Some Lecture Notes on Auctions

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1 Preliminaries

• Perhaps the most fruitful area for the application of optimal screening contracts is in auction theory

• 1 principle, many bidders.

• The goal might be to allocate efficiently (and maximize revenues while doing so)

• Or it might just be to maximize revenues.

The main result we will establish is:

Revenue Equivalence Theorem (RET): Assume each of a given number of risk-neutral potential buyers has a privately-known valuation independently drawn from a strictly-increasing atomless distribution, and that no buyer wants more than one of the \( k \) identical indivisible prizes.

Then any mechanism in which

- (i) the prizes always go to the \( k \) buyers with the highest valuations and

- (ii) any bidder with the lowest feasible valuation expects zero surplus,

yields the same expected revenue (and results in each bidder making the same expected payment as a function of her valuation).

1.1 Model

Throughout, we study the archetypal auction model:

- \( n \) ex ante identical potential bidders

- independent private values, \( v_i \), drawn from some atomless distribution \( F \) on \([0, 1] \).

- Single object to be auctioned.

- Seller has commonly known valuation \( v_0 \).
Game form
Seller chooses the “contract” (i.e. auction form).
Bidders bid.
Payoffs are realized.

2 Second Price Auction

Consider an auction where the winning bidder pays the second highest bid (introduced by Vickrey)

Proposition 1 Suppose bidders have private values, then bidding one’s valuation is a weakly dominant strategy.

Proof. General and informal.
Suppose you bid $b > v_1$, then the outcome compared to the putative eqm strategy is changed only when highest (other than 1’s) is between $v_1$ and $b$. But in these circumstances you incur losses.

Suppose you bid $b < v_1$, then the outcome compared to the putative eqm strategy is changed only when highest bid (other than 1’s) is between $b$ and $v_1$. In these circumstances you miss out on profits. ■

Less general, more mathematical:
Suppose everyone else is bidding according to the increasing bidding strategy $\beta (v)$. Let $F_1^{(n-1)} (y)$ be the distribution function of the highest of $n - 1$ draws from $F$.
That is
$$y = \max \{v_2, v_3, ..., v_n\}$$
Bidder 1 wins if $b \geq \beta (y)$, so
$$E\pi_1 (v_1, b) = \int_0^{\beta^{-1} (b)} [v_1 - \beta (y)] dF_1^{(n-1)} (y).$$

Differentiating wrt $b$,
$$[v_1 - b] f_1^{(n-1)} (\beta^{-1} (b)) \frac{d}{db} [\beta^{-1} (b)] = 0$$
Since $f_1^{(n-1)} (\beta^{-1} (b)) \neq 0$ and $\frac{d}{db} [\beta^{-1} (b)] \neq 0$, it then follows that $b = v_1$.
It then follows that:

Proposition 2 The Vickrey auction is efficient when bidders have private values.

- Experimental findings sometimes differ from this.
Revenues
The seller’s expected revenue is simple $E \left[ Y_{2}^{(n)} \right]$, i.e., the second highest of $n$ draws from $F$.
With uniform distributions, this becomes
$$E \left[ Y_{2}^{(n)} \right] = \frac{n - 1}{n + 1}$$
so it’s obvious that revenues are increasing in $n$ and converge to where the seller obtains all the surplus.
We can get at revenues a different way:
What is the expected payment of a bidder with valuation $x$?

$$P_{II}(x) = \Pr \{ \text{win} \} \times E \{ \text{payment|win} \}$$
$$= \Pr \{ \text{win} \} \times \frac{E \{ \text{payment} \times I_{\text{win}} \}}{\Pr \{ \text{win} \}}$$
$$= \int_{y \leq x} ydF_{1}^{(n-1)}(y)$$
Under the uniform distribution
$$= \int_{0}^{x} yd \left[ y^{n-1} \right]$$
$$= \frac{n - 1}{n} x^{n}$$
The ex ante expected payment is then
$$P_{II} = \int_{0}^{1} P_{II}(x) dF(x)$$
$$= \frac{n - 1}{n(n + 1)}$$
Expected revenue
$$ER = nP_{II}$$
$$= \frac{n - 1}{n + 1}$$
More generally
$$P_{II}(x) = \int_{y \leq x} ydF_{1}^{(n-1)}(y)$$
Integrate by parts
$$P_{II}(x) = xF_{1}^{(n-1)}(x) - \int_{0}^{x} F_{1}^{(n-1)}(y) dy$$
$$= xF^{n-1} - \int_{0}^{x} F^{n-1} dy$$
3 First Price Auctions

Bidder 1, given valuation \( v_1 \), chooses \( b \) to maximize

\[
E \pi_1 (v_1, b) = \int_0^{\beta^{-1}(b)} [v_1 - b] \, dF_1^{(n-1)}(y)
\]

\[
= [v_1 - b] \, F_1^{(n-1)} (\beta^{-1} (b))
\]

