Some Lecture Notes on Auctions

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1 Preliminaries

- Perhaps the most fruitful area for the application of optimal screening contracts is in auction theory
- 1 principle, many bidders.
- The goal might be to allocate efficiently (and maximize revenues while doing so)
- Or it might just be to maximize revenues.

The main result we will establish is:

Revenue Equivalence Theorem (RET): Assume each of a given number of risk-neutral potential buyers has a privately-known valuation independently drawn from a strictly-increasing atomless distribution, and that no buyer wants more than one of the k identical indivisible prizes.

Then any mechanism in which

- (i) the prizes always go to the k buyers with the highest valuations and
- (ii) any bidder with the lowest feasible valuation expects zero surplus,

yields *the same expected revenue* (and results in each bidder making the same expected payment as a function of her valuation).

1.1 Model

Throughout, we study the archetypal auction model:

- n ex ante identical potential bidders
- independent private values, v_i, drawn from some atomless distribution F on $[0, 1]$.
- Single object to be auctioned.
- Seller has commonly known valuation v_0.

Game form
Seller chooses the "contract" (i.e. auction form).
Bidders bid.
Payoffs are realized.

2 Second Price Auction

Consider an auction where the winning bidder pays the second highest bid (introduced by Vickrey)

Proposition 1 *Suppose bidders have private values, then bidding one's valuation is a weakly dominant strategy.*

Proof. General and informal.

Suppose you bid $b > v_1$, then the outcome compared to the putative eqm strategy is changed only when highest (other than 1's) is between v_1 and b. But in these circumstances you incur losses.

Suppose you bid $b < v_1$, then the outcome compared to the putative eqm strategy is changed only when highest bid (other than 1's) is between b and v_1. In these circumstances you miss out on profits. ■

Less general, more mathematical:

Suppose everyone else is bidding according to the increasing bidding strategy $\beta(v)$ .Let $F_1^{(n-1)}(y)$ be the distribution function of the highest of $n-1$ draws from F.

That is

$$y = \max\{v_2, v_3, ..., v_n\}$$

Bidder 1 wins if $b \geq \beta(y)$, so

$$E\pi_1(v_1, b) = \int_0^{\beta^{-1}(b)} [v_1 - \beta(y)]\, dF_1^{(n-1)}(y).$$

Differentiating wrt b.

$$[v_1 - b]\, f_1^{(n-1)}\left(\beta^{-1}(b)\right) \frac{d}{db}\left[\beta^{-1}(b)\right] = 0$$

Since $f_1^{(n-1)}\left(\beta^{-1}(b)\right) \neq 0$ and $\frac{d}{db}\left[\beta^{-1}(b)\right] \neq 0$, it then follows that $b = v_1$.

It then follows that:

Proposition 2 *The Vickrey auction is efficient when bidders have private values.*

- Experimental findings sometimes differ from this.

Revenues

The seller's expected revenue is simple $E\left[Y_2^{(n)}\right]$, i.e., the second highest of n draws from F.

With uniform distributions, this becomes

$$E\left[Y_2^{(n)}\right] = \frac{n-1}{n+1}$$

so it's obvious that revenues are increasing in n and converge to where the seller obtains all the surplus.

We can get at revenues a different way:

What is the expected payment of a bidder with valuation x?

$$\begin{aligned} P_{II}(x) &= \Pr\{win\} \times E\{payment|win\} \\ &= \Pr\{win\} \times \frac{E\{payment \times I_{win}\}}{\Pr\{win\}} \\ &= \int_{y \le x} y dF_1^{(n-1)}(y) \end{aligned}$$

Under the uniform distribution

$$\begin{aligned} &= \int_0^x y d\left[y^{n-1}\right] \\ &= \frac{n-1}{n} x^n \end{aligned}$$

The ex ante expected payment is then

$$\begin{aligned} P_{II} &= \int_0^1 P_{II}(x)\, dF(x) \\ &= \frac{n-1}{n(n+1)} \end{aligned}$$

Expected revenue

$$\begin{aligned} ER &= nP_{II} \\ &= \frac{n-1}{n+1} \end{aligned}$$

More generally

$$P_{II}(x) = \int_{y \le x} y dF_1^{(n-1)}(y)$$

Integrate by parts

$$\begin{aligned} P_{II}(x) &= xF_1^{(n-1)}(x) - \int_0^x F_1^{(n-1)}(y)\, dy \\ &= xF^{n-1} - \int_0^x F^{n-1} dy \end{aligned}$$

3 First Price Auctions

Bidder 1, given valuation v_1, chooses b to maximize

$$\begin{aligned} E\pi_1 (v_1, b) &= \int_0^{\beta^{-1}(b)} [v_1 - b] \, dF_1^{(n-1)} (y) \\ &= [v_1 - b] \, F_1^{(n-1)} \left(\beta^{-1} (b)\right) \end{aligned}$$

