Economics 160
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Stanford University
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Notes for Section: Week 3

Notes prepared by Paul Riskind (pnr@stanford.edu). Let me know if you spot errors or have questions about these notes.

1 Questions and Comments

You're always welcome to ask questions in section, during my office hours, or by email. If you want to talk but can't attend my office hours, you're welcome to schedule an appointment at another time. If I don't know the answer to your question off the top of my head, I'll try to figure it out and get back to you.

Help me improve my teaching by offering feedback. If you have suggestions for improving section, you may talk to me, email me, or leave an anonymous note in my mailbox in the economics department (on the second floor, near the stairwell).

2 Symmetric Games Can Have Asymmetric Equilibria

In office hours (and apparently in answers to problem set 2) several students suggested that Nash equilibria of games with symmetric payoff functions must be symmetric. You've already seen an example, however, of a game with symmetric payoff functions and asymmetric Nash equilibria: the game of "split the sushi" in problem set 2. Moreover, even the weaker statement that games with symmetric payoffs always have a symmetric equilibrium is false. We can provide a counter-example by modifying "split the sushi" to rule out symmetric equilibria.

Bottom line: Since games with symmetric payoff functions can have asymmetric equilibria, you should not assume symmetric solutions for such games.

3 Mixed Strategy Using Undominated Pure Strategies Can Be Strictly Dominated

Suppose σ_i is a mixed strategy that assigns positive probability to some strategy, s_i, that's strictly dominated by $\widehat{s}_i$. It's easy to show that σ_i is strictly dominated by the mixed strategy $\widehat{\sigma}_i$ obtained by modifying σ_i so that i plays $\widehat{s}_i$ whenever σ_i would have i play s_i. So any mixed strategy in which you play a strictly dominated strategy with positive probability is strictly dominated.

What if instead, the support of σ_i contains only strategies that aren't (weakly or strictly) dominated? Can we say that σ_i isn't dominated, either? Surprisingly, the answer is NO.

The following game (from Fudenberg and Tirole, p. 7) shows that a mixed strategy can be strictly dominated even if none of the strategies in its support is even weakly dominated.

		You	
		L	R
	U	$1,3$	$-2,0$
Your Dog	M	$-2,0$	$1,3$
	D	$0,1$	$0,1$

As usual, denote mixed strategies by vectors of probabilities. Clearly, neither U nor M is a strictly or weakly dominated strategy for your dog. Nonetheless, D strictly dominates the mixed strategy $\sigma_D = (1/2, 1/2, 0)$, in which your dog only plays undominated strategies!

In this game, a pure strategy dominates a (nontrivially) mixed strategy. We'll see examples later of mixed strategies that dominate pure strategies.

4 Mixed Strategies Can Change The Set of Rationalizable Strategies

Recall the idea behind rationalizability: A strategy is rationalizable if it's a best response given a reasonable belief you have about how the other players will play. We can derive the set of rationalizable strategies in a game by iteratively removing any strategies that cannot be best responses to any profiles of the other players' remaining strategies.

I want to highlight a point made at the end of Tuesday's class. A pure strategy may be a best response to some mixed strategies even though it was never a best response to any pure strategy. Thus, **allowing mixed strategies may change the set of rationalizable strategies**. (Of course, mixed strategies can contract as well as expand the set of rationalizable strategies.) Here's an example to illustrate this point:

			You	
		L	M	R
Your Dog	T	$4,6$	$0,0$	$2,5$
	B	$0,0$	$4,6$	$2,5$

Your R strategy isn't a best response to either of your dog's pure strategies – L is better against T, and M is better against B. Accordingly, R cannot be rationalizable if only pure strategies are allowed in this game. But against many mixed strategies, including $\sigma_D = (0.5, 0.5)$, R is your best response. And since you make your dog indifferent between his pure strategies when you play R, your dog is willing to use mixed strategies because either T or B is a best response for him. Thus, the strategy profile (σ_D, R) is a Nash equilibrium, and since all Nash equilibrium strategies are rationalizable, R must be rationalizable. When we allow mixed strategies, R is a best response to mixed strategies that your dog could rationally play. We cannot rule out R using rationalizability as we could when we only allowed pure strategies.

In our next example, we'll see that mixed strategies also can change the set of strategies that survive IESDS.

5 Finding Nash Equilibrium With Mixed Strategies

In the next two examples, we'll use two common tricks for finding Nash equilibria in mixed strategies.

