

Some Notes on Timing in Games

John Morgan

University of California, Berkeley

The Main Result If given the chance, it is better to move first than to move at the same time as others; that is $\text{IGOUGO} > \text{WEGO}$ from player 1's perspective. This does not, however, say that going first is better than going second. It may be the case that player 1 would prefer UGOIGO to IGOUGO but, regardless of which of these orders is best, IGOUGO still beats WEGO .

Abstract Argument We now show why the statement is true using a general, and somewhat abstract, argument.

Consider a situation where two firms compete in a one shot game of complete information. This means that the strategies of each of the players are known as are the payoffs associated with each pair of strategies. Define x to be a generic strategy for player 1 and y to be a generic strategy for player 2. We will write $BR_1(y)$ to denote player 1's best response when player 2 chooses strategy y , or, more compactly we will write a best response as a function $x(y)$. Likewise, we write player 2's best response as $y(x)$. We write $\pi_1(x, y)$ to be player 1's payoff when she chooses strategy x and player 2 chooses strategy y . Let $\pi_2(x, y)$ be likewise defined.

WEGO: First consider a Nash equilibrium of a WEGO game. In any such equilibrium, a (pure strategy) equilibrium outcome (x^*, y^*) consists of mutual best responses.

Hence $x^* = x(y^*)$ and $y^* = y(x^*)$. Hence, we may write player 1's payoff in such a game as $\pi_1(x^*, y(x^*))$. If the game has multiple pure strategy equilibrium, let (x^*, y^*) denote the equilibrium of the WEGO game that player 1 likes best, i.e. the one yielding the highest payoff from among the set of equilibrium outcomes.

IGOUGO: Now suppose that player 1 moves first, followed by player 2. Moreover, player 1's move is observed perfectly by 2. Then, for whatever strategy, x , player 1 chooses, we know that player 2 will choose his best response $y(x)$. Thus, the payoffs to player 1 from choosing a strategy x are $\pi_1(x, y(x))$. Notice that, if player 1 chooses $x = x^*$, his resulting payoff will be $\pi_1(x^*, y(x^*))$, exactly as in the WEGO game.

But player 1 can choose whatever she wishes in terms of x . So, if x^* is truly the highest she can possibly achieve, she simply chooses x^* in the IGOUGO game and earns the *same* payoff as in the WEGO game. But, for almost all games (formally, for *generic* games), there exists some value of x , call it x' , such that

$$\pi_1(x', y(x')) > \pi_1(x^*, y(x^*))$$

Note that $x' \neq x^*$ is, by definition, not an equilibrium of the WEGO game since we chose x^* precisely because it was the highest possible payoff to player 1 occurring in any equilibrium. What this means is that x' is not a best response to $y(x')$. Formally $BR_1(y(x')) \neq x'$.

How do we know that there will almost always be a value of $x' \neq x^*$ yielding a higher profit under IGOUGO than under WEGO? To see this, suppose that strategies are continuous and differentiable (we'll do an example having these properties below). Then for x^* to be the best that player 1 can do, this would require that x^* is a global maximum, i.e. it is at the very top of the payoff hill. A necessary condition for this

is:

$$\frac{d}{dx}\pi_1(x^*, y(x^*)) = 0$$

While there may be a set of payoffs for the game where this is true, for all other payoffs it is false. In other words, it would be a complete miracle if this condition happened to hold. When the condition fails, there is, of necessity, some $x' \neq x^*$ that raises profits. Moreover, there was nothing special about having only two players. The same is true regardless of the number of players so long as, in the IGOUGO game, all of the other players 2, 3, 4, ..., n move at the same time following player 1. To summarize, we have shown:

Theorem 1 *In one-shot games of complete information, player 1 can always do better in an IGOUGO game than in a WEGO game.*

Example Finally, we illustrate the main result via a worked example. In fact, ours is the earliest possible example of this result. Somewhat representing the fraught relations of French and Germans at the close of the 19th century, it was first shown by Baron von Stackelberg, a German, as counter to an earlier result on WEGO games worked out by Cournot, a Frenchman.

Two firms compete by choosing how much of a commodity good to place on the market. The price then rises or falls to clear the market. Suppose that demand is $Q_D = 100 - P$, where P is the price and Q_D is the quantity demanded. The inverse demand curve is, obviously, $P = 100 - Q_D$. Let x be the quantity produced by player 1 and y the quantity produced by player 2. Thus, the quantity supplied is simply $Q_S = x + y$, and since, under market clearing $Q_S = Q_D$, it then follows that price satisfies:

$$P = 100 - x - y$$

Suppose that each firm has variable costs equal to c per unit of output and no fixed costs. Then, player 1 earns

$$\pi_1 = (P - c)x$$

and similarly for player 2.