Differentiate

\[
[v_1 - b] \, dF_1^{(n-1)} (\beta^{-1} (b)) \frac{d}{db} (\beta^{-1} (b)) - F_1^{(n-1)} (\beta^{-1} (b)) = 0
\]

Under a symmetric equilibrium, \( b = \beta (v_1) \); hence

\[
[v - \beta (v)] \, f_1^{(n-1)} (v) \frac{1}{\beta' (v)} - F_1^{(n-1)} (v) = 0
\]

Rewriting

\[
\beta' + \beta \times A (v) = vA (v)
\]

where \( A (v) = \frac{f_1^{(n-1)} (v)}{F_1^{(n-1)} (v)} \).

Multiply by \( e^{\int A} \):

\[
\beta' e^{\int A} + \beta A e^{\int A} = vA e^{\int A}
\]

\[
\frac{d}{dv} \left( \beta e^{\int A} \right) = vA e^{\int A}
\]

Hence

\[
b e^{\int A} = \int vA e^{\int A} \, dv + c
\]

\[
b (v) = e^{-\int A} \left( \int vA e^{\int A} \, dv + c \right)
\]

Since \( b (0) = 0 \), then

\[
b (v) = e^{-\int A} \left( \int_0^v tA (t) \, e^{\int A} \, dt \right)
\]

Now

\[
e^{\int A} = e^{\int^{(n-1)f(x)}_0 \frac{\int^{n-2} f(x) \, dx}{(\int f(x) \, dx)^{n-1}}} \, dx
\]

\[
= e^{\int^{(n-1)f(x)}_0 \frac{\int f(x) \, dx}{F(x)} \, dx}
\]

\[
= e^{(n-1) \ln F (x)}
\]

\[
= F (x)^{n-1}
\]

4
Hence

\[ b(v) = \frac{1}{F(v)^{n-1}} \int_0^v (n-1)x \frac{f(x)}{F(x)} F(x)^{n-1} \, dx \]

\[ = \frac{1}{F^{n-1}} \int_0^v xd[F^{n-1}] \]

Integrate by parts

\[ b(v) = v - \frac{1}{F^{n-1}} \int_0^v F^{n-1} \, dx \]

Compute expected payment:

\[ P_I(x) = b(x) F^{n-1}(x) \]

\[ = xF^{n-1} - \int_0^x F^{n-1} \, dy \]

Notice that it is exactly the same as the Second price auction.

Thus, the expected revenue to the seller in either of these auctions is simply

\[ ER = E^{(n)}_2[v] \]

i.e. the expectation of the second highest of \( n \) draws.

Thus, both types of auctions are equally good (Vickrey 1962).

- Strategic equivalence with other auction forms.
  - Difference between strat equivalence in first-price versus second-price case.
- But experiments do not bear this out.

4 The Revenue Equivalence Theorem

Consider the following set of auction contracts.

- Announce a minimum opening bid \( b_0 \).
- High bidder wins
- Rules are anonymous
- Strictly increasing symmetric bidding strategy in auction.
- Non-negative returns to bidding.
Use the revelation principle to restrict attention to direct mechanisms.

**Bidder 1’s Problem**

Given a valuation $v$, choose a message $\hat{v}$ to maximize

$$\pi(\hat{v}, v) = v F^{n-1}(\hat{v}) - P_A(\hat{v})$$

where $P_A$ is the expected payment from pretending to be a $\hat{v}$ type in auction form $A$.

Optimize with respect to $\hat{v}$:

$$\pi_1(\hat{v}, v) = v (n-1) F^{n-2}(\hat{v}) f(\hat{v}) - P_A'(\hat{v})$$

In equilibrium, $v = \hat{v}$

$$v (n-1) F^{n-2}(v) f(v) = P_A'(v)$$

Boundary condition: Find a type $v_*$ solving

$$P_A(v_*) = v_* F(v_*)^{n-1}$$

Now solve the differential equation

$$\int_{v_*}^{v} P_A(x) \, dx = \int_{v_*}^{v} x F^{n-1}(x) \, dx$$

Notice that the RHS is independent of the auction form!

With algebra

$$P_A(v) = v F^{n-1}(v) - \int_{v_*}^{v} F^{n-1}(x) \, dx$$

for all $v \geq v_*$

Expected revenue

$$ER = n E_v[P_A(v)]$$

$$= n \int_{v_*}^{1} [v f(v) + F(v) - 1] F^{n-1}(v) \, dv$$

So we have proved the revenue equivalence theorem!

## 5 Applications

Knowing the RET can be helpful in directly computing bidding strategies.

