Pr[Piv_A] \int_0^1 v dG_A(v) = \lambda v_A \Pr[Piv_A] \\ q_B &= (1-\lambda) \Pr[Piv_B] \int_0^1 v dG_B(v) = (1-\lambda) v_B \Pr[Piv_B] \end{aligned}$$

where v_A is the expected welfare of an A supporter from electing A rather than B and v_B is the expected welfare of a B supporter from electing B rather than A.

Rewriting these expressions as a ratio, we have

$$\frac{mq_A}{mq_B} = \frac{\lambda v_A}{(1-\lambda)\, v_B} \times \frac{\Pr[Piv_A]}{\Pr[Piv_B]} \tag{7}$$

The left-hand side of (7) is simply the ratio of the expected number of A versus B votes. The first term on the right-hand side is the ratio of the welfare from choosing A versus B and is greater (resp. less) than 1 when A (resp. B) is the utilitarian choice. The second term on the right-hand side is the ratio of pivot probabilities, which is linked to the left-hand side via Proposition 1.

This linkage, we claim, implies that the utilitarian choice enjoys a higher expected vote share than the non-utilitarian choice. To see this, suppose that A is the utilitarian choice, but $q_A < q_B$. Proposition 1 implies that a vote for A is more likely to be pivotal than a vote for B and hence $\Pr[Piv_A] / \Pr[Piv_B] > 1$. In that case, both expressions on the right-hand side of (7) exceed 1 while the left-hand side is fractional, a clear contradiction. A similar argument establishes the result when candidate B is the utilitarian choice. Thus, we have:

Proposition 3 *Suppose voting costs are uniformly distributed. In any equilibrium, the expected number of votes for A exceeds the expected number of votes for B if and only if A is the utilitarian choice. Precisely,* $q_A > q_B$ *if and only if* $\lambda v_A > (1-\lambda)\, v_B$.

The following example illustrates Proposition 3.

Example 2 *Suppose that the population follows a Poisson distribution with mean* $m = 1000$ *and that voting costs are uniform. Figure 2 depicts the equilibrium ratio of the expected number of votes for A versus B,* q_A/q_B*, as a function of the welfare ratio,* $\lambda v_A / (1-\lambda)\, v_B$.

As Example 2 illustrates, the utilitarian candidate always receives the higher vote share, even for finite sized electorated. Moreover, elections are not necessarily close—when the welfare ratio strongly favors one side or the other, equilibrium voting produces a landslide.

Proposition 3 applies to *all* equilibria and to all electorate sizes. While it shows that the utilitarian choice is more likely to be elected than the alternative, it does

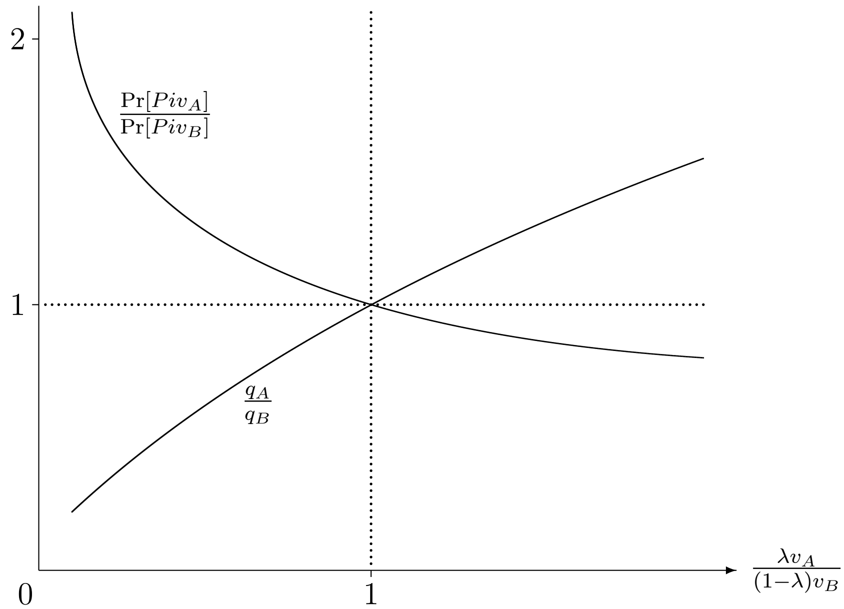

Figure 2: Ratios of Votes and Pivot Probabilities

not say that this happens for sure, i.e., with probability one, nor does it say what happens when voting costs are not uniform. We turn to these issues in the next two sections.

4 Large Elections

We now examine turnout in large elections, and show that the expected numbers of voters on both sides of the election become unbounded and are of the same magnitude. Our main result shows that the vote share advantage of the utilitarian candidate, highlighted in Proposition 3 when voting costs are uniform, extends to all voting cost distributions. Before proceeding, we define what we mean by *large* elections.

Definition 1 *The sequence of distributions π_m^* is* asymptotically large *if for all M,*

$$\lim_{m \to \infty} \sum_{n=M}^{\infty} \pi_m^* (n) = 1 \tag{8}$$

Condition (8) requires that, for large m, the distribution π_m^* places almost all the weight on large populations. In what follows, we consider a sequence of such distributions, and when we speak of a "large election," we mean that m is large. The fixed population specification satisfies (8), of course, as does the Poisson population specification: $\pi_m^*(n) = e^{-m}m^n/n!$. But other interesting specifications satisfy the condition as well. These include the sequence of negative binomial distributions with mean m or the sequence of hypergeometric distributions, again with mean m.

It is easily verified that if the sequence of objective distributions, π_m^*, is asymptotically large, then so is the sequence of subjective distributions, π_m^{**}. The same is true of the sequence π_m of subjective distributions of "other" eligible voters.

Consider a sequence of equilibria, one for each m. Let $p_A(m)$ and $p_B(m)$ be the sequence of equilibrium participation rates of A and B supporters, respectively. Proposition 4 below says that in large elections, both participation rates tend to zero, but at a rate slower than $1/m$. As a result, the *expected* number of voters of each type is unbounded. Establishing that total turnout (the expected number of voters) cannot be finite is straightforward. Were this the case, voters would have strictly positive benefits from voting and hence participate at strictly positive rates, leading to unbounded turnout. Moreover, turnouts for the two candidates cannot become too unbalanced since the side with the lower turnout would enjoy disproportionately higher benefits from voting, a contradiction. Thus, the expected number of A and B voters must be infinite and of the same magnitude.

Proposition 4 *In any sequence of equilibria, the participation rates $p_A(m)$ and $p_B(m)$ tend to zero, and at the same rate, while the expected number of voters $mp_A(m)$ and $mp_B(m)$ tend to infinity.*

Proof. See Lemmas A.8 and A.9. ■

Information about limiting participation rates and turnouts permits us to extend Proposition 3 to arbitrary cost distributions. Since cost thresholds go to zero in the limit only local properties of the cost distribution in this neighborhood matter. The key property, approximate linearity of the cdf, is shared by all cost distributions with positive densities, so voting behavior mirrors the uniform case. Formally,

Proposition 5 *Suppose voting costs are distributed according to a continuous distribution F satisfying $F(0) = 0$ and $F'(0) > 0$. In any equilibrium of a large election, the expected number of votes for A exceeds the expected number of votes for B if and only if A is the utilitarian choice. Precisely, $q_A > q_B$ if and only if $\lambda v_A > (1-\lambda) v_B$.*

Proof. For any cost distribution F satisfying $F'(0) > 0$, let $q_A(m) = \lambda p_A(m)$ and $q_B(m) = (1-\lambda) p_B(m)$ be a sequence of equilibrium voting propensities. Proposition 4 implies that p_A and p_B go to zero as m increases. The pivotal probabilities go to zero as well, which implies that for all v, the cost thresholds $c_A(v)$ and $c_B(v)$ also go to zero. Thus, for large m, the equilibrium conditions (5) and (6) imply[13]

$$
\begin{aligned}
q_A &\approx \lambda \int_0^1 F'(0)\, v \Pr[Piv_A]\, dG_A(v) = F'(0)\, \lambda v_A \Pr[Piv_A] \\
q_B &\approx (1-\lambda) \int_0^1 F'(0)\, v \Pr[Piv_B]\, dG_B(v) = F'(0)\,(1-\lambda)\, v_B \Pr[Piv_B]
\end{aligned}
$$

In ratio form, we have

$$\frac{q_A}{q_B} \approx \frac{\lambda v_A}{(1-\lambda) v_B} \times \frac{\Pr[Piv_A]}{\Pr[Piv_B]}$$

which is asymptotically identical to the case of uniform costs and we know from Proposition 3 that $\lambda v_A > (1-\lambda) v_B$ implies $q_A > q_B$. Thus, in large elections we have that $\lambda v_A > (1-\lambda) v_B$ implies $q_A > q_B$. ■

5 Welfare

Proposition 5 shows that in large elections, the *expected* vote totals always favor the utilitarian choice, but this alone does not guarantee that the utilitarian candidate *always* wins in a large elections (i.e., with probability approaching one). Since the realized vote total is random, dispersion in the size of the electorate may create circumstances where the wrong candidate wins with strictly positive probability. Precisely, we wish to explore the following:

[13] We write $x_n \approx y_n$ to denote that $\lim_{n\to\infty} (x_n/y_n) = 1$.

