**Spring 2006** **Steve Tadelis**

# Econ 206 - Problem Set 3

Due 05/04/06

## Question 1

An Entrepreneur ($E$) has a project which needs an initial investment of $k$. The project's random output, $\widetilde{x} \geqslant 0$ depends on $E$'s choice of effort as follows: $\widetilde{x} = e\widetilde{y}$, where $e \geqslant 0$ is $E$'s choice of effort and $\widetilde{y}$ is a random variable distributed uniformly on $[0, 2]$. $E$'s private cost of effort is $g(e) = \frac{1}{2}e^2$, and this effort is unobservable, while the output is observable and verifiable. $E$ makes a take-it-or-leave-it offer to an investor ($I$). Assume that E has no starting capital, so that $I$ will have to pay the start-up cost $k$. The contract also specifies the sharing rule of output, so that when output is $x$, $E$ leaves $w(x)$ to himself and the rest goes to $I$. $E$ has limited liability which constrains $w(x) \geqslant 0$ for all $x$. Both parties are risk neutral, and the market interest rate is normalized to zero.

**(a)** Suppose that $E$'s effort is observable and verifiable. Solve for the first-best level of effort. For what values of $k$ would the project be worth undertaking if there were no moral hazard?

**(b)** From now on assume that the project is worth undertaking. Show that despite the unobservability of effort by $E$, an optimal contract implements the first-best level of effort. Hint: Consider contracts of the form

$$w(x) = \begin{cases} 0 & \text{if } x < a \\ \overline{w} & \text{if } x \geqslant a \end{cases}$$

**(c)** Now suppose that before output is observed by outsiders (including the principal), $E$ can destroy or borrow output at no cost. What restrictions does this impose on the contract? Solve for the form of the optimal contract in this case. (Hint: You can use integration by parts to express the objective function and constraints through $w(0)$ and $w'(\cdot)$.)

**(d)** Interpret the resulting contract in financial terms.

## Question 2

A Principal has 2 agents, who simultaneously choose one of two possible efforts, high ($H$) or low ($L$). The efforts taken are unobservable by the principal. Each agent $i$ produces output $x_i \in \{0, 1\}$. If both agents choose $H$, then $(x_1, x_2) = (0, 0)$ with probability $\frac{1}{4}$, and $(x_1, x_2) = (1, 1)$ with probability $\frac{3}{4}$. If both agents choose $L$, then $(x_1, x_2) = (0, 0)$ with probability $\frac{3}{4}$, and $(x_1, x_2) = (1, 1)$ with probability $\frac{1}{4}$ .Finally, if agent $i$ chooses $H$ and agent $j$ chooses $L$, then $(x_1, x_2) = (0, 1)$ with probability $\frac{1}{2}$, and $(x_1, x_2) = (1, 0)$ with probability $\frac{1}{2}$. Each agent's expected utility is $Eu(\widetilde{w}) - g(e)$ where $\widetilde{w}$ is a (possible) random income, $u(\cdot)$ is a concave vNM utility function whose range is all the real numbers, and $g(e)$ is the cost of effort $e$, with $g(H) > g(L)$.

**(a)** Suppose the principal offers the agents a compensation scheme of the form $v_1(x_1, x_2)$, $v_2(x_1, x_2)$, where $v_i$ is the utility-equivalent payoff to agent $i$ when the realization of outcomes is $(x_1, x_2)$. Find the least-cost scheme that implements $(H, H)$ as a Nash equilibrium of the game in which the agents choose actions simultaneously.

**(b)** Show that for any least-cost compensation scheme $v_1(x_1, x_2)$, $v_2(x_1, x_2)$ that implements $(H, H)$ as a Nash equilibrium, there is another Nash equilibrium that which Pareto dominates $(H, H)$ from the agent's point of view.

**(c)** Let agent 1 send a message to the principal, simultaneously with his action choice, and let the agents' compensation depend on this message, as well as on the realization of outputs. Design a compensation scheme of this kind where both agents choose $H$ in the *unique* Nash equilibrium, and which has the same cost as the scheme in (a) above. (Hint: you can consider agent 1's message to take on only two possible values.)

## Question 3

Consider an agency problem where the agent has two tasks: $a_1$ is his effort which affects the productivity of an asset and $a_2$ is his effort which affects the value of the asset. The agent is indifferent to the allocation of his effort, and given total effort level $a = a_1 + a_2$, the agent's cost of effort is $c(a) = \frac{a^2}{2} - 4a$ (therefore, if the agent is willing to exert an effort level of $\hat{a}$, he will be willing to follow the principal's request of allocating $\hat{a}$ between $a_1$ and $a_2$). Given a wage of $w$ and an effort level of $a$, the agent's utility is given by $u(w, a) = -e^{-\frac{1}{2}[w - c(a)]}$. The agent's reservation utility has a certainty equivalent of zero. (throughout this question you can restrict yourself to linear contracts by assuming that $s_1$ is a result of a Brownian motion as in Holmström-Milgrom 87).

**a)** Assume that the principal's expected value of efforts $(a_1, a_2)$ is given by $b(a_1, a_2) = a_1^{\frac{1}{4}} \cdot a_2^{\frac{1}{2}}$. However, the only measurable variable is a noisy signal $s_1 = a_1 + \epsilon_1$, where $\epsilon_1 \sim N(0, 1)$. Find the optimal contract which the principal will choose.

**b)** Now assume that the principal's expected value of efforts $(a_1, a_2)$ is given by $b + v$, where $b = 18 \ln a_1$ is the expected value of the output, and $v = 6 \ln(1 + a_2)$ is the expected change in the value of the asset. The actual change in the value of the asset is random and is given by $\tilde{v} = v + \epsilon_2$, where $\epsilon_2 \sim N(0, \sigma^2)$. Once again, the only verifiable signal is $s_1$ as described in part **a)** above. Find the optimal contract which the principal will choose.

**c)** Observe the slope of the incentive scheme you calculated in **b)** above. Does this violate everything you have learned about optimal schemes?

**d)** Briefly compare your result in **b)** above to the results in Holmström-Milgrom 1991 (JLEO).