Dissolving a partnership (un)fairly*

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Summary. In an incomplete information, common values setting with risk-neutral agents, we consider mechanisms for allocating the assets of a dissolving partnership where the mechanism designer has no information about the distribution of signals of the agents. We find that the divide and choose mechanism systematically favors the chooser and hence fails on the grounds of fairness. We also examine the fairness properties of the winning and losing bid auctions and show that they systematically favor winning (resp. losing) bidder in ex post allocation of surplus. Finally, we show that a binding arbitration mechanism implements fair allocations.

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1 Introduction

In dissolving a partnership, adjudicating a divorce settlement, or otherwise determining the "fair" allocation of assets which were previously jointly held, interested parties are faced with the difficulty of finding an acceptable mechanism for achieving a property settlements. In many partnerships, client lists and accumulated goodwill constitute the bulk of the assets; moreover, the value of these items in terms of future revenue potential is probably quite similar for all the parties, but is not likely to be known with precision by any of them. Thus, we consider allocation problems when the asset to be divided has a value common to all the parties, each of whom receives a noisy signal about the true value of the asset.[1] In particular, we focus on whether simple mechanisms lead to fair allocations.[2]

There are several reasons why one might be concerned with fairness in partnership dissolutions. First, because of the well-documented inequality aversion (see Fehr and Schmidt (2000), Bolton and Ockenfels (2001)) in laboratory settings (see, for instance, Roth, Prasnikar, Okuno-Fujiwara, and Zamir (1991)). Individuals anticipating an unfair mechanism used in the case that their partnership dissolves may be reluctant to enter into the partnership in the first place, thereby foregoing economic gains from the partnership arrangement.

A second motivating factor derives from the courts' expressed concerns with fairness. This concern for fairness in settlements is perhaps most strongly manifested in the area of family law, where the introduction of no-fault divorce was intended to redress perceived unfairness in the distribution of marital assets. *Equitable* distribution is the name commonly used to describe divorce law where the judge has discretion to divide marital property.[3] For community property in a divorce settlement, the goals of equitable distribution are clear: the judge is to divide the property to give equal value to

[1] While it is unlikely that the value of client lists or other assets is exactly common to all the parties, we believe that treating the value as common and focusing on fairness rather than efficiency is consistent with the legal basis for property settlement where achieving a "just" allocation is clearly the goal.

[2] We use "simple mechanism" in the same sense as McAfee (1992). These are mechanisms where the mechanism designer has no information about the distribution of player types, but the players have common knowledge about each other's utility and the distribution of types.

[3] According to Matz (2001, fn. 8), a typical example of an equitable distribution state may be found under NY Dom. Rel. Law Sec. 236B(6)(a).

the two parties (see, for instance, Matz (2001) and American Law Institute (2002). It is precisely this notion of considering division schemes rendering equal value that is the primary concern of this paper.

Our model represents a natural incomplete information extension of the fair division problems (with a homogeneous object to be divided) long considered by mathematicians and economists (see Brams (1996) for a survey). In this literature, two notions of fairness are commonly used: (1) *proportionality*, each claimant receives a portion of the asset (or the monetary equivalent) equal to her ex ante share of the asset multiplied by her value of the asset; and (2) *envy freeness,* each claimant should prefer her allocation to allocations received by all other claimants. Each of these notions adapt straightforwardly to incomplete information environments; however, analogous to individual rationality constraints in standard mechanism design problems, the fairness of a mechanism may depend on the amount of information held by the claimant at the time the criterion is applied.

Our analysis of bidding in auctions where the winning bidder's bid is redistributed to other bidders is related to Englebrecht-Wiggans (1994), Bulow, Huang, and Klemperer (1999), Engers and McManus (2000), and Goeree and Turner (2001, 2002). All of these papers are primarily concerned with efficiency and revenues arising in redistributive auctions rather than with fairness properties of these auctions. Dasgupta, Tsui, and Zhu (forthcoming) study auctions where bidders have cross-shareholdings; thus a portion of the surplus accruing to the winning bidder is redistributed to those holding shares of the winner. Their concern is mainly with comparing revenues across auction forms. Unlike our paper, where the bid amount paid by the winner represents a pure redistribution of surplus between the bidders, in their paper the bid paid by the winning bidder is simply surplus lost to all bidders .In that sense, our analysis is also related to the literature on knockout auctions (see, for instance, Graham, Marshall, and Richard (1990) and Deltas (2002) in that the knockout phase represents a "dissolution" of the partnership of colluding bidders, but obviously the concerns of these papers do not center on fairness.