Conjecture 1 *In large elections with costly voting, majority rule produces utilitarian outcomes with probability one.*

The electorate size models used in the extant literature prove the conjecture by relying on the law of large numbers: In the fixed voter model, the conjecture holds since the actual vote total is arbitrarily close to the expected vote total. Myerson (2000) argues that the same is true in the Poisson model since population dispersion vanishes in the limit, so all voting outcomes approach the mean outcome. This seems to suggest that the absence of limiting population dispersion is important, and perhaps necessary, for the conjecture to hold.

We will show, however, that majority rule is utilitarian even with substantial population dispersion. The key is that turnout endogenously accounts for the distribution of electorate sizes. To see this starkly, suppose that we examine welfare from a single voter's perspective, using the *subjective* probability distribution of electorate sizes. Then we have:

Proposition 6 *In large elections, the subjective probability that the utilitarian candidate is elected approaches one.*

Proposition 6 shows that, from an individual voter's perspective, the conjecture holds generally. Roughly the reason is the following. Suppose A is the utilitarian choice. Then the probability that B wins the election is of the same order of magnitude as the probability that a vote for B is pivotal. The properties of equilibrium guarantee that this pivot probability goes to zero and thus so the does the probability of electing the wrong candidate.

But does the *objective* probability of the utilitarian choice—that is, from an outside observer's perspective—also approach one? Formally, suppose $\lambda p_A > (1 - \lambda) p_B$. Then does the fact that

$$\sum_{n=0}^{\infty} \pi_m^{**}(n) \Pr[A \text{ wins} \mid n] \to 1$$

as $m \to \infty$ imply that

$$\sum_{n=0}^{\infty} \pi_m^{*}(n) \Pr[A \text{ wins} \mid n] \to 1$$

as well? The following example sharply illustrates that the answer is no.

Example 3 *Suppose that $v_A = v_B = 1$ for all voters and costs are uniform. Let $\lambda = 2/3$, so A is the utilitarian choice. The electorate may be "small," $n = \sqrt{m}$, or "big," $n = 2m$, with probability distribution such that the mean is m. The objective probability of a big electorate becomes one-half as m gets large, but the subjective probability of a big electorate goes to one. Turnout adjusts to produce the utilitarian outcome when the electorate is big, but not when it is small, where the limiting turnout is zero, producing a tied election.*

The example illustrates the possibility that π_m^* and π_m^{**} might diverge in the limit, thereby wrecking the utilitarian property of majority voting. Thus, the "subjective" result of Proposition 6 does not imply its "objective" counterpart—as stated, the conjecture above is false in general.

For Proposition 6 to extend to objective probabilities, these two probability distributions must be sufficiently close. Uniform convergence would seem the natural strengthening, but even this turns out to be insufficient. To see why, let us return to Example 3, but with an electorate characterized by a family of *logarithmic* distributions.[14] Here, the subjective and objective distributions converge uniformly, but "slowly." Under the objective distribution, π_m^*, the chance that the electorate is $\sqrt{m}$ or less is approximately $\frac{1}{2}$ when m is large. The slow convergence implies that this same chance is zero under the subjective distribution. As in Example 3, the resulting participation is so low that the election ends in a tie (almost surely in the limit) when the electorate size is small (i.e., $\sqrt{m}$ or less).

The example demonstrates that convergence of π_m^* and π_m^{**} alone is not enough, even when this convergence is uniform. Rather, the speed of convergence proves a critical consideration. A sufficient condition is:

Condition 1 *The sequence of population distributions π_m^* satisfies the* concordance

[14]Specifically, $\pi_m^*(n) = \left(1 - \frac{1}{\phi(m)}\right)^n \div n \ln \phi(m)$ where $t = \phi(m)$ solves $m = (t-1)/\ln t$.

condition[15]*:*

$$\|\pi_m^* - \pi_m^{**}\|_\infty = O\left(1/m\right)$$

This condition requires the objective and subjective probability distributions to approach each other at a rate faster than $1/m$. As a practical matter, it is not a very restrictive condition. The two common population distributions used in voting theory, the fixed and Poisson population models, satisfy it. With a fixed population $\pi_m^* = \pi_m^{**}$ for all m, and, with a Poisson population, the subjective and objective distributions approach each other at rate $1/m^2$. Other major families of discrete distributions, including the negative binomial family[16] and the hypergeometric family,[17] also satisfy the condition. Moreover, these distributions exhibit considerable limiting population dispersion. Many *ad hoc* distributions—for instance, a uniform distribution on the integers between $m/2$ and $3m/2$—satisfy the condition as well. At last, we can formalize the conjecture as:

Theorem 1 *Suppose that the concordance condition holds. In large elections with costly voting, majority rule produces utilitarian outcomes with probability one.*

While the theorem places some restrictions on population distributions, it makes no other demands. Even if directional preferences lop-sidedly favor the non-utilitarian choice, the logic of equilibrium turnout produces, in a large election, the correct result. Given the ordinal nature of majority rule, this is quite remarkable. The key is voluntary participation—voters vote with their "feet" as well as with their ballots, thereby registering, not just the direction, but the intensity of their preferences as well—that produces the utilitarian outcome.

To summarize, utilitarianism under majority rule requires the following key ingredients: Elections must be large. The lower support of the voting cost distribution

[15] $\|\cdot\|_\infty$ denotes the *sup norm* and so the "big O" condition says that there exists a $K > 0$ such that for all m and n, $|\pi_m^*(n) - \pi_m^{**}(n)| \leq K/m$.

[16] Specifically, for any $r \geq 1$, $\pi_m^*(n) = \frac{\Gamma(n+r)}{\Gamma(n+1)\Gamma(r)}\left(\frac{r}{m+r}\right)^r\left(\frac{m}{m+r}\right)^n$. This family has a limiting coefficient of variation of $1/\sqrt{r}$.

[17] For example, if $\pi_m^*(n) = \binom{2m}{n}\binom{2m}{2m-n} \div \binom{4m}{2m}$.

must be zero. For cost distributions bounded (strictly) above zero, analysis of large elections is analogous to having fixed voting costs, which offer insufficient flexibility in the "prices" of votes to produce the utilitarian outcome. When a positive mass of voters have negative voting costs, then, since these individuals will always come to the polls, the analysis is analogous to compulsory voting with its attendant problems. While voting costs may differ by candidate preference, the density of costs for A and B voters must be identical in the neighborhood of zero. If not, then majority rule would maximize a *weighted* utilitarian welfare function, where these densities determine these weights. Finally, the distribution of electorate sizes must satisfy concordance.

