The Winner-Take-All Principle in Small Tournaments

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1 Introduction

The idea that an employee's compensation should, at least to some extent, be based on his or her performance relative to that of others is a familiar one. It is almost an axiom that by spurring competition among workers such compensation schemes have beneficial incentive effects. Companies offer bonuses to the "sales-person of the year," law firms reward "rain-makers" and universities reward researchers for writing the "best paper." The compensation of professional athletes, of course, offers the clearest example of such schemes.

In a provocative paper, Lazear and Rosen (1981) analyzed compensation schemes based *solely* on how a worker was ranked in terms of output, calling these rank-order tournaments. In a simple two worker moral hazard model, they delineated circumstances under which the optimal labor contract was a rank-order tournament. Their analysis has since been generalized and extended in many directions by Holmstrom (1982), Green and Stokey (1983), Nalebuff and Stiglitz (1983), Mookherjee (1984) and Bhattacharya and Guasch (1988). This later work suggests that while relative performance is an important component of optimal labor contracts, in most cases a contract based on relative performance alone is not optimal.

In this paper we study the optimal structure of tournaments. We do not address the question of whether tournaments are optimal in the class of all possible compensation schemes, taking as given the fact that tournaments are commonly observed. Instead, we ask what the prize structure of a tournament should be: What proportion of the total purse should go the winner? What proportion should go to the runner-up or the person finishing in third-place? And what, if anything, should the person finishing last get?

A central tension in the model is the trade-off between choosing very unequal payment schemes in order to encourage effort on the part of workers with the need to balance risk-sharing considerations which make such unequal schemes fairly unattractive. To illustrate these trade-offs while still retaining simplicity and tractability, we restrict attention to cases with small numbers of workers competing against one another. Specifically, two, three, or four identical workers compete in unobservable effort for some fixed purse that is split into shares based on an ordinal ranking of the observable performances of the workers. We assume that, due to some form of limited liability, the employer cannot levy monetary penalties against workers who are ranked low.

In this framework, we show that, regardless of risk preferences, winner-take-all tournaments are optimal for two or three competing workers (Propositions 1 and 2, respectively). In the case of four workers, however, whether winner-take-all tournaments are optimal or not depends on risk preferences of the workers. Specifically, we show that, regardless of risk preferences, paying only the winner and the runner-up is optimal and that the winner's share is greater than the runner-up's share. (Proposition 3). In the case of risk-neutral workers, the share differential is extreme and winner-take-all is once again optimal (Proposition 4). Interestingly, all of the above results do not depend on the particular form of the cost of effort.

We also show that winner-take-all results are sensitive to the form of the relative performance scheme being employed. An alternative to a rank-order tournament is an *elimination tournament* first studied by Rosen (1986). With four workers this consists of two round one "matches" each of which consists of a two worker tournament. The two winners of the first round matches are then pitted against one another in a second round tournament. The winner of the second round is obviously the overall winner. The loser of the second round is the overall runner-up and the two losers from the first round share third place. In such set-up, we show that, as in the four worker rank-order tournament, paying only the winner and the runner-up is optimal (Proposition 5). With risk neutral workers, however, the two formats differ: the optimal elimination tournament need not be winner-take-all whereas the optimal rank-order tournament is.

Thus, our paper indicates that both risk sharing considerations as well as the structure of the relative compensation scheme itself can lead to outcomes in which the winner-take-all principle is not optimal.

There is an extensive literature on the incentive properties of relative compensation schemes (see Lazear (1995) and McLaughlin (1988) for surveys) that is closely related to our model. In particular, our model retains most of the features of the tournament schemes considered in the literature, but differs significantly in the following way: We require that prize structures introduce limited liability as a constraint on the set of contracts which may be offered by firms. Perhaps not surprisingly, the upshot of this constraint is that, in contrast to Lazear and Rosen (1981) where optimal rank order tournaments induce first-best effort levels on the part of risk-neutral workers, we find that optimal tournaments induce *less* than first-best effort levels with limited liability.

A second difference, mentioned above, is that we are interested only in the optimal prize structure of tournaments and not whether tournaments are optimal in the class of all labor contracts. Less significantly, our paper also differs from the existing literature in its focus on optimal prize structures for a *given* purse and in examining tournaments without assuming that competitive labor markets force firms to zero profits.