Cramton, Gibbons, and Klemperer (1987), McAfee (1992) and Guth and van Damme (1986) have all considered partnership dissolution problems under incomplete information with independent private values. Later work has allowed for the possibility of interdependent values (Fieseler, Kittsteiner, and Moldovanu (2000); Kittsteiner (2000); and (Jehiel and Pauzner (2001)) and explored conditions in which efficient mechanisms exist as well as analyzing their properties. Moldovanu (2002) presents a unifying framework for the conclusions drawn in this line. The key distinction between this work and the present paper is our focus on *fairness* rather than efficiency. Indeed, we abstract away from efficiency considerations entirely by adopting a pure common values framework.[4] Along these same lines, we assume that indivisible good suffers no loss in economic value from its transfer from joint control to one of the parties.

We examine four allocation mechanisms: (1) the divide and choose mechanism, (2) a first-price (or winning bid) auction, (3) a second-price (or losing bid) auction, and (4) a simultaneous *binding arbitration* mechanism. Consistent with McAfee (1992), the four allocation mechanisms we consider are "simple" in the sense that they do not require knowledge of the underlying distribution of signals about the value of the asset nor do they require knowledge of the utility functions of the participants. The main contributions are as follows: First, we show that any pure strategy equilibrium of the the divide and choose mechanism is unfair. Next, we derive symmetric equilibria in the winning and losing bid auctions and show that these also fail to achieve fair allocations under various definitions of fairness. Finally, we establish the existence of a fully revealing equilibrium in the binding arbitration mechanism, and show that this *always* implements fair allocations.

The remainder of the paper proceeds as follows: In section 2, we develop the basic model. Section 3 examines the fairness properties of the mechanisms. Finally, section 4 concludes.

2 Preliminaries

There are 2 claimants, $i = \{1, 2\}$ with equal claims to an indivisible asset with unknown common value. Claimants each receive a real valued signal $X_i \in [0, 1]$ prior to the division of the asset. The random variables X_1 and X_2 are assumed to be independently drawn from the atomless distribution $F(\cdot)$.

Let the expected common value of the object conditional on signals x_1 and x_2 be defined by

$$E[V|X_1 = x_1, X_2 = x_2] \equiv v(x_1, x_2)$$

[4] Of course, with common values, all allocations are efficient.

We assume that $v(\cdot,\cdot)$ is strictly increasing in both of its arguments, and $v(0,0)=0$.

For much of the paper, we shall assume that agents are risk neutral, earning utility $U_i = v(x_1,x_2)+m_i$ when signals x_1, x_2 are realized, the object is awarded to claimant i, and claimant i has money wealth m_i. For simplicity, we normalize money wealth for all agents at zero, but assume that claimants are able to make any transfers called for by the mechanisms.

2.1 Fairness

In incomplete information settings, notions of fairness, much like participation constraints, depend upon the time at which they are applied. We consider two points in time in assessing fairness:

Definition 1 *An allocation mechanism is* ex ante fair *if the mechanism leads to equal expected utility of all claimants prior to observing their signals.*

Definition 2 *An allocation mechanism is* ex post fair *if the mechanism leads to equal expected utility conditional on all signals received by all claimants.*

3 Simple Allocation Mechanisms

We examine the fairness properties of the four simple allocation mechanisms restricting attention to *pure strategy* perfect Bayesian equilibria. Further, since the last three mechanisms are simultaneous and claimants are ex ante identical, we study symmetric pure strategy equilibria for these mechanisms.

3.1 Divide and Choose

We begin by considering the familiar divide and choose mechanism. Under complete information with a common values, this mechanism is known to lead to a fair allocation. Here we examine fairness in the analogous incomplete information setting.

Claimant 1 specifies a price p possibly depending on her private information. Claimant 2 then decides whether to buy the asset at that price or sell the asset to claimant 1 at the price specified.