**First-price auction**

$$P_I(v) = \Pr\{y \leq v\} b(v)$$

$$v F^{n-1}(v) - \int_{v_*}^{v} F^{n-1}(x) \, dx = F^{n-1}(v) b(v)$$
So
\[ b(v) = v - \frac{1}{F^{n-1}(v)} \int_{v_*}^{v} F^{n-1}(x) \, dx. \]

All-pay Auction
\[ P_{AP}(v) = \gamma(v) \]
so
\[ \gamma(v) = v F^{n-1}(v) - \int_{v_*}^{v} F^{n-1}(x) \, dx \]

And so on.

6 Empirical Tests of the RET
- Turkish treasury auctions
- Structural estimation literature
- EBay experiments

7 Revenue Maximization
Using the RET, the principal’s problem is to choose \( v_* \) to maximize
\[
ER = v_0 F(v_*)^n + n \int_{v_*}^{1} \left[ v f(v) + F(v) - 1 \right] F^{n-1}(v) \, dv
\]
Rewrite this
\[
ER = v_0 F(v_*)^n + n \int_{v_*}^{1} \left[ v - \frac{1 - F(v)}{f(v)} \right] dF^n(v) \, dv
\]
call the term in the square brackets the marginal revenue to the seller.
Optimizing
\[
v_0 dF(v_*)^n - \left[ v_* - \frac{1 - F(v_*)}{f(v_*)} \right] dF(v_*)^n = 0
\]
Which implies
\[ v_0 = \left[ v_* - \frac{1 - F(v_*)}{f(v_*)} \right] \]
or MR=MC.
Further
\[ v_* = v_0 + \frac{1 - F(v_*)}{f(v_*)} \]
so revenue maximization and allocative efficiency are in conflict.
• For first and second price auctions this means that the optimal auction is simply to choose an opening bid of \( v \).

• This opening bid is equal for these two auction forms

• An entry fee will do the trick as well (same entry fee for all auction forms).

• A small increase in the reserve above its lowest level always raises revenues.

• Optimal reserve price is independent of \( n \).

**Negotiation**

To see the monopoly interpretation, consider the case where \( n = 1 \). In this case, the auctioneer is simply a monopolist facing the problem of choosing an offer to maximize

\[
E\pi = p (1 - F(p)) + v_0 F(p)
\]

so \( 1 - F \) is the demand curve.

Optimizing

\[
(1 - F) - pf + v_0 f = 0
\]

Dividing and rearranging

\[
p - \frac{1 - F}{f} = v_0
\]

or

\[
MR = MC
\]

which is of course the same as the optimal reserve price in an \( n \) player auction.

### 8 Tort Reform

• US society is too litigious

• How to reduce incentives to sue?

Toy model: Both parties privately observe the value of winning the case. Each decides how much to spend on lawyers. Higher spender wins.

• This is just the all-pay auction we analyzed earlier.

• European system: Loser pays winner’s expenses.
  
  – Notice that the expected payoff from the lowest type is negative
  – So the RET does not hold
  – So what happens?
Bidder’s problem is to choose a bid $b$ to maximize

$$E\pi (\hat{v}, v) = vF (\hat{v}) - \beta (\hat{v}) (1 - F (\hat{v})) - \int_{\hat{v}}^{1} \beta (t) f (t) \, dt$$

Optimizing

$$vf (\hat{v}) - \beta' (\hat{v}) (1 - F (\hat{v})) + \beta (\hat{v}) f (\hat{v}) + \beta (\hat{v}) f (\hat{v}) = 0$$

In equilibrium, $\hat{v} = v$

$$vf - \beta' (1 - F) + 2\beta f = 0$$

Rewriting

$$\beta' + \beta P (v) = Q (v)$$

where $P (v) = - \frac{f'}{f}$ and $Q (v) = v \frac{f'}{f}$.

Usual trick:

$$\beta (v) = e^{-\int P} \int Q e^{\int P} \, dx$$

and since

$$\int P = \ln (1 - F).$$

Then

$$\beta (v) = \frac{1}{1 - F} \int_{0}^{v} xf (x) \, dx$$

No upper bound on bidding.

Expected payment

$$EP_{EUR} (v) = \beta (v) (1 - F (v)) + \int_{v}^{1} \beta (t) f (t) \, dt$$

$$= \int_{0}^{v} xf (x) \, dx + \int_{v}^{1} \left( \int_{0}^{t} xf (x) \, dx \right) \frac{f (t)}{1 - F} \, dt$$

Recall that expected payment in a Vickrey auction is

$$EP_{II} (v) = \int_{0}^{v} xf (x) \, dx$$

Thus, for all $v$

$$EP_{EUR} (v) > EP_{II} (v)$$

Hence

$$EP_{EUR} > EP_{II}$$

so the European legal system is more expensive than the American system.

- Quayle plan: Losing party pays the winner an amount equal to his own expenses.

Here the RET applies. So the Quayle plan costs exactly the same as the current plan.

- Same incentives to initiate litigation.