The remainder of the paper proceeds as follows: Section 2 outlines the model and introduces some notation on order statistics. In section 3, we characterize symmetric equilibria in an n worker tournament and use this characterization to deduce an optimal prize structure with two, three, and four workers. Section 4 then examines whether the winner-take-all results of the previous sections extend to a relative compensation scheme in which workers participate in a single elimination tournament. Finally, section 5 concludes.

2 Preliminaries

We examine the Lazear and Rosen (1981) model of rank-order tournaments in a setting where a single firm employs n identical workers. Each worker chooses an effort level e_i which results in publicly observable output $Q_i = e_i + X_i$ where X_i is a noise term distributed according to $F(\cdot)$ with associated density $f(\cdot)$ that is continuous on the support of F, the upper end of which is denoted by $a \leq \infty$.

Each worker's noise term, X_i is identically and independently distributed. We assume that $F(\cdot)$ is a symmetric distribution; that is, $F(x) = 1 - F(-x)$ for all x. It follows that the density function, $f(\cdot)$, then satisfies $f(x) = f(-x)$. We also assume that f is unimodal and satisfies $xf'(x) \leq 0$.

A rank-order tournament with total purse V and a prize structure $\alpha_1, \alpha_2, ..., \alpha_n$ (where $\sum_{i=1}^{n} \alpha_i = 1$) is an incentive scheme in which only relative performance matters. The worker producing the highest output receives $\alpha_1 V$; likewise, the worker with the second highest output receives the prize $\alpha_2 V$, and so on. Throughout the paper, we shall focus on optimal tournaments for a fixed purse.

Workers are assumed to be identical and have vNM utility functions of the form:

$$u(w) - C(e)$$

where w denotes monetary wealth and e denotes effort. We make the standard assumptions that $u' > 0$, $u'' \leq 0$, $C(0) = 0, C' > 0$, $C'' > 0$.

In contrast to most work in this area, we also assume that all workers begin with no wealth and that workers have limited liability to firms. Thus, tournament prize structures are constrained to offer *non-negative* prizes for all ranks. Formally, for fixed V, the set of possible tournaments is

$$\Delta = \left\{ \boldsymbol{\alpha} \in \mathbf{R}^n : \alpha_i \geq 0, \sum \alpha_i = 1 \right\}.$$

2.1 Order Statistics

In order to analyze equilibria of the tournament and its properties it is necessary to introduce some notation and terminology about order statistics.

Suppose $X_1, X_2, ..., X_m$ are identically and independently distributed according to the distribution function F with associated density f. Let $Y_i^{(m)}$ denote the ith order statistic of m variables. (For instance, $Y_1^{(m)}$ is the highest of $X_1, X_2, ..., X_m$, $Y_2^{(m)}$ is the second-highest, etc.)

Let $F_i^{(m)}$ denote the distribution of $Y_i^{(m)}$ with associated density $f_i^{(m)}$. We know that:

$$f_i^{(m)}(x) = m\binom{m-1}{i-1}(1-F(x))^{i-1}(F(x))^{m-i} f(x). \tag{1}$$

We adopt the convention that $f_{m+1}^{(m)} \equiv 0$ and $f_0^{(m)} \equiv 0$.

If F is symmetric then from (1) it follows easily that

$$f_i^{(m)}(-x) = f_{m-i+1}^{(m)}(x). \tag{2}$$

3 Rank-Order Tournaments

Suppose all workers except 1 exert the effort level $\overline{e}$. The probability that worker 1 with $X_1 = x$ will finish in rank i when she exerts effort e is:

$$\begin{aligned} \Pr[\text{rank } i \mid x] &= \Pr[\overline{e} + Y_i^{(n-1)} < e + x < \overline{e} + Y_{i-1}^{(n-1)}] \\ &= \Pr[Y_i^{(n-1)} < e + x - \overline{e} < Y_{i-1}^{(n-1)}] \\ &= F_i^{(n-1)}(e + x - \overline{e}) - F_{i-1}^{(n-1)}(e + x - \overline{e}) \end{aligned} \tag{3}$$

where we adopt the convention that $F_0^{(n-1)} = 0$ and $F_n^{(n-1)} = 1$.

