Auctions in the **wild**: Bidding with securities

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Structure of presentation

• Brief introduction to auction theory
  – First- and second-price auctions
  – Revenue Equivalence Theorem
• The paper: *Bidding with securities*
  – Set up
  – The model for formal and informal auctions
  – Analysis and key results
• Comments and conclusion
Auctions 101

• Many different forms of auctions; two traditional types for auctioning a single item include:
  – First-price auction;
  – Second-price auction (introduced by Vickrey).

• These share features of the archetypical auction model:
  – $n$ ex ante identical potential bidders;
  – Independent private values, $v_i$, drawn from atomless distribution $F \in [0,1]$;
  – Seller has commonly known valuation $v_0$.
  – Sequence: (1) Seller chooses the “contract”; (2) Bidders bid; (3) Payoffs are realized.
Properties of first- & second-price auctions

- **Second-price auction:**
  - Weak dominance of truth-telling
    - Proof (informal): Consider bidding \( b > v_1 \) and \( b < v_1 \). In both cases, can do no better than bidding \( b = v_1 \).
  - Seller’s expected revenue is the 2\(^{nd}\) highest of \( n \) draws from \( F: E[Y_2^{(n)}] \).
  - Bidder’s (with valuation \( x \)) expected payment:
    \[
    \int_{y \leq x} y dF_1^{n-1}(y) = xF^n_1 - \int_0^x F^{n-1} dy \quad \text{(Integration by parts)}
    \]

- **First-price auction:**
  - Bidder’s (with valuation \( x \)) expected payment:
    \[
    xF^n_1 - \int_0^x F^{n-1} dy \quad \text{(yes, exactly the same)}
    \]
  - Thus, seller’s ER is equivalent to 2\(^{nd}\)-price auction: \( E[Y_2^{(n)}] \).
Revenue Equivalence Theorem (RET)

RET: Given certain conditions (see auction notes p. 5), any auction mechanism that results in the same outcome (i.e., allocates items to the same bidders) also has the same expected revenue.

- Bidder 1 w/ valuation \( v \) will choose message \( \hat{v} \) to maximize:
  \[
  \pi(\hat{v}, v) = v F^{n-1}(\hat{v}) - P_A(\hat{v})
  \]
  where \( P_A \) is the expected payment from pretending to be a \( \hat{v} \) type in auction form \( A \).

- With optimization & algebra, becomes:
  \[
  P_A(v) = v F^{n-1}(v) - \int_{v^*}^{v} F^{n-1}(x)dx \quad \forall v \geq \hat{v}
  \]

- Expected revenue:
  \[
  nE_v[P_A(v)]
  \]
  \[
  = n\int_{v^*}^{1} [vf(v) + F(v) - 1] F^{n-1}(v)dv
  \]
Securities as bids

- Initial fixed investment of $1M required.
  - Expects project would on average yield cash flows of $4M.
  - Expects project would on average yield cash flows of only $3M.

- Standard 2nd-price auction: Dominant strategy for both to bid their reservation values.

  \[
  \frac{2}{3} \times 4M = 2.67M
  \]

  \[
  \frac{2}{3} > 2M
  \]

  \[
  \text{Winner!}
  \]

- Abhay bids 4 – 1 = $3M.
  - Laura bids 3-1 = $2M.
  \[
  \text{Abhay pays Laura’s bid, } 2M.
  \]

- Now, bidders offer a fraction of future revenues.

  \[
  \frac{3}{4} \text{ of future cash flows.}
  \]

  \[
  \frac{2}{3} \text{ of future cash flows.}
  \]

  \[
  \text{Winner!}
  \]
Formats of auctions studied

• Formal auction: Consists of both an auction format and a security design.
  – E.G.: government sales of oil leases, wireless spectrum, highway building contracts, etc.

• Informal auction: Lacks formal auction rules.
  – E.G.: Authors selling publishing rights, entrepreneurs soliciting venture capital, etc.

