Strategy in Contests – An Introduction

presentation for PHDBA 279B by Aisling Scott and GianCarlo Moschini

* Now available as a book:
Introduction

• What is a contest?
  • Game where players have opportunity to expend resources in order to affect probabilities of winning prizes

• A lot of theoretical progress
  • Less empirical work and open questions

• Importance: apply auction theory to many different contexts

• In all 3 contests
  • Players exert costly irreversible efforts while competing for a prize
  • A player’s probability of winning the prize depends on the players’ relative expenditures

• Each contest defines the exact probability differently
  • Contest Success Function (CSF) maps vector of contestants’ expenditures to probability of winning
A contest

• Strategic game among $n$ contestants
• A prize is allocated, contestants value the prize $v_i > 0$, $i = 1, 2, ..., n$
• Each player makes an “effort” $x_i \geq 0$ at a cost $C_i(x_i)$
• The vector of all efforts $x = (x_1, x_2, ..., x_n)$ determines contestants’ probability of winning
• Payoff to contestants: $\pi_i(x_1, x_2, ..., x_n) = p_i(x_1, x_2, ..., x_n)v_i - C_i(x_i)$
• The “contest success function” $p_i(x_1, x_2, ..., x_n)$ can take alternative forms
3 Contests

• 3 Canonical Contests:
  • All-pay Auction
    • Lobby and Military
  • Tournament
    • Principal-Agent, Labor, and contract design
  • Tullock (or lottery) Contest
    • R&D races, Political contests, and Rent-seeking competitions

• Tullock and Tournaments usually have pure strategy equilibria
  • Exceptions exist

• All-pay usually has non-degenerate mixed strategy equilibria

• Different cases seek to maximize or minimize contest effort expenditure
Examples

• Promotional competition (Friedman 1958)
  • Expend resources on advertising in order to compete
  • CSF: share in total market
  • Can change the total market size indicating that efforts increase public good

• Internal labor markets
  • Relative performance schemes (Lazear and Rosen 1981 and Rosen 1986)

• Litigation
  • Legal system determines rules of the game, but each group allocates effort
  • Entry decision, rule of fees, information asymmetries

• Beauty Contests
  • Olympic games (Steward and Wu 1997)

• Rent Seeking and influence activities
  • Status seeking (Congleton 1989, Konrad 1990 and Konrad 1992)
...and Examples

• Campaigning and committee bribing
  • Different from advertising because payoffs are discontinuous

• Sports (Hoehn and Szymanski 1999 and Szymanski 2003)
  • Role of entry fees, homogeneous participants, and prize number and structure
  • Intra and Inter group contests

• Military conflict
  • Relationship between numerical superiority and battle success is loose
    • Informational asymmetries (Clark and Konrad 2006)

• R&D contests (Fullerton and McAfee 1999)
  • Firms expend effort to get monopoly rent from patent or first developing product
First-Price All-Pay Auction (complete information)

- Recall contestants’ payoff: \( \pi_i(x_1, x_2, \ldots, x_n) = p_i(x_1, x_2, \ldots, x_n)v_i - C_i(x_i) \)

- Illustrate with special case of two players with \( v_1 \geq v_2 > 0 \), \( C(x_i) = x_i \) and
  
  \[
  p_i(x_i, x_j) = \begin{cases}
  1 & \text{if } x_i > x_j \\
  1/2 & \text{if } x_i = x_j \\
  0 & \text{if } x_i < x_j
  \end{cases}
  \]

- There is no equilibrium in pure strategies:

- There is a mixed-strategy equilibrium described by cumulative distribution functions \( F_1(x_1) \) and \( F_2(x_2) \), on the support \([0, v_2]\), with the property that
  
  \[
  \pi_1(x_1) = F_2(x_1)v_1 - x_1 = v_1 - v_2 , \quad \forall x_1 \in [0, v_2] \\
  \pi_2(x_2) = F_1(x_2)v_2 - x_2 = 0 , \quad \forall x_2 \in [0, v_2]
  \]
First-Price All-Pay Auction

• Equilibrium mixed strategies:

\[
F_1(x_1) = \begin{cases} 
\frac{x_1}{v_2} & \text{for } x_1 \in [0, v_2] \\ 
1 & \text{for } x_1 > v_2 
\end{cases}
\]

\[
F_2(x_2) = \begin{cases} 
\left[1 - \frac{v_2}{v_1}\right] + \frac{x_2}{v_1} & \text{for } x_2 \in [0, v_2] \\ 
1 & \text{for } x_2 > v_2 
\end{cases}
\]