The main result of this section is:

Proposition 1 *Using the divide and choose mechanism to dissolve a partnership is unfair and it ex ante favors the chooser..*

Proof. We establish that if either a pooling or separating equilibrium exist, it leads to an allocation that is unfair ex ante and ex post. Consider a possible pooling equilibrium arising in the divide and choose mechanism. In such an equilibrium, claimant 1 chooses a price $p \geq 0$ regardless of his signal x_1. Thus, after observing claimant 1 choose the equilibrium price p, claimant 2 should acquire the asset at the price specified if and only if her signal x_2 satisfies

$$\int_0^1 v\left(x_1, x_2\right) dF\left(x_1\right) - p \geq p.$$

Since v is increasing in each of its arguments and $v\left(0,0\right) = 0$, it follows that there is a unique signal $x_2 = x_2^*$ where

$$\frac{1}{2}\int_0^1 v\left(x_1, x_2^*\right) dF\left(x_1\right) = p. \tag{1}$$

In such an equilibrium, claimant 2 will buy the asset if and only if $x_2 \geq x_2^*$.

We are now in a position to write down the expected utility to each player *ex ante* under such an equilibrium. Claimant 1 earns:

$$EU_1 = \int_0^1 \int_0^{x_2^*} v\left(x_1, x_2\right) dF\left(x_2\right) dF\left(x_1\right) - pF\left(x_2^*\right) + p\left(1 + F\left(x_2^*\right)\right)$$

while claimant 2 earns

$$EU_2 = \int_0^1 \int_{x_2^*}^1 v\left(x_1, x_2\right) dF\left(x_2\right) dF\left(x_1\right) - p\left(1 - F\left(x_2^*\right)\right) + pF\left(x_2^*\right)$$

Substituting for p using the left-hand side of equation 1 and differencing the expected payoffs for the two claimants yields

$$\begin{aligned} EU_2 - EU_1 &= \int_0^1 \left(\begin{array}{c} \int_0^{x_2^*} \left(v\left(x_1, x_2^*\right) - v\left(x_1, x_2\right)\right) dF\left(x_2\right) \\ + \int_{x_2^*}^1 \left(v\left(x_1, x_2\right) - v\left(x_1, x_2^*\right)\right) dF\left(x_2\right) \end{array} \right) dF\left(x_1\right) \\ &> 0 \end{aligned}$$

where the inequality follows from the fact that v is strictly increasing in its arguments.

Thus, we have shown that if a pooling equilibrium exists, the divide and choose mechanism systematically favors the chooser, claimant 2. That is, the mechanism is ex ante unfair.

To show that divide and choose is ex post unfair under a pooling equilibrium, it is sufficient to show that for a positive measure of signal realizations, the allocation is ex post unfair. To see this, fix (x_1', x_2^*) such that $v(x_1', x_2^*) = 2p$. Now, if we consider an $x_1 \in (x_1', x_1' + \varepsilon)$ for $\varepsilon > 0$, we have that claimant 2 continues to purchase the asset at price p, but claimant 1 receives less than half of the surplus.

Next, suppose that a separating equilibrium arises under the divide and choose mechanism. Let $\pi(x_1)$ be the equilibrium price offered when claimant 1 receives signal x_1. In such an equilibrium, claimant 2 will purchase the asset after observing an equilibrium price p if and only if

$$v\left(\pi^{-1}(p), x_2\right) - p \geq p.$$

Thus, in such an equilibrium, the claimant's ex post equilibrium payoffs are

$$U_1 = \min\left(v(x_1, x_2) - p(x_1),\ p(x_1)\right)$$

and

$$U_2 = \max\left(v(x_1, x_2) - p(x_1),\ p(x_1)\right).$$

Since for almost all x_2, $v(x_1, x_2) \neq 2p(x_1)$, it then follows that $U_2 > U_1$ almost always. This immediately implies that under any separating equilibrium, if one exists, the divide and choose mechanism yields a smaller share of the surplus to claimant 1 than to claimant 2. That is, it is neither ex ante nor ex post fair.

Combining the two arguments also rules out equilibria where for some set of signals claimant 1 pools while for others he separates.[5]

This establishes the result in Proposition 1. ■

Discussion The intuition for this result is that, unlike in the complete information case, the informational positions of the two claimants are not symmetric. In particular, claimant 1, in deciding what price to offer, faces a winner's curse problem. Claimant 2 will sell 1 the asset in cases where 2's signal is low; thus, claimant 1 has to update his expectations about the value of the asset conditional on the information contained in clamant 2's decision

[5] We thank an anonymous referee for pointing this extension out.

to sell it to him. Thus, the natural analog to the complete information pricing rule would be to choose a price

$$p(x_1) = \int_0^\infty \frac{1}{2} v(x_1, x_2)\, dF(x_2)$$

When would claimant 2 sell 1 the asset?

$$\begin{aligned} v(x_1, x_2) - p(x_1) &< p(x_1) \\ v(x_1, x_2) &< \int_0^\infty v(x_1, x_2)\, dF(x_2) \end{aligned}$$

that is, precisely when the value of the asset conditional on both signals was less than the expected value of the asset conditional only on the signal x_1. Hence, the complete information pricing rule always leads to claimant 1 receiving less than half the value of the asset. Accounting for the winner's curse can counteract this effect to some extent, but the disadvantage of the adverse selection problem faced by claimant 1 results in divide and chose leading to ex ante unfair allocations.