• Major difference: Seller’s level of commitment.
The model

• Players:
  • $n$ bidders: risk neutral, all competing for an asset
    • *can think of the asset as the rights to a project*
  • 1 seller, also risk neutral, unable to undertake project independently

• Payoffs:
  • Winner makes investment $X > 0$. $X$ is known and equal across bidders *(can think of $X$ as the resources required by the project or a minimal amount of $\$ the seller must raise)*
  • Conditional on being undertaken by bidder $i$, project yields a **stochastic** future payoff $Z_i$
  • Bidders have private signals regarding $Z_i$, denoted $V_i$
    • Example of a possible form of $Z_i$ consistent with the model: $Z_i = \theta (X + V_i)$, where project risk $\theta$ is independent of $V$ and log-normal.
The bids: Shares of the final payoff

• Bidders compete for project by offering seller a share of final payoff
• Bids $\equiv S(z)$
  • Bids are in terms of derivative securities, in which underlying asset is the future payoff of the project $Z_i$
  • Function $S(z)$ indicates the payment to the seller when the project has final payoff $z$.
• Assumption about bids:
  • A feasible security bid’s function, $S(z)$ is weakly increasing in $z$ and $0 \leq S(z) \leq z$
  • Only the underlying asset is used to pay seller
• Both seller’s and bidder’s payment are weakly increasing in the payoff of the project.
Types of securities

- **Equity**: The seller receives some fraction $\alpha \in [0, 1]$ of the payoff: $S(z) = \alpha z$.

- **Debt**: The seller is promised a face value $d \geq 0$, secured by the project: $S(z) = \min(z, d)$.

- **Convertible debt**: The seller is promised a face value $d \geq 0$, secured by the project, or a fraction $\alpha \in [0, 1]$ of the payoff: $S(z) = \max(\alpha z, \min(z, d))$.

- **Levered equity**: The seller receives a fraction $\alpha \in [0, 1]$ of the payoff, after debt with face valued $d \geq 0$ is paid: $S(z) = \alpha \max(z-d, 0)$.

- **Call option**: The seller receives a call option on the firm with strike price $k$: $S(z) = \max(z-k, 0)$. Higher bids correspond to lower strike prices.
Expected value of a security

• Given any security $S$:
  
  $$ES(v) \text{ is defined as } E[S(Z_i) | V_i = v]$$

• Bidder’s expected payoff is $V_i - ES(V_i)$

  $\rightarrow$ Key difference from standard auction is that the seller does not know the value of the bids, but only the security, $S$ (must infer security’s value). Since monotone, value is incr. with $V_i$

**LEMMA 1**: The value of the security $ES(v)$ is twice differentiable. For $S \neq 0$, $ES'(v) > 0$, and for $S \neq Z$, $ES'(v) < 1$. 
Analysis – Formal Auctions
Case 1: Formal Auction w/ Ordered Securities

- Bidders compete by offering a higher security.
- A formal auction is described by an ordered set of contracts/securities $S(s, z)$ and an auction format.
- Easy to generalize auctions to this setting:
  1) **First-Price Auction**: Each agent submits a security. Highest bidder (highest $s$) wins and pays according to his security.
  2) **Second-Price Auction**: Each agent submits a security. Highest bidder (highest $s$) wins and pays according to second-highest security $s$. 
Are the equilibria in these auction formats efficient (i.e., highest bidder wins)?

Answer for 2\textsuperscript{nd}-price auctions is straightforward:

**LEMMA 2:** The unique EQM in the 2\textsuperscript{nd}-price auction is for a bidder $i$ who has value $V_i = v$ to submit security $S(v)$ s.t. $ES(s(v), v) = v$ (ie, each bidder submits securities according to his true value). The EQM strategy $s(v)$ is increasing in $v$. 
First Price Auctions

• By IC constraint, no bidder mimics another type, so security bid $s(v)$ satisfies

$$U(v) = \max_{\hat{v}} F^{n-1}(\hat{v})(v - ES(s(\hat{v}, v)) = F^{n-1}(v)(v - ES(s(v), v))$$

