Game Theory (MBA 217)
Final Paper

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Introduction

The end of a basketball game is when legends are made or hearts are broken. It is what Michael Jordan passionately deemed "Winning Time." Coaching decisions made during this period are critical to a team's chances of success. It is also the most heated topic of discussion for fans. A typical sports talk show could field calls from opposing opinions, both seemingly right. So, in this paper, game theory will be used to find the optimal decisions in two particular end-of-game basketball situations. In both situations a team has the ball with less time left in the game than the shot clock, and must choose to attempt a two point or three point shot. In the first situation the team with the ball is down by two, and in the second situation the team is down by three.

Situation 1: Down by Two with Less Time Than the Shot Clock Left in Game

The first situation analyzed is that of a team losing by two points, in possession of the ball and working with less time left in the game than the length of the shot clock. In this scenario coaches usually instruct their team not to shoot until shortly before the expiration of the game clock so as to not give the other team a chance to score again, although they may call for a shot that leaves enough time on the clock for a put-back off of an offensive rebound. However, the general implication is that the losing team, hereafter referred to as the "shooting team", is working on taking the last shot in regulation, and the success or failure of that shot will determine the outcome of the game. The coach of the shooting team in this situation will typically call a time out to set up a designed play. The coach must decide whether to try for a two point shot, which would tie the game and send it to overtime, or to try for a three point shot that would win the game.

During the same time out the coach of the team ahead on the scoreboard, hereafter referred to as the "defending team", must instruct his players how to set the defense. In general, if the defending coach expects a three point shot attempt, he can instruct his players to defend the perimeter vigorously, but at the risk of leaving an entry pass for a relatively high-percentage two point shot open. Likewise, he could set his defense against an interior pass but leave his team more vulnerable to a three point shot attempt. This section applies the tenets of game theory to analyze the options of each coach and hypothesizes an equilibrium strategy.

Simple Sequential Game Analysis

The simplest way to model this zero-sum game with two players is as a simple sequential game, where the shooting team chooses to either shoot a two or three point shot. The game tree is provided in **Appendix 1A.** For this analysis, overall shot percentages that are roughly consistent with high-level basketball are assumed, namely 50% for two point shots and 33% for three point shots. As the diagram shows, applying the "look forward and reason back" technique to this model suggests that shooting a three point shot is the dominant strategy for the shooting team. Since the three point shot should succeed approximately one third of the time and this will win the game outright, the expected value is greater than for the 50% chance of a two point shot succeeding multiplied times the 50% probability of winning in overtime.

This simple analysis can be intuitively grasped by casual basketball fans, so many expect the coaches to set up three point shot plays consistently when faced with this situation. However, this analysis oversimplifies the decision in one key regard. It only takes into account the actions of the shooting team and not the effect of the strategic choices of the defending team. The defending team can choose to play aggressively on the perimeter, double covering the shooting team's best three point shooters, or it can play a more balanced defense that also covers the inside players who usually only take close-in two point shots. The overall shot percentages are based on a large number of shots taken at different levels of defensive pressure, from an easy lay-up with near 100% success to a contested three point shot with a success rate of under 15%. Where the defense concentrates its efforts has a significant effect on the chances of a shot succeeding, so a more detailed analysis of both teams' actions is needed.

Simultaneous Game Analysis

This situation can also be modeled as a simultaneous game table, since both coaches decide on and implement their strategy during the time out without knowing the strategy of the other coach. A table modeling this situation as a simultaneous game is included in **Appendix 1B.** The key new inputs for constructing a simultaneous game table are estimations of how the defending team's choices will affect the shooting team's shot percentages. In this example it is assumed that shooting a three point shot against a defense geared to stopping the three pointer will yield a shooting percentage of only 15%, while this aggressive perimeter defense will leave open entry passes for high-percentage two point shots that succeed at a rate of 70%. Likewise, defending against the two pointer lowers the two point success rate to 33% while increasing the chances of a successful three point shot to 50%.

The simultaneous game table yields no single dominant strategy for either team. For the defending team, the best response to a three point shot attempt is to defend against the three, and the best response to a two point shot attempt is to do the opposite. Likewise, the best response for the shooting team is to shoot the three pointer when the defending team is defending against the two pointer, and vice versa. Clearly, any consistent or predictable strategy by one team could be effectively countered by the other team, so the equilibrium must be some kind of mixed strategy.