McAfee (1992) showed that the divide and choose rule did not lead to efficient allocations with independent private values. Here, we see that it also does not lead to fair allocations with common values. Obviously, this is cause for concern if the intuition from the complete information case is being relied upon to justify the desirable allocative characteristics of this mechanism.

3.2 Winning bid Auction

Clearly, the sequential nature of the divide and choose mechanism was in part responsible for some of the fairness problems of the mechanism. We next consider the use of a first-price sealed bid auctions (hereafter referred to as a winning bid auction or WBA) as a means of allocating the asset.[6] Since this mechanism is simultaneous then with symmetric equilibrium bidding strategies, ex ante fairness will be immediately obtained. Here we will focus on ex post fairness.

[6] McAfee (1992) refers to this mechanism as a winning bid auction to distinguish it form a conventional first-price auction. The distinction is required because the proceeds of the auction accrue to the losing bidder rather than the auctioneer as in a conventional first-price auction.

Unlike a conventional first-price sealed bid auction where the proceeds of the bids accrue to the auctioneer, the WBA requires that the proceeds of the high bid be paid to the losing bidder.

We begin with a heuristic derivation of symmetric equilibrium strategies. Temporarily suppose that claimant 2 is using the strictly increasing bidding strategy $\beta(x_2)$. In this case, bidder 1 (say) chooses a bid b to maximize

$$U_1(b|x_1) = \int_0^{\beta^{-1}(b)} (v(x_1, x_2) - b) f(x_2) \, dx_2 + \int_{\beta^{-1}(b)}^{\infty} \beta(x_2) f(x_2) \, dx_2$$

Differentiating with respect to b yields the first-order condition:

$$\frac{\left(v\left(x_1, \beta^{-1}(b)\right) - b\right)}{\beta'\left(\beta^{-1}(b)\right)} f\left(\beta^{-1}(b)\right) - F\left(\beta^{-1}(b)\right) - \frac{bf\left(\beta^{-1}(b)\right)}{\beta'\left(\beta^{-1}(b)\right)} = 0$$

Imposing symmetry, $b = \beta(x_1)$,

$$\frac{(v(x_1, x_1) - \beta(x_1))}{\beta'(x_1)} f(x_1) - F(x_1) - \frac{\beta(x_1) f(x_1)}{\beta'(x_1)} = 0$$

which we may write as a first order linear differential equation

$$\beta'(x_1) = (v(x_1, x_1) - 2\beta(x_1)) \frac{f(x_1)}{F(x_1)} \tag{2}$$

Of course, equation (2) is only a necessary condition for equilibrium. We also require $v(x_1, x_1) - \beta(x_1) \geq 0$; hence, $v(0,0) - \beta(0) = 0$. This provides an endpoint condition for (2).

Proposition 2 *A symmetric increasing equilibrium bidding strategy in the WBA is:*

$$\beta(x) = \int_0^x v(\alpha, \alpha) \left[\exp\left(-2\int_\alpha^x \frac{f(s)}{F(s)} ds\right) \frac{f(\alpha)}{F(\alpha)} d\alpha\right] \tag{3}$$

Proof. The proof closely parallels Theorem 14 of Milgrom and Weber (1982). Define

$$L(\alpha|x) = \exp\left(-2\int_\alpha^x \frac{f(s)}{F(s)} ds\right)$$

Differentiating with respect to x,

$$\begin{aligned}\frac{\partial}{\partial x}L\left(\alpha|x\right) &= -2\frac{f\left(x\right)}{F\left(x\right)}\exp\left(-2\int_{\alpha}^{x}\frac{f\left(s\right)}{F\left(s\right)}ds\right)\\ &< 0\end{aligned}$$

Thus, $L\left(\alpha|x\right)$, regarded as a probability distribution is stochastically increasing in x. The expression is square brackets in equation (3) is simply, $\frac{1}{2}\frac{\partial}{\partial\alpha}L\left(\alpha|x\right)$. Since $v\left(\alpha,\alpha\right)$ is increasing in its arguments, then $\beta\left(x\right)$ is increasing.