• NOTE: ◊ Inefficiency: highest valuation contestant does not always get the prize

◊ Expected efforts are \( E_{X_1} = \frac{v_2}{2} \) and \( E_{X_2} = \frac{v_2^2}{2v_1} \) \( \Rightarrow \) \( E_{X_1} + E_{X_2} < v_2 \)

• More than 2 contestants: \( v_1 \geq v_2 > v_3 \geq \ldots \geq v_n > 0 \) \( \Rightarrow \) same as case of \( n = 2 \)

\( v_1 = v_2 = \ldots = v_j > v_{j+1} \geq \ldots \geq v_n > 0 \) \( \Rightarrow \) continuum of equilibria
All-Pay Auction with liquidity constraints

- Contestants might be constrained in their efforts (e.g., limits on campaign contributions)
- Application to R&D contests with liquidity constraints

$n$ contestant all value the prize equally at $v$

each contestant’s effort limited by the fund availability $w$. Assume $w_1 > w_2 > w_3 \geq \ldots \geq w_n$

Equilibrium mixed strategies:

\[
F_1(x_1) = \begin{cases} 
\frac{x_1}{v} & \text{for } x_1 \in [0,w_2) \\
1 & \text{for } x_1 \geq w_2
\end{cases}
\]

\[
F_2(x_2) = \begin{cases} 
\left[1 - \frac{w_2}{v}\right] + \frac{x_2}{v} & \text{for } x_2 \in [0,w_2) \\
1 & \text{for } x_2 \geq w_2
\end{cases}
\]
All-Pay Auction with incomplete information

• Contestants know their own valuation and the distribution $F(v)$ from which all valuations are drawn. Example: case of $n = 2$ and $v_i \in [0,1]$.

With bid functions $x = \xi(v)$, the contestant’s payoff is $\pi_i(x_i) = F\left( \xi^{-1}(x_i) \right) v_i - x_i$

FOC: $\pi'_i(x_i) = 0 \rightarrow F'\left( \xi^{-1}(x_i) \right) \frac{d\xi^{-1}}{dx_i} v_i - 1 = 0$

• If $F(v)$ is the uniform distribution, the symmetric equilibrium is: $x = \frac{v^2}{2}$

• Note: here contestant with highest valuation always wins prize (inefficiency noted earlier disappears)
All-Pay Auction and models of litigation

• Can nest alternative legal fee shifting rules in the Contest framework. Example:

\[
\pi_i(x_i, x_j, v_i) = \begin{cases} 
  v_i - bx_i - dx_j & \text{if } i \text{ wins} \\
  -ax_i - tx_j & \text{if } j \text{ wins}
\end{cases}
\]

• American system: \( a = b = 1 \) and \( d = t = 0 \) (standard contest)

• British rule (loser pays all costs): \( a = t = 1 \) and \( b = d = 0 \)

• In an all-pay auction with incomplete information, the expected total effort by contestants is higher under the British system than the American system

-- see prof. Morgan’s lecture notes on Auctions
All-Pay Auction with additive noise (tournament)

• Includes a random element (additive noise) to the contest success function

Effort produces a “performance” \( y_i = x_i + \varepsilon_i \); success function (for \( n=2 \)) is

\[
p_i(x_i, x_j) = \begin{cases} 
1 & \text{if } x_i + \varepsilon_i > x_j + \varepsilon_j \\
1/2 & \text{if } x_i + \varepsilon_i = x_j + \varepsilon_j \\
0 & \text{if } x_i + \varepsilon_i < x_j + \varepsilon_j 
\end{cases}
\]

• Define \( \varepsilon = \varepsilon_2 - \varepsilon_1 \) and assume it has distribution \( G(\varepsilon) \) on \([-e, e]\)

Suppose both contenders value winning equally \( (v_i = b_W) \) and loser gets \( b_L \)

payoff to contestants:

\[
\pi_i = p_i(x_i, x_j)b_W + \left[ 1 - p_i(x_i, x_j) \right]b_L - C(x_i)
\]

optimality:

\[
\frac{\partial p_i(x_i, x_j)}{\partial x_i} \left( b_W - b_L \right) = C'(x_i) \quad \text{where} \quad \frac{\partial p_i(x_i, x_j)}{\partial x_i} = \frac{\partial G(x_i - x_j)}{\partial x_i}
\]
• In a symmetric NE both contestant spend same effort and win with prob $1/2$
  the equilibrium solves $G'(0) \times (b_W - b_L) = C'(x^*)$
  