6 Supermajority Rules

While majority rule is the most common voting rule, many situations use supermajority rules. California and Arizona require legislative supermajorities for any tax increase.[18]. Florida and Illinois require a supermajority to pass constitutional amendments. We have shown that majority rule is utilitarian in large elections, but since turnout adjusts based on the voting rule itself, perhaps supermajority rules are utilitarian as well. Such a "rule irrelevance" result occurs in Condorcet models, where Feddersen and Pesendorfer (1998) showed that all supermajority voting rules (save for unanimity) aggregate information in large elections. Rule irrelevance does not hold in our model—only simple majority rule is utilitarian

To study supermajority rules, let Candidate B be the default choice while A needs a fraction $\phi \geq \frac{1}{2}$ of the votes cast in order to be elected. We will assume that ϕ is a rational number and so will write $\phi = a/(a+b)$, where a and b are positive integers which are relatively prime (have no common factors) and such that $a \geq b$. In the event of a tie—a situation in which A obtains exactly n proportion of the votes—A

[18]The required legislative supermajorities differ across states. Arizona and California, among others, require a 2/3 majority. Arkansas and Oklahoma require a 3/4 majority for certain types of tax increases while Florida and Oregon require a 3/5 majority.

is chosen with probability t and B with probability $1 - t$.[19]

In this section, the population of voters is Poisson with mean m. We show that, unless the voting rule is majoritarian ($a = b = 1$), the outcome of a large election will not coincide with the utilitarian choice. Since the result is negative, it suffices to establish the non-utilitarian property of strict supermajority rules for the Poisson case only. The key to our analysis is Proposition 7 which extends the underdog result (Proposition 1) to supermajority rules when the electorate is Poisson. This proposition shows that in large elections, if A is on the losing side, i.e., the ratio of voting propensities, q_A/q_B falls short of the required a/b, then the pivot ratio $\Pr[Piv_A] / \Pr[Piv_B]$ exceeds b/a. Formally,

Proposition 7 *If for all m large, $\frac{q_A(m)}{q_B(m)} \geq \frac{a}{b}$, then $\limsup \frac{\mathcal{P}[Piv_A]}{\mathcal{P}[Piv_B]} \leq \frac{b}{a}$. Similarly, if for all m large, $\frac{q_A(m)}{q_B(m)} \leq \frac{a}{b}$, then $\liminf \frac{\mathcal{P}[Piv_A]}{\mathcal{P}[Piv_B]} \geq \frac{b}{a}$.*

Proof. See Appendix C. ■

The workings of Proposition 7 are easily seen under a 2/3 supermajority rule. If the vote ratio is less than the required $2:1$ for A, then a vote for B is *twice* as likely to be pivotal as a vote for A. This multiple derives from an asymmetry in pivotal events under strict supermajority rules. Votes for A and B are both pivotal when the election is tied or where one additional vote will lead to a tie. But a vote for B can also "flip" the election and swing the outcome from a sure loss to a sure win. This occurs when the vote count is of the form $(2k-1, k-1)$. The chance of such events is approximately equal to the chance of a tie (or near tie); however, the flip events receive twice the weight since they do not lead to or break a tie. As a consequence, $\Pr[Piv_B]$ is approximately twice as large as $\Pr[Piv_A]$.

The asymmetry highlighted in Proposition 7 creates a wedge between election outcomes and utilitarianism, as the main result of this section shows.

Theorem 2 *Among all supermajority rules only majority rule is utilitarian. Specifically, in a $\frac{a}{a+b}$ supermajority election with a large Poisson population, if*

$$\lambda v_A > \left(\frac{a}{b}\right)^2 (1-\lambda) v_B$$

[19] Majority rule is a nested case where $a = b = 1$ and $t = 1/2$.

then A wins with probability one. If the reverse inequality holds strictly, then B wins with probability one.

Proof. Suppose that $\lambda v_A > \left(\frac{a}{b}\right)^2 (1-\lambda) v_B$. We first claim that for all large m, $\frac{q_A}{q_B} > \frac{a}{b}$; that is, the vote shares favor A.

Suppose to the contrary that there is a sequence of equilibria along which $\frac{q_A}{q_B} \leq \frac{a}{b}$ and so by Proposition 7, along this sequence $\frac{\mathcal{P}[Piv_A]}{\mathcal{P}[Piv_B]} \geq \frac{b}{a}$. If voting costs are uniform, the equilibrium conditions imply

$$\frac{q_A}{q_B} = \frac{\lambda v_A}{(1-\lambda) v_B} \frac{\mathcal{P}[Piv_A]}{\mathcal{P}[Piv_B]}$$

But since the left-hand side is less than or equal to a/b while the right-hand side is strictly greater than a/b, this is a contradiction.

The remainder of the proof, showing that when $\frac{q_A}{q_B} > \frac{a}{b}$ holds for all large m, it is the case that $\Pr[A \text{ wins}] \to 1$, is the same as in Theorem 1 and is omitted. ■

Theorem 2 shows that (strict) supermajority rules bias the election in favor of the default alternative, as intuition would suggest. Thus, one might conjecture that the outcome of a large supermajority election maximizes a *weighted* utilitarian welfare function proportional to the required vote share ratio, a/b. Theorem 2, however, says that supermajority rules exaggerate the welfare weight given to the default, in effect giving a weight $(a/b)^2$ to B relative to A. In the $2/3$ supermajority rule, even though A must obtain *twice* as many votes as B, outcomes correspond to a welfare function that places *four* times the weight on B compared to A. The "squaring property" arises from a combination of the asymmetry in the rule and the additional pivotal events this asymmetry creates. Since each of these effects has a factor a/b, their combination squares this term in implied the welfare weight. While Theorem 2 only delineates outcomes in large elections, the following example suggests that the asymptotic results are well-approximated even when the size of the electorate is relatively small.

Example 4 *Consider the $2/3$ majority rule. Suppose that the expected size of the population $m = 1000$ and that voting costs are uniform. Figure 3 depicts the equilibrium ratio of the expected number of votes for A versus B, q_A/q_B, as a function of*

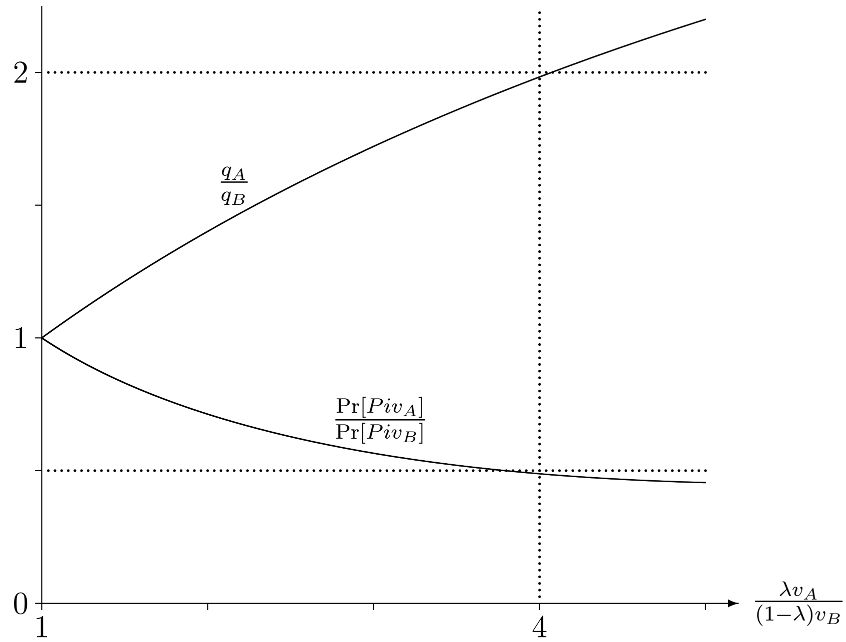

Figure 3: Ratios of Votes and Pivot Probabilities: 2/3 Rule

the welfare ratio, $\lambda v_A / (1-\lambda) v_B$. *Even with a small number of voters, it is (approximately) the case that* A *obtains the required* $2:1$ *vote ratio if and only if the welfare ratio is at least* $4:1$.

7 Conclusion

Majority rule is, perhaps, the most common means of group decision making. Whether it be mundane problems, like where a group should go to lunch, or deeply consequential decisions like whom to elect as president, the same rule is used. Its ubiquitousness stems from its simplicity and perceived fairness. However majority rule is perceived to suffer from a key defect. As a counting rule, it only reflects the direction and not the intensity of preferences.

When voting is voluntary, we show that preference intensity is encoded via endogenous participation rates, and these, in turn, influence election outcomes. Specifically, we showed that, under majority rule, the utilitarian choice always enjoys the higher vote share in large elections. This is true regardless of the distribution of values, costs, and numbers of eligible voters.