• Assuming the bidder’s profit function is log-supermodular (A.C), the unique EQM is the solution to the differential equation:

$$s'(v) = \frac{(n - 1)f(v)}{F(v)} \times \frac{[v - ES(s(v), v)]}{ES_1(s(v), v)}$$
General Symmetric Mechanism

- 1\textsuperscript{st}- & 2\textsuperscript{nd}-price auctions are standard mechanisms in which highest bid wins & only the winner pays
- DKS define a (more) general class of mechanisms with these properties - a **General Symmetric Mechanism** (GSM):
  - A GSM is a symmetric incentive compatible mechanism in which the highest type wins and pays a security chosen at random from a given set S.
  - The randomization can depend on the realization of types, but not on the identity of the bidders (provides symmetry)
- First-price auction is a GSM with no randomization (since security is a function of winner’s type)
- Second-price auction is a GSM with randomization, since security depends on realization of second-highest type
Incentive Compatibility in GSMs

• Lemma 4: Incentive compatibility in a GSM implies the existence of securities \( \hat{S}_v \) in the convex hull of \( S \) s.t.

\[
v \in \arg \max_{v'} \, F^{n-1}(v')(v - E\hat{S}_{v'}(v))\]

Thus, it is equal to a GSM in which the winner pays the non-random security \( \hat{S}_v \).

• This condition will be useful when we compare auction formats.
Ranking Security Designs (Prop. 1)

• Revenues of security designs depend on the “steepness” of securities
  – But what is “steepness”? 
• Note that in the case of equity vs. debt bids, the slopes of $S(Z)$ are ranked differently for different levels of $Z$:
  – Debt has a higher slope than equity for low realizations of $Z$ but the opposite is true for high realizations of $Z$. 

![Payoff Diagrams for Call Options, Equity, and Debt](image-url)
Ranking Security Designs

• Standard sets of contracts/securities can be ranked under the assumption of SMLRP → what matters is the relative slope of the securities at the point they cross

  – **Definition: Strict Crossing**: An ordered set of securities $S_1$ is steeper than an ordered set $S_2$ if, for all $s_1$ and $s_2$ from two sets, $ES_1(v^*) = ES_2(v^*)$ implies that $ES_1'(v^*) > ES_2'(v^*)$.

    • If true we say that “$S_1$ strictly crosses $S_2$ from below.”
Ranking Security Designs

• **Single Crossing Property** is a sufficient condition for a security to strictly cross another security:

  **Lemma 5 (Single Crossing):** $S_1$ *strictly crosses* $S_2$ *from below* if for $S_1 \neq S_2$, there exists $z^*$ such that $S_1(z) \leq S_2(z)$ for $z < z^*$ and $S_1(z) \geq S_2(z)$ for $z > z^*$.
Proposition 1: Suppose the ordered set of contracts/securities $S_A$ is steeper than $S_B$. Then for either a first-price or a second-price auction, for any realization of types (almost surely), the seller's revenues are higher using $S_A$ than using $S_B$.

- Note the slight change of notation for clarity
Proof of Proposition 1 (for 2\textsuperscript{nd}-price auction)

• Consider a second-price auction,
  – In EQM, the winning bidder with type $V_1$ pays the security bid by the second highest type $V_2$: $\text{ES}(s(V_2), V_1)$
  – And, since bidders bid their reservation value in a second-price auction, $\text{ES}(s(V_2), V_2) = V_2$

• Thus, the security design impacts revenues only through the difference $\text{ES}(s(V_2), V_1) - \text{ES}(s(V_2), V_2)$, which is just the sensitivity of the security to the true type.
  – So to compare the revenues of securities A and B we need to compare their slopes at $V_2$. 
Proof of Proposition 1 (for 2\textsuperscript{nd}-price auction)

• B/c steeper securities are more sensitive at the crossing point, they lead to higher revenues.
  – By steepness of $S_A$, the $ES_A$ (expected payoff from security a) is increasing faster than $ES_B$ as we increase the type from $z^{(2)}$ to $z^{(1)}$ → thus $S_A$ leads to higher revenues for seller