Suppose that claimant 2 is using $\beta\left(x_2\right)$. Claimant 1 receives a signal x and submits a bid associated with signal z to maximize:

$$U_1\left(z|x\right)=\int_0^z\left(v\left(x,x_2\right)-\beta\left(z\right)\right)f\left(x_2\right)dx_2+\int_z^1\beta\left(x_2\right)f\left(x_2\right)dx_2$$

Differentiating with respect to z

$$\begin{aligned}\frac{\partial}{\partial z}U_1\left(z|x\right) &= \left(v\left(x,z\right)-2\beta\left(z\right)\right)f\left(z\right)-\beta'\left(z\right)F\left(z\right)\\ &= \left(v\left(x,z\right)-2\beta\left(z\right)\right)f\left(z\right)-\left(v\left(z,z\right)-2\beta\left(z\right)\right)\frac{f\left(z\right)}{F\left(z\right)}F\left(z\right)\\ &= \left(v\left(x,z\right)-v\left(z,z\right)\right)f\left(z\right) \qquad (4)\end{aligned}$$

At $z=x$, equation (4) is zero, and since $v\left(\cdot,\cdot\right)$ is increasing in its arguments, (4) is negative for $z<x$ and positive for $z>x$. ■

It is useful to contrast this bidding strategy to that of a conventional first-price auction with two bidders. In this case, Milgrom and Weber (1982) show that a symmetric equilibrium bidding strategy, $\gamma\left(x\right)$ is given by

$$\gamma'\left(x\right)=\left(v\left(x,x\right)-\gamma\left(x\right)\right)\frac{f\left(x\right)}{F\left(x\right)}$$

with the initial condition $\gamma\left(0\right)=0$.

It is of some interest to notice that

Remark 1 *For every realization $\left(x_1,x_2\right)$,the winning bid in the WBA is less than that in a conventional first-price auction.*

To see this, notice that differencing the bidding functions, we have

$$\begin{aligned}
\gamma(x)-\beta(x) &= \int_0^x v(\alpha,\alpha)\left[\exp\left(-\int_\alpha^x \frac{f(s)}{F(s)}ds\right)\frac{f(\alpha)}{F(\alpha)}d\alpha\right]- \\
&\quad \int_0^x v(\alpha,\alpha)\left[\exp\left(-2\int_\alpha^x \frac{f(s)}{F(s)}ds\right)\frac{f(\alpha)}{F(\alpha)}d\alpha\right] \\
&= \int_0^x v(\alpha,\alpha)\frac{f(\alpha)}{F(\alpha)}\left(\exp\left(-\int_\alpha^x \frac{f(s)}{F(s)}ds\right)-\exp\left(-2\int_\alpha^x \frac{f(s)}{F(s)}ds\right)\right)d\alpha \\
&> 0
\end{aligned}$$

where the inequality follows from the fact that all of the terms in the integrand are positive. Hence, bids in conventional first price auctions are more aggressive than WBAs. It then follows directly that for every realization (x_1, x_2) the price an object is sold for in a WBA is less than in a first-price auction.

Intuitively, the WBA provides higher surplus to losing bidders than does the conventional first-price auction. As a result, the need to bid aggressively to win the auction is tempered by the surplus forgone in losing the auction less often in the WBA; hence, bidding is less aggressive since losing is a more desirable outcome.

Does this mechanism lead to fair allocations?

Proposition 3 *Suppose $x_1 > x_2$, then there exists a neighborhood $N(x_1) = (x_1-\varepsilon, x_1)$ such that equilibrium allocations arising in the WBA for $x_2 \in N(x_1)$ favor the winning bidder.*

Proof. Suppose that claimant 1 receives signal x_1 and claimant 2 receives signal x_2 where $x_1 \geq x_2$. Then, claimant 1 earns expected utility of

$$U_1(x_1,x_2) = v(x_1,x_2) - \beta(x_1)$$

whereas claimant 2 receives

$$U_2(x_1,x_2) = \beta(x_1)$$

Comparing $U_1 - U_2$:

$$\begin{aligned}
U_1(x_1,x_2) - U_2(x_1,x_2) &= v(x_1,x_2) - 2\beta(x_1) \\
&\leq v(x_1,x_1) - 2\beta(x_1) \\
&= \frac{F(x_1)}{f(x_1)}\beta'(x_1)
\end{aligned}$$

As $x_2 \to x_1$, then $U_1(x_1,x_2) - U_2(x_1,x_2) = v(x_1,x_1) - 2\beta(x_1) = \frac{F(x_1)}{f(x_1)}\beta'(x_1) > 0$. Hence for x_2 in the neighborhood of x_1, the allocation of claimant 1 is strictly preferred to that assigned claimant 2. ■

Notice that as the signals become closer to one another, the premium to the winning bidder grows larger.