In standard models of voting, which assume that the number of eligible voters is either fixed or Poisson distributed, the expected vote share alone determines election outcomes and hence welfare in large elections. The reason is that these models assume there is no limiting dispersion in the number of voters and hence vote totals are arbitrarily close to their expected values. we showed that this assumption was consequential—with limiting dispersion, the link between expected vote share and welfare becomes more tenuous and may be broken entirely.

The key condition guaranteeing that this link is not severed is that the objective and subjective distributions of the number of voters must converge sufficiently quickly. The objective distribution reflects the view of an outside observer whereas the subjective view reflects that of an eligible voter, and these views differ since a given voter is more likely to eligible in a large electorate than a small one. While not all distributions satisfy the convergence condition, many commonly used distributions with limiting dispersion do, including the negative binomial and hypergeometric distributions. Thus, it is not limiting dispersion that is, *per se*, the problem but rather the impact of this dispersion on the views of insiders and outsiders to an election.

We also showed that when the convergence condition holds, majority rule is the *only* rule having the utilitarian property. All other supermajority rules distort participation so as to grossly overweight the issue or candidate favored by the rule.

The complexity of pivotal calculations and the microscopic nature of pivot probabilities in large elections are causes for worry in this regard. But since the performance of majority rule hinges on *relative* participation rates, the conclusions are robust to misperceptions of these probabilities so long as the magnitude of the errors is consistent across voters. For instance, if all voters overestimated pivotal probabilities by a factor of 10,000, for instance, our conclusions would be unaltered. There is also growing evidence that the rational voter paradigm is, in fact, descriptive of behavior. For instance, Levine and Palfrey (2007) conduct "elections," consisting of 3 to 51 individuals, with costly and voluntary voting in a controlled laboratory setting. Their main findings support the rational model—the underdog principle is strongly observed in the data and turnout adjusts to changes in the fraction of A supporters and the size of the electorate in the direction predicted by theory.

A Turnout

The purpose of this appendix is to provide a proof of Proposition 4. This is done via Lemmas A.8 and A.9 below.

When studying the asymptotic behavior of the pivotal probabilities, it is useful to rewrite these in a more convenient form.

A.1 Roots of Unity Formulae

For $n > 1$, let $\omega = \exp(2\pi i/n)$. Since $\omega^n = e^{2\pi i} = 1$, ω is an nth (complex) *root of unity*. Note that $\sum_{r=0}^{n-1} \omega^r = (1-\omega^n)/(1-\omega) = 0$.

Lemma A.1 *For x, y, z positive,*

$$\sum_{k=0}^{n} \binom{n}{k,k} x^k y^k z^{n-2k} = -x^n - y^n + \frac{1}{n}\sum_{r=0}^{n-1} \left(\omega^r x + \omega^{-r} y + z\right)^n$$

Proof. Using the trinomial formula, for $r < n$,[20]

$$\left(\omega^r x + \omega^{-r} y + z\right)^n = \sum_{k=0}^{n}\sum_{l=0}^{n} \binom{n}{k,l} \omega^{rk} x^k \omega^{-rl} y^l z^{n-k-l}$$

and so, averaging over $r = 0, 1, ..., n-1$,

$$\begin{aligned}
\frac{1}{n}\sum_{r=0}^{n-1} \left(\omega^r x + \omega^{-r} y + z\right)^n &= \frac{1}{n}\sum_{r=0}^{n-1}\sum_{k=0}^{n}\sum_{l=0}^{n} \binom{n}{k,l} \omega^{r(k-l)} x^k y^l z^{n-k-l} \\
&= \frac{1}{n}\sum_{k=0}^{n}\sum_{l=0}^{n} \binom{n}{k,l} \left(\sum_{r=0}^{n-1} \omega^{r(k-l)}\right) x^k y^l z^{n-k-l} \\
&= \frac{1}{n} x^n \left(\sum_{r=0}^{n-1} \omega^{rn}\right) + \frac{1}{n} y^n \left(\sum_{r=0}^{n-1} \omega^{-rn}\right) \\
&\quad + \frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} \binom{n}{k,l} \left(\sum_{r=0}^{n-1} \omega^{r(k-l)}\right) x^k y^l z^{n-k-l}
\end{aligned}$$

[20] Recall the convention that if $m < k + l$, then $\binom{m}{k,l} = 0$.

Now observe that since $\omega^n = 1$,

$$\sum_{r=0}^{n-1} \omega^{r(k-l)} = \begin{cases} n & \text{if } k = n \text{ or } l = n \\ n & \text{if } k = l \\ \frac{1-\omega^{n(k-l)}}{1-\omega^{(k-l)}} = 0 & \text{otherwise} \end{cases}$$

Thus,

$$\frac{1}{n}\sum_{r=0}^{n-1} \left(\omega^r x + \omega^{-r} y + z\right)^n = x^n + y^n + \sum_{k=0}^{n} \binom{n}{k,k} x^k y^k z^{n-2k}$$

■

Lemma A.2 *For x, y, z positive,*

$$\sum_{k=0}^{n} \binom{n}{k,k+1} x^k y^{k+1} z^{n-2k-1} = -nx^{n-1}z + \frac{1}{n}\sum_{r=0}^{n-1} \omega^r \left(\omega^r x + \omega^{-r} y + z\right)^n$$

Proof. The proof is almost the same as that of Lemma A.1 and is omitted. ■

The following lemma studies asymptotic properties of the pivotal probabilities when the propensities to vote and abstain remain fixed as m increases.

Lemma A.3 *For x, y, z positive, satisfying $x + y + z = 1$,*

$$\lim_n \frac{1}{n}\sum_{r=0}^{n-1} \left(\omega^r x + \omega^{-r} y + z\right)^n = 0$$

Proof. First, note that since $|\omega^r| = 1 = |\omega^{-r}|$,

$$\left|\omega^r x + \omega^{-r} y + z\right| \leq |\omega^r|\, x + \left|\omega^{-r}\right| y + z = 1$$

Fix a K. Then for all $n \geq K$

$$\begin{aligned} \frac{1}{n}\sum_{r=0}^{n-1} \left(\omega^r x + \omega^{-r} y + z\right)^n &\leq \frac{1}{n}\sum_{r=0}^{n-1} \left|\omega^r x + \omega^{-r} y + z\right|^n \\ &\leq \frac{1}{n}\sum_{r=0}^{n-1} \left|\omega^r x + \omega^{-r} y + z\right|^K \end{aligned}$$

and thus

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\left(\omega^r x+\omega^{-r}y+z\right)^n &\leq \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\left|\omega^r x+\omega^{-r}y+z\right|^K \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}\left|x\exp\left(2\pi i\tfrac{r}{n}\right)+y\exp\left(-2\pi i\tfrac{r}{n}\right)+z\right|^K \\
&= \int_0^1 \left|x\exp\left(2\pi it\right)+y\exp\left(-2\pi it\right)+z\right|^K dt \qquad (9)
\end{aligned}$$

using the definition of the Riemann integral. Since

$$\left|x\exp\left(2\pi it\right)+\exp\left(-2\pi it\right)y+z\right| \leq x\left|\exp\left(2\pi it\right)\right|+y\left|\exp\left(-2\pi it\right)\right|+z=1$$

with a strict inequality unless $t=0$ or $t=1$. To see this, first note that the inequality above is strict for $t=\frac{1}{2}$. For all $t\neq 0,\frac{1}{2},1$ observe that

$$\left|x\exp\left(2\pi it\right)+y\exp\left(-2\pi it\right)\right|=\sqrt{x^2+y^2+2xy\cos\left(4\pi t\right)}<x+y$$

Thus, for all $t\neq 0$ or 1, $\left|x\exp\left(2\pi it\right)+y\exp\left(-2\pi it\right)+z\right|<1$.

Since the inequality in (9) holds for all K and the integral on the right-hand side is decreasing in K and goes to zero as $K\to\infty$, the left-hand side must be zero. ■

Lemma A.4 *For x,y,z positive, satisfying $x+y+z=1$,*

$$\lim_{n}\frac{1}{n}\sum_{r=0}^{n-1}\omega^r\left(\omega^r x+\omega^{-r}y+z\right)^n=0$$

Proof. Since $|\omega^r|=1$,

$$\frac{1}{n}\sum_{r=0}^{n-1}\left|\omega^r\left(\omega^r x+\omega^{-r}y+z\right)^n\right|=\frac{1}{n}\sum_{r=0}^{n-1}\left|\left(\omega^r x+\omega^{-r}y+z\right)^n\right|$$

and the result now follows by applying the previous lemma. ■

Lemma A.5 *For all $q<\frac{1}{2}$, the function*

$$\phi\left(q\right)\equiv\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k,k}q^{2k}\left(1-2q\right)^{n-2k}+\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k,k+1}q^{2k+1}\left(1-2q\right)^{n-2k-1}$$

is decreasing in q.