5.1 Example 1: Using Strict Dominance

Let's find all Nash equilibria – including equilibria in mixed strategies – of the following game (adapted from Watson, p. 107):

		You		
		L	M	R
Your Dog	U	$8, 3$	$3, 5$	$6, 3$
	C	$3, 3$	$5, 5$	$4, 8$
	D	$5, 2$	$3, 7$	$4, 9$

Finding all mixed strategy equilibria of a 3x3 game would be tedious without a shortcut. Fortunately, we can use iterated elimination of strictly dominated strategies (IESDS) to simplify the game. We can use IESDS when finding Nash equilibria in mixed strategies because the pure strategies eliminated by IESDS can never be played with any positive probability in a Nash equilibrium. If at some stage of IESDS, s_i strictly dominates $\widetilde{s}_i$, then in a Nash equilibrium we can never place any positive probability on $\widetilde{s}_i$ because player i has some strategy remaining that *always* generates a higher payoff than $\widetilde{s}_i$. Player i could increase her payoff by never playing $\widetilde{s}_i$ and increasing the probability with which she plays s_i or some other strategy that always beats $\widetilde{s}_i$.

In the first round of IESDS, we eliminate your L strategy since M strictly dominates L. The game becomes:

		You	
		M	R
	U	$3,5$	$6,3$
Your Dog	C	$5,5$	$4,8$
	D	$3,7$	$4,9$

The second round of IESDS is harder. None of the remaining pure strategies for either player strictly dominates any of the others. We can, however, use mixed strategies to eliminate your dog's D strategy. Suppose your dog plays both U and C with strictly positive probability, but not D, so his mixed strategy takes the form $\sigma_D = (p, 1-p, 0)$, where $p \in (0,1)$. His payoffs from this strategy are:

$$
\begin{aligned}
u_D(\sigma_D, M) &= 3p + 5(1-p) = 5 - 2p > 3 = u_D(D, M) \\
u_D(\sigma_D, R) &= 6p + 4(1-p) = 4 + 2p > 4 = u_D(D, R)
\end{aligned}
$$

Thus, the mixed strategy σ_D strictly dominates D and we can eliminate D. The game becomes:

		You	
		M	R
Your Dog	U	$3,5$	$6,3$
	C	$5,5$	$4,8$

We can no longer eliminate strategies using IESDS. Note that adding mixed strategies has changed the set of strategy profiles that survive IESDS. Without mixed strategies, we could not have eliminated your dog's D strategy as we just did.

We next check for Nash equilibria in pure strategies, and find none.

We now look for equilibria in mixed strategies that use only the remaining pure strategies with positive probability. Suppose your dog plays $\sigma_D = (p, 1-p)$, and you play $\sigma_Y = (q, 1-q)$, where $p, q \in (0,1)$. We need to find the value of p that makes you indifferent between playing M and R, and the value of q that makes your dog indifferent between playing U and C.

The payoffs for your remaining pure strategies are:

$$
\begin{aligned}
u_Y(\sigma_D, M) &= 5 \\
u_Y(\sigma_D, R) &= 3p + 8(1-p) = 8 - 5p
\end{aligned}
$$

The payoffs to M and R are equal and you're indifferent between the two when $p = 0.6$.

The payoffs to your dog's remaining pure strategies are:

$$
\begin{aligned}
u_D(U, \sigma_Y) &= 3q + 6(1-q) = 6 - 3q \\
u_D(C, \sigma_Y) &= 5q + 4(1-q) = 4 + q
\end{aligned}
$$

The payoffs to U and C are equal when $q = 0.5$.

We conclude that the only Nash equilibrium for the original game is the pair of mixed strategies $\sigma_D = (0.6, 0.4, 0)$, $\sigma_Y = (0, 0.5, 0.5)$.

Two lessons to draw from this example: (1) mixed strategies can change the outcome of IESDS; and (2) dominance by a mixed strategy can greatly simplify our hunt for Nash equilibria.

5.2 Example 2: Finding Best Responses Graphically

Now, we'll find all Nash equilibria for the following game (taken from Osborne, p. 141):

		You		
		L	M	R
Your Dog	T	$2,2$	$0,3$	$1,3$
	B	$3,2$	$1,1$	$0,2$

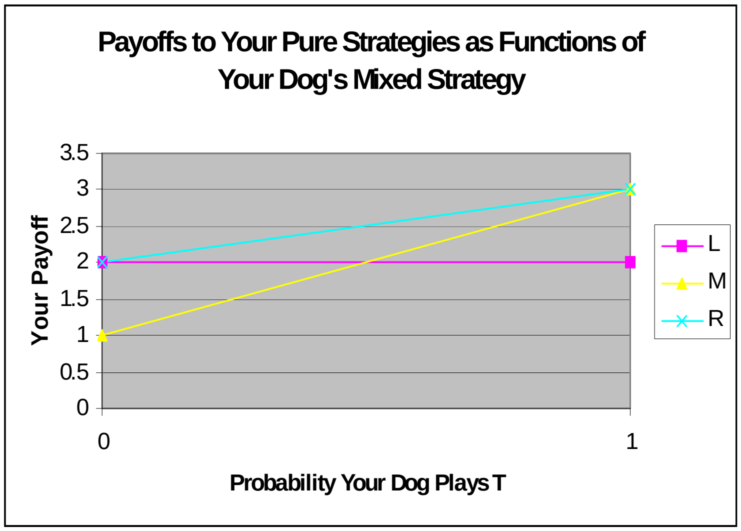
Unfortunately, we can't eliminate any strategies using strict dominance. We can quickly spot two Nash equilibria in pure strategies: (B, L) and (T, R).