Differentiate

$$[v_1 - b] \, dF_1^{(n-1)} \left(\beta^{-1} (b)\right) \frac{d}{db} \left(\beta^{-1} (b)\right) - F_1^{(n-1)} \left(\beta^{-1} (b)\right) = 0$$

Under a symmetric equilibrium, $b = \beta (v_1)$; hence

$$[v - \beta (v)] \, f_1^{(n-1)} (v) \frac{1}{\beta' (v)} - F_1^{(n-1)} (v) = 0$$

Rewriting

$$\beta' + \beta \times A (v) = v A (v)$$

where $A (v) = \frac{f_1^{(n-1)} (v)}{F_1^{(n-1)} (v)}$.

Multiply by $e^{\int A}$:

$$\begin{aligned} \beta' e^{\int A} + \beta A e^{\int A} &= v A v e^{\int A} \\ \frac{d}{dv} \left(\beta e^{\int A}\right) &= v A v e^{\int A} \end{aligned}$$

Hence

$$\begin{aligned} b e^{\int A} &= \int v A e^{\int A} dv + c \\ b (v) &= e^{-\int A} \left(\int v A e^{\int A} dv + c \right) \end{aligned}$$

Since $b (0) = 0$, then

$$b (v) = e^{-\int A} \left(\int_0^v t A (t) \, e^{\int A} dt \right)$$

Now

$$\begin{aligned} e^{\int A} &= e^{\int \frac{(n-1) F(x)^{n-2} f(x)}{(F(x))^{n-1}} dx} \\ &= e^{\int \frac{(n-1) f(x)}{F(x)} dx} \\ &= e^{(n-1) \ln F(x)} \\ &= F (x)^{n-1} \end{aligned}$$

Hence

$$\begin{aligned} b(v) &= \frac{1}{F(v)^{n-1}} \int_0^v (n-1)\, x \frac{f(x)}{F(x)} F(x)^{n-1}\, dx \\ &= \frac{1}{F^{n-1}} \int_0^v x d\left[F^{n-1}\right] \end{aligned}$$

Integrate by parts

$$b(v) = v - \frac{1}{F^{n-1}} \int_0^v F^{n-1} dx$$

Compute expected payment:

$$\begin{aligned} P_I(x) &= b(x)\, F^{n-1}(x) \\ &= xF^{n-1} - \int_0^x F^{n-1} dy \end{aligned}$$

Notice that it is exactly the same as the Second price auction.

Thus, the expected revenue to the seller in either of these auctions is simply

$$ER = E_2^{(n)}[v]$$

i.e. the expectation of the second highest of n draws.

Thus, both types of auctions are equally good (Vickrey 1962).

- Strategic equivalence with other auction forms.
  - Difference between strat equivalence in first-price versus second-price case.
- But experiments do not bear this out.

4 The Revenue Equivalence Theorem

Consider the following set of auction contracts.

- Announce a minimum opening bid b_0.
- High bidder wins
- Rules are anonymous
- Strictly increasing symmetric bidding strategy in auction.
- Non-negative returns to bidding.

Use the revelation principle to restrict attention to direct mechanisms.

Bidder 1's Problem

Given a valuation v, choose a message $\hat{v}$ to maximize

$$\pi(\hat{v}, v) = vF^{n-1}(\hat{v}) - P_A(\hat{v})$$

where P_A is the expected payment from pretending to be a $\hat{v}$ type in auction form A

Optimize with respect to $\hat{v}$:

$$\pi_1(\hat{v}, v) = v(n-1)F^{n-2}(\hat{v}) f(\hat{v}) - P'_A(\hat{v})$$

In equilibrium, $v = \hat{v}$

$$v(n-1)F^{n-2}(v) f(v) = P'_A(v)$$

Boundary condition: Find a type v_* solving

$$P_A(v_*) = v_* F(v_*)^{n-1}$$

Now solve the differential equation

$$\int_{v_*}^{v} P'_A(x)\, dx = \int_{v_*}^{v} x dF^{n-1}(x)$$

Notice that the RHS is independent of the auction form!

With algebra

$$P_A(v) = vF^{n-1}(v) - \int_{v_*}^{v} F^{n-1}(x)\, dx$$

for all $v \geq v_*$

Expected revenue

$$\begin{aligned} ER &= nE_v [P_A(v)] \\ &= n \int_{v_*}^{1} [vf(v) + F(v) - 1] F^{n-1}(v)\, dv \end{aligned}$$

So we have proved the revenue equivalence theorem!

5 Applications

Knowing the RET can be helpful in directly computing bidding sstrategies.

