Voluntary Voting: Costs and Benefits*

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Abstract

We compare voluntary and compulsory voting in a Condorcet-type model in which voters have identical preferences but differential information. With voluntary voting, all equilibria involve sincere voting and positive participation. Thus, in contrast to situations with compulsory voting, there is no conflict between strategic and sincere behavior. When voting is costless, voluntary voting is welfare superior to compulsory voting. Even when voting is costly, participation rates are such that, in the limit, the correct candidate is elected—that is, information fully aggregates.

1 Introduction

Should voting be a right or a duty? Faced with declining turnouts in elections, many countries have concluded that voting should be a duty—a requirement to be enforced by sanction—rather than a right.[1] On a smaller scale, voting is considered a duty in most committees as well. Attendance is usually required, and members are encouraged to "make their voice heard" through actually casting votes rather than abstaining. In some committees, abstentions count as "no" votes.

Advocates of voting as a duty offer several arguments in support. First, high turnout may confer legitimacy to those elected. Second, compulsory voting may give greater voice to poorer sections of society who would otherwise not participate (Lijphart, 1997). Third, by aggregating the opinions of more individuals, compulsory voting may have informational benefits. In this paper, we do not directly address the first two arguments. Rather we examine the right versus duty question on informational grounds. Which regime produces better decisions?

The analysis of voting on informational grounds begins with Condorcet's celebrated Jury Theorem which states that, when voters have common interests but

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[1] Over 40 countries—for instance, Australia, Belgium, and many countries in Latin America—have compulsory voting laws.

differential information, sincere voting under majority rule produces the correct outcome in large elections. There are two key components to the theorem. First, it postulates that voting is sincere—that is, voters vote solely according to their private information. Second, that voter turnout is high.

Recent work shows, however, that sincerity is inconsistent with rationality—it is typically not an equilibrium to vote sincerely. The reason is that rational voters will make inferences about others' information and, as a result, will have the incentive to vote against their own private information (Austen-Smith and Banks, 1996). Equilibrium voting behavior involves the use of *mixed* strategies—with positive probability, voters vote against their private information. Surprisingly, this does not overturn the conclusion of the Jury Theorem: In large elections, there exist equilibria in which the correct candidate is always chosen despite insincere voting (Feddersen and Pesendorfer, 1998).[2] These convergence results, while powerful, rest on equilibrium behavior that may be deemed implausible. Voting is not only insincere but random. Moreover, some voters have *negative* returns to voting—they would rather not vote at all—this is a manifestation of the "swing voter's curse" (Feddersen and Pesendorfer, 1996).

These generalizations of the Jury Theorem rely on the assumption that voter turnout is high. Indeed it is implicitly assumed that voting is compulsory, so all eligible voters show up and vote. When voting is voluntary and costly, however, there is reason to doubt that voters will turn out in large enough numbers to guarantee correct choices. Indeed, even if there were no swing voter's curse, rational voters would correctly realize that a single vote is unlikely to affect the outcome, so there is little benefit to voting. This is the "paradox of not voting" (Downs, 1957).

In this paper, we revisit the classic Condorcet Jury model but relax the assumption that voting is compulsory (i.e., it is not possible to abstain). We study two variants of the model: in one, voting is costless but abstention is possible; in the other, voters incur private costs of voting and may avoid these by abstaining. Voters in our model are fully rational, so the twin problems of strategic voting and the paradox of not voting are present.

For our analysis, we adopt the Poisson model introduced by Myerson (1998 & 2000). In this model, the size of the electorate is random. As Myerson (1998) has demonstrated the qualitative predictions of Poisson voting models are identical to those with a fixed electorate. The analysis is, however, much simpler.

We find:

1. If voting is sincere, full participation is not optimal. A planner would like to restrict participation even with a relatively small number of voters (Proposition 1).

2. With voluntary voting, there is no conflict between rationality and sincerity—all equilibria involve sincere voting and positive participation (Theorem 1). This result holds regardless of the size of the electorate.

[2] Bhattacharya (2008) identifies conditions, in a model with more general preferences, where compulsory voting does not lead to information aggregation.

3. When voting is costless, voluntary voting is superior to compulsory voting (Proposition 7). Again, this result holds regardless of the size of the electorate.
4. Even when voting is costly, the correct candidate always wins in *large* elections under voluntary voting—that is, information fully aggregates (Proposition 9). While this is also true of compulsory voting, voluntary voting economizes on costs and so is superior (Proposition 10).

To summarize, our results point to the advantages of voting as a right over voting as duty. Welfare is higher. Moreover, equilibrium behavior under the voluntary scheme is simple and intuitive. Strategic behavior is no longer at odds with sincerity.

The following example may be used to illustrate our main results.[3] Three voters must decide between two candidates, A and B. Voters have equal priors over who is the better candidate but receive private signals. When A is best, each voter receives an a signal for sure. When B is best, however, a voter receives a b signal only with probability s strictly between $\frac{1}{2}$ and 1. Notice that a single b signal indicates that B is the best candidate for sure.

First, suppose that all voters participate and vote sincerely. While this leads to the correct outcome when A is best, it produces errors when B is best. The most likely error occurs when two voters receive a signals and only one receives a b signal (this is more likely than the event that all three receive a signals). The situation improves if a voters were to participate at slightly lower rates. The first order effect of this change is to reduce the errors when B is best without affecting the error rate when A is best. Thus, full participation with sincere voting is not optimal.

Next, suppose that voting is compulsory. If the other two voters voted sincerely, a voter with an a signal would correctly reason that she is decisive only when the vote is split. But this can only happen if one of the other voters has a b signal. And since even one b signal predicts perfectly that B is the better candidate, it is optimal to vote for B. Therefore, such an a voter would be well-advised to vote insincerely.

In contrast, under voluntary voting, voters with a signals would come to the polls less often than those with b signals. This is because b voters are certain that B is the best candidate while a voters are unsure. How should an a voter vote if she does decide to come to the polls? She is decisive in two cases—on a split vote when B is best and when she is the only voter and A is best. If the participation rates are such that an a voter rates the latter case as more likely, she would vote sincerely, that is, for A. Our results will show that the participation rates are indeed such that they induce sincere voting. Finally, since full participation with sincere voting is not optimal, this reduction in participation may have the beneficial effect of also reducing the error rate. Indeed, in equilibrium, we show that it minimizes the error rate when voting is costless.

Related literature Early work on the Condorcet Jury Theorem viewed it as a purely statistical phenomenon—an expression of the law of large numbers. Perhaps

[3] For purposes of exposition, in the example there is a fixed number of voters. In the model we study, the number of voters is random.

this was the way that Condorcet himself viewed it. Game theoretic analyses of the Jury Theorem originate in the work of Austen-Smith and Banks (1996). They show that sincere voting is generally not consistent with equilibrium behavior.

Feddersen and Pesendorfer (1998) derive the ("insincere") equilibria of the voting games specified above—these involve mixed strategies—and then study their limiting properties. They show that, despite the fact that sincere voting is not an equilibrium, large elections still aggregate information correctly. Using a mechanism design approach, Costinot and Kartik (2009) investigate optimal voting rules under a variety of behavioral assumptions including strategic and sincere voting. They show that there is a unique voting rule, independent of voter behavior, that aggregates information. McLennan (1998) views such voting games, in the abstract, as games of common interest and argues on that basis that there are always Pareto efficient equilibria of such games. Apart from the fact that voting is voluntary, and perhaps costly, our basic setting is the same as that in these papers—there are two candidates, voters have common interests but differential information (sometimes referred to as a setting with "common values").

A separate strand of the literature is concerned with costly voting and endogenous participation but in settings in which voter preferences are diverse (sometimes referred to as "private values"). Palfrey and Rosenthal (1985) consider costly voting with privately known costs but where preferences over outcomes are commonly known (see also Palfrey and Rosenthal, 1983 and Ledyard, 1984 for models in which the costs are also common knowledge). These papers are interested in formalizing Downs' paradox of not voting. Börgers (2004) studies majority rule in a costly voting model with private values—that is, with diverse rather than common preferences. He compares voluntary and compulsory voting and argues that individual decisions to vote or not do not properly take into account a "pivot externality"—the casting of a single vote decreases the value of voting for others. As a result, participation rates are too high relative to the optimum and a law that makes voting compulsory would only worsen matters. Krasa and Polborn (2009) show that the externality identified by Börgers' is sensitive to his assumption that the prior distribution of voter preferences is 50-50. With unequal priors, under some conditions, the externality goes in the opposite direction and there are social benefits to encouraging increased turnout via fines for not voting.

Ghosal and Lockwood (2009) reexamine Börgers' result when voters have more general preferences—including common values—and show that it is sensitive to the private values assumption. Finally, Feddersen and Pesendorfer (1996) examine abstention in a common values model when voting is costless. The number of voters is random, some are informed of the state, while others have no information whatsoever. Abstention arises in their model as a result of the aforementioned swing voter's curse—in equilibrium, a fraction of the uninformed voters do not participate. McMurray (2010) studies a similar model in which the information that voters have differs in quality. In large elections, a positive fraction of voters with imprecise information continue to vote even though there are voters with more precise information.

Much of this work postulates a fixed and commonly known population of voters.

Myerson (1998 & 2000) argues that precise knowledge of the number of eligible voters is an idealization at best, and suggests an alternative model in which the size of the electorate is a Poisson random variable. This approach has the important advantage of considerably simplifying the analysis of pivotal events. Myerson illustrates this by deriving the mixed equilibrium for the majority rule in *large* elections (in a setting where signal precisions are asymmetric). He then studies its limiting properties as the number of expected voters increases, exhibiting information aggregation results parallel to those derived in the known population models. We also find it convenient to adopt Myerson's Poisson game technology but are able to show that there is a *sincere* voting equilibrium for any (expected) size electorate.

Feddersen and Pesendorfer (1999) use the Poisson framework to study abstention when voting is costless but preferences are diverse—voters differ in the intensity of their preferences, given the state. In large elections, a positive fraction of the voters abstain even though voting is costless. Nevertheless, information aggregates. different voters have different with pure common values but Herrera and Morelli (2009) also use a diverse preference Poisson model to compare turnout rates in proportional and winner-take-all parliamentary elections. In our model, we allow for costly voting and also compare voluntary voting with a system in which everyone votes.

The paper is organized as follows. In Section 2 we introduce the basic environment and Myerson's Poisson model. As a benchmark, in Section 3 we first consider the model with compulsory voting and establish (a) even if voting is sincere, full participation is not optimal; and (b) under full participation, sincere voting is not an equilibrium. In Section 4, we introduce the model with voluntary voting. We show that all equilibria entail sincere voting and positive participation. Section 5 compares the performance of voluntary and compulsory voting schemes when voting is costless. Our main finding is that voluntary voting produces the correct outcome more often than compulsory voting and hence is preferred. Section 6 studies the limiting properties of the equilibria when voting is costly. We show that, despite the "paradox of not voting," in the limit, information fully aggregates and the correct candidate is elected with probability one under voluntary voting. Compulsory voting also produces the correct outcome in the limit, but at higher cost; hence, voluntary voting is again superior.

Omitted proofs are collected in the appendices.

2 The Model

There are two candidates, named A and B, who are competing in an election decided by majority voting.[4] There are two equally likely states of nature, α and β.[5] Candidate A is the better choice in state α while candidate B is the better choice in state β. Specifically, in state α the payoff of any citizen is 1 if A is elected and 0 if B is elected. In state β, the roles of A and B are reversed.

[4] In the event of a tied vote, the winning candidate is chosen by a fair coin toss.

[5] The analysis is unchanged if the states are not equally likely. We study the simple case only for notational ease.

The size of the electorate is a random variable which is distributed according to a *Poisson* distribution with mean n. Thus the probability that there are exactly m eligible voters (or *citizens*) is $e^{-n}n^m/m!$.[6]

Prior to voting, every citizen receives a private signal S_i regarding the true state of nature. The signal can take on one of two values, a or b. The probability of receiving a particular signal depends on the true state of nature. Specifically, each voter receives a conditionally independent signal where

$$\Pr[a \mid \alpha] = r \text{ and } \Pr[b \mid \beta] = s$$

We suppose that both r and s are greater than $\frac{1}{2}$, so that the signals are informative and less than 1, so that they are noisy. Thus, signal a is associated with state α while the signal b is associated with β. The posterior probabilities of the states after receiving signals are

$$q(\alpha \mid a) = \frac{r}{r + (1-s)} \text{ and } q(\beta \mid b) = \frac{s}{s + (1-r)}$$

We assume, without loss of generality, that $r > s$. It may be verified that

$$q(\alpha \mid a) < q(\beta \mid b)$$

Thus the posterior probability of state α given signal a is smaller than the posterior probability of state β given signal b even though the "correct" signal is more likely in state α.

Pivotal Events An *event* is a pair of vote totals (j, k) such that there are j votes for A and k votes for B. An event is *pivotal* for A if a single additional vote for A will affect the outcome of the election. We denote the set of such events by Piv_A. One additional vote for A makes a difference only if either (i) there is a tie; or (ii) A has one vote less than B. Let $T = \{(k, k) : k \geq 0\}$ denote the set of ties and let $T_{-1} = \{(k-1, k) : k \geq 1\}$ denote the set of events in which A is one vote short of a tie. Similarly, Piv_B is defined to be the set of events which are pivotal for B. This set consists of the set T of ties together with events in which A has one vote more than B. Let $T_{+1} = \{(k, k-1) : k \geq 1\}$ denote the set of events in which A is ahead by one vote.

Let σ_A be the *expected* number of votes for A in state α and let σ_B be the expected number of votes for B in state α. Analogously, let τ_A and τ_B be the expected number of votes for A and B, respectively, in state β. Since it may be possible for voters to abstain, it is only required that $\sigma_A + \sigma_B \leq n$ and $\tau_A + \tau_B \leq n$.