• More generally, steepness enhances competition between bidders since even with the same bid, a higher type will pay more (since $Z$ increases as $V$ increases)
Proof of Proposition 1 (for 1st Price auction)

• Recall that in first price auction the winner pays his/her bid.
  – Incentive compatibility implies that no bidder gains by lying about his/her type \( \Rightarrow \) thus security \( s(v) \) satisfies:
    \[
    U(v) = \max_{\hat{v}} F^{n-1}(\hat{v})(v - ES(s(\hat{v}, v)))
    = F^{n-1}(v)(v - ES(s(v), v))
    \]

If \( U^1(v) = U^2(v) \), then \( ES^1(v) = ES^2(v) \) (where superscript denotes type)
Proof of Proposition 1
(for 1st Price auction)

• Now assume $S_1$ is steeper $S_2 \rightarrow$ thus $ES_{v}^{1'}(v) > ES_{v}^{2'}(v)$ (by single crossing property)
• Thus, using the envelope theorem and that in EQM, bidder reports his true type, we get that
  
  $U^{1'} = F^{n-1}(v)(v - ES^{1'}(v))$,
  
  which is less than
  
  $F^{n-1}(v)\left(v - ES^{1'}(v)\right) = U^{2'}(v)$

  Thus,
  
  $U^{1'}(v) < U^{2'}(v)$

• Since bidder’s payoffs are lower, seller’s expected revenue is higher for reach realization of winning type under $S_1$ (since the total expected surplus is the same in the 2 auction formats, the ranking of expected seller revenues is opposite to ranking of bidder surpluses)
Revenue Equivalence Theorem & Security-based auctions

We have shown that (fixing the auction format) steeper securities lead to higher revenues for the seller.

So, what about mechanism (auction) format? Does it matter, or does the RET hold?

If it does matter (i.e., RET does not hold), then what impacts revenues more, security design or auction format?
Sub- and super-convex sets of securities

- First, must understand the differences across security sets in the difference in steepness between the set of securities and its convex hull.
- Definition: An ordered set of securities \( S = \{S(s, \cdot) : s \in [s_0, s_1,]\} \) is super-convex if it is steeper than any nontrivial convex combination of the securities in \( S \). It is sub-convex if any nontrivial convex combination of the securities in \( S \) is steeper than \( S \).
Consider a couple of cases

• For debt securities, consider any feasible security $S_2$. If $S_2(z) > \min(d, z)$, then $z > d$ and so $S_2(z') > \min(d, z') \forall z' > z$. Hence, $\min(d, z)$ crosses $S_2$ from above.

  → The set of standard debt contracts is sub-convex.

• For levered equity, a convex combination of these securities for different levels of leverage is a security $S_2(z)$ that is convex with a maximum slope $\alpha$. Thus, any levered security crosses $S_2$ from below.

  → The set of levered equity contracts is super-convex.

  → Similar arguments apply for convertible debt and call options. (This argument is Lemma #6)
Ranking 1\textsuperscript{st}- & 2\textsuperscript{nd}-price auctions

Proposition 2:

• If the ordered set of securities is sub-convex, then the first-price auction yields lower expected revenues than the second-price auction.

• If the ordered set of securities is super-convex, the first-price auction yields higher expected revenues than the second-price auction.
Proof of Proposition 2

• Consider the direct revelation game corresponding to the two auctions.

• Let $S_v^1$ be the security bid in the first-price auction, and let $S_v^2$ be the expected security-payment in the second-price auction for a winner with type $v$, defined as in Lemma #4 ($v \in \arg \max_{v'} F^{n-1}(v')(v - E\hat{S}_{v'}(v))$).

• If the set of securities is super-convex, $S_v^1$ crosses $S_v^2$ from below.

• Now, can use the proof from Prop. 1. At the point where they cross, $S_v^1$ is steeper than $S_v^2$, implying that the utility to the bidder will be lower for $S_v^1$.

• Proof for sub-convex sets is identical, with inequalities reversed.
But what happens if super-convex is too super, and sub-convex is too sub?