Thus the expected utility of worker 1 when she exerts effort e is:

$$U(e,\overline{e}) = \sum_{i=1}^{n} u(\alpha_i V) \int_{-\infty}^{\infty} \left[F_i^{(n-1)}(e + x - \overline{e}) - F_{i-1}^{(n-1)}(e + x - \overline{e}) \right] f(x)\, dx - C(e).$$

Maximizing U with respect to e results in the first-order condition:

$$C'(e) = \sum_{i=1}^{n} u(\alpha_i V) \int_{-\infty}^{\infty} \left[f_i^{(n-1)}(e + x - \overline{e}) - f_{i-1}^{(n-1)}(e + x - \overline{e}) \right] f(x)\, dx.$$

At a symmetric equilibrium, the optimal $e = \overline{e}$ and thus, a necessary condition for $\overline{e}$ to be a symmetric equilibrium is:

$$C'(e) = \sum_{i=1}^{n} u(\alpha_i V) \int_{-\infty}^{\infty} \left[f_i^{(n-1)}(x) - f_{i-1}^{(n-1)}(x) \right] f(x)\, dx \qquad (4)$$

recalling our convention that $f_n^{(n-1)}(x) = f_0^{(n-1)}(x) = 0$.

The existence of a symmetric pure-strategy equilibrium is guaranteed if $U(\cdot, \overline{e})$ is concave. Concavity holds if F is "dispersed enough" since the first term in the expression for $U(e, \overline{e})$ then becomes negligible. In this paper, we will assume that this holds.

It is useful to define

$$\beta_i = \int_{-\infty}^{\infty} \left[f_i^{(n-1)}(x) - f_{i-1}^{(n-1)}(x) \right] f(x)\, dx$$

as the coefficient of $u(\alpha_i V)$ in (4). The coefficient β_i measures the marginal change in the ex-ante probability of being in rank i induced by an increase in effort. To see this, notice that an increase in effort induces a change in the probabilities of the various ranks and from (3),

$$\frac{\partial}{\partial e} \Pr[\text{rank } i \mid x] = f_i^{(n-1)}(e + x - \overline{e}) - f_{i-1}^{(n-1)}(e + x - \overline{e})$$

and thus we express β_i as:

$$\beta_i = \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial e} \Pr\left[\text{rank } i \mid x\right] \bigg|_{e=\bar{e}} \right] f(x)\, dx.$$

It is apparent that an increase in effort unambiguously improves the chances of finishing first and decreases the chances of finishing last. Formally,

$$\beta_1 = \int_{-\infty}^{\infty} f_1^{(n-1)}(x)\, f(x)\, dx > 0$$

whereas

$$\begin{aligned} \beta_n &= -\int_{-\infty}^{\infty} f_{n-1}^{(n-1)}(x)\, f(x)\, dx \\ &= -\int_{-\infty}^{\infty} f_1^{(n-1)}(x)\, f(x)\, dx \\ &= -\beta_1 \\ &< 0 \end{aligned}$$

where we have used (2).

Since the firm's objective is to maximize e and since $C''(e) > 0$, the firm will choose $\boldsymbol{\alpha} \in \Delta$ to maximize the right-hand side of (4). Thus, in the optimal tournament $\alpha_n = 0$; that is, the worker who finishes last gets no reward. This immediately implies

Proposition 1 *With two workers the optimal tournament is winner-take-all.*

In the case of two workers, risk-sharing considerations in determining the prize structure are effectively absent. Due to limited liability, all tournament payment schemes satisfy individual rationality and since offering payments to last-finishing workers simply undermines effort incentives, it is clear that these incentives dominate regardless of risk preferences.

The fact that individual rationality constraints are not binding on workers with limited liability plays a crucial role in affecting the effort incentives of an optimal tournament. Specifically, limited liability constraints do not permit the firm to levy penalties on workers; however, with risk-neutral workers, the fact that $\beta_n < 0$ implies that a profit maximizing firm would wish to assess a penalty against the worker who finishes last. Indeed, the results of Lazear and Rosen (1981) that rank order tournaments yield first-best effort levels when firms employ two risk-neutral workers rely essentially on firms' ability to do this. The imposition of limited liability can result in such tournaments yielding efforts below first-best levels even when firms are allowed to choose the prize structure and the purse.