Consider an event where (other than voter 1) the realized electorate is of size m and there are k votes in favor of A and l votes in favor of B. The number of

[6] The Poisson assumption should be viewed as an analytic convenience—the conclusions derived in the Poisson model are the same as those in a model in which the number of voters is fixed and commonly known. The precise calculations (especially of the pivotal probabilities) are, however, simpler, and the limiting properties of the two models are identical.

abstentions is thus $m-k-l$. The probability of this event in state α is

$$\Pr\left[(k,l;m)\mid\alpha\right]=\frac{e^{-n}}{m!}\binom{m}{k+l}\binom{k+l}{k}\left(n-\sigma_A-\sigma_B\right)^{m-k-l}\sigma_A^k\sigma_B^l$$

It is useful to rearrange the expression as follows:

$$\begin{aligned}\Pr\left[(k,l;m)\mid\alpha\right] &= e^{-(n-\sigma_A-\sigma_B)}\frac{(n-\sigma_A-\sigma_B)^{m-k-l}}{(m-k-l)!}\\ &\quad\times e^{-\sigma_A}\frac{\sigma_A^k}{k!}e^{-\sigma_B}\frac{\sigma_B^l}{l!}\end{aligned}$$

Of course, the size of the electorate is unknown to voter 1. The probability of the event (k,l), irrespective of the size of the electorate, is

$$\begin{aligned}\Pr\left[(k,l)\mid\alpha\right] &= \sum_{m=k+l}^{\infty}\Pr\left[(k,l;m)\mid\alpha\right]\\ &= e^{-\sigma_A}\frac{\sigma_A^k}{k!}e^{-\sigma_B}\frac{\sigma_B^l}{l!}\end{aligned}\tag{1}$$

The probability of the event (k,l) in state β may similarly be obtained by replacing σ with τ.

The probability of a tie in state α is

$$\Pr\left[T\mid\alpha\right]=e^{-\sigma_A-\sigma_B}\sum_{k=0}^{\infty}\frac{\sigma_A^k}{k!}\frac{\sigma_B^k}{k!}\tag{2}$$

while the probability that A falls one vote short in state α is

$$\Pr\left[T_{-1}\mid\alpha\right]=e^{-\sigma_A-\sigma_B}\sum_{k=1}^{\infty}\frac{\sigma_A^{k-1}}{(k-1)!}\frac{\sigma_B^k}{k!}\tag{3}$$

The probability $\Pr\left[T_{+1}\mid\alpha\right]$ that A is ahead by one vote may be written by exchanging σ_A and σ_B in (3). The corresponding probabilities in state β are obtained by substituting τ for σ. Let Piv_A be the set of events where one additional vote for A is decisive. Then, in state α

$$\Pr\left[Piv_A\mid\alpha\right]=\tfrac{1}{2}\Pr\left[T\mid\alpha\right]+\tfrac{1}{2}\Pr\left[T_{-1}\mid\alpha\right]$$

where the coefficient $\frac{1}{2}$ arises since, in the first case, the additional vote for A breaks a tie while, in the second, it leads to a tie. A similar expression applies for state β as well.

Likewise, define Piv_B to be the set of events where one additional vote for B is decisive. Hence, in state β,

$$\Pr\left[Piv_B\mid\beta\right]=\tfrac{1}{2}\Pr\left[T\mid\beta\right]+\tfrac{1}{2}\Pr\left[T_{+1}\mid\beta\right]$$

and, again, a similar expression holds when the state is α.

In what follows, it will be useful to rewrite the pivot probabilities in terms of *modified Bessel functions* (see Abramowitz and Stegun, 1965), defined by

$$\begin{aligned} I_0(z) &= \sum_{k=0}^{\infty} \frac{\left(\frac{z}{2}\right)^k}{k!} \frac{\left(\frac{z}{2}\right)^k}{k!} \\ I_1(z) &= \sum_{k=1}^{\infty} \frac{\left(\frac{z}{2}\right)^{k-1}}{(k-1)!} \frac{\left(\frac{z}{2}\right)^k}{k!} \end{aligned}$$

In terms of modified Bessel functions, we can rewrite the probabilities associated with close elections as

$$\begin{aligned} \Pr[T \mid \alpha] &= e^{-\sigma_A - \sigma_B} I_0\left(2\sqrt{\sigma_A \sigma_B}\right) \\ \Pr[T_{\pm 1} \mid \alpha] &= e^{-\sigma_A - \sigma_B} \left(\frac{\sigma_A}{\sigma_B}\right)^{\pm \frac{1}{2}} I_1\left(2\sqrt{\sigma_A \sigma_B}\right) \end{aligned} \tag{4}$$

Again, the corresponding probabilities in state β are found by substituting τ for σ.

For our asymptotic results it is useful to note that when z is large, the modified Bessel functions can be approximated as follows[7] (see Abramowitz and Stegun, 1965, p. 377)

$$I_0(z) \approx \frac{e^z}{\sqrt{2\pi z}} \approx I_1(z) \tag{5}$$

3 Compulsory Voting

We begin by examining equilibrium voting behavior under compulsory voting. By *compulsory* voting we mean that each voter must cast a vote for either A or B. While many countries have compulsory voting laws, these can, at best, only compel voters to come to the polls; under most voting systems, they are still free to cast "spoilt" or "blank" ballots. But this is not the intent of compulsory voting laws—it only highlights the conflict between compulsory voting and the right to a secret ballot. Moreover, it would be hard to find an advocate of compulsory voting who argues for the right to cast a blank ballot. At the committee level, where attendance is typically required in any event, the obligation to cast a vote can occur in several ways. Some committees simply exclude abstention as an option. Still others, such as university promotion and tenure committees, treat abstentions as "no" votes since they require a "yes" by majority or supermajority of those present. Thus, these settings have, in effect, a system of compulsory voting as modelled here. Without calling attention to the compulsory aspect, this is also the usual model studied in the literature.

As a first step, consider a planner who wishes to maximize the ex ante probability that the right candidate is chosen. To simplify matters, suppose that voters are not strategic and voting is *sincere*—each voter votes as if he were the only decision maker. In our model, this means that those with a signal of a vote for A and those with a

[7] $X(n) \approx Y(n)$ means that $\lim_{n\to\infty}(X(n)/Y(n)) = 1$.

signal of b vote for B. The planner then chooses the participation levels of the two types of voters. The optimal policy seems obvious: Surely the planner should make voting compulsory. After all, this uses all of society's available information in making a choice and, presumably more information leads to better choices.

While this intuition seems compelling, it is, in fact, incorrect. The flaw is that, while the average informational contribution of a voter is positive, the marginal contribution need not be. To see this, consider the supposedly ideal situation where everyone is participating and, as assumed, voting sincerely. What happens if the participation by voters with a signals decreases by a small amount? The welfare impact of decreased participation comes only from tied or near-tied outcomes. Since signals are more dispersed in state β, ties and near-ties are more likely.[8] If voters with a signals participate less, this increases the error rate in state α, but reduces it in state β. Since the latter is more likely, the net effect of a *decrease* in participation by voters with a signals is to *increase* welfare.

In short, when voting is sincere, compulsory voting is not optimal. Proposition 1 formalizes this argument.

Define

$$n(r,s) = \frac{1}{2}\frac{\ln\sqrt{\frac{1-s}{s}} + \ln\sqrt{\frac{1-r}{r}}}{\sqrt{r(1-r)} - \sqrt{s(1-s)}} \tag{6}$$

Proposition 1 *Suppose that $n \geq n(r,s)$. Under sincere voting, full participation is not optimal.*

Proof. The welfare in state α (the probability that A wins) is

$$W(\alpha) = \tfrac{1}{2}\Pr[T \mid \alpha] + \sum_{m=1}^{\infty}\Pr[T_{+m} \mid \alpha]$$

where T_{+m} denotes the set of events in which A beats B by m votes. Similarly, the probability that B wins in state β is

$$W(\beta) = \tfrac{1}{2}\Pr[T \mid \beta] + \sum_{m=1}^{\infty}\Pr[T_{-m} \mid \beta]$$

where T_{-m} denotes the set of events in which B beats A by m votes. Let $W = \frac{1}{2}W(\alpha) + \frac{1}{2}W(\beta)$ denote the ex ante probability that the right candidate wins.

Let p_a and p_b be the participation rates of the two types of voters. Under sincere voting, this means that the expected vote totals are $\sigma_A = nrp_a$; $\sigma_B = n(1-r)p_b$; $\tau_A = n(1-s)p_a$; and $\tau_B = nsp_b$. We will argue that when $p_a = 1$ and $p_b = 1$; that is, there is full participation,

$$\frac{\partial W}{\partial p_a} < 0$$

[8] This requires a modestly large number of voters. Obviously, if there is only a single voter, ties are equally likely in both states. A precise definition of "modestly large" is offered in the proposition below.

To see this note that, using equation (1), we obtain that for all m,

$$\begin{aligned}\frac{\partial \Pr[T_{+m} \mid \alpha]}{\partial p_a} &= nr\left(\Pr[T_{m-1} \mid \alpha] - \Pr[T_m \mid \alpha]\right) \\ \frac{\partial \Pr[T_{-m} \mid \beta]}{\partial p_a} &= n(1-s)\left(\Pr[T_{-m-1} \mid \beta] - \Pr[T_{-m} \mid \beta]\right)\end{aligned}$$

and some routine calculations using the formulae for $W(\alpha)$ and $W(\beta)$ show that

$$\begin{aligned}\frac{\partial W}{\partial p_a} &= \tfrac{1}{2}nr\left(\Pr[T \mid \alpha] + \Pr[T_{-1} \mid \alpha]\right) - \tfrac{1}{2}n(1-s)\left(\Pr[T \mid \beta] + \Pr[T_{-1} \mid \beta]\right) \\ &= nr\Pr[Piv_A \mid \alpha] - n(1-s)\Pr[Piv_A \mid \beta]\end{aligned}$$

Next observe that when $p_a = 1$ and $p_b = 1$,

$$nr\Pr[Piv_A \mid \alpha] = \tfrac{1}{2}e^{-n}n\left(rI_0(2nr^*) + r^*I_1(2nr^*)\right) \tag{7}$$

where $r^* = \sqrt{r(1-r)}$ is the geometric mean of r and $1-r$. Similarly,

$$n(1-s)\Pr[Piv_A \mid \beta] = \tfrac{1}{2}e^{-n}n\left((1-s)I_0(2ns^*) + s^*I_1(2ns^*)\right) \tag{8}$$

where $s^* = \sqrt{s(1-s)}$. Notice that since $r > s > \frac{1}{2}$, $s^* > r^*$ and I_1 is an increasing function, the second term in (8) is greater than the second term in (7). A sufficient condition for the first term in (8) to be at least as large as the first term in (7) is that

$$n \geq \frac{1}{2}\frac{\ln\sqrt{\frac{1-s}{s}} + \ln\sqrt{\frac{1-r}{r}}}{\sqrt{r(1-r)} - \sqrt{s(1-s)}}$$

This last inequality is derived as follows. First, note that $e^{-x}xI_0(x)$ is an increasing function for all $x > 0$. Since $s^* > r^*$, this implies that $e^{-s^*}s^*I_0(s^*) > e^{-r^*}r^*I_0(r^*)$. This inequality now follows. ■

Proposition 1 shows that compulsory sincere voting is not optimal when the size of the voting body is sufficiently large. As a practical matter, for reasonable parameter values the proposition has force even for modest sized voting bodies.

Remark 1 *The lower bound on n in the proposition is not "too large." For instance, if $r = 0.65$ and $s = 0.6$ then $n(r,s) \leq 20$. Thus even if the degree of asymmetry is small, it is optimal to restrict participation for relatively small electorates.*

The proposition carries with it a surprising implication: Policies or even informal rules suggesting that all voters "offer their voice" (in the form of a vote) produces worse outcomes than allowing some voters to opt out—even when all voters are equally well-informed on the issue under consideration.[9]

[9] The exception occurs when signals are symmetric, i.e., $r = s$, and, consequently $n(r,s) = \infty$. Here, full participation can be optimal even accounting for voting costs. See Example 2 of Ghosal and Lockwood (2009).

Of course, the proposition requires that the planner restrict participation *differentially* for voters with a and b signals. Simply reducing the participation rate across the board produces no advantage. Since the planner has no information about voter types, it would seem that the optimal policy is difficult to implement. We will show below, however, that a voluntary voting policy will in fact accomplish the task.

The imposition of compulsory voting brings with it another difficulty—the possibility of strategic voting. Austen-Smith and Banks (1996) showed that sincere voting does not constitute an equilibrium in a model with a fixed number of voters. Here, we show that this conclusion extends to the Poisson framework as well (Myerson, 1998).

Recall that under sincere and compulsory voting the expected vote totals in state α are $\sigma_A = nr$ and $\sigma_B = n(1-r)$. Similarly, the expected vote totals in state β are $\tau_A = n(1-s)$ and $\tau_B = ns$. As n increases, both $\sigma \to \infty$ and $\tau \to \infty$, and so the formulae in (5) imply that for large n,

$$\frac{\Pr[Piv_A \mid \alpha] + \Pr[Piv_B \mid \alpha]}{\Pr[Piv_A \mid \beta] + \Pr[Piv_B \mid \beta]} \approx \frac{e^{2n\sqrt{r(1-r)}}}{e^{2n\sqrt{s(1-s)}}} \times K(r,s) \tag{9}$$

where $K(r,s)$ is positive and independent of n. Since $r > s > \frac{1}{2}$, $s(1-s) > r(1-r)$ and so the expression in (9) goes to zero as n increases. This implies that, when n is large and a voter is pivotal, state β is infinitely more likely than state α. Thus, voters with a signals will not wish to vote sincerely.[10] It then follows that:

Proposition 2 *If voting is compulsory, sincere voting is not an equilibrium in large elections.*

To summarize, compulsory voting suffers from two problems: (1) when voting is sincere, full participation is not optimal; and (2) under full participation, sincere voting is not an equilibrium. We next examine equilibrium behavior under voluntary voting.