Proof. Using the formulae in Lemmas A.1 and A.2 (set $x = y = q$ and $z = 1 - 2q$), routine calculations show that

$$\phi(q) = -\frac{1}{2}q^n - \frac{1}{2}nq^{n-1}(1-2q) + \frac{1}{2n}\sum_{r=0}^{n-1}\left(\cos\left(2\pi\frac{r}{n}\right)+1\right)\left(2q\cos\left(2\pi\frac{r}{n}\right)+1-2q\right)^n$$

and so

$$\phi'(q) = -\frac{1}{2}n(n-1)q^{n-2}(1-2q) + \sum_{r=0}^{n-1}\left(\cos^2\left(2\pi\frac{r}{n}\right)-1\right)\left(2q\cos\left(2\pi\frac{r}{n}\right)+1-2q\right)^{n-1}$$

Now if n is odd, $\phi'(q)$ is clearly negative. When n is even, the first term on the right-hand side of the expression is clearly negative, so it remains to consider the term containing the sum. Notice that the terms corresponding to $r = 0$ and $r = n/2$ are zero; thus, the sum can be rewritten as:

$$2\sum_{r=1}^{\frac{n}{2}-1}\left(\cos^2\left(2\pi\frac{r}{n}\right)-1\right)\left(2q\cos\left(2\pi\frac{r}{n}\right)+1-2q\right)^{n-1}$$

using the fact that for all $r < \frac{n}{2}$, $\cos\left(2\pi\frac{r}{n}\right) = \cos\left(2\pi\frac{n-r}{n}\right)$. It may be easily verified that for all $q < \frac{1}{2}$, for $r < \frac{n}{4}$, the sum of the rth term and the $\left(\frac{n}{2} - r\right)$th term is negative. If n is not a multiple of 4, this accounts for all of the terms; however, if n is a multiple of 4, the remaining term, $r = n/4$, is clearly negative. Thus the sum is negative and so is $\phi'(q)$. ■

A.2 Asymptotic Participation Rates

We begin with a lemma that shows that both aggregate participation rates cannot remain positive in the limit.

Lemma A.6 *Along any sequence of equilibria,* $\lim p_A = 0$ *or* $\lim p_B = 0$ *(or both).*

Proof. Suppose to the contrary that neither is zero. Then there exists a subsequence such that $\lim p_A(m) = p_A^* > 0$ and $\lim p_B(m) = p_B^* > 0$. Define $q_A^* = \lambda p_A^*$ and $q_B^* = (1-\lambda)p_B^*$. Choose $\delta > 0$ such that the closed ball B^δ of radius δ around (q_A^*, q_B^*) lies in $\mathbb{R}_{++}^2$ and every element $(q_A, q_B) \in B^\delta$ satisfies $q_A + q_B < 1$. Let m_1 be such that for all $m > m_1$, $(q_A(m), q_B(m)) = (\lambda p_A(m), (1-\lambda)p_B(m)) \in B^\delta$.

Now Lemma 1 (ii) in Taylor and Yildirim (2010) shows that

$$\Pr\left[Piv_A \mid n, q_A, q_B\right] \leq \Pr\left[Piv_A \mid n, q_A, q_A\right]$$

Lemma A.5 shows that the right-hand side is decreasing in q_A, and so if $\underline{q} = \min_{B^\delta}\{q_A\}$, we have that for all $m > m_1$,

$$\Pr\left[Piv_A \mid n, q_A(m), q_B(m)\right] \leq \Pr\left[Piv_A \mid n, \underline{q}, \underline{q}\right]$$

Lemmas A.3 and A.4 imply that $\lim_{n\to\infty} \Pr\left[Piv_A \mid n, \underline{q}, \underline{q}\right] = 0$. Thus, for all ε, there exists an n_0 such that for all $n > n_0$ and for all $m > m_1$,

$$\Pr\left[Piv_A \mid n, q_A(m), q_B(m)\right] < \varepsilon$$

As a result,

$$\begin{aligned}
\Pr\left[Piv_A \mid q_A(m), q_B(m)\right] &= \sum_{n\leq n_0} \pi_m(n) \Pr\left[Piv_A \mid n, q_A(m), q_B(m)\right] \\
&\quad + \sum_{n>n_0} \pi_m(n) \Pr\left[Piv_A^m \mid n, q_A(m), q_B(m)\right] \\
&< \sum_{n\leq n_0} \pi_m(n) + \varepsilon \sum_{n>n_0} \pi_m(n)
\end{aligned}$$

But since $\lim_{m\to\infty} \sum_{n\leq n_0} \pi_m(n) = 0$, there exists an m_2 such that for all $m > m_2$, $\sum_{n\leq n_0} \pi_m(n) < \varepsilon$. Thus, we have shown that for all ε, there exists an $m_0 = \max\{m_1, m_2\}$, such that for all $m > m_{0,}$

$$\Pr\left[Piv_A \mid q_A(m), q_B(m)\right] < 2\varepsilon$$

A similar argument shows that $\lim \Pr\left[Piv_B\right] = 0$ as well. But the equilibrium conditions (5) and (6) now imply that along the subsequence, $\lim q_A(m) = 0$ and $\lim q_B(m) = 0$, contradicting the initial supposition. ■

Next we show that in the limit, the participation rates are of the same magnitude.

Lemma A.7 *Along any sequence of equilibria,* $0 < \liminf \frac{p_A}{p_B} \leq \limsup \frac{p_A}{p_B} < \infty$.

Proof. Suppose that for some subsequence, $\lim \frac{q_A}{q_B} = 0$. This implies that for all m large enough, along the subsequence, $q_A = \lambda p_A < (1-\lambda) p_B = q_B$ and so from Lemma 1, $\Pr[Piv_A] > \Pr[Piv_B]$. The equilibrium conditions: for all v,

$$c_A(v) = v \Pr[Piv_A] \text{ and } c_B(v) = v \Pr[Piv_B]$$

imply that when m is large enough, for all v, $c_A(v) > c_B(v)$, and hence, for all v, $p_A(v) > p_B(v)$ as well.

The fact that $\lim \frac{p_A}{p_B} = 0$ implies that $\lim p_A = 0$ and since $p_A = \int_0^1 p_A(v)\, dG_A(v)$, for almost all values of v, $\lim p_A(v) = 0$. Since $p_A(v)$ is continuous in v, we have that for all v, $\lim p_A(v) = 0$. Now because $p_A(v) > p_B(v)$, it is the case that $\lim p_B(v) = 0$ as well. This in turn implies that $\lim c_A(v) = 0 = \lim c_B(v)$.

Thus, along the subsequence, when m is large enough,

$$p_A = \int_0^1 F(c_A(v))\, dG_A(v) \approx \int_0^1 F'(0)\, v \Pr[Piv_A]\, dG_A(v) = F'(0) \Pr[Piv_A]\, v_A$$

Similarly, $p_B \approx F'(0) \Pr[Piv_B]\, v_B$. Thus, for all large m,

$$\frac{p_A(m)}{p_B(m)} \approx \frac{\Pr[Piv_A]}{\Pr[Piv_B]} \frac{v_A}{v_B} > \frac{v_A}{v_B}$$

since $\Pr[Piv_A] > \Pr[Piv_B]$. Since the right-hand side of the inequality above is independent of m, this contradicts the assumption that $\lim \frac{p_A}{p_B} = 0$. ■

Lemma A.8 *In any sequence of equilibria, the participation rates $p_A(m)$ and $p_B(m)$ tend to zero, and at the same rate.*

Proof. Lemmas A.6 and A.7 together complete the proof of Lemma A.8. ■

Lemma A.9 *In any sequence of equilibria, the expected number of voters $mp_A(m)$ and $mp_B(m)$ tend to infinity.*

Proof. [21]Suppose to the contrary that there is a sequence of equilibria in which, say, $\lim mp_A < \infty$. Lemma A.7 then implies that $\lim mp_B < \infty$ as well.