This is easily illustrated in the following example. Suppose that there are two risk-neutral workers, $C(e) = \frac{1}{3}e^3$, and that noise is distributed uniformly on $[-1, 1]$. One can show that under limited liability the optimal purse is $V = \frac{1}{2}$. This yields equilibrium effort $e = \frac{1}{2}$ which is less than the first-best effort $(e = 1)$.

3.1 Three Worker Case

One may well surmise that the inequitable prize structures seen in the two person tournament will be mitigated by the trade-off between risk-sharing and incentive aspects of prize awards for larger numbers of workers. In the following section we examine this trade-off for the cases of three and four workers. In both of these cases, providing payments to a last finishing worker simply undermines effort incentives without providing any compensating benefit. However, there is a trade-off in maximizing the effort incentives of workers not to finish last by balancing the prizes offered to higher ranked workers versus the disincentive effects such balancing has on the incentives of workers to finish first.

The case of three workers is the simplest setting in which this tension might arise. In this case,

$$\begin{aligned}
\beta_2 &= \int_{-\infty}^{\infty} \left[f_2^{(2)}(x) - f_1^{(2)}(x) \right] f(x)\,dx \\
&= \int_{-\infty}^{\infty} \left[2(1 - F(x)) f(x) - 2F(x) f(x) \right] f(x)\,dx \\
&= \int_{0}^{\infty} \left[2(1 - F(-x)) - 2F(-x) \right] (f(-x))^2\,dx \\
&\quad + \int_{0}^{\infty} \left[2(1 - F(x)) - 2F(x) \right] (f(x))^2\,dx \\
&= \int_{0}^{\infty} \left[2(F(x)) - 2(1 - F(x)) \right] (f(x))^2\,dx \\
&\quad + \int_{0}^{\infty} \left[2(1 - F(x)) - 2F(x) \right] (f(x))^2\,dx \\
&= 0
\end{aligned}$$

where we have used the fact that since F is a symmetric distribution function.

Now (4) reduces to

$$C'(e) = u(\alpha V)\,\beta_1$$

and clearly in the optimal tournament $\alpha = 1$.

Proposition 2 *With three workers the optimal tournament is winner-take-*

all.

Thus, we find that for the case of three workers, the desirable incentive aspects of the winner-take-all prize structure *always* outweigh any risk-sharing considerations. Since the coefficient β_2 is zero, there is simply no gain in equilibrium effort by awarding a second prize. Risk-sharing considerations only arise from the trade-off in marginal utilities of wealth among prizes where the β coefficients are positive. Put another way, the presence of two or more positive β coefficients is a necessary condition for risk-sharing considerations to affect the prize structure of an optimal tournament.

An additional implication of this result is that, for a fixed purse V, the equilibrium effort level undertaken by each worker in a three worker tournament is identical to that undertaken in a two worker tournament.

3.2 Four Worker Case

In cases of two and three workers, risk preferences do not affect the structure of the optimal tournament. Below we show that the four worker case is the smallest number of workers in which the trade-off between risk sharing and effort incentives affects tournament prize structures.

In the case of four workers,

$$\beta_2 = \int_{-\infty}^{\infty} \left[f_2^{(3)}(x) - f_1^{(3)}(x) \right] f(x)\, dx$$

and

$$\beta_3 = \int_{-\infty}^{\infty} \left[f_3^{(3)}(x) - f_2^{(3)}(x) \right] f(x)\, dx.$$

Now using (2) we can write:

$$\int_{-\infty}^{\infty} f_1^{(3)}(x) f(x) \, dx = \int_{-\infty}^{\infty} f_3^{(3)}(x) f(x) \, dx.$$

and thus

$$\begin{aligned} \beta_3 &= \int_{-\infty}^{\infty} \left[f_3^{(3)}(x) - f_2^{(3)}(x) \right] f(x) \, dx \\ &= \int_{-\infty}^{\infty} \left[f_1^{(3)}(x) - f_2^{(3)}(x) \right] f(x) \, dx \\ &= -\beta_2. \end{aligned}$$

We now show that β_2 is non-negative.