4 Voluntary Voting

In this section, we replace the compulsory voting assumption with that of voluntary voting. We now allow for the possibility of abstention—every citizen need not vote. In effect, this gives voters a third option. A second aspect of our model concerns whether or not voters incur costs of voting. We study two separate models. In the *costless* voting model, voters incur no costs of going to the polls. In the *costly* voting model, they have heterogeneous costs of going to the polls, which can be avoided by staying at home. The costless voting model seems appropriate in settings where all voters must participate in the process, such as in committees, but have the option to

[10] If $r = s$, then the ratio of the pivot probabilities is always 1 and incentive compatibility holds. This corresponds to one of the non-generic cases identified by Austen-Smith and Banks (1996) in a fixed n model. See also Myerson (1998).

abstain (and abstention is not counted as being in favor of one or the other option). The costly voting model seems more appropriate for elections.

We begin by analyzing behavior in the costly voting model. The analysis of costless voting then follows in a straightforward manner.

4.1 Costly Voting

A citizen's cost of voting is private information and determined by an independent realization from a continuous probability distribution F with support $[0, 1]$. We suppose that F admits a density f that is strictly positive on $(0, 1)$. Finally, we assume that voting costs are independent of the signal as to who is the better candidate.

Thus, prior to the voting decision, each citizen has two pieces of private information—his cost of voting and a signal regarding the state. We will show that there exists an equilibrium of the voting game with the following features.

1. There exists a pair of positive *threshold costs*, c_a and c_b, such that a citizen with a cost realization c and who receives a signal $i = a, b$ votes if and only if $c \leq c_i$. The threshold costs determine differential *participation rates* $F(c_a) = p_a$ and $F(c_b) = p_b$.

2. All those who vote do so sincerely—that is, all those with a signal of a vote for A and those with a signal of b vote for B.

In the model with voluntary and costly voting, our main result is

Theorem 1 *Under voluntary voting, in any equilibrium, both types participate and vote sincerely.*

The result is established in three steps. First, we consider only the participation decision. Under the assumption of sincere voting, we establish the existence of positive threshold costs and the corresponding participation rates. Second, we show that given the participation rates determined in the first step, it is indeed an equilibrium to vote sincerely. Third, we show that all equilibria involve sincere voting.

4.1.1 Equilibrium Participation Rates

We now show that when all those who vote do so sincerely, there is an equilibrium in cutoff strategies. That is, there exists a threshold cost $c_a > 0$ such that all voters receiving a signal of a and having a cost $c \leq c_a$ go to the polls and vote for A. Analogously, there exists a threshold cost $c_b > 0$ for voters with a signal of b. Equivalently, one can think of a participation probability, $p_a = F(c_a)$ that a voter with an a signal goes to the polls and a probability $p_b = F(c_b)$ that a voter with a b signal goes to the polls.

Under these conditions, a given voter will vote for A in state α only if he receives the signal a (which happens with probability r) and has a voting cost lower than c_a (which happens with probability p_a). Thus the expected number of votes for A

in state α is $\sigma_A = nrp_a$. Similarly, the expected number of votes for B in state α is $\sigma_B = n(1-r)p_b$. The expected number of votes for A and B in state β are $\tau_A = n(1-s)p_a$ and $\tau_B = nsp_b$, respectively.

We look for participation rates p_a and p_b such that a voter with signal a and cost $c_a = F^{-1}(p_a)$ is indifferent between going to the polls and staying home. Formally, this amounts to the condition that

$$U_a(p_a, p_b) \equiv q(\alpha \mid a)\Pr[Piv_A \mid \alpha] - q(\beta \mid a)\Pr[Piv_A \mid \beta] = F^{-1}(p_a) \qquad \text{(IRa)}$$

where the pivot probabilities are determined using the expected vote totals σ and τ as above. Likewise, a voter with signal b and cost $c_b = F^{-1}(p_b)$ must also be indifferent.

$$U_b(p_a, p_b) \equiv q(\beta \mid b)\Pr[Piv_B \mid \beta] - q(\alpha \mid b)\Pr[Piv_B \mid \alpha] = F^{-1}(p_b) \qquad \text{(IRb)}$$

Proposition 3 *There exist participation rates $p_a^* \in (0,1)$ and $p_b^* \in (0,1)$ that simultaneously satisfy IRa and IRb.*

To see why there are positive participation rates, suppose to the contrary that voters with a signals, say, do not participate at all. Consider a citizen with signal a. Since no other voters with a signals vote, the only circumstance in which he will be pivotal is either if no voters with b signals show up or if only one such voter shows up. Conditional on being pivotal, the likelihood ratio of the states is simply the ratio of the pivot probabilities, that is,

$$\frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} = \frac{e^{-n(1-r)p_b}}{e^{-nsp_b}} \times \frac{1 + n(1-r)p_b}{1 + nsp_b}$$

Notice that the ratio of the exponential terms favors state α while the ratio of the linear terms favors state β. It turns out that the exponential terms always dominate. (Formally, this follows from the fact that the function $e^{-x}(1+x)$ is strictly decreasing for $x > 0$ and that $s > 1-r$.) Since state α is perceived more likely than β by a voter with an a signal who is pivotal, the payoff from voting is positive.

The next result shows that voters with a signals are more likely to show up at the polls than those with b signals.

Lemma 1 *If $r > s$, then any solution to IRa and IRb satisfies $p_a^* < p_b^*$.*

To see why the result holds, consider the case where the participation rates are the same for both types. In that case, no inference may be drawn from the overall level of turnout, only from the vote totals. Consider a particular voter. When the votes of the others are equal in number, it is clear that a tie among the other voters is more likely in state β than in state α (since signals are more dispersed in state β and everyone is voting sincerely), and this is true whether the voter has an a signal or a b signal. Now consider a voter with an a signal. When the votes of the others are such that A is one behind, then once the voter includes his own a signal (and votes sincerely), the overall vote is tied and by the same reasoning as above, an overall tie is more likely in state β than in α. Finally, consider a voter with a b signal. When the votes of the others are such that B is one behind, then once the voter includes his own b signal (and again votes sincerely), the overall vote is tied once more. Again, this is more likely in β than in α.

Thus if participation rates are equal, chances of being pivotal are greater in state β than in state α. This implies that voting is more valuable for someone with a b signal than for someone with an a signal. But then the participation rates cannot be equal. The formal proof (in Appendix A) runs along the same lines but applies to all situations in which $p_a \geq p_b$.

The workings of the proposition may be seen in the following example.

Example 1 *Consider an expected electorate* $n = 100$. *Suppose the signal precisions* $r = \frac{3}{4}$ *and* $s = \frac{2}{3}$ *and that the voting costs are distributed according to* $F(c) = c^{\frac{1}{3}}$. *Then* $p_a^* = 0.15$ *and* $p_b^* = 0.18$.

Figure 1 depicts the IRa and IRb curves for this example. Notice that neither curve defines a function. In particular, for some values of p_b, there are multiple solutions to IRa. To see why this is the case, notice that for a fixed p_b, when p_a is small there is little chance of a close election outcome and hence little benefit to voters with a signals of voting. As the proportion of voters with a signals who vote increases, the chances of a close election also increase and hence the benefits from voting rise. However, once p_a becomes relatively large, the chances of a close election start falling and, consequently, so do the benefits from voting.

4.1.2 Sincere Voting

In this subsection we establish that given the participation rates as determined above, it is a best-response for every voter to vote sincerely. As before, by this we mean that upon entering the voting booth, each voter behaves as if he were the only decision maker. Of course, as seen above, an individual's participation decision itself is influenced by the overall participation rates in the population.

Likelihood Ratios The following result is key in establishing this—it compares the likelihood ratio of α to β conditional on the event Piv_B to that conditional on the event Piv_A. It requires only that the voting behavior is such that expected number of votes for A is greater in state α than in state β and the reverse is true for B. While the lemma is more general, it is easy to see that sincere voting behavior satisfies the assumptions of the lemma.

Lemma 2 (Likelihood Ratio) *If voting behavior is such that* $\sigma_A > \tau_A$ *and* $\sigma_B < \tau_B$, *then*

$$\frac{\Pr[Piv_B \mid \alpha]}{\Pr[Piv_B \mid \beta]} > \frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} \tag{10}$$

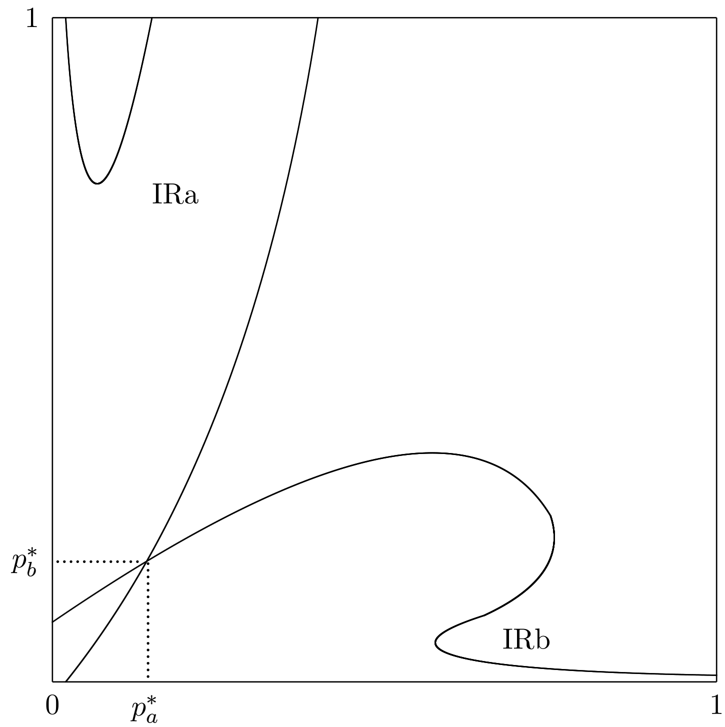

Figure 1: Equilibrium Participation Rates

Since $\sigma_A > \tau_A$ and $\sigma_B < \tau_B$, then, on "average," the ratio of A to B votes is higher in state α than in state β. Of course, voters' decisions do not depend on the average outcome, but rather on pivotal outcomes. The lemma shows that, even when one considers the set of "marginal" events where the vote totals are close (and a voter is pivotal), it is still the case that A is more likely to be leading in state α and more likely to be trailing in state β (details are provided in Appendix A).

Incentive Compatibility With the Likelihood Ratio Lemma in hand, we now examine the incentives to vote sincerely. Let (p_a^*, p_b^*) be equilibrium participation rates. A voter with signal a and cost $c_a^* = F^{-1}(p_a^*)$ is just indifferent between voting and staying home, that is,

$$q(\alpha \mid a) \Pr[Piv_A \mid \alpha] - q(\beta \mid a) \Pr[Piv_A \mid \beta] = F^{-1}(p_a^*) \tag{IRa}$$

We want to show that sincere voting is optimal for a voter with an a signal if others are voting sincerely. That is,

$$\begin{aligned} & q(\alpha \mid a)(\Pr[Piv_A \mid \alpha] - q(\beta \mid a) \Pr[Piv_A \mid \beta]) \\ & \geq q(\beta \mid a) \Pr[Piv_B \mid \beta] - q(\alpha \mid a) \Pr[Piv_B \mid \alpha] \end{aligned} \tag{ICa}$$

The left-hand side is the payoff from voting for A whereas the right-hand side is the payoff to voting for B.

Now notice that since $p_a^* > 0$, IRa implies

$$\frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} > \frac{q(\beta \mid a)}{q(\alpha \mid a)}$$

and so applying Lemma 2 it follows that,

$$\frac{\Pr[Piv_B \mid \alpha]}{\Pr[Piv_B \mid \beta]} > \frac{q(\beta \mid a)}{q(\alpha \mid a)}$$

which is equivalent to

$$q(\beta \mid a)\Pr[Piv_B \mid \beta] - q(\alpha \mid a)\Pr[Piv_B \mid \alpha] < 0$$

and so the payoff from voting for B with a signal of a is negative. Thus ICa holds. We have argued that if (p_a^*, p_b^*) are such that a voter with signal a and cost $F^{-1}(p_a^*)$ is just indifferent between participating or not, then all voters with a signals who have lower costs, have the incentive to vote sincerely. Recall that this was not the case under compulsory voting.

What about voters with b signals? Again, since (p_a^*, p_b^*) are equilibrium participation rates, then a voter with signal b and cost $c_b^* = F^{-1}(p_b^*)$ is just indifferent between voting and staying home, that is,

$$q(\beta \mid b)\Pr[Piv_B \mid \beta] - q(\alpha \mid b)\Pr[Piv_B \mid \alpha] = F^{-1}(p_b^*) \tag{IRb}$$

We want to show that a voter with signal b is better off voting for B over A, that is

$$\begin{aligned} &q(\beta \mid b)\Pr[Piv_B \mid \beta] - q(\alpha \mid b)\Pr[Piv_B \mid \alpha] \\ &\geq q(\alpha \mid b)\Pr[Piv_A \mid \alpha] - q(\beta \mid b)\Pr[Piv_A \mid \beta] \end{aligned} \tag{ICb}$$

As above, since $p_b^* > 0$, the left-hand side of ICb is strictly positive and Lemma 2 implies that the right-hand side is negative.