- There’s a set in the middle that’s just right: An ordered set of securities that is convex (equal to its convex hull). Examples: Equity & cash.

Convex sets & Proposition #3

• Each security in a convex set is a convex combination of the lowest security $s_0$ and the highest security $s_1$.

• **Proposition 3** (Revenue Equivalence): *Every efficient EQM of a GSM with securities from an ordered convex set yields the same expected revenues.*
Proof of Proposition 3

• In GSM, winner pays according to a random security.
• Again uses Lemma 4: the expected payment by type \( v \) reporting \( v' \) is \( E\hat{S}_{v'}(v) \), where \( \hat{S}_{v'} \) is in the convex hull of the ordered set of securities \( S \).
• Since \( S \) is convex, we can define \( s^*(v') \) s.t.

\[
S(s^*(v'), \bullet) = \hat{S}_{v'}(\bullet)
\]

• \( S \) is ordered, so incentive compatibility implies \( s^*(v) \) must be increasing. Thus, it defines an efficient EQM for a 1\textsuperscript{st}-price auction.
• Result then follows from uniqueness of EQM.
Pulling Props #1-3 together

• **Proposition 4** says:
  – *Winner for best security design and format combination (in terms of expected revenues among all GSMs)*...

  **First-price auction with call options!**

  – *Loser for security design and format combination*...

  **First-price auction with standard debt!**
Final comments on formal auctions

• Design of the security is more important than the specific auction format.
• For any feasible set, if there is a steepest set of securities which is (super-) convex, then a (first-price) auction using this set yields the highest possible revenues and intensifies competition among bidders.
• If there is a flattest set which is (sub-)convex, then a (first-price) auction using this set yields the lowest possible revenues.
A few words on informal auctions

They are messy.
Informal auctions in one slide

• Seller considers all bids, chooses most attractive bid ex post.
• Security design is in the hands of the bidders. Cheapest way for bidders to win?
  – Flattest securities.
• When mimicking a lower type, it is cheaper for a higher type to use a flatter security.
• If we impose a weak refinement of the notion of strategic stability (D1 refinement), can characterize EQM:
  – Prop. 5: Given symmetric strategies, there is a unique EQM of the informal auction satisfying D1. EQM is equivalent to that of a 1st-price auction in which players bid with the possible flattest securities.
Extensions: Relaxing Liquidity Constraints

• So far, DKS have assumed that the seller in an auction is liquidity-constrained: $X$ as amount the seller needs to raise.

• What if seller has surplus cash?
  – Let $X =$ resources required for the project.
  – Define securities $S(v)$ as a reimbursement to the winner for a portion of the initial investment (so $S(0) < 0$).
Extensions: Relaxing Liquidity Constraints

• These securities can be steeper than call options and so increase revenues...to the point of full extraction. To see this:
  – Suppose seller auctions off the rights to a fraction of the cash flow and reimburses the winner directly for investment in the project (say, $(1 - \varepsilon)X$)
  – Making arbitrarily small, seller can extract the entire surplus

• Another example: Take an equity auction with investment required for a project to be $X$. From the paper (p. 945), we know the dominant strategy is to bid $\propto (v) = \left[\frac{v}{v+X}\right]$.  
  – However, as $X \to 0$, the bidding function converges to 100%!

• Highlights how “steepness” of a securities contract can be taken to the extreme
Extensions: Moral Hazard

• But what if the winner’s investment of $X$ in the project is not fully verifiable (i.e., the winner’s investment is not fully contractible)?
  – If winner receives only a small fraction of future revenues, he may underinvest

• Example: Suppose auction takes place, then winner (agent $i$) can decide whether to invest $X$
  – If $X$ is invested, the payoff of the project is $Z_i$ as before, and his payment to the seller is $S(Z_i)$.
  – If $X$ is not invested, the payoff is 0 - bidder’s payment to the seller is $S(0)$. 
Extensions: Moral Hazard