Mixed Strategy Equilibrium

In situations where there is no dominant strategy for either side, each team needs to mix its plays in an unpredictable fashion so as to keep the other side from systematically exploiting its tendencies. However, since the payoffs are asymmetric, each team must determine the right mix of strategies to truly keep its opponent off guard. In particular, the shooting team must go for the three pointer frequently enough for the defending team to consider defending it, but rarely enough so that the defending team does not defend against the three pointer every time. Similarly, the defending team needs to randomize its defense in a proportion that does not make it profitable to always choose either the two point or three point strategy.

There is a mathematical equilibrium point where each team is randomizing its strategies so that the other team does not have a dominant strategy and must also mix its strategies in an optimum percentage. In other words the optimal strategy mix should leave the opponent indifferent between its two courses of action. The mathematical determination of this equilibrium point is in **Appendix 2** and a graphical illustration of the mutual best responses is contained in **Appendix 3.** Under the set of assumptions used in this analysis, the equilibrium point is for the shooting team to attempt a three point shot approximately 35% of the time, while the defending team will defend against the three point shot with 63% frequency. This equilibrium results in an expected value of .28 for the shooting team, meaning the shooting team should win the game 28% of the time, with the defending team winning the other 72% of the time.

Implications

The equilibrium point is somewhat counter-intuitive in light of the dominant strategy implied by the simple sequential game analysis, as the shooting team will only attempt the three point shot a little over a third of the time even though a successful three point shot ends the game instead of risking overtime. However, this fact is well-known to the defending team, and indeed the distance involved in a three point shot means that it is very difficult to successfully convert a three point shot against a tight perimeter defense. The most common outcome for this end-of-game situation will be the defending team tightly guarding the perimeter and the shooting team attempting a higher-percentage two point shot to send the game to overtime. While the preceding analysis simplifies many aspects of the game, including the abilities of individual players, the players' ability to dynamically change strategies depending on the defense, the chance of fouls and the opportunities presented by offensive rebounding, the general principle probably holds unless the shot percentages assumed are off by a significant degree.

The idea that a team will usually go for the two pointer and overtime instead of the three point shot and the win also has implications for the reputations and careers of coaches. The mixed strategy equilibrium assumes many repetitions of the same game, but for an individual game the strategies will only be observed once. Seeing the coach trying to send the game to overtime with a two point shot opens up the coach to criticisms of "not playing to win" or "playing not to lose", which if made by the team managers in the NBA or the athletic directors in the NCAA could result in harm to coach's career as well as grumbling in the stands by the fans of the team. External career-related concerns could lead coaches to undertake sub-optimal strategies.

Situation 2: Down by Three with Less Time than the Shot Clock Left in the Game

The next scenario examined offers more complexity thanks to a greater score differential and a resulting expansion in strategic possibilities. For convenience, the conventions and assumptions presented in the first scenario (e.g. "shooting team", shot success rates, etc.) will be used. As before, the strategic situation is one team in possession of the ball, losing and working with less time left in the game than the length of the shot clock. This time the team down will be trailing by three. Once again, the coach will call a timeout to implement a designed play to either: 1) attempt a three point shot to tie and send the game into overtime, or 2) attempt to sink a quick two point shot with the intention of fouling the defending team on their possession, hope they miss their foul shots, and get the ball back with just enough time to sink a bucket for the win. As in Scenario 1, the defending team has the option to defend the two or the three aggressively. A similar analysis will be applied to this scenario in order to develop a hypothesis for an equilibrium strategy. The second option is more complex, but it offers the only option to win the game outright within regulation time. Since there is no direct "one-shot" move to win the game, many fans and "arm-chair" coaches would opt for the three-pointer to put the game into overtime. As illustrated in the previous section, intuition is not always the best selector of strategy.