  -- note: increased dispersion of noise or reduction in win reward reduce effort

• Contest design problem: choose $b_W$ and $b_L$ to max obj function, e.g., $\varphi(2x) - b_W - b_L$

• Participation constraint at symmetric NE requires $\frac{1}{2}b_W + \frac{1}{2}b_L \geq C(x^*)$

• Hence designer problem is to max $\varphi(2x) - 2C(x)$

• Can implement the efficient solution satisfying $\varphi'(2x^*) = C'(x^*)$

• Note role of randomness in all the settings considered so far
Standard Tullock

• Contest success function:

\[ p_i(x_1, \ldots x_n) = \begin{cases} \frac{x_i^r}{\sum_{j=1}^{n} x_j^r} & \text{if } \max\{x_1, \ldots x_n\} > 0 \\ 1/n & \text{otherwise.} \end{cases} \]

\[ C_i(x_i) = x_i \]

• If \( r=1 \) lottery contest
• If \( r>0 \), allows for general types of contests
• As \( r \to \infty \) the above function converges to contest success function converges to no noise tournament
• Imperfectly discriminating
Existence and Uniqueness

• In a symmetric contest with n contestants there exists a pure strategy Nash Equilibrium if
  \[ r \leq \frac{n}{n-1} \]

• Otherwise a mixed strategy Nash Equilibrium exists
  • It is possible for total effort to exceed the valuation of the prize
  • Expected total effort cannot exceed the value of the prize
Two Players

• FOCs

\[ \frac{rx_1^{r-1}x_2^r}{(x_1^r + x_2^r)^2} v_1 = 1 \quad \text{and} \quad \frac{rx_2^{r-1}x_1^r}{(x_1^r + x_2^r)^2} v_2 = 1 \]

• Solve for the symmetric case:

Note: \( V_i = v \ \forall i \) and \( x_1 = x_2 = x^* \) by symmetry

\[ \frac{r(x^*)^{(2r-1)}}{4(x^*)^{2r}} v = 1 \]

\[ x^* = \frac{rv}{4} \]
When does rent dissipation exist?
Since \( 2x^* = 2 \frac{rv}{4} \)
\[
2v = 2x^*
\]
\[
r = 2
\]

Thus when \( r = 2 \) all rent is dissipated.

Let \( r = 1 \) so the best reaction functions are:
\[
BR_1 = x_1 = \sqrt{x_2v} - x_2
\]
\[
BR_2 = x_2 = \sqrt{x_1v} - x_1
\]
Best Reaction Functions

Note $v_1 = 2, v_2 = 1, r = 1$
Nash Equilibrium

\[ x_1^* = r \frac{v^r_2 v^{1+r}_1}{(v^r_1 + v^r_2)^2} \quad \text{and} \quad x_2^* = r \frac{v^{1+r}_2 v^r_1}{(v^r_1 + v^r_2)^2} \]

\[ p_1^* = \frac{v^r_1}{v^r_1 + v^r_2} \quad \text{and} \quad p_2^* = \frac{v^r_2}{v^r_1 + v^r_2} \]

\[ \pi_1^* = \frac{v^{r+1}_1 (v^r_1 + v^r_2 (1 - r))}{(v^r_1 + v^r_2)^2} \quad \text{and} \quad \pi_2^* = \frac{v^{r+1}_2 (v^r_2 + v^r_1 (1 - r))}{(v^r_1 + v^r_2)^2} \]
Many Participants

- General Contest Success Function: \( f(x_i) / \sum f(x_j) \)
- Denote \( X = \sum_{i=1}^{n} x_i \) and let \( p_i(x) = x_i / X \)
- Yields FOC: \( x_i = \max[0, X(1 - \frac{1}{v_i})] \)
- Thus we can characterize total efforts of participants that expend nonzero efforts as:

\[
\sum_{i=1}^{n} x_i^* = X^* = \frac{n-1}{n} \sum_{i=1}^{n} \frac{1}{v_i}
\]
Why is Tullock (Lottery) model so popular?