Remark 2 *When signals are far apart, it may be the case that the losing bidder receives more of the surplus than the winning bidder in the WBA.*

To see this, suppose that each bidder receives a signal drawn from the uniform distribution over the unit interval and that the common value of the object is simply the average of the bidders' signals. Using Proposition 2, a symmetric equilibrium bidding strategy is $\beta(x) = \frac{1}{3}x$. Suppose $x_1 \geq x_2$,then claimant 1 obtains utility: $U_1(x_1,x_2) = \frac{x_1}{6} + \frac{x_2}{2}$. It then follows that if $x_2 < \frac{1}{3}x_1$, claimant 1 receives less than half of the surplus.

3.3 Losing bid auction

We now turn to second-price sealed bid auctions (hereafter 'losing bid auctions' or 'LBAs') as a means of allocating the asset. With the redistribution of the second highest bid to the losing bidders in the LBA, the property that full revelation is weakly dominant no longer holds. Again, we look for increasing symmetric strategies and begin with a heuristic derivation. With two claimants, bidder 1 chooses a bid b to maximize

$$U_1(b|x_1) = \int_0^{\beta^{-1}(b)} (v(x_1,x_2) - \beta(x_2)) f(x_2)\,dx_2 + \int_{\beta^{-1}(b)}^1 bf(x_2)\,dx_2$$

Differentiating

$$\frac{\left(v\left(x_1,\beta^{-1}(b)\right) - \beta\left(\beta^{-1}(b)\right)\right)}{\beta'\left(\beta^{-1}(b)\right)} f\left(\beta^{-1}(b)\right) - \frac{bf\left(\beta^{-1}(b)\right)}{\beta'\left(\beta^{-1}(b)\right)} + \left(1 - F\left(\beta^{-1}(b)\right)\right) = 0$$

Imposing symmetry

$$\frac{(v(x_1,x_1) - \beta(x_1))}{\beta'(x_1)} f(x_1) - \frac{\beta(x_1) f(x_1)}{\beta'(x_1)} + (1 - F(x_1)) = 0$$

which we may write as:

$$\begin{aligned}\beta'(x_1) &= (v(x_1,x_1)-2\beta(x_1))\frac{f(x_1)}{F(x_1)-1}\\ \beta'(x_1) &= (2\beta(x_1)-v(x_1,x_1))\lambda(x_1)\end{aligned}$$

where λ is the hazard rate of F. We may use this to construct a pair of equilibrium bidding strategies.

Proposition 4 *A symmetric increasing equilibrium in the LBA is given by the bidding strategy:*

$$\beta(x) = \int_x^1 v(\alpha,\alpha)\exp\left(-2\int_x^\alpha \lambda(s)\,ds\right)\lambda(\alpha)\,d\alpha$$

Proof. Define

$$L(\alpha|x) = \exp\left(-2\int_x^\alpha \lambda(s)\,ds\right)$$

then

$$\frac{\partial}{\partial x}L(\alpha|x) = 2\lambda(x)\exp\left(-2\int_x^\alpha \lambda(s)\,ds\right) > 0$$

so higher values of x are stochastically dominated by lower values of x. Since v is increasing in its arguments, it then follows that β is increasing.

Suppose that when claimant 1 has a signal x, he acts as though his signal were z. Then

$$U_1(z|x) = \int_0^z (v(x,x_2)-\beta(x_2))f(x_2)\,dx_2 + \int_z^1 \beta(z)f(x_2)\,dx_2$$

Differentiating with respect to z yields:

$$\begin{aligned}\frac{d}{dz}U_1(z|x) &= (v(x,z)-2\beta(z))f(z)+(1-F(z))\beta'(z)\\ &= (v(x,z)-2\beta(z))f(z)+(2\beta(z)-v(z,z))f(z)\\ &= (v(x,z)-v(z,z))f(z)\end{aligned}$$

hence, by the same arguments as in the WBA, $z=x$ is an equilibrium. ■

Notice that the bidding strategy in the LBA is identical to that in the WBA except for the endpoint forming the boundary condition. In the case of the LBA, this is the upper end of the support.