[21]We thank Ramazan Bora for suggesting this proof.

First, recall that for all q_A, q_B

$$\Pr[T \mid q_A, q_B] = \sum_{n=0}^{\infty} \pi_m(n) \Pr[T \mid n, q_A, q_B]$$

Also, Roos (1999) has shown that for all n, q_A, q_B

$$|\Pr[T \mid n, q_A, q_B] - \mathcal{P}[T \mid n, q_A, q_B]| \leq q_A + q_B$$

where $\mathcal{P}[Piv_A \mid n, q_A, q_B]$ is the probability of Piv_A calculated according to a Poisson multinomial distribution with an expected population size of n (see Appendix D). Combining these, we can write

$$\left| \sum_{n=0}^{\infty} \pi_m(n) \Pr[T \mid n, q_A, q_B] - \sum_{n=0}^{\infty} \pi_m(n) \mathcal{P}[T \mid n, q_A, q_B] \right| \leq q_A + q_B \tag{10}$$

Second, we claim that $\lim_{m\to\infty} \inf \sum_{n=0}^{\infty} \pi_m(n) \mathcal{P}[T \mid n, q_A(m), q_B(m)] > 0$. To see this, notice first that for all n,

$$\mathcal{P}[T \mid n, q_A, q_B] > \mathcal{P}[(0,0) \mid n, q_A, q_B] = e^{-n(q_A+q_B)}$$

Define $E_\pi[N] = m'$ and note that since $m' \geq m - 1$, as $m \to \infty$, $m' \to \infty$ as well. Next,

$$\begin{aligned} \sum_{n=0}^{\infty} \pi_m(n) \mathcal{P}[T \mid n, q_A, q_B] &\geq \sum_{n \leq 2m'} \pi_m(n) \mathcal{P}[T \mid n, q_A, q_B] \\ &\geq \sum_{n \leq 2m'} \pi_m(n) e^{-n(q_A+q_B)} \\ &\geq \sum_{n \leq 2m'} \pi_m(n) e^{-2m(q_A+q_B)} \end{aligned}$$

And since $E_\pi[N] = m'$, it is the case that[22] $\sum_{n \leq 2m'} \pi_m(n) > \frac{1}{2}$ and so for all m,

$$\sum_{n=0}^{\infty} \pi_m(n) \mathcal{P}[T \mid n, q_A, q_B] \geq \frac{1}{2} e^{-2m'(q_A+q_B)}$$

But since $\lim m'(q_A + q_B) = \lim m'(\lambda p_A + (1-\lambda) p_B) \equiv Q^* < \infty$ (say),

$$\lim_{m\to\infty} \inf \sum_{n=0}^{\infty} \pi_m(n) \mathcal{P}[T \mid n, q_A, q_B] \geq \frac{1}{2} e^{2Q^*}$$

and since q_A and q_B both go to zero, $\lim \Pr[Piv_A] > 0$ as well. The equilibrium conditions now imply that $\lim p_A > 0$, contradicting Lemma A.8. ■

[22]Every distribution function F with non-negative support and mean $\mu > 0$ satisfies $F(2\mu) > \frac{1}{2}$.

B Welfare

Lemma B.1 *Suppose* $\lambda v_A > (1-\lambda) v_B$. *Then* $\lim_{m\to\infty} \sup \frac{q_B}{q_A} < 1$.

Proof. We have already shown that if $\lambda v_A > (1-\lambda) v_B$, then when n is large $q_A > q_B$. Suppose, to the contrary, that there is a subsequence $(q_A(m), q_B(m))$ such that

$$\lim_{m\to\infty} \frac{q_B}{q_A} = 1$$

The equilibrium conditions imply that

$$\frac{q_B}{q_A} \times \frac{\Pr[Piv_A]}{\Pr[Piv_B]} \approx \frac{(1-\lambda) v_B}{\lambda v_A} \tag{11}$$

Since $\lim \frac{q_B}{q_A} = 1$, for all $\varepsilon > 0$, there exists an n_0 such that for all $n > n_0$, $\frac{q_B}{q_A} > 1 - \varepsilon$. Now observe that for all $n > n_0$, it is also the case that

$$\begin{aligned}
\frac{\Pr[Piv_A]}{\Pr[Piv_B]} &= \frac{\Pr[T] + \Pr[T_{-1}]}{\Pr[T] + \Pr[T_{+1}]} \\
&= \frac{\Pr[T] + \frac{q_B}{q_A}\Pr[T_{+1}]}{\Pr[T] + \Pr[T_{+1}]} \\
&> \frac{\Pr[T] + (1-\varepsilon)\Pr[T_{+1}]}{\Pr[T] + \Pr[T_{+1}]} \\
&> 1 - \varepsilon
\end{aligned}$$

and hence for all $n > n_0$,

$$\frac{q_B}{q_A} \times \frac{\Pr[Piv_A]}{\Pr[Piv_B]} > (1-\varepsilon)^2$$

But this is impossible since the limit of left-hand side of (11) is greater than 1 while the right-hand side is strictly less than 1. This completes the proof. ■

Proof of Proposition 6. Suppose that $\lambda v_A > (1-\lambda) v_B$ so that A is the utilitarian choice (the case when B is the utilitarian choice is analogous). Proposition 5 implies that in any sequence of equilibria, for all large m, $q_A > q_B$. We now show that as m increases without bound, from any single voter's perspective, the subjective probability that A is elected approaches 1, or equivalently, the probability that B

is elected approaches 0. We argue by contradiction. So suppose that there is a subsequence $(q_A(m), q_B(m))$ such that along this subsequence

$$\lim_{m\to\infty}\sum_{n=1}^{\infty}\pi_m^{**}(n)\Pr[B\text{ wins}\mid n]>0.$$

Let T_{-k} be the event that A loses by k votes. Then, the probability that B wins when the realized electorate is of size n,

$$\Pr[B\text{ wins}\mid n]=\tfrac{1}{2}\Pr[T\mid n]+\sum_{k=1}^{n}\Pr[T_{-k}\mid n]<\sum_{k=0}^{n}\Pr[T_{-k}\mid n]$$

In order to estimate the subjective probability that B wins, we use a result of Roos (1999) that the probability $\Pr[S\mid n]$ of *any* event $S\subset\mathbb{Z}_+^2$ in the multinomial model with population n is well-approximated by the corresponding probability $\mathcal{P}[S\mid n]$ in the Poisson model with a *mean* population of n (see Appendix D). In particular, given the voting propensities q_A and q_B,

$$|\Pr[B\text{ wins}\mid n]-\mathcal{P}[B\text{ wins}\mid n]|\le q_A+q_B$$

and observe that the bound on the right-hand side does not depend on n. Thus,

$$\left|\sum_{n=1}^{\infty}\pi_m^{**}(n)\Pr[B\text{ wins}\mid n]-\sum_{n=1}^{\infty}\pi_m^{**}(n)\mathcal{P}[B\text{ wins}\mid n]\right|\le q_A+q_B \tag{12}$$

We have assumed that the limit of the first term is positive, and since $q_A+q_B\to 0$, the limit of the second term is positive as well.

To compute the second term, observe that the probability that A loses by k votes using the Poisson distribution with mean population n is

$$\begin{aligned}\mathcal{P}[T_{-k}\mid n] &= e^{-n(q_A+q_B)}\sum_{j=0}^{\infty}\frac{(nq_A)^j}{j!}\frac{(nq_B)^{j+k}}{(j+k)!}\\ &= e^{-n(q_A+q_B)}I_k\left(2n\sqrt{q_Aq_B}\right)\left(\sqrt{\tfrac{q_B}{q_A}}\right)^k\\ &< e^{-n(q_A+q_B)}I_0\left(2n\sqrt{q_Aq_B}\right)\left(\sqrt{\tfrac{q_B}{q_A}}\right)^k\end{aligned}$$

where I_k is the kth order modified Bessel function of the first kind.[23] The last inequality follows from the fact that when $z>0$, then for all $k>0$, $I_k(z)<I_0(z)$,

[23]This is defined as $I_k(z)=\sum_{j=0}^{\infty}\frac{(z/2)^j}{j!}\frac{(z/2)^{j+k}}{(j+k)!}$ (see Abramowitz and Stegum, 1965).