• Axiomatic Reasoning
  • Individual payoffs depend only on own preferences and aggregated values of other players’ efforts
  • For r=1 for a given aggregate effort of all contestants, the probability of winning is linear in contestant i’s own effort
  • Belle, Keeney, and Little (1975) prove that this is the only contest success function that has this property

• Information aspects
  • Tullock (1980) assumed contestants know everything about each other

• Microeconomic underpinnings
  • Simple search model
  • Ratio of efforts as a probability for winning the contest
Experimental

• Most experiment find support for contest theory
• Overbidding occurs (aggregate effort exceeds Nash equilibrium predictions)
  • Exception: rank-order tournaments
    • Why is this the case?
  • Open question: What contributes to overbidding?
Robust Results

• 1. One player’s increased efforts creates a negative externality for another
  • More heterogeneous contestants, usually reduces aggregate equilibrium effort and increases net payoffs

• 2. Contestant with higher valuation is usually more likely to win
  • Also expends more effort
  • Close one-to-one relationship for differences in valuation and differences in individual cost of production given effort levels
Sequential choices

- Stage 1: contestant simultaneously choose whether to exert effort “early” \((E)\) or “late” \((L)\)

- Stage 2: simultaneous contest if 1\(^{st}\) stage equilibrium is \((E,E)\) or \((L,L)\)
  sequential contest if 1\(^{st}\) stage equilibrium is \((E,L)\) or \((L,E)\)

- Illustrate with Tullock contest \((n=2)\)

- Also, \(v_1 > v_2\)

- If \((E,L)\) in stage 1, then \(S_1\) in stage 2

- If \((L,E)\) in stage 1, then \(S_2\) in stage 2

- If \((E,E)\) or \((L,L)\) in stage 1, then \(N\) in stage 2

- The SPNE is \((L,E)\) and \(S_2\)

- The “weaker” player moves first and commits to a lower effort
Voluntary participation

- Contests may require an entry fee (or, contestant have an opportunity cost)
- If contest fully dissipates rent, why should contestants want to incur an entry fee?
- 2-stage game with $n$ possible contestants with same valuation $v > 0$; entry fee of $D$
  
  Stage 1: contestants decide whether or not to participate
  
  Stage 2: All-pay auction among contestants who entered
- Asymmetric equilibria in pure strategy with only one contestant who enters
- Symmetric mixed strategy equilibrium: each enters with probability $q^* = 1 - (D/v)^{1/(n-1)}$
- Does all this constitute an “entry puzzle”?
- Possible resolution: players’ cost in any given contest may be a random draw
Exclusion

• Contest designer’s objective function assumed to care about aggregate effort
• In general, high contestants’ valuation (low costs) increases aggregate effort
• Also, homogeneity of contestants tend to increase aggregate effort
• Counter-intuitive result: it might be desirable to exclude contestant who value the prize most highly
• Illustrate with all-pay auction, \( n=3 \), and \( v_1 > v_2 = v_3 \)
• Equilibria where expected aggregate effort \( X \) is highest have \( X = \frac{v_2}{2} \left( 1 + \frac{v_2}{v_1} \right) < v_2 \)
• But if contestant 1 is excluded, \( X = v_2 \)
Delegation

• Contestants act as “principals” and hire “agents” to do their bidding
• Illustrate with all-pay auction and two contestants
• Contract is \((\varphi_i, b_i)\), where \(\varphi_i = \) down payment, and \(b_i \in [0, \bar{b}]\) is the “delegated value”
• Payoff to agent \(\pi_i^A = -\varphi_i + p_i(x_i, x_j)b_i - x_i\)
• Payoff to principal \(\pi_i^P = \varphi_i + p_i(x_i, x_j)(v_i - b_i) \rightarrow \pi_i^P = p_i^* v_i - E x_i^*\) (with \(E \pi_i^A = 0\) by IR)
  \[
  \begin{align*}
  (b_1^* = v/2, b_2^* = \bar{b}) & \quad (b_1^* = \bar{b}, b_2^* = v/2)
  \end{align*}
  \]
• Two pure strategy equilibria:
• Contestants use delegation to generate asymmetry between the two agents, thereby leading to a smaller dissipation of the prize
• Note the role of “homogeneity” vs “heterogeneity” of players
• However, the delegation contract is not renegotiation-proof
Externalities

• Joint Ownership
  • Firms are no longer indifferent about who wins the prize
  • Minority Shareholdings can reduce rents generated by the allocation of the prize and the firms’ aggregate equilibrium efforts