We have thus established,

Proposition 4 *Under voluntary participation, sincere voting is incentive compatible.*

Proposition 4 shows that it is optimal for each participating voter to vote according to his or her own private signal alone, provided that others are doing so. One may speculate that equilibrium participation rates are such that, conditional on being pivotal, the posterior assessment of α and β is 50-50. Thus, a voter's own signal "breaks the tie" and sincere voting is optimal. This simple intuition turns out to be incorrect, however. In Example 1, for instance, this posterior assessment favors state β slightly; that is, $\Pr[\alpha \mid Piv_A \cup Piv_B] < \frac{1}{2}$. But once a voter with an a signal takes his own information also into account, the posterior assessment favors α, that is, $\Pr[\alpha \mid a, Piv_A \cup Piv_B] > \frac{1}{2}$.

We have argued above that there exists an equilibrium with positive participation rates and sincere voting. We now show that *all* equilibria have these features. Our task is made somewhat easier since Poisson games have the nice feature that *all* equilibria must be symmetric (see Myerson, 1998). Asymmetric equilibria in voting models require common knowledge about the population size—a feature which is absent from Poisson models.

Now suppose that there is a symmetric equilibrium. It is easy to see that either those with a signals or those with b signals must vote sincerely. Let $U(A, a)$ denote payoff to a voter with an a signal from voting for A. Similarly, define $U(B, a)$, $U(A, b)$ and $U(B, b)$. If voters with b signals vote for A, then we have

$$U(A, a) > U(A, b) \geq U(B, b)$$

where the first inequality follows from the fact that all else being equal, voting for A must be better having received a signal in favor of A than a signal in favor of B. At the same time, if voters with a signals vote for B, we also have

$$U(B, b) > U(B, a) \geq U(A, a)$$

and the two inequalities contradict each other. To show that, in fact, both voter with both signals vote sincerely, notice that the Likelihood Ratio Lemma applies even when voting is insincere. However, the Likelihood Ratio condition then shows that it cannot be a best response for voters with either signal to vote insincerely (Lemma 7 in Appendix A). Thus all equilibria involve sincere voting.[11]

Proposition 5 *In any equilibrium under costly voting, there is positive participation and sincere voting.*

Unlike the case of compulsory voting, where strategic voters do not vote sincerely, under voluntary voting there is no tension between voting strategically and voting sincerely—voting is always sincere in equilibrium. Of course, there is a "price" to be paid for this sincerity—limited participation.

4.2 Costless Voting

The analysis of the costly voting model extends quite simply to the costless voting model. Now voters have three choices: vote for A, vote for B and abstain. None of these have any consequences as far as costs are concerned.

Notice that Proposition 4 still applies. As long as there is positive participation, all those who show up at the polls vote sincerely. Only the participation decisions differ. The Likelihood Ratio Lemma (Lemma 2) guarantees that if the payoff from voting for those with a signals is nonnegative, then the payoff from voting for those with b signals is *strictly* positive. It is easy to argue that full participation by both types cannot be an equilibrium (it is sufficient for this that n exceed the bound in

[11] We know of no examples with multiple sincere voting equilibria. Later we will establish that when n is large, there is indeed a unique equilibrium.

Proposition 1). Moreover, it is also easy to argue that it is never the case that only voters with b signals participate and those with a signals do not (the argument for positive participation is the same as in the proof of Proposition 3). Thus we have

Proposition 6 *In any equilibrium under costless voting in which there is full participation by voters with b signals, partial participation by those with a signals and sincere voting.*

In the next section, we compare welfare under voluntary and compulsory voting.

5 Welfare

The informational comparison between voluntary and compulsory voting is influenced by the following trade-off. Under voluntary voting, (i) not everyone votes; but (ii) everyone who votes, does so sincerely. On the other hand, under compulsory voting, (i) everyone votes; but (ii) voters do not vote sincerely (see Proposition 2). Put another way, under voluntary voting, there is less information provided but it is accurate. Under compulsory voting, there is more information provided but it is inaccurate. In what follows, we study this trade-off between the quality and quantity of information. We will show that the trade-off is always resolved in favor of quality over quantity—voluntary voting is welfare superior to compulsory voting.

In this section, we will suppose that voting costs are *zero*. Introducing voting costs adds an additional factor, a selection effect, which favors voluntary voting. This is because under voluntary voting only those with low realized costs turn out to vote and incur these costs whereas under compulsory voting *all* voters incur voting costs. Since we will show that voluntary voting is superior even when there are no costs, the ranking will obviously be unchanged if we introduce small voting costs. Specifically, for fixed n if we let the distribution of voting costs $F \to 0$, voluntary voting will remain superior.

When voting costs are zero, incentives of all voters are perfectly aligned—that is, voluntary voting results in a common interest game. In such games there always exists an efficient equilibrium (McLennan, 1998). In other words, there exists an equilibrium under voluntary voting (abstention) that induces participation and voting behavior that maximizes welfare.[12] Thus, the abstention option can never be harmful. We show below that, provided that n is large enough so that the abstention option is exercised, allowing abstention *strictly* improves welfare.

Proposition 7 *Suppose voting is costless and* $n \geq \frac{1}{2}\frac{\ln\sqrt{\frac{1-s}{s}}+\ln\sqrt{\frac{1-r}{r}}}{\sqrt{r(1-r)}-\sqrt{s(1-s)}}$. *Then the efficient equilibrium under voluntary voting (abstention) is strictly superior to any equilibrium under compulsory voting.*

[12] In large elections with voluntary voting, the efficient equilibrium is the only equilibrium. This can be shown using the calculations in Appendix 6.2.

Proof. First, note that under compulsory voting, all equilibria have the following structure: all voters with b signals vote for B while those with a signals vote for A with probability $\mu \leq 1$. Second, compulsory voting also results in a game of common interest but with the additional constraint that abstention is not permitted. We will show that if abstention is permitted then those with a signals will be strictly better off if they exercise this option. Thus, welfare with the abstention option will be strictly higher.

If $\mu < 1$, then it must be that those with a signals are indifferent between voting for A and voting for B. Thus we must have

$$\begin{aligned} &q(\alpha \mid a)\left(\Pr\left[Piv_A \mid \alpha\right] - q(\beta \mid a)\Pr\left[Piv_A \mid \beta\right]\right) \\ &= q(\beta \mid a)\Pr\left[Piv_B \mid \beta\right] - q(\alpha \mid a)\Pr\left[Piv_B \mid \alpha\right] \end{aligned}$$

or equivalently,

$$\frac{\Pr\left[Piv_A \mid \alpha\right] + \Pr\left[Piv_B \mid \alpha\right]}{\Pr\left[Piv_A \mid \beta\right] + \Pr\left[Piv_B \mid \beta\right]} = \frac{q(\beta \mid a)}{q(\alpha \mid a)}$$

It is routine to verify that $\sigma_A > \tau_A$ and $\sigma_B < \tau_B$. The Likelihood Ratio Lemma (Lemma 2) now implies that

$$\frac{\Pr\left[Piv_A \mid \alpha\right]}{\Pr\left[Piv_A \mid \beta\right]} < \frac{q(\beta \mid a)}{q(\alpha \mid a)}$$

and this may be rewritten as

$$q(\alpha \mid a)\Pr\left[Piv_A \mid \alpha\right] - q(\beta \mid a)\Pr\left[Piv_A \mid \beta\right] < 0$$

showing that if those with a signals mix, then they would rather abstain.

If $\mu = 1$, that is, if those with a signals also vote sincerely, then Proposition 1 implies that the constraint on abstention is binding. Again, welfare with the abstention option is strictly higher. ■

The proposition shows that decreased participation with sincere voting leads to better outcomes than full participation with insincere (strategic) voting. Indeed, the abstention option offers two benefits–it removes the conflict between strategic and sincere voting and, more importantly, it improves decision quality. For small committees, the welfare gains of permitting abstention can be substantial as the following example illustrates.

Example 2 *Suppose the signal precisions are $r = 0.9$ and $s = 0.6$. For $n \leq 25$, Figure 2 compares the welfare (the probability of a correct decision) from the unique equilibrium under voluntary voting to the welfare from the unique equilibrium under compulsory voting.*

For all $n \geq 2$, the welfare under voluntary voting is strictly higher than that under compulsory voting. For moderate sized committees, on the order of 5 to 15 members, voluntary voting raises the chance that the committee "gets it right" by

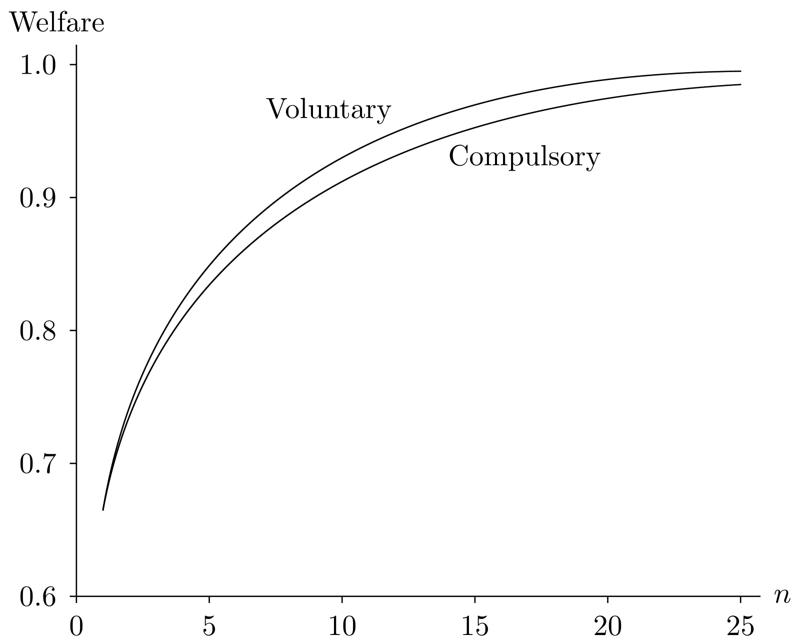

Figure 2: Welfare Comparison in Example 2

about 3%. The example also shows that the sufficient condition on n identified in the proposition is far from necessary. While the bound identified there is $n \geq 13$, the welfare improvement holds even for smaller n.

6 Large Elections

The previous section showed that when voting is costless, voluntary voting is superior to compulsory voting. However, when voting is costly, voluntary voting may suffer from another potential problem—the diminishing incentives to vote at all. As the size of the electorate grows large, each voter (correctly) perceives the chance of being pivotal as becoming vanishingly small and hence very little benefit to voting. While it is known that compulsory voting produces the correct outcome asymptotically despite the presence of strategic voting, the same need not be true of voluntary voting. One possibility is that, as n grows large, expected turnout out under voluntary voting could get "stuck" at some finite level. Still another possibility is that even if the expected turnout grows large, the turnout proportions of the two types may produce inefficient outcomes. Does voluntary voting aggregate information in the limit? Specifically, for a fixed distribution of costs, F, does information aggregate as $n \to \infty$?

We begin with participation rates. Together, Propositions 3 and 4 show that there exist a pair of positive equilibrium participation rates which induce sincere voting. We will show that although the participation rates go to zero as n increases, they do so sufficiently slowly so that the expected number of voters goes to infinity. As a first step we have[13]

Lemma 3 *In any sequence of equilibria, the participation rates tend to zero; that is,* $\limsup p_a(n) = \limsup p_b(n) = 0$.

To see why this is the case, suppose to the contrary that one or both types of voters participated at positive rates even in the limit. Then an infinite number of voters would turn out and the gross benefit to voting would go to zero since there is no chance that an individual's vote would be pivotal. Since voting is costly, a voter would be better off staying at home than voting under these circumstances. Of course, this contradicts the notion that participation rates are positive in the limit.

While Lemma 3 shows that, for a fixed cost distribution F, participation rates go to zero as the number of potential voters goes to infinity, there is, in fact, a race between the shrinking participation rates and the growing size of the electorate. A common intuition is that the outcome of this race depends on the shape of the cost distribution—particularly in the neighborhood of 0. As we show below, however, equilibria have the property that the number of voters (with either signal) becomes unbounded regardless of the shape of the cost distribution. In other words, the problem of too little participation does not arise in the limit—even though voting is voluntary and costly. Formally,

Proposition 8 *In any sequence of equilibria, the expected number of voters with either signal tends to infinity; that is,*

$$\liminf np_a(n) = \infty = \liminf np_b(n)$$

Proof. The proof is a direct consequence of Lemmas 10 and 11 in Appendix B. ■

On its face, the result seems intuitive. If there is only a finite turnout in expectation, then there is a positive probability that a voter is pivotal and, one might guess, this would mean that there is a positive benefit from voting; thus contradicting the idea that the cost thresholds go to zero in the limit. However, the mere fact of being pivotal with positive probability is no guarantee of a positive benefit from voting. It may well be that, conditional on being pivotal, the likelihood ratio is exactly 50-50 under sincere voting. In that case, there would be no benefit from voting whatsoever and hence the cost threshold would, appropriately, go to zero.

To gain some intuition for why this is never the case, it is helpful to consider what happens when a and b signals are equally precise, that is, when $r = s$. It is easy to see that in that case, the participation rates for a and b voters will be the same, and hence the likelihood of a given state will depend only on the relative vote

[13] Unless otherwise specified, all limits are taken as $n \to \infty$.

totals. Consider a voter with an a signal when aggregate turnout is finite. This voter is pivotal under two circumstances—when A is behind by a vote and when the vote total is tied. When A is behind by a vote, the inclusion of the voter's own a signal leads to a 50-50 likelihood of α versus β. In other words, when the voter includes her own signal, these events are not decisive as to the likelihood of α versus β. When the vote total is tied, the likelihood ratio favors α. Thus, the overall likelihood ratio favors α.