(see, for instance, Nåsell, 1974). Thus, for all $k > 0$,

$$\mathcal{P}\left[T_{-k} \mid n\right] < \mathcal{P}\left[T \mid n\right]\left(\sqrt{\tfrac{q_B}{q_A}}\right)^k$$

and so the Poisson probability that B wins when the mean population is n

$$\begin{aligned}
\mathcal{P}\left[B \text{ wins} \mid n\right] &< \sum_{k=0}^{\infty} \mathcal{P}\left[T_{-k} \mid n\right] \\
&< \sum_{k=0}^{\infty} \mathcal{P}\left[T \mid n\right]\left(\sqrt{\tfrac{q_B}{q_A}}\right)^k \\
&= \mathcal{P}\left[T \mid n\right]\left(1 - \sqrt{\tfrac{q_B}{q_A}}\right)^{-1} \qquad (13)
\end{aligned}$$

Thus, writing $\gamma_m = \left(1 - \sqrt{\frac{q_B}{q_A}}\right)^{-1}$, we have that for all large m,

$$\begin{aligned}
\sum_{n=1}^{\infty} \pi_m^{**}(n)\, \mathcal{P}\left[B \text{ wins} \mid n\right] &< \gamma_m \sum_{n=1}^{\infty} \pi_m^{**}(n)\, \mathcal{P}\left[T \mid n\right] \\
&= \gamma_m \sum_{n=0}^{\infty} \pi_m(n)\, \mathcal{P}\left[T \mid n+1\right] \\
&< \gamma_m \sum_{n=0}^{\infty} \pi_m(n)\, \mathcal{P}\left[T \mid n\right]
\end{aligned}$$

where the last inequality follows from the fact that for all n, $\mathcal{P}\left[T \mid n+1\right] < \mathcal{P}\left[T \mid n\right]$ (since for $x > y$, the function $e^{-xn} I_0(yn)$ is decreasing in n).

Now Lemma B.1 implies that $\gamma = \lim_m \gamma_m < \infty$, and so

$$\lim_{m\to\infty} \sum_{n=1}^{\infty} \pi_m^{**}(n)\, \mathcal{P}\left[B \text{ wins} \mid n\right] \le \gamma \lim_{m\to\infty} \sup \sum_{n=0}^{\infty} \pi_m(n)\, \mathcal{P}\left[T \mid n\right]$$

and since the left-hand side is positive, so is the right-hand side.

Finally, the inequality (again see Appendix D),

$$\left| \sum_{n=0}^{\infty} \pi_m(n) \Pr\left[T \mid n\right] - \sum_{n=0}^{\infty} \pi_m(n)\, \mathcal{P}\left[T \mid n\right] \right| \le q_A + q_B$$

implies that

$$\lim \sum_{n=0}^{\infty} \pi_m(n) \Pr\left[T \mid n\right] > 0$$

and since $\Pr\left[Piv_A\right] = \frac{1}{2}\Pr\left[T\right] + \frac{1}{2}\Pr\left[T_{-1}\right]$, it is also the case that $\lim \Pr\left[Piv_A\right] > 0$.

But this is impossible since if $\lim \Pr\left[Piv_A\right]$ were positive, the limiting turnout p_A would be positive as well, contradicting Proposition 4. This completes the proof. ■

Proof of Theorem 1. Suppose $\lambda v_A > (1-\lambda) v_B$ (the other case is analogous). Exactly as in the proof of Proposition 6 (simply use π^* instead of π^{**}), it suffices to prove that

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} \pi_m^* (n) \mathcal{P}[T \mid n] = 0$$

Now since for all $n \geq 1$, $\pi_m^{**}(n) = \pi_m^*(n) \frac{n}{m}$, the concordance condition implies that there exists a constant $K > 0$ such that for all m and $1 \leq n \leq m$,

$$\pi_m^*(n) \leq \pi_m^{**}(n) + \frac{K}{m}$$

Now note that

$$\begin{aligned}
\sum_{n=0}^{\infty} \pi_m^*(n) \mathcal{P}[T \mid n] &\leq \pi_m^*(0) + \sum_{n=1}^{m} \pi_m^*(n) \mathcal{P}[T \mid n] + \sum_{n=m+1}^{\infty} \pi_m^{**}(n) \mathcal{P}[T \mid n] \\
&\leq \pi_m^*(0) + \frac{K}{m} \sum_{n=1}^{m} \mathcal{P}[T \mid n] + \sum_{n=1}^{m} \pi_m^{**}(n) \mathcal{P}[T \mid n] + \sum_{n=m+1}^{\infty} \pi_m^{**}(n) \mathcal{P}[T \mid n] \\
&= \pi_m^*(0) + \frac{K}{m} \sum_{n=1}^{m} \mathcal{P}[T \mid n] + \sum_{n=1}^{\infty} \pi_m^{**}(n) \mathcal{P}[T \mid n]
\end{aligned}$$

since for all $n > m$, $\pi_m^{**}(n) > \pi_m^*(n)$. Proposition 6 implies that the second sum in the expression above goes to zero. Now observe that

$$\begin{aligned}
\sum_{n=1}^{m} \mathcal{P}[T \mid n] &= \sum_{n=1}^{m} e^{-n(q_A+q_B)} I_0 \left(2n\sqrt{q_A q_B}\right) \\
&\leq \sum_{n=1}^{m} e^{-2n\sqrt{q_A q_B}} I_0 \left(2n\sqrt{q_A q_B}\right) \\
&\leq \sum_{n=1}^{m} \int_{n-1}^{n} e^{-2x\sqrt{q_A q_B}} I_0 \left(2x\sqrt{q_A q_B}\right) dx \\
&= \int_{0}^{m} e^{-2x\sqrt{q_A q_B}} I_0 \left(2x\sqrt{q_A q_B}\right) dx
\end{aligned}$$

where we have used the fact that $e^{-2x\sqrt{q_A q_B}} I_0 \left(2x\sqrt{q_A q_B}\right)$ is decreasing in x. To evaluate the integral in the last step, write $r = 2\sqrt{q_A q_B}$ and notice that

$$\int_0^m e^{-rx} I_0(rx)\, dx = \frac{1}{r} \int_0^{mr} e^{-y} I_0(y)\, dy$$

by changing the variable of integration from x to $y = rx$. Since[24]

$$\int e^{-y} I_0(y)\, dy = e^{-y} y \left(I_0(y) + I_1(y)\right)$$

we obtain

$$\begin{aligned} \frac{1}{r}\int_0^{mr} e^{-y} I_0(y)\, dy &= \frac{1}{r} e^{-mr} mr \left(I_0(mr) + I_1(mr)\right) \\ &\leq 2e^{-mr} m I_0(mr) \end{aligned}$$

Thus,

$$\frac{1}{m}\sum_{n=0}^{m} \mathcal{P}[T \mid n] \leq \frac{1}{m} + 2e^{-2m\sqrt{q_A q_B}} I_0\left(2m\sqrt{q_A q_B}\right)$$

Since $mq_A \to \infty$ and $mq_B \to \infty$ (Lemma A.9), $2m\sqrt{q_A q_B} \to \infty$ as well and the fact that $\lim_{x\to\infty} e^{-x} I_0(x) = 0$ implies that

$$\lim_{m\to\infty} \frac{1}{m}\sum_{n=0}^{m} \mathcal{P}[T \mid n] = 0$$

This completes the proof. ■

C Supermajority Rules

This appendix provides a proof of Theorem 2. Throughout, we assume that the population is *Poisson* distributed with mean m.

Pivot Probabilities As before, an event (j, k) is *pivotal for* A if a single additional vote for A will affect the outcome of the election and denote the set of such events by Piv_A. The events in Piv_A can be classified into three separate categories:

A1. There is a tie and so a single vote for A will result in A winning. A tie can occur only if the number of voters is a multiple of $a + b$. The set of ties is thus

$$T = \{(la, lb) : l \geq 0\} \tag{14}$$

[24] This may be verified using $I_0'(x) = I_1'(x)$ and $(xI_1(x))' = xI_0(x)$.

