Some notes on first-price auctions against kinked bidding strategies

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Suppose that the rival bids according to the strategy

$$\beta(v) = \begin{cases} \alpha v & \text{if} \quad v < v' \\ \alpha v' + \gamma(v - v') & \text{if} \quad v \geq v' \end{cases}$$

where v' is the value where the kink in the bidding function occurs. Here $1 > \alpha > \gamma > 0$.

In that case, the probability of winning with a bid b is

$$\Pr[win] = \begin{cases} \frac{b}{\alpha} & \text{if} \quad b < \alpha v' \\ v' + \frac{b - \alpha v'}{\gamma} & \text{if} \quad b \geq \alpha v' \end{cases}$$

First, we will solve for the bidding strategy for low values. This is identical to the earlier analysis, therefore, we know that the optimal bidding strategy is simply $b = v/2$.

The threshold value where the low value strategy ends occurs where

$$\frac{v}{2} = \alpha v'$$

or

$$v = 2\alpha v'$$

When $v > 2\alpha v'$, then the optimal bid solves

$$\pi = (v - b)\left(v' + \frac{b - \alpha v'}{\gamma}\right)$$

Differentiating with respect to b yields the first-order condition

$$-\left(v' + \frac{b - \alpha v'}{\gamma}\right) + \frac{v - b}{\gamma} = 0$$

Solving for b reveals

$$b = \frac{1}{2}\left(v + (\alpha - \gamma)\, v'\right)$$

We will check that this exceeds $\alpha v'$. Since this strategy is increasing in v, it suffices to check at the lowest possible value of v, $v = 2\alpha v'$. Substituting, we have

$$\begin{aligned} b &= \frac{1}{2}\left(2\alpha v' + (\alpha - \gamma)\, v'\right) \\ &= \alpha v' + \frac{1}{2}(\alpha - \gamma)\, v' \end{aligned}$$

and since $\alpha > \gamma > 0$, we then observe that $b\,(2\alpha v') > \alpha v'$.

Thus, we have shown:

Proposition 1 *The optimal bidding strategy against a rival with a kinked bidding strategy is*

$$b\,(v) = \begin{cases} v/2 & \textit{if} \quad v < 2\alpha v' \\ \frac{1}{2}\left(v + (\alpha - \gamma)\, v'\right) & \textit{if} \quad v \geq 2\alpha v' \end{cases}$$

Notice that the slope of the bidding function is one-half in both cases; however there is a jump in the bid at the critical value $2\alpha v'$. The size of the jump is higher the larger is the gap between α and γ and the higher is the kink point.