Simple Sequential Game Analysis

To get a better feel for the strategic situation the scenario is mapped as a sequential move, zero-sum game. **Appendix 4a** outlines the basic decision tree with the standard shooting percentages represented at the various chance nodes. The shot success percentages are as presented in Scenario 1. Upon inspection it is clear that the upper node, representing the two point shot option is incomplete; the foul situation must be mapped out in order to develop an expected value (EV) for this node. **Appendix 4b** displays the decision tree for the possible foul scenarios. As in **Appendix 4a**, the foul shots are displayed with their appropriate success probabilities. It is clear from this tree that the two point shot plus foul scenario is indeed complicated. To reduce the complexity college rules are assumed, so the foul shots are modeled as a one-and-one situation, without the bonus shot. The mandatory two penalty shots awarded in the pros, and the bonus shot in college, would expand the lower node in **Appendix 4b** into two decision nodes, similar to the upper node. After a successful two point bucket by the shooting team, the success of the defending team at the line will determine whether the shooters are down by one, two or three points. This in turn will determine the strategic choices available.

Upon further inspection, it is apparent the middle node of **Appendix 4b** ("down by two") is equivalent to Scenario 1. The EV from the earlier analysis can represent this node. The lower node, illustrating a one point differential between the teams, is new. Finally, the upper node appears to represent the original "down by three" scenario. This is not quite the case; following the foul and subsequent free-throws, the shooting team will have limited time to execute another play. For this reason, there would not be enough time to shoot for two and repeat the foul strategy outlined previously. As such, the outcome of the two point shot is modeled as a loss. Since the shooting team will not pursue this option, the decision tree is greatly simplified with the three point shot as the only valid choice.

Solving the decision tree for the foul situation, and using the resulting expected value (EV) to solve the tree presented in **Appendix 4a**, the decision would favor the "two point shot plus foul" strategy (EV of 0.28 vs. 0.165; calculations not shown here but fairly straightforward from the numbers provided). This does not appear completely irrational since this strategy does offer the only possibilities to win within regulation time. Though complex, the average fan would generally view this as a challenge of execution that can be successfully carried out by a well-coached, well-executing team. However, as before, the simple analysis neglects the interactive nature of the game and does not adjust the shooting percentages to account for the impact of the defensive schemes utilized. Will this strategic solution hold when the problem is approached as a simultaneous game?

Simultaneous Game Analysis

The correct way to model this situation is as a simultaneous game. Since this scenario is considerably more complex, involving several intermediate decisions, solutions must be derived for each of the intermediate nodes in order to develop an appropriate strategy for the whole scenario. To begin breaking this model down, the analysis begins with the foul scenario and will once again "look forward and reason back". Once an overall EV has been generated for the foul scenario and each of its intermediate decisions, this EV can be applied to the overall decision illustrated in **Appendix 4a**.

The first leg of the model solved is the "down by three" situation following the foul shots (see upper node of **Appendix 4b**). The EV to the shooting team for this decision is a relatively straightforward calculation due to the simplifying assumptions detailed above. Overtime is assigned a value of 0.5, and assuming that both teams view the three point shot as the only valid option, the appropriate probability of success is 15% (shooting three with the defense guarding the three). The result is an EV of 0.075 for this decision, the first piece of our puzzle.

As demonstrated earlier, the "down by two" scenario illustrated in the second node (see **Appendix 4b**) required a mixed strategy solution. The equilibrium solution placed the shooting team at the three-point line 35% of the time and the defending team covering the three 63% of the time. The details of the calculations are shown in **Appendix 2**. The shooting team's EV is 0.28, and this is the second piece of our puzzle.

The last step to determining the value of the foul scenario, is to develop a strategy for the lower node in **Appendix 4b**. The simultaneous game table for this decision is presented in **Appendix 5**. The game table shows that there is no dominant pure strategy leading to a Nash equilibrium, so a must develop a mixed strategy solution must be developed. Calculations for this equilibrium are presented in **Appendix 5** as well. The equilibrium in this situation has the defending team covering the three point shot 24% of the time because of the disproportionate payoffs to the two pointer (see game table in **Appendix 5**). This stands in contrast to the findings in Scenario 1 because both shots, if successful, will result in a win, therefore the probabilities of success drive the equilibrium. On the other hand, the shooting team will split their shots almost evenly between the two and three pointer. For the shooter, the EV = 0.416.

With all of the intermediate values, an overall value for the foul situation can be calculated. Since the foul shots themselves don't require a strategy, the EVs for each of the intermediate decisions are used and applied the shooting probabilities to generate an EV = 0.199 for entire foul node. **Appendix 6** displays the foul strategy decision tree with the appropriate EVs from the intermediate mixed strategy solutions added to the figure.