We now consider fairness.

Proposition 5 *Suppose $x_1 > x_2$, then there exists a neighborhood $N(x_2) = (x_2, x_2 + \varepsilon)$ such that equilibrium allocations arising in the LBA for $x_1 \in N(x_2)$ favor the losing bidder.*

Proof. Since bidding strategies are increasing, claimant 1 receives the asset and earns utility of

$$U_1(x_1, x_2) = v(x_1, x_2) - \beta(x_2)$$

whereas claimant 2 obtains

$$U_2(x_1, x_2) = \beta(x_2)$$

Differencing

$$\begin{aligned} U_2(x_1, x_2) - U_1(x_1, x_2) &= 2\beta(x_2) - v(x_1, x_2) \\ &\leq 2\beta(x_2) - v(x_2, x_2) \\ &= \beta'(x_2)\frac{1}{\lambda(x_2)} \end{aligned}$$

hence there exists a neighborhood $N(x_2) = (x_2, x_2 + \varepsilon)$ such that for all $x_1 \in N(x_2)$, $U_2(x_1, x_2) > U_1(x_1, x_2)$. ■

Again, it is not the case that for all realizations, the losing bidder does better than the winning bidder.

Remark 3 *When signals are far apart, it may be the case that the winning bidder receives more of the surplus than the losing bidder in the LBA.*

To see this, suppose bidders receive uniformly distributed signals, and the valuation of the object is the average of the signals. Using Proposition 4, a symmetric increasing equilibrium strategy in the LBA is $\beta(x) = \frac{1}{3}x + \frac{1}{6}$. Hence, for $x_1 > x_2$, claimant 1 earns $U_1(x_1, x_2) = \frac{1}{2}x_1 + \frac{1}{6}x_2 - \frac{1}{6}$. If $x_1 > \frac{1}{3}x_2 + \frac{2}{3}$ then the winning bidder obtains more than 50% of the surplus.

3.3.1 Comparing the Auction Mechanisms

To obtain some intuition for the differences in the allocation of the surplus between the two mechanisms, it is useful to compare their equilibrium bidding strategies. In the WBA, we have

$$\begin{aligned}
\beta_{WBA}(x) &= \int_0^x v(\alpha,\alpha)\left[\exp\left(-2\int_\alpha^x \frac{f(s)}{F(s)}ds\right)\frac{f(\alpha)}{F(\alpha)}d\alpha\right] \\
&\leq v(x,x)\int_0^x \left[\exp\left(-2\int_\alpha^x \frac{f(s)}{F(s)}ds\right)\frac{f(\alpha)}{F(\alpha)}d\alpha\right] \\
&= \frac{1}{2}v(x,x)
\end{aligned}$$

Comparing this to the bidding strategy in the LBA.

$$\begin{aligned}
\beta_{LBA}(x) &= \int_x^1 v(\alpha,\alpha)\exp\left(-2\int_x^\alpha \lambda(s)\,ds\right)\lambda(\alpha)\,d\alpha \\
&\geq v(x,x)\int_x^1 \exp\left(-2\int_x^\alpha \lambda(s)\,ds\right)\lambda(\alpha)\,d\alpha \\
&= \frac{1}{2}v(x,x)
\end{aligned}$$

Remark 4 *Bidding in the LBA is more aggressive than bidding in the WBA.*

In the LBA, bidders never pay their bids, but do receive their bid amounts when they lose the auction; whereas in the WBA, bidders pay their bids and *do not* receive their bids if they lose. Hence, it is intuitive that the LBA leads to more aggressive bidding. When signals are close to one another, this aggressiveness manifests itself in an increased share of the surplus being allocated to the losing bidder. Thus, if both bidders have signals $x_1 = x_2 = x$,then in the WBA, the winning bidder receives:

$$v(x,x) - \beta_{WBA}(x) \geq \frac{1}{2}v(x,x)$$

that is, the winning bidder receives more than half of the available surplus. In the LBA, the results are reversed.

$$v(x,x) - \beta_{LBA}(x) \leq \frac{1}{2}v(x,x)$$

Here, more aggressive bidding shrinks the surplus available to the winning bidder and allocates it instead to the losing bidder.