Next, we have to hunt for equilibria in mixed strategies. It would be great to know what these equilibria look like. For example, will you play all your pure strategies with positive probability, or only some of them? We can get some intuition by graphing the payoffs to your pure strategies as functions of your dog's mixed strategy.

As usual, we denote mixed strategies with vectors of probabilities. Suppose your dog plays $\sigma_D = (p, 1-p)$, with $p \in [0, 1]$. The payoffs for your pure strategies are:

$$\begin{aligned} u_Y(\sigma_D, L) &= 2 \\ u_Y(\sigma_D, M) &= 3p + (1-p) = 1 + 2p \\ u_Y(\sigma_D, R) &= 3p + 2(1-p) = 2 + p \end{aligned}$$

If we graph these payoffs as functions of p, we get:



Payoffs to Your Pure Strategies

The graph demonstrates that L, R, and any mixed strategy including only L and R in its support are all best responses when $p = 0$. When $p = 1$, M, R, and any mixed strategy including only these pure strategies in its support are all best responses. When $p \in (0, 1)$, only R is a best response. So in equilibrium, you can never play all three of your strategies with positive probability. We also observe that you're only indifferent between pure strategies – and therefore willing to play a mixed strategy – when your dog plays a pure strategy. In equilibrium, your dog can never play a mixed strategy. Why? In any Nash equilibrium where your dog mixes, you must play R, but your dog is not willing to mix if you play R, because then he strictly prefers T to B.

We've found two possible classes of equilibria in mixed strategies: (1) you mix between L and R and your dog plays B; and (2) you mix between M and R and your dog plays T. To finish solving for the equilibria in mixed strategies, we need to find out which of your mixed strategies will make your dog willing to play the appropriate pure strategy.

Start with the case where you play only L and R with positive probability, so your strategy takes the form $\sigma_Y = (q, 0, 1 - q)$, where $q \in [0, 1]$. Your dog's payoffs for his pure strategies are:

$$\begin{aligned} u_D(T, \sigma_Y) &= 2q + (1 - q) = 1 + q \\ u_D(B, \sigma_Y) &= 3q \end{aligned}$$

We said you'd only be willing to mix between L and R if your dog plays B. Your dog will be willing to play B so long as $u_D(B, \sigma_Y) \geq u_D(T, \sigma_Y)$, or $3q \geq 1 + q$.

Thus, he'll play B if $q \geq 1/2$. (Note that we only need to make sure your dog doesn't strictly prefer T to B. We don't need to check whether he'd prefer some mixed strategy to B because if he weakly prefers B to T, then he also weakly prefers B to any mix between T and B. So we've shown your dog is playing a best response.) We conclude that there's one Nash equilibrium in mixed strategies where your dog plays B and you play $\sigma_Y = (q, 0, 1-q)$, with $q \in [0.5, 1]$.

In the second possible case, you play only M and R with positive probability, so your strategy takes the form $\sigma_Y = (0, r, 1-r)$, where $r \in [0, 1]$. Your dog's payoffs for his pure strategies are:

$$\begin{aligned} u_D(T, \sigma_Y) &= 1 - r \\ u_D(B, \sigma_Y) &= r \end{aligned}$$

We said you'd only be willing to mix between M and R if your dog plays T. Your dog will be willing to play T so long as $u_D(T, \sigma_Y) \geq u_D(B, \sigma_Y)$, or $1 - r \geq r$. Thus, he'll play T if $r \leq 1/2$. (Note again that we only need to make sure your dog doesn't like some other pure strategy better than T; in doing so, we also show he wouldn't strictly prefer any mixed strategy. Again, we've shown your dog is playing a best response in this strategy profile.) So there's a second Nash equilibrium in mixed strategies where your dog plays T and you play $\sigma_Y = (0, r, 1-r)$, with $r \in [0, 0.5]$.

No other Nash equilibria in mixed strategies are possible because we've ruled out all other cases.

The main lesson to take away from this example is that **when you're solving for an equilibrium in mixed strategies, graphing payoffs can be an easy way to rule out some types of mixed strategies and thereby simplify the problem**. But if the graph is complicated, you should confirm your graphical analysis algebraically.