• If \( S(0) \geq 0 \), the bidder’s payoff is non-positive without investment, and so the option not to invest is irrelevant
• BUT: what if a bid with \( S(0) < 0 \) is accepted by the seller?
  – Then every bidder, including the lowest type, can earn positive profits by making such a bid and not investing.
• Yet if bidders do not invest and earn \( S(0) < 0 \), the seller loses money – thus seller would choose not to accept such securities
Extensions: Moral Hazard

• Thus, when an investment X by buyers is not contractible, we can rule out seller reimbursement: it will either not occur in EQM or not be in the seller’s best interest
  – This is Proposition 6 in DKS
Extensions: Moral Hazard

Prop. 6: Suppose that the seller is not liquidity constrained and the investment $X$ is not contractible. In a first- and second-price formal auction (with an ordered set of securities):

a) If a security without reimbursement is allowed, then with probability 1 the winning bid satisfies $S(0) > 0$. That is, competition between bidders rules out reimbursement.

b) If all securities involve reimbursement, then all bidders bid the highest allowed security and do not invest, leading to negative revenues for the seller.
Extensions: Moral Hazard

Proof of Prop. 6:
- Let $s'$ be a bid that wins with positive probability such that $S(s', 0) < 0$.
- This bid will earn strictly positive profits for any type, since any type can simply not invest and collect $-S(s', 0)$ in a first-price auction (even more in a second-price auction since the second highest bid is below $s'$).
  - Thus, by incentive compatibility, all equilibrium bids earn positive profits
Extensions: Moral Hazard

• Define $\_s$ as the lowest submitted bid.
  – Previous slide implies this bid will win with positive probability.
• Then raising bid slightly would lead to a discrete jump in the probability of winning (and profits as well).
• Incentive compatibility therefore implies $\_s = s_1$ (the highest bid).
• If $S(s^1, 0) \geq 0$, this contradicts the existence of $s'$ (reimbursement ruled out)
• If $S(s^1, 0) < 0$, then all types bid $s_1$. But at $s_1$, all types lose money if they invest in the project. Thus, all types bid $s_1$ but do not invest and collect the subsidy $-S(s', 0) > 0$ from the seller (i.e., negative returns for the seller)
Where has the literature gone from here?

• Importance of information asymmetries:
  – Moral hazard (Kogan & Morgan, 2008)
  – Adverse selection (Kogan & Morgan, 2008),
  – (Che & Kim, 2010)

• Che & Kim develop a similar model to DKS and add a caveat to their analysis:
  – A steeper security is more vulnerable to adverse selection, and could result in poorer revenue performance, than a flatter security.
Korgan & Morgan (2010): Auctions with Contingent Payments in Practice

- Auctions with contingent payments are common in the private sector:
  - One potential disadvantage of using steep securities/contracts is creation of moral hazard: when the winner of the auction is not a residual claimant of $Z_i$, it may not take profit-maximizing actions.
  - Using contingent payments creates a tradeoff: extracting a larger fraction of the surplus from the winning bidder vs. creating a distortion in effort that reduces total surplus.
Appendix
Assumptions

- **A**: Private signals $V = (V_1,\ldots,V_n)$ and payoffs $Z = (Z_1,\ldots,Z_n)$ satisfy:
  
  a) Private signals $V_i$ are i.i.d. w/ density $f(v)$ & support $[v_L, v_H]$  
  
  b) Conditional on $V=v$, the payoff $Z_i$ has density $h(z | v_i)$ with full support $[0, \infty)$  
  
  c) $(Z_i, V_i)$ satisfy the strict Monotone Likelihood Ratio Property (SMLRP); that is, the likelihood ratio $h(z | v)/h(z | v')$ is increasing in $z$ if $v > v'$  

  (Equivalently, $h$ is log supermodular)  

- Given assumption A, normalize (WLOG) the private signals s.t. $E[Z_i | V_i] - X = V_i$  

  We can interpret the signal as the NPV of the project
Add’l Assumptions

• B: The conditional density function $h(z|v)$ is 2x differentiable in $z$ & $v$. In addition, the functions $zh(z|v)$, $zh_v(z|v)$, and $zh_{vv}(z|v)$ are integrable for $z$ exists in $(0, \infty)$