First-price auction

$$\begin{aligned} P_I(v) &= \Pr\{y \leq v\} b(v) \\ vF^{n-1}(v) - \int_{v_*}^{v} F^{n-1}(x)\, dx &= F^{n-1}(v) b(v) \end{aligned}$$

So

$$b(v) = v - \frac{1}{F^{n-1}(v)} \int_{v_*}^{v} F^{n-1}(x)\,dx.$$

All-pay Auction

$$P_{AP}(v) = \gamma(v)$$

so

$$\gamma(v) = vF^{n-1}(v) - \int_{v_*}^{v} F^{n-1}(x)\,dx$$

And so on.

6 Empirical Tests of the RET

- Turkish treasury auctions
- Structural estimation literature
- EBay experiments

7 Revenue Maximization

Using the RET, the principal's problem is to choose v_* to maximize

$$ER = v_0 F(v_*)^n + n \int_{v_*}^{1} [vf(v) + F(v) - 1]\, F^{n-1}(v)\,dv$$

Rewrite this

$$ER = v_0 F(v_*)^n + n \int_{v_*}^{1} \left[v - \frac{1 - F(v)}{f(v)}\right] dF^n(v)\,dv$$

call the term in the square brackets the marginal revenue to the seller.

Optimizing

$$v_0 dF(v_*)^n - \left[v_* - \frac{1 - F(v_*)}{f(v_*)}\right] dF(v_*)^n = 0$$

Which implies

$$v_0 = \left[v_* - \frac{1 - F(v_*)}{f(v_*)}\right]$$

or MR=MC.

Further

$$v_* = v_0 + \frac{1 - F(v_*)}{f(v_*)}$$

so revenue maximization and allocative efficiency are in conflict.

- For first and second price auctions this means that the optimal auction is simply to choose an opening bid of v_*.
- This opening bid is equal for these two auction forms
- An entry fee will do the trick as well (same entry fee for all auction forms).
- A small increase in the reserve above its lowest level always raises revenues.
- Optimal reserve price is indenpendent of n.

Negotiation

To see the monopoly interpretation, consider the case where $n = 1$.

In this case, the auctioneer is simply a monopolist facing the problem of choosing an offer to maximize

$$E\pi = p\left(1 - F\left(p\right)\right) + v_0 F\left(p\right)$$

so $1 - F$ is the demand curve.

Optimizing

$$(1 - F) - pf + v_0 f = 0$$

Dividing and rearranging

$$p - \frac{1 - F}{f} = v_0$$

or

$$MR = MC$$

which is of course the same as the optimal reserve price in an n player auction.

8 Tort Reform

- US society is too litigious
- How to reduce incentives to sue?

Toy model: Both parties privately observe the value of winning the case. Each decides how much to spend on lawyers. Higher spender wins.

- This is just the all-pay auction we analyzed earlier.
- European system: Loser pays winner's expenses.
  - Notice that the expected payoff from the lowest type is negative
  - So the RET does not hold
  - So what happens?

Bidder's problem is to choose a bid b to maximize

$$E\pi(\hat{v},v) = vF(\hat{v}) - \beta(\hat{v})(1-F(\hat{v})) - \int_{\hat{v}}^{1} \beta(t) f(t)\,dt$$

Optimizing

$$vf(\hat{v}) - \beta'(\hat{v})(1-F(\hat{v})) + \beta(\hat{v}) f(\hat{v}) + \beta(\hat{v}) f(\hat{v}) = 0$$

In equilibrium, $\hat{v} = v$

$$vf - \beta'(1-F) + 2\beta f = 0$$

Rewriting

$$\beta' + \beta P(v) = Q(v)$$

where $P(v) = -\frac{f}{1-F}$ and $Q(v) = v\frac{f}{1-F}$.

Usual trick:

$$\beta(v) = e^{-\int P} \int Q e^{\int P} dx$$

and since

$$\int P = \ln(1-F).$$

Then

$$\beta(v) = \frac{1}{1-F}\int_0^v xf(x)\,dx$$

No upper bound on bidding.

Expected payment

$$\begin{aligned} EP_{EUR}(v) &= \beta(v)(1-F(v)) + \int_v^1 \beta(t) f(t)\,dt \\ &= \int_0^v xf(x)\,dx + \int_v^1 \left(\int_0^t xf(x)\,dx\right)\frac{f(t)}{1-F}dt \end{aligned}$$

Recall that expected payment in a Vickrey auction is

$$EP_{II}(v) = \int_0^v xf(x)\,dx$$

Thus, for all v

$$EP_{EUR}(v) > EP_{II}(v)$$

Hence

$$EP_{EUR} > EP_{II}$$

so the European legal system is more expensive than the American system.

- Quayle plan: Losing party pays the winner an amount equal to his own expenses.

Here the RET applies. So the Quayle plan costs exactly the same as the current plan.

- Same incentives to initiate litigation.