Of course, when signal precisions are not the same, turnout rates are no longer equal and the inference from the vote totals is more complicated. However, when voting is efficient (that is, A is more likely to win in state α), then the same basic intuition obtains. Voters endogenously participate in such a way that the likelihood ratios turn on the tie events rather than on the events in which A is either ahead or behind by one vote. As a consequence, the likelihood ratio for a voter with an a signal favors α and hence there is a strictly positive benefit to voting. This, in turn, implies that the expected number of voters becomes unbounded. For the inefficient case, the argument is more delicate. The formal proof, which is somewhat involved, shows, however, that the likelihood ratio cannot be 50-50 for both sides.

We now turn to the question of whether the equilibrium is efficient under costly voting. In other words, is it the case that in large elections, the "right" candidate is elected? One may have thought that we have, in effect, already answered this question (in the affirmative) by showing that voting is sincere and expected participation is unbounded in large elections. However, this ignores that the fact that voters with different signals turn out at different rates. If turnout is too lop-sided in favor of B versus A, then even with sincere voting, the election could still fail to choose the "right" candidate.

6.1 Information Aggregation

In large elections, candidate A is chosen in state α if and only if $rp_a > (1-r)\,p_b$ and candidate B is chosen in state β if and only if $(1-s)\,p_a < sp_b$. Information aggregation thus requires that for large n, the equilibrium participation rates satisfy

$$\frac{1-r}{r} < \frac{p_a}{p_b} < \frac{s}{1-s} \tag{11}$$

First, recall from Lemma 1 that any solution to the threshold equations satisfies $p_a < p_b$. Thus in large elections, in equilibrium, voters with b signals turn out to vote at higher rates than do those with a signals. Since $s > \frac{1}{2}$, this implies that the second inequality holds and so in large elections, B wins in state β with probability 1.

In state α, however, the larger turnout for B is detrimental. We now argue that in large elections, the first inequality also holds.

First, note that since with sincere voting, it follows from Lemma 11 (in Appendix B) that

$$\limsup \left(\frac{\sigma_A}{\sigma_B}\right)^{\pm\frac{1}{2}} < \infty \text{ and } \limsup \left(\frac{\tau_A}{\tau_B}\right)^{\pm\frac{1}{2}} < \infty$$

Second, the approximation in (5) implies that if $\sqrt{\sigma_A \sigma_B} \to \infty$, as $n \to \infty$ then, for large n

$$\Pr[T \mid \alpha] \approx \frac{e^{-\left(\sigma_A+\sigma_B-2\sqrt{\sigma_A\sigma_B}\right)}}{\sqrt{4\pi\sqrt{\sigma_A\sigma_B}}} = \frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{\sqrt{4\pi\sqrt{\sigma_A\sigma_B}}} \tag{12}$$

Also, the probability of "offset" events of the form T_{+1} or T_{-1} can be approximated as follows

$$\Pr[T_{\pm 1} \mid \alpha] \approx \Pr[T \mid \alpha] \times \left(\frac{\sigma_A}{\sigma_B}\right)^{\pm\frac{1}{2}} \tag{13}$$

And of course, the corresponding probabilities in state β can again be approximated by substituting τ for σ. The probabilities of the pivotal events defined in Section 2 can then be approximated by using (12) and (13).[14] In state α,

$$\Pr[Piv_A \mid \alpha] \approx \tfrac{1}{2}\Pr[T \mid \alpha] \times \left(1+\sqrt{\tfrac{\sigma_B}{\sigma_A}}\right) \tag{14}$$

$$\Pr[Piv_B \mid \alpha] \approx \tfrac{1}{2}\Pr[T \mid \alpha] \times \left(1+\sqrt{\tfrac{\sigma_A}{\sigma_B}}\right) \tag{15}$$

Again, the pivot probabilities in state β can similarly be obtained by substituting τ for σ.

Hence in the expressions for the pivot probabilities (specifically, (14) and the corresponding formula in state β), the exponential terms dominate in the limit. Thus we have

$$\frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} = \frac{e^{-\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2}}{e^{-\left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2}} \times K(\sigma_A, \sigma_B, \tau_A, \tau_B)$$

where K is a function that stays finite in the limit. Thus it must be the case that in the limit

$$\left(\sqrt{\sigma_A}-\sqrt{\sigma_B}\right)^2 = \left(\sqrt{\tau_A}-\sqrt{\tau_B}\right)^2 \tag{16}$$

In particular, suppose that the left-hand side of (16) was greater than the right-hand side. In that case,

$$\lim \frac{\Pr[Piv_A \mid \alpha]}{\Pr[Piv_A \mid \beta]} = 0$$

and it would then follow that state β is infinitely more likely in the event Piv_A than is state α. This, however, would imply that the gross benefit to a voter with signal a from voting is negative, which contradicts Lemma 3. Similarly, if the left-hand side was smaller then it would then follow that state α is infinitely more likely in the event Piv_B than is state β. This, however, would then imply that the gross benefit to a voter with signal b from voting is negative, which also contradicts Lemma 3. Thus (16) must hold in the limit.

Under sincere voting $\sigma_A = nrp_a$; $\sigma_B = n(1-r)p_b$; $\tau_A = n(1-s)p_a$ and $\tau_B = nsp_b$, and so (16) can be rewritten as

$$\sqrt{s} - \sqrt{1-s}\sqrt{\frac{p_a}{p_b}} \approx \pm\left(\sqrt{r}\sqrt{\frac{p_a}{p_b}} - \sqrt{1-r}\right)$$

[14] The approximation formulae for the pivot probabilities also follow from Myerson (2000).

and the left-hand side is positive since $p_b > p_a$. Now observe that if $(1-r)\,p_b \geq rp_a$, then we have

$$\sqrt{s} - \sqrt{1-s}\sqrt{\tfrac{p_a}{p_b}} \approx \sqrt{1-r} - \sqrt{r}\sqrt{\tfrac{p_a}{p_b}}$$

and this is impossible since both r and s are greater than $\frac{1}{2}$ (Lemma 11 in Appendix B ensures that $\frac{p_a}{p_b}$ is bounded). Thus we must have, that for large n, $rp_a > (1-r)\,p_b$.

We have thus shown that information fully aggregates in large elections.

Proposition 9 *In any sequence of equilibria, the probability that the right candidate is elected in each state (A in state α and B in β) goes to one.*

We are now in a position to make a welfare comparison between voluntary and compulsory voting when voting is costly. Under both systems, when n is large, the right candidate is elected in each state. But voluntary voting has a second obvious benefit. It economizes on costs—only those with low voting costs show up to vote. Compulsory voting imposes costs on all voters. Since there is no difference in outcomes, the cost component dominates. Thus, we have

Proposition 10 *Under costly voting, when n is large, voluntary voting is superior to compulsory voting.*

While it is nice to know that voluntary voting selects the correct candidate in large elections, it is useful to know the margin of victory of the better candidate. If the margin is small, then perhaps some unmodeled aspects of the voting process might interfere and tip the outcome in favor of the wrong candidate, thereby altering the welfare ordering. The equilibrium condition for voluntary voting in large elections implies that

$$\sqrt{s} - \sqrt{1-s}\sqrt{\tfrac{p_a}{p_b}} \approx \sqrt{r}\sqrt{\tfrac{p_a}{p_b}} - \sqrt{1-r}$$

and so we obtain that ratio of the participation probabilities satisfies

$$\lim \sqrt{\frac{p_a}{p_b}} = \frac{\sqrt{1-r} + \sqrt{s}}{\sqrt{r} + \sqrt{1-s}} \tag{17}$$

This estimate can be used to determine the margin of victory for the better candidate in large elections under voluntary voting. In state α, the ratio of the vote totals in large elections is easily shown to be

$$\frac{rp_a}{(1-r)\,p_b} \approx \left(\frac{1+\sqrt{\frac{s}{1-r}}}{1+\sqrt{\frac{1-s}{r}}}\right)^2$$

whereas in state β it is

$$\frac{sp_b}{(1-s)\,p_a} \approx \left(\frac{1+\sqrt{\frac{r}{1-s}}}{1+\sqrt{\frac{1-r}{s}}}\right)^2$$

Thus the election is not close in either state. The more precise are the signals, the larger is the margin of victory.

Clearly, our asymptotic results rely on the assumption that the lower support of the distribution of costs is zero. The results do not, however, depend on the shape of the distribution F of costs in the neighborhood of zero. In particular, one may have surmised that it was important that there was a sufficient mass of voters with costs close to zero, as in other voting models (for instance, Martinelli, 2006). Obviously, if the lower support of F, say $\underline{c}$, is strictly positive, for the usual "paradox of voting" reasons, turnout will be limited, and there is no hope that information will aggregate. The correct policy response in this case is not to impose compulsory voting, rather the optimal policy is to impose a fine of $\underline{c}$ for not voting with the proceeds redistributed in lump-sum fashion. This effectively shifts the lower support of F to zero and hence results in full information aggregation in large elections. Compared to voluntary voting, welfare is enhanced because the election outcomes are now efficient and the *per capita* costs of voting are close to zero.

6.2 Uniqueness

Equilibrium multiplicity is a common difficulty encountered in voting models. In this section, we show that with voluntary and costly voting, there is a unique equilibrium when n is large. Recall that the equilibrium derived in the previous sections has the following features: (i) voting is sincere; and (ii) the cost thresholds are determined by IRa and IRb.

We have already shown that all equilibria involve sincere voting (Proposition 5). The final step is to show that when n is large, there is a unique solution to the cost thresholds. We know that in the limit, all sincere voting equilibria are efficient: A wins in state α and B wins in state β. Thus, for large n, the equilibrium participation probabilities satisfy (11). It can be shown that for any pair of participation probabilities satisfying (11), the IRa curve is steeper than the IRb curve (Lemma 12 in Appendix C). Thus they can intersect only once and so we obtain,

Proposition 11 *In large elections with costly voting, there is a unique equilibrium.*

Proof. See Appendix C. ■

7 Partisans

Under compulsory voting, the asymmetry of signals in our model ($r \neq s$) leads voters to vote insincerely (Proposition 2). Voluntary voting, by permitting voters to express preferences with their feet as well as at the polls, restores the incentives to vote sincerely (Proposition 4). Other kinds of asymmetry have the same effect under compulsory voting. For instance, introducing partisans—voters favoring one candidate (say A) regardless of the state—also destroys the incentives to vote sincerely under compulsory voting even when signals are symmetric. In this section, we briefly discuss whether voluntary voting can correct incentives in this case as well.

Consider the following amendment to our basic model. With probability λ, a voter is an A-partisan, who gets a payoff of 1 if candidate A is elected, regardless of the state. Partisans With probability $1-\lambda$, a voter has payoffs as specified in earlier sections. Other than their preferences, partisans are like other voters—they receive signals and draw costs from the same distribution. Let $\overline{p}_a$ and $\overline{p}_b$ denote the turnout rates for A-partisans who receive a signal of a and b, respectively. Now the four equilibrium turnout rates, p_a and p_b for non-partisans and $\overline{p}_a$ and $\overline{p}_b$ for partisans, are determined by four threshold conditions: (IRa), (IRb) for the nonpartisans and

$$\begin{aligned} q(\alpha \mid a)\Pr[Piv_A \mid \alpha] + q(\beta \mid a)\Pr[Piv_A \mid \beta] &= F^{-1}(\overline{p}_a) \\ q(\alpha \mid b)\Pr[Piv_A \mid \alpha] + q(\beta \mid b)\Pr[Piv_A \mid \beta] &= F^{-1}(\overline{p}_b) \end{aligned}$$

for the partisans. Notice that, when they are pivotal for A, partisans get a positive payoff from voting for A even if the state is β.

Here we do not carry out a full analysis of a model with partisan voters, largely because of the additional complexity this entails. But as the following variant of Example 1 indicates, sincere voting occurs even in the presence of partisans.

Example 3 *As in Example 1, suppose* $n = 100$*,* $r = \frac{3}{4}$ *and* $s = \frac{2}{3}$ *and that the voting costs are distributed according to* $F(c) = c^{\frac{1}{3}}$*. Suppose that the probability of being a partisan is* $\lambda = 0.05$*. Then there exists an equilibrium in which all non-partisans vote sincerely and turn out at rates* $p_a^* = 0.14$ *and* $p_b^* = 0.20$*. Partisans turn out at rates* $\overline{p}_a^* = 0.28$ *and* $\overline{p}_b^* = 0.31$*.*

8 Conclusions

In situations where informed voters have a common interest in making the right decision, we have shown that mandatory voting requirements and the elimination or suppression of the option to abstain are positively harmful. On informational grounds, voting should be a right rather than a duty. Many situations involve common interests: In committee-like settings there are votes by corporate boards of directors with a shared interest in the profitability of the company, votes for hiring and promotion in university settings, and votes taken in judiciary bodies such as state or federal supreme courts. In election settings, there are votes to retain or select judges, votes for administrative functions such as comptroller or solicitor, and votes on various ballot initiatives such as bond measures.

Of course, there are other situations in which the common interests assumption may not hold. In most elections for legislative office, the ideology of the candidates is an important consideration over which voters are unlikely to hold common interests. When ideology plays only a modest role in payoffs, our results are still valid. Even when ideology plays a large role, voluntary voting continues to be best (Krishna and Morgan, 2010). In particular, the key intuition that differential participation provides an important channel facilitating information aggregation holds quite generally.