Final Mixed Strategy Equilibrium

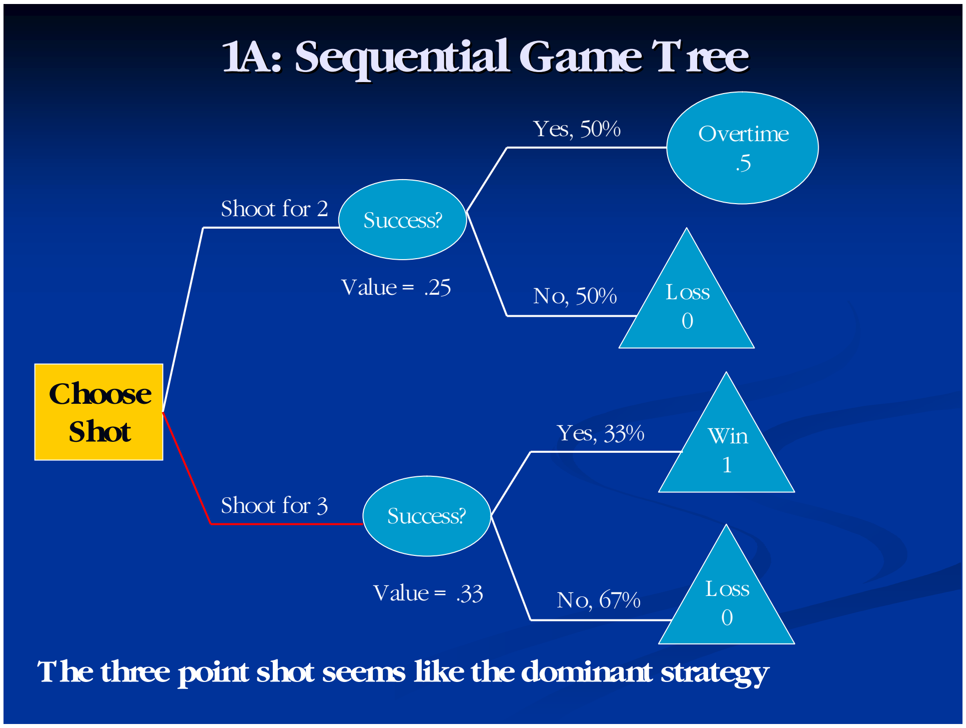
Referring again to **Appendix 4a**, all the pieces are in place to generate an overall solution to the scenario. Using the EV developed in the previous analysis for the foul scenario, a simultaneous game table with the defense adjusted payoffs for the full "down by three" scenario is generated (see **Appendix 7**). A quick perusal of the game table indicates that there is no dominant strategy, so once again a mixed strategy must be found. Solving as before finds an equilibrium with the shooting team shooting the three pointer about 30% of the time and the defending team defending the three about 74% of the time. **Appendix 8** contains a graphical representation of the equilibrium. In line with the prediction provided by the simple analysis, this equilibrium suggests that the complicated, shoot-foul-shoot again strategy holds more value for the shooting team than the more simple "tie-overtime" strategy. Moreover, examining the EVs (0.12 and 0.88 for the shooting team and defending team, respectively) demonstrates that the equilibrium strongly favors the defending team.

Implications

Similar to the equilibrium developed for Scenario 1, the shooting team chooses the three only about a third of the time. However, the reasons are slightly different in this case, since the shooting team is choosing to forego the more simple (from the perspective of execution) three point shot to pursue the chance to win outright. Presumably, the defense prefers to force the shooting team to choose the more complex two point path, in spite of the chance it might present to win outright. The defending team knows that the chance to shoot high percentage free throws and defend against an almost certain three point shot biases the outcome in their favor.

As in Scenario 1, the analysis and resulting equilibria developed here assume many repetitions with identical circumstances and the general simplifying assumptions regarding the individual players, rebound opportunities, etc. This scenario opens the coach up to the same criticisms as Scenario 1, and presents a situation where exogenous pressures related to career and owner-manager-coach-fan dynamics could lead to sub-optimal strategic choices.