A2. Candidate A is one vote short of a tie. The set of such events is[25]

$$T-(1,0)=\{(la-1,lb):l\geq 1\}$$

A3. A is losing but a single additional vote will result in A winning. For any integer k such that $1\leq k<b$, events in sets of the form

$$T-(\left\lceil \frac{a}{b}k\right\rceil ,k)=\left\{(la-\left\lceil \frac{a}{b}k\right\rceil ,lb-k):l\geq 1\right\}$$

have the required property.[26] This is because for any $k<b$ the condition that

$$\frac{la-\left\lceil \frac{a}{b}k\right\rceil}{lb-k}<\frac{a}{b}<\frac{la-\left\lceil \frac{a}{b}k\right\rceil+1}{lb-k}$$

is equivalent to

$$\left\lceil \frac{a}{b}k\right\rceil>\frac{a}{b}k>\left\lceil \frac{a}{b}k\right\rceil-1$$

Similarly, events that are *pivotal for* B can also be classified into three categories:

B1. There is a tie and so a single vote for B will result in B winning. This occurs for vote totals in the set T as defined above in (14).

B2. Candidate B is one vote short of a tie. The set of such events is

$$T-(0,1)=\{(la,lb-1):l\geq 1\}$$

B3. B is losing but a single additional vote will result in B winning. For any integer j such that $1\leq j<a$, events in sets of the form

$$T-(j,\left\lceil \tfrac{b}{a}j\right\rceil)=\left\{(la-j,lb-\left\lceil \tfrac{b}{a}j\right\rceil):l\geq 1\right\}$$

have the required property. This is because for any $j<a$, the condition that

$$\frac{la-j}{lb-\left\lceil \frac{b}{a}j\right\rceil}>\frac{a}{b}>\frac{la-j}{lb-\left\lceil \frac{b}{a}j\right\rceil+1}$$

is equivalent to

$$\left\lceil \frac{b}{a}j\right\rceil-1<\frac{b}{a}j<\left\lceil \frac{b}{a}j\right\rceil$$

[25] Of course, the number of votes cast is nonnegative, so the point $(-1,0)$ is excluded from this set.

[26] $\lceil z\rceil$ denotes the smallest integer greater than z.

(Under majority rule there are no events of the kind listed in A3. and B3.)

As usual, let q_A be the probability of a vote for A and q_B the probability of a vote for B. Under the $\frac{a}{a+b}$-supermajority rule, the Poisson probability of a tie is

$$\mathcal{P}[T] = \sum_{k=0}^{\infty} e^{-mq_A} \frac{(mq_A)^{ka}}{(ka)!} e^{-mq_B} \frac{(mq_B)^{kb}}{(kb)!} \tag{15}$$

Approximations Now suppose that we have a sequence $(q_A(m), q_B(m))$ such that both $mq_A(m) \to \infty$ and $mq_B(m) \to \infty$. Myerson (2000) has shown that, for large m, the Poisson probability of a tie, given in (15), can be approximated as follows:

$$\mathcal{P}[T] \approx \frac{\exp\left((a+b)\left(\frac{mq_A}{a}\right)^{\frac{a}{a+b}}\left(\frac{mq_B}{b}\right)^{\frac{b}{a+b}} - mq_A - mq_B\right)}{\left(2\pi(a+b)\left(\frac{mq_A}{a}\right)^{\frac{a}{a+b}}\left(\frac{mq_B}{b}\right)^{\frac{b}{a+b}}\right)^{\frac{1}{2}}(ab)^{\frac{1}{2}}} \tag{16}$$

Second, Myerson (2000) has also shown that the probability of "offset" events of the form $T-(j,k)$ can be approximated as follows:

$$\mathcal{P}[T-(j,k)] \approx \mathcal{P}[T] \times x^{bj-ak} \tag{17}$$

where $x = \left(\frac{q_B}{q_A}\frac{a}{b}\right)^{\frac{1}{a+b}}$.

The pivotal probabilities can then be approximated by using (16) and (17):

$$\begin{aligned}\mathcal{P}[Piv_A] &\approx \mathcal{P}[T] \times \left(1-t+tx^b+\sum_{k=1}^{b-1} x^{b\left\lceil\frac{a}{b}k\right\rceil - ak}\right)\\ \mathcal{P}[Piv_B] &\approx \mathcal{P}[T] \times \left(t+(1-t)\,x^{-a}+\sum_{j=1}^{a-1} x^{bj-a\left\lceil\frac{b}{a}j\right\rceil}\right)\end{aligned}$$

where t is the probability that a tie is resolved in favor of A.

Since it is the case that $\left\{b\left\lceil\frac{a}{b}k\right\rceil - ak : k = 1,2,...,b-1\right\} = \{1,2,...,b-1\}$ and similarly, $\left\{a\left\lceil\frac{b}{a}j\right\rceil - bj : j = 1,2,...,a-1\right\} = \{1,2,...,a-1\}$, we have

$$\mathcal{P}[Piv_A] \approx \mathcal{P}[T] \times \left(1-t+tx^b+\sum_{k=1}^{b-1} x^k\right) \tag{18}$$

$$\mathcal{P}[Piv_B] \approx \mathcal{P}[T] \times \left(t+(1-t)\,x^{-a}+\sum_{j=1}^{a-1} x^{-j}\right) \tag{19}$$

Proof of Proposition 7. Using the formulae in (18) and (19), the ratio

$$\frac{\mathcal{P}\left[Piv_A\right]}{\mathcal{P}\left[Piv_B\right]} \approx \frac{1-t+tx^b+\sum_{k=1}^{b-1} x^k}{t+(1-t)\,x^{-a}+\sum_{j=1}^{a-1} x^{-j}}$$

The numerator is increasing in x, while the denominator is decreasing. Thus, the ratio of the pivotal probabilities is increasing in x. When $x=1$,

$$\frac{\mathcal{P}\left[Piv_A\right]}{\mathcal{P}\left[Piv_B\right]} \approx \frac{b}{a}$$

If, for all m large, $\frac{q_A(m)}{q_B(m)} > \frac{a}{b}$, then $x<1$ and so for all m large, $\frac{\mathcal{P}[Piv_A]}{\mathcal{P}[Piv_B]} < \frac{b}{a}$. If there is a subsequence along which $\frac{q_A}{q_B} = \frac{a}{b}$ and along this subsequence $\lim \frac{\mathcal{P}[Piv_A]}{\mathcal{P}[Piv_B]} > \frac{b}{a}$, then this contradicts the fact that $x=1$ implies $\frac{\mathcal{P}[Piv_A]}{\mathcal{P}[Piv_B]} \approx \frac{b}{a}$. Thus, if for all m large, $\frac{q_A(m)}{q_B(m)} \geq \frac{a}{b}$, then $\limsup \frac{\mathcal{P}[Piv_A]}{\mathcal{P}[Piv_B]} \leq \frac{b}{a}$.

The other case is analogous. ■

D Poisson Approximations of the Multinomial

We are interested in the distribution of the sum of independent Bernoulli vector variables (X_A, X_B) where $\Pr\left[(X_A, X_B) = (1,0)\right] = q_A$; $\Pr\left[(X_A, X_B) = (0,1)\right] = q_B$ and $\Pr\left[(X_A, X_B) = (0,0)\right] = q_0 = 1 - q_A - q_B$. The probability that after n draws, the sum of the variables (X_A, X_B) is (k,l) is

$$\Pr\left[(k,l) \mid n\right] = \binom{n}{k,l} \left(q_A\right)^k \left(q_B\right)^l \left(q_0\right)^{n-k-l}$$

Now consider a bivariate Poisson distribution with means nq_A and nq_B. The Poisson probability $\mathcal{P}\left[(k,l)\right]$ that the number of occurrences of A and B will be k and l, respectively, is

$$\mathcal{P}\left[(k,l) \mid n\right] = e^{-nq_A - nq_B} \frac{\left(nq_A\right)^k}{k!} \frac{\left(nq_B\right)^l}{l!}$$

Roos (1999, p. 122) has shown that

$$\sup_{S \subset \mathbb{Z}_+^2} \left|\Pr\left[S \mid n\right] - \mathcal{P}\left[S \mid n\right]\right| \leq q_A + q_B$$

Note that the bound on the right-hand side does not depend on n.

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