3.4 Binding Arbitration

Finally, we consider a version of binding arbitration. In this mechanism both claimants simultaneously submit messages m_i of what their signal about the value of the object is to an arbitrator. The arbitrator sets a price $p(m_1, m_2) = \frac{v(m_1,m_2)}{2}$ and randomly selects a claimant to buy the asset from the other at this price. The other claimant then receives the proceeds of the sale. Arbitration is binding in the sense that, after submitting a message, the claimant may no longer decline to participate in the mechanism.

Suppose that claimant 2 chooses truth-telling, then, if claimant 1 receives a signal x and reports z, she earns:

$$\begin{aligned} U(z|x) &= \frac{1}{2}\int_0^1 \frac{1}{2}v(z,x_2) f(x_2)\, dx_2 + \frac{1}{2}\int_0^1 v(x,x_2) - \frac{1}{2}v(z,x_2) f(x_2)\, dx_2 \\ &= \frac{1}{2}\int_0^1 v(x,x_2) f(x_2)\, dx_2 \end{aligned}$$

and truthful revelation is a weak best-response by claimant 1 as his payment is independent of z.

If the claimants were characterized by even minor departures from risk neutrality (and were instead slightly risk averse), then truthful revelation is a strict best response to truthful revelation by claimant 2. To see this, suppose that claimants are risk averse with vNM utility function $V(v(x_1,x_2)+m)$ which is strictly increasing and concave in its argument. If claimant 2 truthfully reveals, we have

$$U(z|x) = \frac{1}{2}V\left(\int_0^1 \frac{1}{2}v(z,x_2) f(x_2)\, dx_2\right) + \frac{1}{2}V\left(\int_0^1 \left(v(x,x_2) - \frac{1}{2}v(z,x_2)\right) f(x_2)\, dx_2\right)$$

Differentiating with respect to z yields

$$\begin{aligned} \frac{\partial U(z|x)}{\partial z} &= \left[V'\left(\int_0^1 \frac{1}{2}v(z,x_2) f(x_2)\, dx_2\right) - V'\left(\int_0^1 \left(v(x,x_2) - \frac{1}{2}v(z,x_2)\right) f(x_2)\, dx_2\right)\right] \times \\ & \left(\frac{1}{2}\int_0^1 v_z(z,x_2) f(x_2)\, dx_2\right) \end{aligned}$$

and this expression equals zero when $z = x$,which completes the proof that truthful revelation is a strict best-response by claimant 1.

By construction, under truthful revelation the binding arbitration mechanism achieves an ex post fair allocation for all realizations of x_1 and x_2.

To sum up:

Proposition 6 *Under the binding arbitration mechanism, truthful revelation is a symmetric increasing equilibrium for risk-neutral or risk averse claimants. Moreover, allocations are ex post and ex ante fair for all realizations of* x_1 *and* x_2.

Recall that the difficulty with the original divide and choose mechanism was the adverse selection problem faced by the dividing claimant. By changing the mechanism to a simultaneous game and introducing a lottery over (in effect) the "chooser" in the second stage, we avoid these difficulties while still preserving the fairness properties of the divide and choose mechanism in the complete information case.

4 Conclusions

We have considered four simple mechanisms for fairly allocating an asset when the precise value of the asset when claimants to the asset have private information about its value. Despite the fact that bidders are ex ante identical and that information is independently and identically distributed, three of the four mechanisms systematically fail at rendering fair allocations.

In the case of the divide and choose mechanism, the desirable fairness properties in the complete information are lost in the presence of incomplete information. Instead, the dividing player is faced with a winner's curse problem which leaves him with less than 50% of the surplus. In the case of the WBA and LBA, under (resp. over) aggressive bidding leads to the systematic favoritism of the winning (resp. losing) bidder in the ex post allocations of each of these mechanisms.

Finally, under binding arbitration, we are able to recapture the desirable fairness properties of the divide and choose mechanism in an incomplete information setting. Specifically, by deciding which side of the market each of the claimants for the asset will be on at random *after* their reports of values, the mechanism achieves an ex post and ex ante fair allocation. As in the complete information version of the divide and choose mechanism, biding arbitration leaves each claimant indifferent about which side of the market they are on, thus resulting in truthful revelation as a weakly dominant strategy.

The lessons about the fairness of these simple mechanisms under an incomplete information framework have obvious applicability in areas of law and dispute resolution. Extensions of this work would allow for the possibility that claimants' signals are affiliated as well as allowing for more than two claimants. While the auctions and the biding arbitration mechanism are readily extendable to the more than two claimant case, the same cannot be said of the divide and choose mechanism. This remains for future research.

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