Appendix 1: Game Diagrams, Scenario 1



1B: Simultaneous Game Table

Shooting Team	Defending Team: Defend 2	Defending Team: Defend 3
Shoot 2	Shooting % = 33% 33% x .5 = .165 (1-.165) = .835	Shooting % = 70% 70% x .5 = .350 (1-.350) = .650
Shoot 3	Shooting % = 50% 50% x 1 = .500 (1-.500) = .500	Shooting % = 15% 15% x 1 = .150 (1-.150) = .850

- No pure strategy equilibria
- Must find mixed strategy

Appendix 2: Mathematical Determination of Mixed Strategy Equilibrium

Shooter's Best Response

Let q = % of time defender defends 3

The expected payoff to the shooter is:

$q \times .15 + (1-q) \times .5$ if shooting a 3

$q \times .35 + (1-q) \times .165$ if shooting a 2

Therefore, the shooter should shoot the 3 if:

$q \times .15 + (1 - q) \times .50 > q \times .35 + (1 - q) \times .165$

$.15q + .5 - .5q > .35q + .165 - .165q$

$-.35q + .5 > .185q + .165$

$-.535q > -.335$

$q < .626$, meaning the shooter should always shoot the three if the defender defends against the three pointer less than 62.6% of the time.

Check:

Payoff for shooting a 3: $.5 \times .374 + .15 \times .626 = .28$

Payoff for shooting a 2: $.165 \times .374 + .35 \times .621 = .28$

Defender's Best Response

Let p = % of time shooter shoots 3

The expected payoff to the defender is:

$p \times .85 + (1 - p) \times .65$ if defending 3

$p \times .50 + (1 - p) \times .835$ if defending 2

Therefore, the defender should defend the three if:

$p \times .85 + (1 - p) \times .65 > p \times .50 + (1 - p) \times .835$, which reduces to:

$.85p + .65 - .65p > .5p + .835 - .835p$

$.20p + .65 > -.335p + .835$

$.535p > .185$

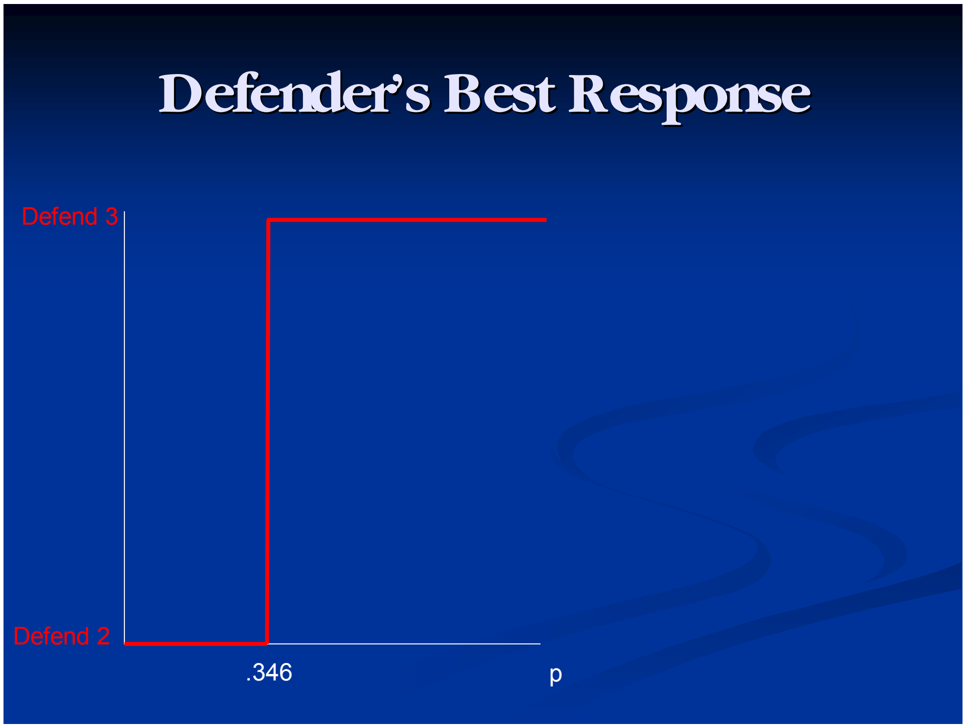
$p > .346$, meaning the defender should always defend the 3 if the shooter shoots the three more than 34.6% of the time

Check:

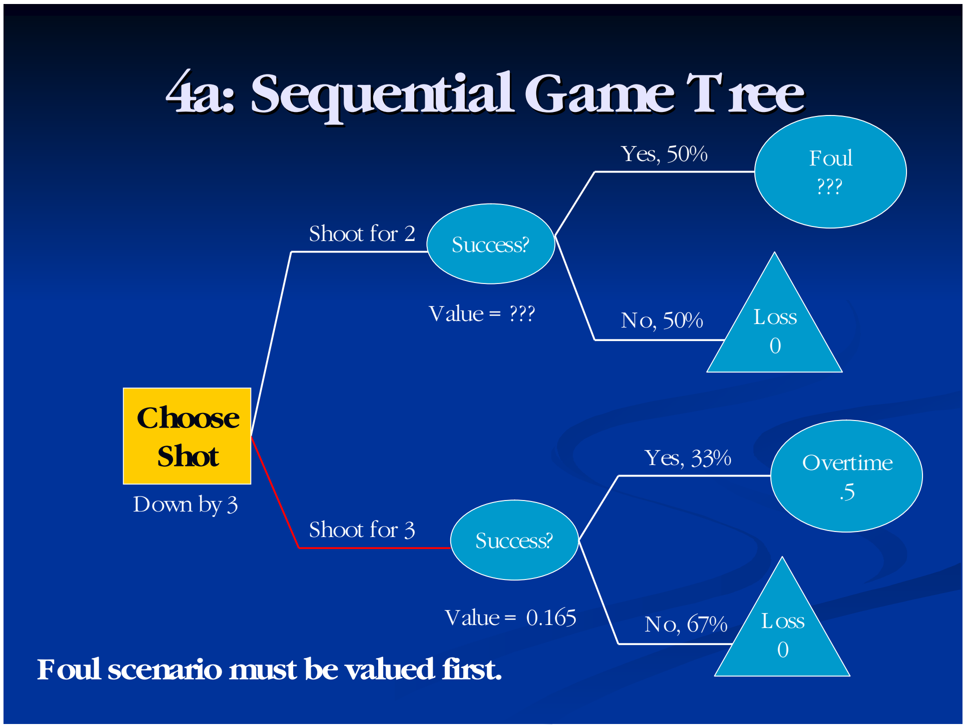
Payoff for defending a 3: $.65 \times (1 - .346) + .85 \times .346 = .72$

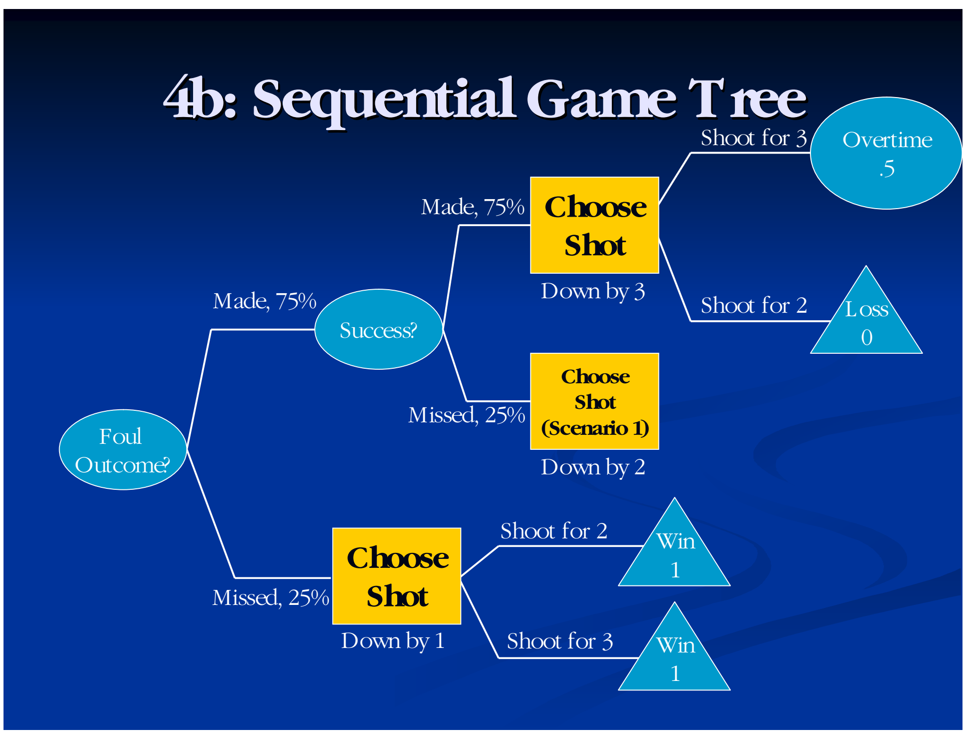
Payoff for defending a 2: $.835 \times (1 - .346) + .5 \times .346 = .72$