• Information externalities
  • Revealing information to uninformed voters
  • Inverse campaigning dissipates rents (Konrad 2004)
Public Goods and Free Riding

• Homogeneous groups
  • Aggregate effort of a group and its win probability are completely independent of number of members in the two groups (Katz et al. 1990)
    • Why? Free-riding might intensify as group gets larger

• Heterogeneous groups
  • Doesn’t effect aggregate equilibrium group effort
    • Only highest group member valuation matters
Prizes

• Endogeneous Prizes
  • Suppose prize values are functions of the vector of efforts
  • Theoretically generates a different set of predictions
  • In some case, all-pay auctions can have pure strategy equilibrium

• Multiple Prizes
  • Sports tournaments often award multiple prizes
  • Nobel Prize exemplifies general non-optimality of a single large prize
  • How does prize structure influence contest outcome?
  • Assumption: Contest designer allocates a fixed amount of money either to one grand prize or split between multiple prizes
  • But how does the designer allocate these prizes?
    • Sequential games or based on a CSF on basis of actual efforts
  • Many cases have been looked at but not all
“Handicapping”

• More heterogeneity between contestants may lead to lower aggregate equilibrium efforts

• Contest designers may want to favor or bias towards the contestant who values the prize lower to increase aggregate efforts in equilibrium

• These concepts have been used to study affirmative action issues

• Handicap effort of strong players or help the efforts of the weak
  • Clark and Riis (2000) show that this increases a bureaucrat's receipts
Grand Contests

• Introduce more detail on how contestants’ efforts translate into win probabilities.

  Three structures are discussed:

• **Nested contests**: contestants are actually groups, once the winning group is determined the prize needs to be allocated to members (another contest)
  -- Inter-group contest and intra-group contest

• **Alliances**: the formation of groups in the inter-group contest may be endogenous

• **Multi-battle contests**: there are several stages to a contest
  -- Race
  -- Elimination tournament
  -- Tug of war
Nested Contests

- Exogenous sharing rule (intra-group allocation): merit vs egalitarianism
  Example: 2 groups, prize $v$ allocated to a group based on the total effort of its members
  Group win probabilities $p_i = \frac{x_i}{x_1 + x_2} v$, $x_i = \sum_{j=1}^{n_j} x_{ij}$
  Sharing rule: the share of the prize to member $j$, if group wins, is $q_{ij} = (1 - \alpha) \frac{x_{ij}}{x_i} + \alpha \frac{1}{n_i}$
  A “merit” rule ($\alpha = 0$) yields more group efforts than an “egalitarian” rule ($\alpha = 1$)

- Group choice of a sharing rule: possible time-consistency problem

- Hierarchical structures may be advantageous. Example: $nm$ homogeneous contestants

- One-stage Tullock contest: expected payoff of winner is $\left(1 - \frac{nm - 1}{nm}\right)v$

- Two-stage contest: 1st stage competition between $n$ groups of $m$ members each; 2nd stage competition between members of winning group. Expected payoff: $\left(1 - \frac{n - 1}{nm^2} - \frac{m - 1}{m}\right)v$
  \(\rightarrow\) two-stage contest dissipates less rent
Multi-battle contests

- “Race”: a contestant needs to accumulated a certain number of wins in sub-contests (battles) in order to win the contest.
  Examples: Tennis, R&D, primary elections
  Illustration: need to win 2 battles to win contest

- “Elimination tournament”: the field of contestants is reduced (e.g., halved) in each stage.
  Examples: tennis tournaments; Jack Welch selection of his successor as CEO of GE

- “Tug of War”: need a certain number of net stage successes (# wins minus # losses) to win the contest.

- “Discouragement effect”: e.g., the fact that most of the prize is dissipated in the final of an elimination tournament discourages contestants in the semi-finals.

Illustration:

- [Diagram showing multi-battle contests with nodes and arrows representing outcomes and winners.
- Each node represents a stage or a battle.
- Arrows indicate the flow of the contest, showing the progression from one stage to the next.
- Nodes labeled with outcomes, such as “player 1 wins” or “player 2 wins.”
- Example scenarios are indicated, such as (2,1) for player 1 winning, (1,1) for equal outcome, and (2,2) for player 2 winning.
- The diagram visually represents the concept of multi-battle contests, illustrating the progression and outcomes through the different stages.