Let us now determine the best responses for each firm. Given output x , player 2 chooses y to maximize

$$\begin{aligned}\pi_2 &= (P - c)y \\ &= (100 - x - y - c)y\end{aligned}$$

Differentiating this expression with respect to y (and recalling that, the derivative of an expression αy^k with respect to y is simply $\alpha k y^{k-1}$), we obtain

$$\frac{d\pi_2}{dy} = 100 - x - c - 2y$$

At an optimum, we know that $\frac{d\pi_2}{dy} = 0$, hence, we may solve for the optimal y given x using the above equation to obtain:

$$y(x) = \frac{1}{2}(100 - c) - \frac{1}{2}x$$

This expression is intuitive. The first part is simply the usual monopoly formula. But, since firm 2 is only a monopoly when $x = 0$, output adjusts with x . The equation shows that, for each additional unit of output by firm 1, firm 2 reduces his own output by 1/2 a unit. This also makes sense. As player/firm 1 produces more, the market clearing price falls, and player/firm 2 wants to produce less output. To him, it is as though the inverse demand curve has shifted downward from its original intercept point at 100.

An identical exercise reveals that player 1's best response is

$$x(y) = \frac{1}{2}(100 - c) - \frac{1}{2}y$$

In equilibrium, we must have mutual best responses. Moreover, since the best response functions are entirely symmetric, so too is the equilibrium, which solves the equation

$$x^* = \frac{1}{2}(100 - c) - \frac{1}{2}x^*$$

or

$$x^* = y^* = \frac{1}{3}(100 - c)$$

So we find that firms produce the same output, which is less than the monopoly output. Together, the firms produce more than the monopoly output and so competition lowers the market price from its monopoly level. The payoffs to player 1 from this equilibrium are:

$$\pi_1(x^*, y(x^*)) = (P^* - c)x^*$$

where $P^* = 100 - x^* - y^*$. If variable costs are negligible ($c = 0$), then $P^* = \frac{1}{3} \times 100$ and $x^* = \frac{1}{3} \times 100$, so firm 1 earns precisely

$$\pi_1(x^*, y(x^*)) = \frac{1}{9} \times 10000$$

Now, suppose that firm 1 produces first. It's profit function now becomes

$$\begin{aligned} \pi_1(x, y(x)) &= (P - c)x \\ &= (100 - x - y(x) - c)x \\ &= \left(100 - c - \left(\frac{1}{2}(100 - c) - \frac{1}{2}x\right) - x\right)x \\ &= \frac{1}{2}(100 - c - x)x \end{aligned}$$

Notice that this expression looks different from the WEGO game. The reason is, in the WEGO game, player 2's choice cannot depend directly on 1's choice at the time 1 selects his value of x . The reason is that, player 2 chooses y based on an

expectation of how much 1 will produce. If 1 unexpectedly produces more or less than this expectation, 2 cannot anticipate this and so does not change his output. By contrast, in the IGOUGO game, 2 *observes* 1's output and does respond directly to it. If firm 1 produces a little more than expected, 2 will produce a little less, and vice-versa. This recognition fundamentally changes 1's strategic position. Now 1 can *credibly* influence 2 by its production decision.

As usual, we differentiate to find the optimal production for firm 1. This process yields

$$\frac{d\pi_1(x, y(x))}{dx} = \frac{1}{2} (100 - c - 2x)$$

Setting this expression equal to zero to find an optimum reveals

$$x' = \frac{1}{2} (100 - c)$$

That is, remarkably, when player 1 goes first, it now *optimally* produces more than it did in the WEGO game. Since $x' \neq x^*$, we know that firm 1 must do better but we can compute this directly:.

$$\begin{aligned} \pi_1(x', y(x')) &= (P - c)x' \\ &= (100 - x' - y(x') - c)x' \\ &= \left(100 - c - \frac{1}{2}(100 - c) - \frac{1}{4}(100 - c)\right) \\ &= \frac{1}{4}(100 - c) \times \frac{1}{2}(100 - c) \\ &= \frac{1}{8}(100 - c)^2 \end{aligned}$$

Setting $c = 0$, this simplifies to

$$\pi_1(x', y(x')) = \frac{1}{8} \times 10000$$

And since $\frac{1}{8} > \frac{1}{9}$, one can readily see that the profits from IGOUGO are higher than under WEGO.

Discussion Why is the result true? The key reason is that, in IGOUGO games, player 1 gains *the power of commitment*. In the WEGO games, firm 1 can claim that it will produce more than the equilibrium amount, but firm 2 will take such an announcement with a very large grain of salt, so much so that it will be entirely disbelieved. Thus, the strategic possibilities available to player 1 in the WEGO game consist of only strategies that produce mutual best responses. All others will be disbelieved by player 2 and, ultimately, not undertaken as a result. In an IGOUGO game, all of these lost strategic possibilities are back on the table. While player 2 can choose to be cynical about player 1's *claims* as to what she will do, but cannot be cynical when presented with the hard evidence of Player 1's actual choices. This additional freedom of maneuver is valuable to player 1, and the main result reflects this value.