Appendix 3: Mixed Strategy Best Response Diagrams



Appendix 4: Game Diagrams, Scenario 2





Appendix 5: Mixed Strategy Solution, "Down by 1", Scenario 2

Shooter's Best Response

Let q = % of time defender defends 3

Shooter should shoot the 3 if:

$q \times .15 + (1 - q) \times .50 > q \times .7 + (1 - q) \times .33$

Solves to shoot 3 when $q < .24$, i.e. the shooter should always shoot the three if the defender guards against the three less than 24% of the time.

EV to Shooter of 0.416

Defender's Best Response

Let p = % of time shooter shoots 3

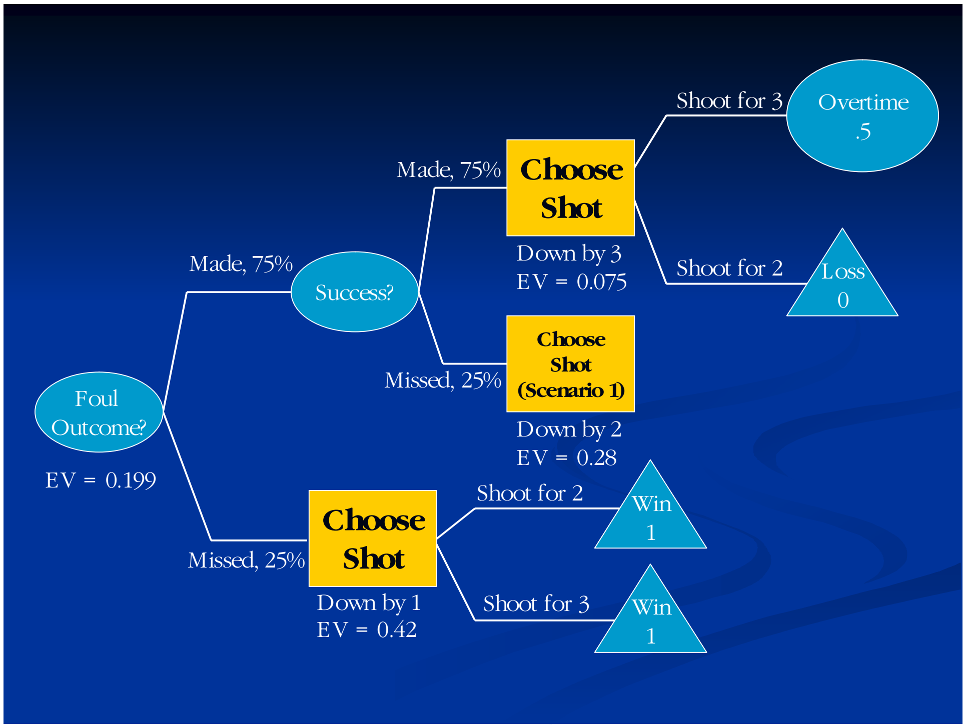
Defender should defend the three if

$p \times .85 + (1 - p) \times .3 > p \times .50 + (1 - p) \times .67$

Solves to defend 3 when $p > .51$, i.e. the defender should always defend against the three if the shooter is shooting the three more than 51% of the time.

EV to Defender of 0.581

Appendix 6: Expected Value for the Foul Node of Scenario 2



Appendix 7: Mixed Strategy Solution, Scenario 2

Shooter's Best Response

Let q = % of time defender defends 3

Shooter should shoot the 3 if:

$q \times .075 + (1 - q) \times .25 > q \times .139 + (1 - q) \times .066$

Solves to shoot 3 when $q < .742$

EV to Shooter of 0.12

Defender's Best Response

Let p = % of time shooter shoots 3

Defender should defend the three if

$p \times .925 + (1 - p) \times .861 > p \times .75 + (1 - p) \times .934$

Solves to defend 3 when $p > .294$

EV to Defender of 0.880

Appendix 8: Best Response Diagram, Scenario 2