Another important consideration outside the scope of the model is the decision by voters to become informed in the first place. It is sometimes argued that compulsory voting provides greater incentives in this regard though we know of no formal model showing this. Whether this is, in fact, the case, is far from clear. Specifically, because of the greater turnout under compulsory voting, the chances that an individual voter is decisive are lower than under voluntary voting. Since the "investment" in information is only valuable in these situations, it might well be the case that voluntary voting offers better incentives in this regard. Endogenizing the information acquisition decision is clearly an important next step, but beyond the scope of the present analysis.[15]

Rational choice models of voting behavior have long been criticized on behavioral grounds. They require voters to employ mixed strategies, they imply that swing voters would prefer not to come to the polls, and when voting is costly, they beg the question as to why anyone should bother to vote at all.

Many of these problems disappear if one amends the standard model to allow for realistic features such as the possibility of abstention and heterogeneous costs of going to the polls. With these additions, there is no longer a conflict between sincere and strategic voting and swing voters willingly participate. Moreover, voting in large elections nearly always produces the "right" outcome.

A Appendix: Equilibrium

Proof of Proposition 3. It is useful to rewrite IRa and IRb in terms of threshold costs rather than participation probabilities. Let $V_a(c_a, c_b)$ denote the payoff to a voter with signal a from voting for A when the two threshold costs are $c_a = F^{-1}(p_a)$ and $c_b = F^{-1}(p_b)$; that is, $V_a(c_a, c_b) \equiv U_a(F(c_a), F(c_b))$. Similarly, let $V_b(c_a, c_b) \equiv U_b(F(c_a), F(c_b))$. We will show that there exist $(c_a, c_b) \in (0,1)^2$ such that $V_a(c_a, c_b) = c_a$ and $V_b(c_a, c_b) = c_b$.

The function $V = (V_a, V_b) : [0,1]^2 \to [-1,1]^2$ maps a pair of threshold costs to a pair of payoffs from voting sincerely. Note that payoffs may be negative.

Consider the function $V^+ : [0,1]^2 \to [0,1]^2$ defined by

$$\begin{aligned} V_a^+(c_a, c_b) &= \max\{0, V_a(c_a, c_b)\} \\ V_b^+(c_a, c_b) &= \max\{0, V_b(c_a, c_b)\} \end{aligned}$$

Since V is a continuous function, V^+ is also continuous and so by Brouwer's Theorem V^+ has a fixed point, say $(c_a^*, c_b^*) \in [0,1]^2$.

We argue that c_a^* and c_b^* are strictly positive. Suppose that $c_a^* = 0$. Then $p_a^* = F(c_a^*)$ is also zero and so there are no voters with a signals who vote. Consider an individual who receives a signal of a. The only events in which a vote for A is pivotal is if either (i) no voters with b signals show up to vote; or (ii) a single voter with a b

[15] See Persico (2004), Martinelli (2006) and Oliveros (2009) for work in this direction.

signal shows up. Thus

$$\begin{aligned}\Pr\left[Piv_A \mid \alpha\right] &= \tfrac{1}{2}e^{-n(1-r)p_b^*}\left(1+n\left(1-r\right)p_b^*\right) \\ \Pr\left[Piv_A \mid \beta\right] &= \tfrac{1}{2}e^{-nsp_b^*}\left(1+nsp_b^*\right)\end{aligned}$$

where $p_b^* = F\left(c_b^*\right)$. We claim that $\Pr\left[Piv_A \mid \alpha\right] > \Pr\left[Piv_A \mid \beta\right]$. This follows from the fact that the function $e^{-x}\left(1+x\right)$ is strictly decreasing for $x > 0$ and $s > 1 - r$. Hence, if $p_a^* = 0$

$$q\left(\alpha \mid a\right)\Pr\left[Piv_A \mid \alpha\right] - q\left(\beta \mid a\right)\Pr\left[Piv_A \mid \beta\right] > 0$$

since $q\left(\alpha \mid a\right) > \frac{1}{2}$. Since $c_a^* = 0$, this is equivalent to

$$V_a^+\left(c_a^*, c_b^*\right) > c_a^*$$

contradicting the assumption that $\left(c_a^*, c_b^*\right)$ was a fixed point. Thus $c_a^* > 0$.

A similar argument shows that $c_b^* > 0$.

Since both c_a^* and c_b^* are strictly positive, we have that

$$V^+\left(c_a^*, c_b^*\right) = V\left(c_a^*, c_b^*\right) = \left(c_a^*, c_b^*\right)$$

Thus $\left(c_a^*, c_b^*\right)$ is also a fixed point of V and so solves IRa and IRb.

Next, notice that at any point $\left(1, p_b\right)$

$$q\left(\alpha \mid a\right)\Pr\left[Piv_A \mid \alpha\right] - q\left(\beta \mid a\right)\Pr\left[Piv_A \mid \beta\right] < 1$$

Thus if $\left(c_a^*, c_b^*\right)$ is a fixed point of V then we also have that both c_a^* and c_b^* are also less than one. ■

Proof of Lemma 1. We claim that if $p_a \geq p_b$, then $U_a\left(p_a, p_b\right) < U_b\left(p_a, p_b\right)$. A rearrangement of the relevant expressions shows that $U_a\left(p_a, p_b\right) < U_b\left(p_a, p_b\right)$ is equivalent to

$$\left(q\left(\alpha \mid a\right) + q\left(\alpha \mid b\right)\right)\Pr\left[T \mid \alpha\right] + q\left(\alpha \mid a\right)\Pr\left[T_{-1} \mid \alpha\right] + q\left(\alpha \mid b\right)\Pr\left[T_{+1} \mid \alpha\right] \tag{18}$$

being less than

$$\left(q\left(\beta \mid b\right) + q\left(\beta \mid a\right)\right)\Pr\left[T \mid \beta\right] + q\left(\beta \mid a\right)\Pr\left[T_{-1} \mid \beta\right] + q\left(\beta \mid b\right)\Pr\left[T_{+1} \mid \beta\right] \tag{19}$$

We will show that each term in (18) is less than the corresponding term in (19).

With sincere voting, $\sigma_A = nrp_a$, $\sigma_B = n\left(1-r\right)p_b$, $\tau_A = n\left(1-s\right)p_a$ and $\tau_B = nsp_b$.

First, since $r > s > \frac{1}{2}$, we have $\sigma_A\sigma_B < \tau_A\tau_B$ and since $p_a \geq p_b$, $\sigma_A + \sigma_B \geq$

$\tau_A + \tau_B$. Thus,

$$\begin{aligned}\Pr[T \mid \alpha] &= e^{-\sigma_A - \sigma_B} \sum_{k=0}^{\infty} \frac{\sigma_A^k}{k!} \frac{\sigma_B^k}{k!} \\ &< e^{-\tau_A - \tau_B} \sum_{k=0}^{\infty} \frac{\tau_A^k}{k!} \frac{\tau_B^k}{k!} \\ &= \Pr[T \mid \beta]\end{aligned}$$

It is also easily verified that $q(\alpha \mid a) + q(\alpha \mid b) < q(\beta \mid b) + q(\beta \mid a)$.

Second, since $r > s > \frac{1}{2}$, we have for all $k \geq 1$, $r\sigma_A^{k-1}\sigma_B^k < (1-s)\tau_A^{k-1}\tau_B^k$. Thus,

$$\begin{aligned}q(\alpha \mid a)\Pr[T_{-1} \mid \alpha] &= e^{-\sigma_A - \sigma_B} \frac{r}{r+1-s} \sum_{k=1}^{\infty} \frac{\sigma_A^{k-1}}{(k-1)!} \frac{\sigma_B^k}{k!} \\ &< e^{-\tau_A - \tau_B} \frac{1-s}{r+1-s} \sum_{k=1}^{\infty} \frac{\tau_A^{k-1}}{(k-1)!} \frac{\tau_B^k}{k!} \\ &= q(\beta \mid a)\Pr[T_{-1} \mid \beta]\end{aligned}$$

Third, a similar argument establishes that

$$q(\alpha \mid b)\Pr[T_{+1} \mid \alpha] < q(\beta \mid b)\Pr[T_{+1} \mid \beta]$$

Combining these three facts establishes that (18) is less than (19)

This means that if $p_a^* \geq p_b^*$, then (p_a^*, p_b^*) cannot satisfy IRa and IRb. Thus $p_a^* < p_b^*$. ■

Proof of Lemma 2. Consider the functions

$$\begin{aligned}G(x,y) &= I_0(z) + \sqrt{\tfrac{y}{x}} I_1(z) \\ H(x,y) &= I_0(z) + \sqrt{\tfrac{x}{y}} I_1(z)\end{aligned}$$

where $z = 2\sqrt{xy}$. Note that inequality (10) is equivalent to

$$\frac{G(\tau_A, \tau_B)}{H(\tau_A, \tau_B)} > \frac{G(\sigma_A, \sigma_B)}{H(\sigma_A, \sigma_B)} \tag{20}$$

We will argue that G/H is decreasing in x and increasing in y. Since $\sigma_A > \tau_A$ and $\sigma_B < \tau_B$, this will establish the inequality (20).

It may be verified that

$$\begin{aligned}
HG_x - GH_x &= \left(I_0(z) + \sqrt{\tfrac{x}{y}} I_1(z)\right)\left(\tfrac{y}{x} I_0(z) + \left(1 - \tfrac{1}{x}\right)\sqrt{\tfrac{y}{x}} I_1(z)\right) \\
&\quad - \left(I_0(z) + \sqrt{\tfrac{y}{x}} I_1(z)\right)^2 \\
&= -\tfrac{1}{x}\left((y-x)\left(I_1(z)^2 - I_0(z)^2\right) + \sqrt{\tfrac{y}{x}} I_0(z) I_1(z) + I_1(z)^2\right) \\
&= -\tfrac{1}{x} g(x,y)
\end{aligned}$$

where

$$g(x,y) = (y-x)\left(I_1(z)^2 - I_0(z)^2\right) + \sqrt{\tfrac{y}{x}} I_0(z) I_1(z) + I_1(z)^2$$

We claim that $g(x,y) > 0$, whenever x and y are positive. Note that for any $y > 0$,

$$\lim_{x \to 0} g(x,y) = 0$$

Some routine calculations show that

$$g_x(x,y) = \left(I_0(z) + \sqrt{\tfrac{y}{x}} I_1(z)\right)^2 + \left(I_0(z)^2 - I_1(z)^2\right) - \frac{1}{x} g(x,y)$$

Thus, if $g(x,y) \leq 0$, then $g_x(x,y) > 0$ (recall that $I_0(z) > I_1(z)$). This implies that for all $x > 0$, $g(x,y) > 0$ and so $HG_x - GH_x < 0$.

It may also be verified that

$$\begin{aligned}
HG_y - GH_y &= \left(I_0(z) + \sqrt{\tfrac{x}{y}} I_1(z)\right)^2 \\
&\quad - \left(I_0(z) + \sqrt{\tfrac{y}{x}} I_1(z)\right)\left(\tfrac{x}{y} I_0(z) + \left(1 - \tfrac{1}{y}\right)\sqrt{\tfrac{x}{y}} I_1(z)\right) \\
&= \tfrac{1}{y}\left((x-y)\left(I_1(z)^2 - I_0(z)^2\right) + \sqrt{\tfrac{x}{y}} I_0(z) I_1(z) + I_1(z)^2\right) \\
&= \tfrac{1}{y} h(x,y)
\end{aligned}$$

where $h(x,y) = g(y,x)$. The same reasoning now shows that so for all $y > 0$, $HG_y - GH_y > 0$.

This completes the proof. ■

Proof of Proposition 5 We first show that all equilibria involve positive participation by both types.

Lemma 4 *In any equilibrium, $p_a > 0$ and $p_b > 0$.*

Proof. Suppose by way of contradiction that $p_a = 0$ and $p_b > 0$. Then from Proposition 3, we know that voting cannot be sincere. Clearly, it cannot be that voters with b signals all vote for A with probability 1. So suppose that voters with b signals vote for A with probability $\mu < 1$.

In that case, the gross payoff for voters with b signals (not including costs of voting) of voting for A, denoted by $U(A,b)$, must be the same as the gross payoff from voting for B, denoted by $U(B,b)$. Since voters with b signals have positive participation levels,

$$U(A,b) = U(B,b) > 0$$

But now the gross payoff to a voter with an a signal from voting for A must be larger, that is,

$$U(A,a) > U(A,b) > 0$$

contradicting the assumption that $p_a = 0$.

The proof for the case when $p_a > 0$ and $p_b = 0$ is analogous. ■

We now turn to voting behavior using the fact that both types participate with positive probability. We first rule out equilibria in which voters with a signals and voters with b signals both vote against their own signals with positive probability.

Lemma 5 *In any equilibrium, either voters with a signals or voters with b signals vote sincerely.*

Proof. Suppose to the contrary that neither vote sincerely.

Let $U(A,a)$ denote the gross payoff (not including costs of voting) of voting for A to a voter with an a signal. Similarly, define $U(B,a)$, $U(A,b)$ and $U(B,b)$.

Then we have that

$$U(A,a) > U(A,b) \geq U(B,b)$$

where the first inequality follows from the fact that all else being equal, a vote of A is more valuable with signal a than with with signal b. The second inequality follows from the fact that, by assumption, voters with b signals vote for A with positive probability.

On the other hand, similar reasoning leads to

$$U(B,b) > U(B,a) \geq U(A,a)$$

and the two inequalities contradict each other. Hence it cannot be that voters with both signals vote insincerely. ■

Lemma 6 *There cannot be an equilibrium in which both types always vote for the same candidate.*

Proof. Suppose that all voters vote for A (say). Then we have that

$$U(A,a) > U(A,b) \geq U(B,b) > U(B,a)$$

Moreover, since voters with b signals participate,

$$\begin{aligned} U(A,b) &= q(\alpha \mid b) \Pr[Piv_A \mid \alpha] - q(\beta \mid b) \Pr[Piv_A \mid \beta] \\ &= q(\alpha \mid b) \tfrac{1}{2} e^{-n(rp_a + (1-r)p_b)} - q(\beta \mid b) \tfrac{1}{2} e^{-n((1-s)p_a + sp_b)} \\ &\geq 0 \end{aligned}$$

since the only circumstances in which a vote for A is pivotal is if no one else shows up.