$$\begin{aligned} \beta_2 &= -\int_{-\infty}^{\infty} \left[F_2^{(3)}(x) - F_1^{(3)}(x) \right] f'(x) \, dx \\ &= -3 \int_{-\infty}^{\infty} (1 - F(x)) (F(x))^2 f'(x) \, dx \\ &= -3 \int_{0}^{\infty} (1 - F(-x)) (F(-x))^2 f'(-x) \, dx \\ &\quad -3 \int_{0}^{\infty} (1 - F(x)) (F(x))^2 f'(x) \, dx \\ &= 3 \int_{0}^{\infty} F(x) (1 - F(x))^2 f'(x) \, dx \\ &\quad -3 \int_{0}^{\infty} (1 - F(x)) (F(x))^2 f'(x) \, dx \\ &= 3 \int_{0}^{\infty} F(x) (1 - F(x)) (1 - 2F(x)) f'(x) \, dx \\ &\geq 0, \end{aligned}$$

since for $x \geq 0$, $F(x) \geq \frac{1}{2}$ and $f'(x) \leq 0$.

This implies that $\beta_3 \leq 0$ and thus in the optimal tournament, $\alpha_3 = 0$. This implies that *with four workers, an optimal tournament pays only the winner and the runner-up.*

We now examine the relative sizes of the prizes awarded to the top two finishers. This depends on the relative magnitudes of β_1 and β_2.

$$\begin{aligned}
\beta_1 &= \int_{-\infty}^{\infty} f_1^{(3)}(x)\, f(x)\, dx \\
&= f(a) - \int_{-\infty}^{\infty} F_1^{(3)}(x)\, f'(x)\, dx \\
&= f(a) - \int_{-\infty}^{0} (F(x))^3 f'(x)\, dx - \int_{0}^{\infty} (F(x))^3 f'(x)\, dx \\
&= f(a) - \int_{0}^{\infty} (F(-x))^3 f'(-x)\, dx - \int_{0}^{\infty} (F(x))^3 f'(x)\, dx \\
&= f(a) + \int_{0}^{\infty} (1 - F(x))^3 f'(x)\, dx - \int_{0}^{\infty} (F(x))^3 f'(x)\, dx \\
&= f(a) + \int_{0}^{\infty} \left[(1 - F(x))^3 - (F(x))^3\right] f'(x)\, dx
\end{aligned}$$

Finally, we show that $\beta_1 > \beta_2$.

$$\begin{aligned}
\beta_1 - \beta_2 &= f(a) + \int_{0}^{\infty} \left[(1 - F(x))^3 - (F(x))^3\right] f'(x)\, dx \\
&\quad -3 \int_{0}^{\infty} F(x)(1 - F(x))(1 - 2F(x))\, f'(x)\, dx \\
&= f(a) - \int_{0}^{\infty} (2F(x) - 1)^3 f'(x)\, dx \\
&> 0.
\end{aligned}$$

Our findings on tournaments with four workers may be summarized as follows:

Proposition 3 *With four workers, an optimal tournament pays only the winner and the runner-up. The winner's share is greater than that of the runner up.*

The intuition for the result is the following. Consider a small increase in the effort of worker 1 holding fixed the other workers at the equilibrium level. Now, as we have shown, the marginal increase in probability of the first rank is greater than the marginal increase in probability of the second rank ($\beta_1 > \beta_2$). This is because if worker 1 increases his effort level the chances of his attaining the highest rank are determined by the likelihood of some *other* worker's noise component being large. Since $f'(x) < 0$ for $x > 0$ this likelihood is smaller for the first rank than for the second rank. Thus an increase in effort has a larger effect on worker 1's chances of attaining the first rank than the second rank.

Absent risk-sharing considerations, this difference in the effort incentives arising from awarding a first versus a second prize immediately implies that all weight should be assigned to the first prize. Formally,

Proposition 4 *With four* ***risk-neutral*** *workers, an optimal tournament is winner-take-all.*

With risk-averse workers, risk-sharing considerations also affect the optimal prize structure. In this case, the optimal scheme equates the product of the β coefficients and the marginal utility for the first and second prizes, respectively.