Since $U(A,a) > U(A,b)$, and voters with both signals vote for A, the gross benefits from voting for those with signal a is higher than the gross benefits for those with signal b. Thus, the participation rate for those with a signals must be higher than for those with b signals. Thus, we have that $p_a > p_b$. Since $r > 1 - s$, this implies that $rp_a + (1-r)p_b > (1-s)p_a + sp_b$ contradicting the inequality above. ■

Lemma 7 *In any equilibrium, voting is sincere.*

Proof. Lemmas 5 and 6 imply that any equilibrium must have the following form: voters with one of the signals vote sincerely and those with the other signal vote according to their signals with positive probability.

First, suppose that voter with a signals vote for A and those with signals b vote for B with probability $\mu < 1$. In this case,

$$\begin{array}{ll} \sigma_A = n\left(rp_a + (1-r)\left(1-\mu\right)p_b\right); & \sigma_B = n\left(1-r\right)\mu p_b \\ \tau_A = n\left(\left(1-s\right)p_a + s\left(1-\mu\right)p_b\right); & \tau_B = ns\mu p_b \end{array} \tag{21}$$

Since those with b signals are indifferent between voting for A and voting for B, we have

$$0 \leq U\left(B,b\right) = U\left(A,b\right) < U\left(A,a\right)$$

where the inequality follows from the fact that, all else being equal, the payoff from voting for A when the signal is a is higher than when the signal is b. Thus the gross payoff of those with b signals is lower than the gross payoff of those with a signals and so $p_b < p_a$. If $p_b < p_a$, then using (21), it is easy to verify that $\sigma_A > \tau_A$ and $\sigma_B < \tau_B$. Hence voting behavior in any such equilibrium satisfies the conditions of the Likelihood Ratio Lemma (Lemma 2). The gross payoff to a voter with a b signal from voting is

$$U\left(B,b\right) = q\left(\beta \mid b\right)\Pr\left[Piv_B \mid \beta\right] - q\left(\alpha \mid b\right)\Pr\left[Piv_B \mid \alpha\right] \geq 0$$

where the pivot probabilities are computed using the expected vote totals in (21). The inequality $U\left(B,b\right) \geq 0$ may be rewritten as

$$\frac{\Pr\left[Piv_B \mid \beta\right]}{\Pr\left[Piv_B \mid \alpha\right]} \geq \frac{q\left(\alpha \mid b\right)}{q\left(\beta \mid b\right)}$$

Lemmas 2 then implies that,

$$\frac{\Pr\left[Piv_A \mid \beta\right]}{\Pr\left[Piv_A \mid \alpha\right]} > \frac{q\left(\alpha \mid b\right)}{q\left(\beta \mid b\right)}$$

which is equivalent to

$$U\left(A,b\right) = q\left(\alpha \mid b\right)\Pr\left[Piv_A \mid \alpha\right] - q\left(\beta \mid b\right)\Pr\left[Piv_A \mid \beta\right] < 0$$

which is a contradiction.

Second, suppose that voters with b signals vote sincerely and voters with a signals vote sincerely with probability $\mu < 1$. In this case,

$$\begin{array}{ll} \sigma_A = nr\mu p_a; & \sigma_B = n\left(r\left(1-\mu\right)p_a + \left(1-r\right)p_b\right) \\ \tau_A = n\left(1-s\right)\mu p_a; & \tau_B = n\left(\left(1-s\right)\left(1-\mu\right)p_a + sp_b\right) \end{array} \tag{22}$$

An analogous argument shows that now $p_b > p_a$ and again the conditions of Lemma 2 are satisfied. As above, this implies that voters with a signals cannot be indifferent.

■

B Appendix: Large Elections

Proof of Lemma 3. Suppose to the contrary, that for some sequence, $\lim c_a\left(n\right) > 0$. In that case, the gross benefits (excluding the costs of voting) to voters with a signals from voting must be positive; that is

$$\lim\left(q\left(\alpha \mid a\right)\Pr\left[Piv_A \mid \alpha\right] - q\left(\beta \mid a\right)\Pr\left[Piv_A \mid \beta\right]\right) > 0$$

where it is understood that the probabilities depend on n.

We know that along the given sequence, $\lim p_a\left(n\right) > 0$. This implies that $\lim \sigma_A\left(n\right) = \lim nrp_a\left(n\right) = \infty$.

First, suppose that there is a subsequence along which $\lim \sqrt{\sigma_A\sigma_B} < \infty$. In that case,

$$\Pr\left[Piv_A \mid \alpha\right] = \tfrac{1}{2}e^{-\sigma_A-\sigma_B}\left(I_0\left(2\sqrt{\sigma_A\sigma_B}\right) + \sqrt{\tfrac{\sigma_B}{\sigma_A}}I_1\left(2\sqrt{\sigma_A\sigma_B}\right)\right)$$

and since $\lim\left(e^{-\sigma_A}/\sqrt{\sigma_A}\right) = 0$ and $\limsup e^{-\sigma_B}\sqrt{\sigma_B} < \infty$, along any such subsequence,

$$\lim \Pr\left[Piv_A \mid \alpha\right] = 0$$

Second, suppose that there is a subsequence along which $\lim \sqrt{\sigma_A\sigma_B} = \infty$. In that case,

$$\Pr\left[Piv_A \mid \alpha\right] \approx \tfrac{1}{2}\frac{e^{-\left(\sigma_A+\sigma_B-2\sqrt{\sigma_A\sigma_B}\right)}}{\sqrt{4\pi\sqrt{\sigma_A\sigma_B}}}\left(1+\sqrt{\tfrac{\sigma_B}{\sigma_A}}\right)$$

Notice that the denominator is unbounded while the numerator is always bounded. Hence, along any such subsequence,

$$\lim \Pr\left[Piv_A \mid \alpha\right] = 0$$

An identical argument applies for $\tau_A\left(n\right)$ and $\tau_B\left(n\right)$. Therefore,

$$\lim \Pr\left[Piv_A \mid \beta\right] = 0$$

But this means that the gross benefit of voting for A when the signal is a tends to zero. This contradicts the assumption that $\lim c_a\left(n\right) > 0$. ■

Proof of Proposition 8. The result is a consequence of a series of lemmas.

Lemma 8 *Suppose that there is a sequence of sincere voting equilibria such that* $\lim np_a(n) = n_a < \infty$ *and* $\lim np_b(n) = n_b < \infty$. *If* $rn_a \geq (1-r)\,n_b$ *and* $U_b = 0$, *then* $U_a > 0$.

Proof. The condition that $U_b = 0$ is equivalent to

$$s\Pr[Piv_B \mid \beta] = (1-r)\Pr[Piv_B \mid \alpha]$$

whereas $U_a > 0$ is equivalent to

$$r\Pr[Piv_A \mid \alpha] > (1-s)\Pr[Piv_A \mid \beta]$$

We will argue that

$$\frac{r\Pr[Piv_A \mid \alpha]}{(1-s)\Pr[Piv_A \mid \beta]} > \frac{(1-r)\Pr[Piv_B \mid \alpha]}{s\Pr[Piv_B \mid \beta]}$$

or equivalently,

$$\frac{rn_a\left(\Pr[T \mid \alpha] + \Pr[T_{-1} \mid \alpha]\right)}{(1-s)\,n_a\left(\Pr[T \mid \beta] + \Pr[T_{-1} \mid \beta]\right)} > \frac{(1-r)\,n_b\left(\Pr[T \mid \alpha] + \Pr[T_{+1} \mid \alpha]\right)}{sn_b\left(\Pr[T \mid \beta] + \Pr[T_{+1} \mid \beta]\right)}$$

Now note that

$$rn_a\Pr[T_{-1} \mid \alpha] = (1-r)\,n_b\Pr[T_{+1} \mid \alpha]$$

and

$$(1-s)\,n_a\Pr[T_{-1} \mid \beta] = sn_b\Pr[T_{+1} \mid \beta]$$

and the required inequality follows from the fact that $rn_a \geq (1-r)\,n_b$ and $(1-s)\,n_a < sn_b$. ■

Lemma 9 *Suppose that there is a sequence of sincere voting equilibria such that* $\lim np_a(n) = n_a < \infty$ *and* $\lim np_b(n) = n_b < \infty$. *If* $rn_a < (1-r)\,n_b$, *then* $U_a > 0$.

Proof. Consider the function

$$K(x,y) = e^{-x-y}\left(xI_0(z) + \tfrac{1}{2}zI_1(z)\right)$$

where $z = 2\sqrt{xy}$.

Note that if $\sigma_A = rn_a$ and $\sigma_B = (1-r)\,n_b$, then

$$\begin{aligned} rn_a\Pr[Piv_A \mid \alpha] &= \tfrac{1}{2}\sigma_A e^{-\sigma_A-\sigma_B}\left(I_0\left(2\sqrt{\sigma_A\sigma_B}\right) + \sqrt{\tfrac{\sigma_B}{\sigma_A}}I_1\left(2\sqrt{\sigma_A\sigma_B}\right)\right) \\ &= \tfrac{1}{2}K(\sigma_A,\sigma_B) \end{aligned}$$

Similarly, if $\tau_A = (1-s)\,n_a$ and $\tau_B = sn_b$, then

$$(1-s)\,n_a\Pr[Piv_A \mid \beta] = \tfrac{1}{2}K(\tau_A,\tau_B)$$

We will show that when $x < y$, $K(x,y)$ is increasing in x and decreasing in y. Observe that

$$\begin{aligned}
K_x(x,y) &= e^{-x-y}\left(I_0(z) + xI_0'(z)\,z_x + \tfrac{1}{2}(zI_1(z))'\,z_x - xI_0(z) - \tfrac{1}{2}zI_1(z)\right) \\
&= e^{-x-y}\left(I_0(z) + xI_1(z)\,z_x + \tfrac{1}{2}zI_0(z)\,z_x - xI_0(z) - \tfrac{1}{2}zI_1(z)\right) \\
&= e^{-x-y}(1+y-x)\,I_0(z) \\
&> 0
\end{aligned}$$

where we have used the fact that $I_0'(z) = I_1(z)$ and $(zI_1(z))' = zI_0(z)$. Also, $xz_x = \frac{1}{2}z$ and $\frac{1}{2}zz_x = y$.

Also,

$$\begin{aligned}
K_y(x,y) &= e^{-x-y}\left(xI_0'(z)\,z_y + \tfrac{1}{2}(zI_1(z))'\,z_y - xI_0(z) - \tfrac{1}{2}zI_1(z)\right) \\
&= e^{-x-y}\left(xI_1(z)\,z_y + \tfrac{1}{2}zI_0(z)\,z_y - xI_0(z) - \tfrac{1}{2}zI_1(z)\right) \\
&= e^{-x-y}\left(xz_y - \tfrac{1}{2}z\right)I_1(z) \\
&< 0
\end{aligned}$$

where we have used the fact that, $z_y z = 2x$ and $z_y = \sqrt{\frac{x}{y}} < 1$ and $x < \frac{1}{2}z$.

Finally, notice that since $rn_a < (1-r)\,n_b$

$$(1-s)\,n_a < rn_a < (1-r)\,n_b < sn_b$$

which is the same as

$$\tau_A < \sigma_A < \sigma_B < \tau_B$$

and since $K_x > 0$ and $K_y < 0$ for $x < y$, we have

$$\frac{r\Pr[Piv_A \mid \alpha]}{(1-s)\Pr[Piv_A \mid \beta]} = \frac{K(\sigma_A, \sigma_B)}{K(\tau_A, \tau_B)} > 1$$

and so

$$U_a = \frac{r}{1+1-s}\Pr[Piv_A \mid \alpha] - \frac{1-s}{r+1-s}\Pr[Piv_A \mid \beta] > 0$$

■

Lemma 10 *In any sequence of sincere voting equilibria, either* $\lim np_a(n) = \infty$ *or* $\lim np_b(n) = \infty$.

Proof. Lemma 3 then implies that both

$$\lim U_a(p_a(n), p_b(n)) = 0 \text{ and } \lim U_b(p_a(n), p_b(n)) = 0$$

Suppose to the contrary that $\lim np_a(n) < \infty$ and $\lim np_b(n) < \infty$. But now Lemmas 8 and 9 lead to a contradiction. ■

Our next lemma shows that in the limit, the participation rates are of the same order of magnitude.

Lemma 11 *In any sequence of sincere voting equilibria, (i)* $\liminf \frac{p_a(n)}{p_b(n)} > 0$*; and (ii)* $\liminf \frac{p_b(n)}{p_a(n)} > 0$.

Proof. To prove part (i), suppose to the contrary that $\liminf \frac{p_a(n)}{p_b(n)} = 0$. Lemma 10 implies that $\liminf np_b(n) = \infty$.

Consider the probability of outcome (k,l) in state α

$$\Pr\left[(k,l) \mid \alpha\right] = e^{-nrp_a} \frac{(nrp_a)^k}{k!} e^{-n(1-r)p_b} \frac{(n(1-r)p_b)^l}{l!}$$

and the corresponding probability $\Pr\left[(k,l) \mid \beta\right]$, which is obtained by substituting $(1-s)$ for r in the expression above.