4 Elimination Tournaments

We now compare rank-order tournaments to *elimination* tournaments (as in Rosen (1986)). Consider a setting in which four identical players compete in a two-round single elimination tournament. As usual, we let each player's performance be given by the sum of costly effort e and a random component

x which is independently drawn from the atomless distribution F for each player and each round.

The total purse for the tournament is V and α_i is the share of the purse given to the ith highest finisher. Naturally, $\alpha_1 + \alpha_2 + 2\alpha_3 = 1$. (We are assuming that the losers in the first round both get $\alpha_3 V$.)

Consider the effort choices of two contestants meeting in the *final* round of the tournament. Since this is the same as a rank order tournament with two workers we obtain that the equilibrium effort in the second round, e^{**}, is given by the condition:

$$C'\left(e^{**}\right) = u\left(\alpha_1 V\right) \int_{-\infty}^{\infty} \left(f\left(x\right)\right)^2 dx - u\left(\alpha_2 V\right) \int_{-\infty}^{\infty} \left(f\left(x\right)\right)^2 dx. \qquad (5)$$

Thus, the "indirect utility" from the second-round equilibrium as a function of the shares is:

$$W\left(\alpha_1, \alpha_2\right) = \frac{1}{2} u\left(\alpha_1 V\right) + \frac{1}{2} u\left(\alpha_2 V\right) - C\left(e^{**}\left(\alpha_1, \alpha_2\right)\right).$$

Now consider the optimal effort choices in the first round. Notice that, conditional on advancing to the finals, each contestant anticipates a surplus of $W\left(\alpha_1, \alpha_2\right)$. Thus, given an effort choice $\widehat{e}$ in the first round by her opponent, player 1 chooses e to maximize:

$$\begin{aligned} U\left(e, \widehat{e}\right) &= W\left(\alpha_1, \alpha_2\right) \int_{-\infty}^{\infty} F\left(e + x - \widehat{e}\right) f\left(x\right) dx \\ &\quad + u\left(\alpha_3 V\right) \int_{-\infty}^{\infty} \left(1 - F\left(e + x - \widehat{e}\right)\right) f\left(x\right) dx - C\left(e\right) \end{aligned}$$

Again, differentiating and imposing symmetry, we obtain that the equilibrium effort level in the first round e^* is given by:

$$C'(e^*) = W(\alpha_1, \alpha_2) \int_{-\infty}^{\infty} (f(x))^2 \, dx - u(\alpha_3 V) \int_{-\infty}^{\infty} (f(x))^2 \, dx \tag{6}$$

Suppose that the firm's objective is to maximize the total expected output (or equivalently, to maximize the average expected output per worker). The firm wishes to choose α_1, α_2 and α_3 to maximize

$$4e^* + 2e^{**}$$

Since the coefficient of $u(\alpha_3 V)$ in (6) is negative it is clear that in the optimal elimination tournament, $\alpha_3 = 0$. Thus we obtain:

Proposition 5 *With four workers, an optimal elimination tournament pays only the winner and the first runner-up.*

In examining the prize structure of an optimal elimination tournament, it is useful to highlight the two effects that the second prize has on equilibrium effort. In the final round of the elimination tournament, we are in a situation identical to a two-worker rank-order tournament. Consequently, offering a second prize unambiguously decreases effort incentives in the round regardless of risk preferences. At the same time, offering rewards to second place finishers increases the surplus, $W(\alpha_1, \alpha_2)$, associated with winning the first round of the elimination tournament. Obviously, equilibrium effort in the first round is increasing in the size of the expected surplus; thus, offering larger second prizes increases equilibrium effort in the first round of the tournament regardless of risk preferences. Since the firm cares about overall effort in both rounds of the tournament, it might be willing to trade-off decreased effort in the second round for increased effort in the first. In the following example, we establish that this trade-off can indeed result in the award of a second prize; that is, winner-take-all elimination tournaments need not be optimal, even with risk-neutral workers.

Example Suppose that $V = 1$, $C(e) = e^2$ and that F is distributed uniformly on $\left[-\frac{3}{8}, \frac{3}{8}\right]$. From Proposition 5, we know that setting $\alpha_3 = 0$ is optimal; hence we restrict attention to schemes where $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$. It is easy to see that in the optimal tournament the winner's share must exceed the runner-up's share and so we suppose $\alpha \geq \frac{1}{2}$. Thus, in the second round of the tournament, (5) reduces to

$$2e^{**} = \frac{4}{3}(2\alpha - 1),$$

and it can be verified that this constitutes an equilibrium. Thus the indirect utility from the second round equilibrium is:

$$W(\alpha) = \frac{1}{2} - \left(\frac{2}{3}(2\alpha - 1)\right)^2.$$

Thus, (6) reduces to:

$$2e^{*} = \frac{4}{3}\left(\frac{1}{2} - \left(\frac{2}{3}(2\alpha - 1)\right)^2\right).$$

Suppose that the firm wishes to maximize total effort for a given purse V. Since there are two first round matches and one second round match, then the total effort expended in the tournament is

$$E(\alpha) = 4e^{*}(\alpha) + 2e^{**}(\alpha).$$

Substituting for e^* and e^{**} yields

$$E(\alpha) = \frac{4}{3}(2\alpha - 1) + \frac{8}{3}\left(\frac{1}{2} - \left(\frac{2}{3}(2\alpha - 1)\right)^2\right).$$

The value of α maximizing this expression is

$$\alpha = \frac{25}{32};$$

thus we have shown that a winner-take-all elimination tournament is not optimal.

5 Conclusion

In their controversial book, *The Winner-Take-All Society*, Frank and Cook (1995) argue that the dramatic rise income inequality in the US over the last twenty years is, at least in part, attributable to the proliferation of "winner-take-all" labor markets; i.e., markets where compensation is determined by relative performance. Frank and Cook conjecture that technological advances in assessing relative performance have led to the observed proliferation of these types of labor markets in fields such as law, education, banking, and technology as well as their continued use in markets such as entertainment and sports. While reductions in the difficulty of monitoring relative performance may indeed make the use of tournament schemes more prevalent, it does not explain the *winner-take-all* payoff structure of these schemes. Indeed, for the proliferation of markets where rewards are determined by relative compensation to be the source of growing income inequality requires the *optimality* of the winner-take-all principle.

Our results on this issue depend on the definition of the winner-take-all principle one chooses to adopt. If one takes "winner-take-all" to literally mean that the highest ranked worker receives the entire purse, then our results show that the winner-take-all principle is not generally optimal. In the case of four workers, balancing risk-sharing considerations or the use of an elimination tournament require that the purse be split among the top two workers. Nonetheless, a weaker form of the winner-take-all principle does seem to hold: If one views winner-take-all as meaning that only the top half of the field is compensated and that higher finishers are paid more than lower finishers, then our results are entirely consistent with the optimality of the winner-take-all principle. In light of this, one may well conjecture that the weak form of the winner-take-all principle holds generally for the n worker case. Investigation of this conjecture remains for future research.

6 References

Bhattacharya, Sudipto and L. Guasch (1988), "Heterogeneity, Tournaments, and Hierarchies." *Journal of Political Economy* 96 : 867-881.

Frank, Robert and Philip Cook (1995), *The Winner-Take-All Society,* New York: The Free Press.

Green, Jerry and Nancy Stokey (1983), "A Comparison of Tournaments and Contracts." *Journal of Political Economy* 91 (June): 349-364.

Holmstrom, Bengt (1982), "Moral Hazard in Teams." *Bell Journal of Economics* 13: 324-340.

Lazear, Edward (1995), *Personnel Economics*, Cambridge: MIT Press.

Lazear, Edward. and Sherwin Rosen (1981), "Rank-Order Tournaments as Optimum Labor Contracts." *Journal of Political Economy* 89 (Oct.): 841-864.

McLaughlin, Kenneth (1988), "Aspects of Tournament Models: A Survey." in *Research in Labor Economics,* Vol. 9, Greenwich: JAI Press.

Mookherjee, Dilip (1984), "Optimal Incentive Schemes with Many Agents." *Review of Economic Studies* 51: 433-446.

Nalebuff, Barry. and Joseph Stiglitz (1983), "Prizes and Incentives: Towards a General Theory of Compensation and Competition." *Bell Journal of Economics* 14 (Spring): 21-43.

Rosen, Sherwin (1986), "Prizes and Incentives in Elimination Tournaments." *American Economic Review* 76 (Sept.): 701-715.