The likelihood ratio

$$\frac{\Pr\left[(k,l) \mid \alpha\right]}{\Pr\left[(k,l) \mid \beta\right]} = e^{np_b(r+s-1)\left(1-\frac{p_a}{p_b}\right)} \times \frac{r}{(1-s)^k} \frac{(1-r)^l}{s}$$

Since along some sequence, $\frac{p_a}{p_b} \to 0$ and $np_b \to \infty$

$$e^{np_b(r+s-1)\left(1-\frac{p_a}{p_b}\right)} \to \infty$$

Moreover, in all events in the set Piv_B, $|k-l| \leq 1$.

Thus, there exists an n_0 such that for all $n \geq n_0$

$$\frac{\Pr\left[Piv_B \mid \alpha\right]}{\Pr\left[Piv_B \mid \beta\right]} > \frac{q(\beta \mid b)}{q(\alpha \mid b)}$$

But this contradicts the fact that for all n, the participation thresholds are positive, that is

$$q(\beta \mid b)\Pr\left[Piv_B \mid \beta\right] - q(\alpha \mid b)\Pr\left[Piv_B \mid \alpha\right] = F^{-1}(p_b) > 0$$

Part (ii) is, of course, an immediate consequence of Lemma 1. ■

C Appendix: Uniqueness

The purpose of this appendix is to provide a proof of Proposition 11.

Proposition 5 establishes that in any equilibrium, there is positive participation and voting behavior is sincere. This now means that all equilibria must be of the kind we have studied—and the only way there could be multiple equilibria is that there are multiple solutions to the equilibrium participation rates. We complete the proof of uniqueness by showing that when n is large, there can be only one pair of equilibrium participation rates.

It remains to show that given sincere voting, there is a unique set of participation rates—that is, there is a unique solution (p_a^*, p_b^*) to IRa and IRb. As we show next, this is also true in large elections.[16]

Lemma 12 *In large elections, there is a unique solution to the cost threshold equations IRa and IRb.*

Proof. Equilibrium cost thresholds are determined by the equations

$$U_a(p_a, p_b) \equiv q(\alpha \mid a) \Pr[Piv_A \mid \alpha] - q(\beta \mid a) \Pr[Piv_A \mid \beta] = F^{-1}(p_a) \tag{IRa}$$

$$U_b(p_a, p_b) \equiv q(\beta \mid b) \Pr[Piv_B \mid \beta] - q(\alpha \mid b) \Pr[Piv_B \mid \alpha] = F^{-1}(p_b) \tag{IRb}$$

We will show that when n is large, at any intersection of the two, the curve determined by IRa is steeper than that determined by IRb, that is,

$$-\left(\frac{\partial U_a}{\partial p_a} - \left(F^{-1}(p_a)\right)'\right) \div \frac{\partial U_a}{\partial p_b} > -\frac{\partial U_b}{\partial p_a} \div \left(\frac{\partial U_b}{\partial p_b} - \left(F^{-1}(p_b)\right)'\right) \tag{23}$$

The calculation of the partial derivatives is facilitated by using the following simple fact. If we write,

$$\Pr[(l,k) \mid \alpha] = e^{-nrp_a} \frac{(nrp_a)^l}{l!} e^{-n(1-r)p_b} \frac{(n(1-r)p_b)^k}{k!}$$

as the probability of outcome (l,k) in state α, then

$$\begin{aligned}
\frac{\partial \Pr[(l,k) \mid \alpha]}{\partial p_a} &= nr \Pr[(l-1,k) \mid \alpha] - nr \Pr[(l,k) \mid \alpha] \\
\frac{\partial \Pr[(l,k) \mid \alpha]}{\partial p_b} &= n(1-r) \Pr[(l,k-1) \mid \alpha] - n(1-r) \Pr[(l,k) \mid \alpha]
\end{aligned}$$

Similar expressions obtain for the partial derivatives of $\Pr[(l,k) \mid \beta]$.

Since the probability of a pivotal term Piv_C where $C = A, B$ is just a sum of terms of the form $\Pr[(l,k) \mid \alpha]$, we obtain

$$\begin{aligned}
\frac{\partial \Pr[Piv_C \mid \alpha]}{\partial p_a} &= nr \Pr[Piv_C - (1,0) \mid \alpha] - nr \Pr[Piv_C \mid \alpha] \\
\frac{\partial \Pr[Piv_C \mid \alpha]}{\partial p_b} &= n(1-r) \Pr[Piv_C - (0,1) \mid \alpha] - n(1-r) \Pr[Piv_C \mid \alpha]
\end{aligned}$$

Again, similar expressions obtain for the partial derivatives of $\Pr[Piv_C \mid \beta]$ where $C = A, B$.

Myerson (2000) has shown that when the expected number of voters is large, the

[16] This result does not hold in a corresponding model of costly voting with a fixed population. Ghosal and Lockwood (2007) provide an example with the majority rule in which there are multiple cost thresholds and hence, multiple equilibria.

probabilities of the "offset" events in state α are

$$\begin{aligned}
\Pr\left[Piv_C - (1,0) \mid \alpha\right] &\approx \Pr\left[Piv_C \mid \alpha\right] x^{\frac{1}{2}} \\
\Pr\left[Piv_C - (0,1) \mid \alpha\right] &\approx \Pr\left[Piv_C \mid \alpha\right] x^{-\frac{1}{2}}
\end{aligned}$$

where

$$x = \frac{1-r}{r}\frac{p_b}{p_a}$$

Similarly, the probabilities of the offset events in state β are

$$\begin{aligned}
\Pr\left[Piv_C - (1,0) \mid \alpha\right] &\approx \Pr\left[Piv_C \mid \beta\right] y^{\frac{1}{2}} \\
\Pr\left[Piv_C - (0,1) \mid \alpha\right] &\approx \Pr\left[Piv_C \mid \beta\right] y^{-\frac{1}{2}}
\end{aligned}$$

where

$$y = \frac{s}{1-s}\frac{p_b}{p_a}$$

Using Myerson's offset formulae it follows that

$$\begin{aligned}
\frac{\partial U_a}{\partial p_a} &\approx nq\left(\alpha \mid a\right) r \Pr\left[Piv_A \mid \alpha\right] (x^{\frac{1}{2}} - 1) - nq\left(\beta \mid a\right) (1-s) \Pr\left[Piv_A \mid \beta\right] (y^{\frac{1}{2}} - 1) \\
\frac{\partial U_a}{\partial p_b} &\approx nq\left(\alpha \mid a\right) (1-r) \Pr\left[Piv_A \mid \alpha\right] (x^{-\frac{1}{2}} - 1) - nq\left(\beta \mid a\right) s \Pr\left[Piv_A \mid \beta\right] (y^{-\frac{1}{2}} - 1)
\end{aligned}$$

and similarly,

$$\begin{aligned}
\frac{\partial U_b}{\partial p_a} &\approx nq\left(\beta \mid b\right) (1-s) \Pr\left[Piv_B \mid \beta\right] (y^{\frac{1}{2}} - 1) - nq\left(\alpha \mid b\right) r \Pr\left[Piv_B \mid \alpha\right] (x^{\frac{1}{2}} - 1) \\
\frac{\partial U_b}{\partial p_b} &\approx nq\left(\beta \mid b\right) s \Pr\left[Piv_B \mid \beta\right] (y^{-\frac{1}{2}} - 1) - nq\left(\alpha \mid b\right) (1-r) \Pr\left[Piv_B \mid \alpha\right] (x^{-\frac{1}{2}} - 1)
\end{aligned}$$

We have argued that when n is large, any point of intersection of IRa and IRb, say (p_a, p_b), results in efficient electoral outcomes—A wins in state α and B wins in state β. This requires that (p_a, p_b) satisfy

$$\frac{1-r}{r}\frac{p_b}{p_a} < 1 \text{ and } \frac{s}{1-s}\frac{p_b}{p_a} > 1 \tag{24}$$

and by definition this is the same as

$$x < 1 \text{ and } y > 1 \tag{25}$$

From this it follows that at any point (p_a, p_b) satisfying (24),

$$\frac{\partial U_a}{\partial p_a} < 0 \text{ and } \frac{\partial U_a}{\partial p_b} > 0$$

and similarly,

$$\frac{\partial U_b}{\partial p_a} > 0 \text{ and } \frac{\partial U_b}{\partial p_b} < 0$$

Thus at any (p_a, p_b) satisfying (24), the curves determined by IRa and IRb are both positively sloped.

Since $\left(F^{-1}(p_a)\right)'$ and $\left(F^{-1}(p_b)\right)'$ are both positive, in order to establish the inequality in (23), it is sufficient to show that

$$\left(-\frac{\partial U_a}{\partial p_a}\right) \div \frac{\partial U_a}{\partial p_b} > \frac{\partial U_b}{\partial p_a} \div \left(-\frac{\partial U_b}{\partial p_b}\right)$$

which is equivalent to

$$\begin{aligned} &\frac{q(\alpha \mid a)\, r \Pr[Piv_A \mid \alpha](1 - x^{\frac{1}{2}}) + q(\beta \mid a)(1-s)\Pr[Piv_A \mid \beta](y^{\frac{1}{2}} - 1)}{q(\alpha \mid a)(1-r)\Pr[Piv_A \mid \alpha](x^{-\frac{1}{2}} - 1) + q(\beta \mid a)\, s \Pr[Piv_A \mid \beta](1 - y^{-\frac{1}{2}})} \\ &\quad > \frac{q(\alpha \mid b)\, r \Pr[Piv_B \mid \alpha](1 - x^{\frac{1}{2}}) + q(\beta \mid b)(1-s)\Pr[Piv_B \mid \beta](y^{\frac{1}{2}} - 1)}{q(\alpha \mid b)(1-r)\Pr[Piv_B \mid \alpha](x^{-\frac{1}{2}} - 1) + q(\beta \mid b)\, s \Pr[Piv_B \mid \beta](1 - y^{-\frac{1}{2}})} \end{aligned}$$

Using

$$q(\alpha \mid a) = \frac{r}{r + (1-s)} \text{ and } q(\beta \mid b) = \frac{s}{s + (1-r)}$$

and writing

$$L_A = \frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_A \mid \alpha]} \text{ and } L_B = \frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]}$$

as the two likelihood ratios, the inequality above is the same as

$$\frac{(r)^2(1 - x^{\frac{1}{2}}) + (1-s)^2(y^{\frac{1}{2}} - 1)L_A}{r(1-r)(x^{-\frac{1}{2}} - 1) + s(1-s)(1 - y^{-\frac{1}{2}})L_A} > \frac{r(1-r)(1 - x^{\frac{1}{2}}) + s(1-s)(y^{\frac{1}{2}} - 1)L_B}{(1-r)^2(x^{-\frac{1}{2}} - 1) + s(1 - y^{-\frac{1}{2}})L_B}$$

Cross-multiplying and cancelling terms, further reduces the inequality to

$$\begin{aligned} &\left(\frac{(1-r)(1-s)}{rs}(x^{-\frac{1}{2}} - 1)(y^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - y^{-\frac{1}{2}})\right) \times L_A \\ > &\left(\frac{(1-r)(1-s)}{rs}(x^{-\frac{1}{2}} - 1)(y^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - y^{-\frac{1}{2}})\right) \times \frac{rs}{(1-r)(1-s)} \times L_B \end{aligned} \tag{26}$$

We claim that for all (p_a, p_b) satisfying (24),

$$\frac{(1-r)(1-s)}{rs}(x^{-\frac{1}{2}} - 1)(y^{\frac{1}{2}} - 1) - (1 - x^{\frac{1}{2}})(1 - y^{-\frac{1}{2}}) < 0 \tag{27}$$

To see this, note that by definition,

$$\begin{aligned} y &= \frac{s}{1-s}\frac{p_b}{p_a} \\ &= \frac{rs}{(1-r)(1-s)}\frac{1-r}{r}\frac{p_b}{p_a} \\ &= \frac{rs}{(1-r)(1-s)}x \\ &= Rx \end{aligned}$$

where $R = \frac{rs}{(1-r)(1-s)}$. Substituting $y = Rx$ we obtain

$$R(x^{-\frac{1}{2}}-1)(y^{\frac{1}{2}}-1)-(1-x^{\frac{1}{2}})(1-y^{-\frac{1}{2}}) = R^{-1}(x^{-\frac{1}{2}}-1)(R^{\frac{1}{2}}x^{\frac{1}{2}}-1)-(1-x^{\frac{1}{2}})(1-R^{-\frac{1}{2}}x^{-\frac{1}{2}})$$

Now consider the function

$$\phi(x) = R^{-1}(x^{-\frac{1}{2}}-1)(R^{\frac{1}{2}}x^{\frac{1}{2}}-1)-(1-x^{\frac{1}{2}})(1-R^{-\frac{1}{2}}x^{-\frac{1}{2}})$$

Since $x < 1 < y = Rx$, we have $R^{-1} < x < 1$. Notice that $\phi(1) = 0 = \phi(R^{-1})$. It is routine to verify that ϕ is convex and so $\phi(x) < 0$ for all $x \in (R^{-1}, 1)$. Thus we have established (27).

Now because of (27), the inequality in (26) reduces to

$$\frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_A \mid \alpha]} < R \times \frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]} \tag{28}$$

Finally, notice that IRa and IRb imply, respectively, that

$$\frac{r}{1-s} = \frac{q(\alpha \mid a)}{q(\beta \mid a)} > \frac{\Pr[Piv_A \mid \beta]}{\Pr[Piv_A \mid \alpha]} \text{ and } \frac{\Pr[Piv_B \mid \beta]}{\Pr[Piv_B \mid \alpha]} > \frac{q(\alpha \mid b)}{q(\beta \mid b)} = \frac{1-r}{s}$$

and this immediately implies (28), thereby completing the proof that at any point of intersection of IRa and IRb, the slope of IRa is greater than the slope of IRb. This means that the curves cannot intersect more than once. ■

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